This is a preprint of: "On a family of polynomial differential equations having at most three limit cycles", Armengol Gasull, Yulin Zhao, *Houston J. Math.*, vol. 39(1), 191–203, 2013.

ON A FAMILY OF POLYNOMIAL DIFFERENTIAL EQUATIONS HAVING AT MOST THREE LIMIT CYCLES

ARMENGOL GASULL AND YULIN ZHAO

Abstract. We prove the existence of at most three limit cycles for a family of planar polynomial differential equations. Moreover we show that this upper bound in sharp. The key point in our approach is that the differential equations of this family can be transformed into Abel differential equations.

1. Introduction and main results

Hilbert's 16th Problem has been one of the main problems in the qualitative theory of ordinary differential equations in the last century, and continues to attract widespread interest. It is concerned with the number and possible configurations of limit cycles for planar polynomial differential systems. This problem has not been solved even for the quadratic case.

In this paper we are interested in the study of the number of limit cycles of polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= x(P_{n-1}(x,y)+P_{n+2m-1}(x,y)+P_{n+3m-1}(x,y))+Q_{n+m}(x,y),\\ \dot{y} &= y(P_{n-1}(x,y)+P_{n+2m-1}(x,y)+P_{n+3m-1}(x,y))+R_{n+m}(x,y), \end{aligned} \tag{1}$$

where the dot denotes the derivative with respect to the time t, n and m are positive natural numbers and $P_k(x,y), Q_k(x,y)$ and $R_k(x,y)$ are homogeneous polynomials of degree k. We introduce the new homogeneous polynomial

$$G_{n+m+1}(x,y)=xR_{n+m}(x,y)-yQ_{n+m}(x,y),$$

and associated to it, the function g_{n+m+1} defined by $g_{n+m+1}(\theta)=G_{n+m+1}(\cos\theta,\sin\theta)$.

Our main result is:

Theorem 1. *Consider system* (1). *Then*

(a) *When $n+m$ is even it has no limit cycles.*
(b) *When $n+m$ is odd and g_{n+m+1} vanishes it has no limit cycles.*
(c) *When $n+m$ is odd, g_{n+m+1} does not vanish and P_{n+3m-1} does not change sign it has at most three limit cycles counting their multiplicities. Moreover:*
   (i) *When n is odd (and so m is even) there exist systems of the form* (1) *satisfying the above hypotheses and having exactly* 0, 1, 2 *or* 3 *limit cycles, taking into account their multiplicities.*
   (ii) *When n is even (and so m is odd), then it has at most one limit cycle and when it exists it is hyperbolic. Furthermore this upper bound is sharp.*
(d) *When $n+m$ is odd, g_{n+m+1} does not vanish and P_{n+3m-1} changes sign then:*

2000 *Mathematics Subject Classification.* Primary 34C07. Secondary: 34A34, 34C25, 37C27.
Key words and phrases. Limit cycle, polynomial differential system, Abel equation, Riccati equation.

(i) *For any n odd and m even there are systems having at least four limit cycles.*
(ii) *For any n even and m odd there are systems having at least two limit cycles.*

Notice that in the literature there are many results about non existence or uniqueness of limit cycles for planar polynomial vector fields. On the other hand in this paper we have been able to give an exact upper bound of three limit cycles for some subcases of system (1). As we will see, our proof shows that when system (1) has limit cycles then its only critical point is the origin and so, all them must surround it.

Observe also that the only situation for which the above theorem gives no an upper bound for the number of limit cycles is the case (d), namely $n+m$ is odd, g_{n+m+1} does not vanish and P_{n+3m-1} changes sign. We do not know which is the maximum number of limit cycles in this case, although, from our proof of item (d), we suspect that it might increase with n and m.

Part (c) of the above theorem extends a result of [15] where the existence of systems of the form (1) having exactly three limit cycles is established in a particular subcase which is Darboux integrable. The integrability of some similar systems has been also studied in [16, 21]. Finally observe that generically the infinity of the Poincaré compactification of system (1) is full of critical points. Systems with degenerate infinity are frequently studied in the literature, see for instance [5, 13, 20]

The above result can be refined when P_{n+3m-1} instead of not changing sign is identically zero.

Theorem 2. *Consider system* (1) *and assume that* $P_{n+3m-1}(x,y) \equiv 0$. *Then*

(a) *When $n+m$ is even it has no limit cycles.*
(b) *When $n+m$ is odd and g_{n+m+1} vanishes it has no limit cycles.*
(c) *When $n+m$ is odd, g_{n+m+1} does not vanish and $P_{n+2m-1}(x,y) \equiv 0$ it has at most one limit cycle and when it exists it is hyperbolic. Moreover:*
  (i) *When n is odd (and so m is even) there exist systems of the form* (1) *satisfying the above hypotheses and having exactly one hyperbolic limit cycle.*
  (ii) *When n is even (and so m is odd) it has no limit cycles.*
(d) *When $n+m$ is odd, g_{n+m+1} does not vanish and $P_{n+2m-1}(x,y) \not\equiv 0$ it has at most two limit cycles counting their multiplicities. Moreover:*
  (i) *When n is odd (and so m is even) there exist systems of the form* (1) *satisfying the above hypotheses and having exactly* 0, 1 *or* 2 *limit cycles, taking into account their multiplicities.*
  (ii) *When n is even (and so m is odd), then indeed it has at most one limit cycle and when it exists it is hyperbolic.*

A key point for proving the above theorems is that when system (1) can have limit cycles, it can be written in some coordinates as an Abel differential equation,

$$\frac{dR}{d\theta} = A_0(\theta) + A_1(\theta)R + A_2(\theta)R^2 + A_3(\theta)R^3. \tag{2}$$

Moreover, if $P_{n+3m-1}(x,y) \equiv 0$ then $A_3(\theta) \equiv 0$ and this Abel differential equation is indeed a Riccati differential equation. The same idea has already been used in several papers, see for instance [4, 6, 9, 12, 15, 14, 17]. As we will see, a main difference between our situation and these papers is that in our Riccati or Abel equations the function A_0 is not necessarily identically zero. This fact provokes that while most of the results of these papers prove the existence of at most two limit cycles for the planar differential equation, we will get a result of existence of at most three limit cycles.

2. Proof of Theorem 1

To prove both theorems we introduce some notations. In polar coordinates (r, θ), defined by $x = r\cos\theta,\ y = r\sin\theta$, system (1) becomes

$$\begin{aligned} \dot r &= f_{n+1}(\theta) r^n + f_{n+m+1}(\theta) r^{n+m} + f_{n+2m+1}(\theta) r^{n+2m} + f_{n+3m+1}(\theta) r^{n+3m}, \\ \dot\theta &= g_{n+m+1}(\theta) r^{n+m-1}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} f_{n+m+1}(\theta) &= \cos\theta Q_{n+m}(\cos\theta,\sin\theta) + \sin\theta R_{n+m}(\cos\theta,\sin\theta), \\ f_{k+1}(\theta) &= P_{k-1}(\cos\theta,\sin\theta),\ k = n,\ n+2m,\ n+3m. \end{aligned} \tag{4}$$

By using the new variable $R = r^m$, when $g_{n+m+1}(\theta) \neq 0$, equation (3) can be written as the Abel equation

$$\frac{dR}{d\theta} = \frac{m}{g_{n+m+1}(\theta)} (f_{n+1}(\theta) + f_{n+m+1}(\theta) R + f_{n+2m+1}(\theta) R^2 + f_{n+3m+1}(\theta) R^3). \tag{5}$$

In the rest of this paper we will denote by $R(\theta, R_0)$ the solution of the Abel equation (5) with the initial condition $R(0, R_0) = R_0$. Notice that, under the above changes of variables, a limit cycle of equation (1) is transformed into a solution of the Abel equation starting at a $R = R_0^* > 0$ and satisfying $R(2\pi, R_0^*) = R_0^*$, which is isolated in the set of solutions satisfying this property. We will also call this kind of solutions, limit cycles of the Abel equation. It is clear that the multiplicity of R_0^* as a zero of the map $R_0 \to R(2\pi, R_0) - R_0$, coincides with the multiplicity of the associated solution of system (5) as a limit cycle of the system.

Proof of Theorem 1.

(a)-(b) By using equation (3) we can deduce that, apart from the origin, all the critical points of (1) are located on the straight lines $\{\theta = \bar\theta\}$, where $\bar\theta$ ranges over all the solutions of the equation $g_{n+m+1}(\theta) = 0$. Clearly, if some of these solutions $\theta = \bar\theta$ exists then this line is invariant by the flow of (1) and no limit cycles of the differential equation can exist. So the proof of item (b) follows. The proof of item (a) is a consequence of the same fact, because when $n+m$ is even the equation $g_{n+m+1}(\theta) = 0$ has always real solutions.

(c) Note that in this case the only singularity of (1) is the origin. Firstly we will prove that equation (1) has at most three limit cycles, taking into account their multiplicities.

We will use the following result, proved in [12]: Let equation (2) be an Abel differential equation with $A_i(\theta)$, for $i = 0, 1, 2, 3$, 2π-periodic smooth functions and $A_3(\theta)$ not changing sign. Then its maximum number of limit cycles, taking into account their multiplicities, is three. In our situation, writing equation (1) as equation (5) we get that $A_3(\theta) = f_{n+3m+1}(\theta)/g_{n+m+1}(\theta)$ does not change sign and so this upper bound follows.

When n is even and m is odd we want to prove that the upper bound of three limit cycles can be reduced to a result of uniqueness of limit cycles. Note that in this situation,

$$\begin{array}{ll} f_{n+1}(\pi+\theta) = -f_{n+1}(\theta), & f_{n+m+1}(\pi+\theta) = f_{n+m+1}(\theta), \\ g_{n+m+1}(\pi+\theta) = g_{n+m+1}(\theta), & f_{n+2m+1}(\pi+\theta) = -f_{n+2m+1}(\theta), \\ f_{n+3m+1}(\pi+\theta) = f_{n+3m+1}(\theta). & \end{array}$$

These equalities imply that, if $R = \rho(\theta)$ is a solution of equation (5), then so is $R = -\rho(\pi+\theta)$. Thus, if system (5) had two limit cycles given by $R = \rho_1(\theta) > 0$ and $R = \rho_2(\theta) > 0$, then $R = -\rho_1(\pi+\theta)$ and $R = -\rho_2(\pi+\theta)$ would also be limit cycles of equation (5). This would imply that equation (5) has at least four limit cycles, which is in contradiction with the upper bound proved in the above paragraph. Similarly, a non hyperbolic limit cycle of equation (1) would give rise to two non hyperbolic limit cycles of the Abel equation, again in contradiction with the result that we have proved in the previous paragraph.

To end the proof of this item we give examples, when n is odd with $0, 1, 2$ or 3 limit cycles (taking into account the multiplicities) and when n is even having 0 or 1 limit cycle.

When n is odd we consider (1) with

$$P_{n+km-1}(x,y) = c_k(x^2+y^2)^{\frac{n+km-1}{2}}, \quad \text{when } k = 0, 2, 3,$$

and

$$Q_{n+m}(x,y) = (c_1x - my)(x^2+y^2)^{\frac{n+m-1}{2}}, \quad R_{n+m}(x,y) = (mx + c_1y)(x^2+y^2)^{\frac{n+m-1}{2}},$$

with $c_i \in \mathbb{R}$. Then (3) writes as

$$\frac{dR}{d\theta} = c_0 + c_1R + c_2R^2 + c_3R^3.$$

Since the constants c_i can be chosen arbitrarily, it is clear that there exist systems with the desired number of limit cycles.

When n is even, in the above example we have to take $c_0 = c_2 = 0$ and thus only examples with 0 or 1 limit cycle can be constructed, as we wanted to prove.

The above examples are essentially the ones appearing in [15].

(d) For $n+m$ odd, we consider the following particular family of systems of de form (1): $P_{n+km-1}(x,y)$, for $k = 2, 3$, arbitrary homogeneous polynomials, $P_{n-1}(x,y) = \frac{1-(-1)^n}{2}c_0(x^2+y^2)^{\frac{n-1}{2}}$ and

$$Q_{n+m}(x,y) = (c_1x - my)(x^2+y^2)^{\frac{n+m-1}{2}}, \quad R_{n+m}(x,y) = (mx + c_1y)(x^2+y^2)^{\frac{n+m-1}{2}},$$

with $c_0, c_1 \in \mathbb{R}$. Then equation (3) writes as

$$\frac{dR}{d\theta} = \tfrac{1-(-1)^n}{2}c_0 + c_1R + f_{n+2m+1}(\theta)R^2 + f_{n+3m+1}(\theta)R^3. \tag{6}$$

We consider first the case $c_0 = c_1 = 0$. We study how many limit cycles bifurcate from $R = 0$.

Given an analytic 2π-periodic Abel equation of the form

$$\frac{dR}{d\theta} = A_2(\theta)R^2 + A_3(\theta)R^3, \tag{7}$$

let $R = r(\theta,\rho)$ be the solution that takes the value ρ when $\theta = 0$. Therefore,

$$r(\theta,\rho) = \rho + u_2(\theta)\rho^2 + u_3(\theta)\rho^3 + \ldots, \text{ with } u_k(0) = 0 \text{ for } \; k \geq 2, \tag{8}$$

where the functions $u_k(\theta)$ satisfy simple differential equations: $u_2'(\theta) = A_2(\theta), u_3'(\theta) = A_3(\theta)+2A_2(\theta)u_2(\theta), \ldots$, see for instance [2]. Clearly the first no zero value $u_\ell(2\pi)$ gives the stability of $R=0$, and the number $V_\ell := u_\ell(2\pi)$, defined when $V_2 = V_3 = \cdots = V_{\ell-1} = 0$ is called the ℓ-th Lyapunov constant of $R=0$. These quantities can be used to study the bifurcations of limit cycles from the origin, as the Lyapunov constants in the usual Andronov-Hopf bifurcations, see [1, 11]. In these papers, the following expressions for V_ℓ are given:

$$V_2 = \int_0^{2\pi} A_2(\theta)\, d\theta, \qquad V_3 = \int_0^{2\pi} A_3(\theta)\, d\theta,$$
$$V_4 = \int_0^{2\pi} \Big(A_3(\theta)(\int_0^\theta A_2(\psi)\, d\psi)\Big)\, d\theta, \quad V_5 = \int_0^{2\pi} \Big(A_3(\theta)(\int_0^\theta A_2(\psi)\, d\psi)^2\Big)\, d\theta.$$

Let us compute some of them for equation (6). We start with the case n odd, and so m even. By convenience we write the real polynomials $A_2(\theta)$ and $A_3(\theta)$ as complex Fourier series:

$$A_2(\theta) = P_{2k}(\cos\theta, \sin\theta) = \sum_{j=-k}^{k} D_{2j} e^{2j\theta i}, \text{ where } D_{2j} \in \mathbb{C} \text{ and } D_{-2j} = \overline{D_{2j}},$$

$$A_3(\theta) = P_{2k+m}(\cos\theta, \sin\theta) = \sum_{j=-(k+m/2)}^{k+m/2} E_{2j} e^{2j\theta i}, \text{ where } E_{2j} \in \mathbb{C} \text{ and } E_{-2j} = \overline{E_{2j}},$$

where $k := (n+2m-1)/2$. Then

$$V_2 = 2D_0\pi \in \mathbb{R}, \quad V_3 = 2E_0\pi \in \mathbb{R} \quad \text{and} \quad V_4 = 2\pi \sum_{j=-k,\, j\neq 0}^{k} \frac{\operatorname{Im}(D_{2j}\overline{E_{2j}})}{j}.$$

By using standard arguments, see for instance [8, Ex. C], we know that taking in equation (6), $V_0 := c_0$, $V_1 := c_1$ and V_ℓ, $\ell = 2,3,4$ as above, satisfying $V_j V_{j+1} < 0$, $j = 0,1,2,3$, and

$$|V_0| \ll |V_1| \ll |V_2| \ll |V_3| \ll |V_4|,$$

we obtain a system of the form (1) with at least 4 limit cycles, all them bifurcating from the origin. Notice that the bifurcation associated to the parameter $V_0 = c_0$ is not an Andronov-Hopf bifurcation, because when $c_0 \neq 0$ the origin is not a focus but a kind of star-shape node, which stability is given by the sign of c_0.

By computing more Lyapunov constants it seems clear that in general we will get more limit cycles. It is important to notice that when $P_{n+3m-1}(\cos\theta, \sin\theta) = f_{n+3m+1}(\theta)$ is not identically zero, and does not change sign, then

$$V_3 = \int_0^{2\pi} A_3(\theta) d\theta = \int_0^{2\pi} f_{n+3m+1}(\theta) d\theta = 2E_0\pi \neq 0$$

and the above procedure can give rise to at most 3 limit cycles, as Theorem 1 asserts.

When n is even (and so m is odd) we obtain that equation (6) writes as

$$\frac{dR}{d\theta} = c_1 R + f_{n+2m+1}(\theta) R^2 + f_{n+3m+1}(\theta) R^3.$$

By doing similar computations that in the above case we get that $V_2 = V_4 = 0$, $V_3 = 2E_0\pi$ and that V_5 can take arbitrary values. Hence by choosing suitable c_1, V_3 and V_5, at least two limit cycles bifurcate from the origin of system (1), as we wanted to prove. Similarly that in the previous case, when $P_{n+3m-1}(\cos\theta, \sin\theta) = f_{n+3m+1}(\theta)$

is not identically zero, and does not change sign, then $V_3 \neq 0$ and at most one limit cycle bifurcates from the origin of system (1). □

The examples given in item (c) of the proof of Theorem 1, showing that our upper bounds for the number of limit cycles are sharp, are integrable systems and moreover their limit cycles are circles. In the next proposition we show that there are more complicated systems having the same number of limit cycles. We remark that the proof of item (d) of Theorem 1 also provides a method to give examples with the maximal number of limit cycles predicted by the theorem.

Proposition 3. *Consider the system*

$$\begin{array}{rcl} \dot{x} & = & -y(x^2+y^2)^{(n+m-1)/2} + \varepsilon \mathcal{P}(x,y), \\ \dot{y} & = & x(x^2+y^2)^{(n+m-1)/2} + \varepsilon \mathcal{Q}(x,y), \end{array} \tag{9}$$

where ε is a small parameter,

$$\mathcal{P}(x,y) = \bar{Q}_{n+m}(x,y) + x(P_{n-1}(x,y) + P_{n+2m-1}(x,y) + P_{n+3m-1}(x,y)),$$

$$\mathcal{Q}(x,y) = \bar{R}_{n+m}(x,y) + y(P_{n-1}(x,y) + P_{n+2m-1}(x,y) + P_{n+3m-1}(x,y)),$$

and $P_k(x,y), \bar{Q}_k(x,y)$ and $\bar{R}_k(x,y)$ are homogeneous polynomials of degree k.

If n is odd and m is even (respectively, n is even and m is odd), then there exist polynomials $\mathcal{P}(x,y)$ and $\mathcal{Q}(x,y)$ such that $P_{n+3m+1}(\theta)$ does no change sign and system (9) *has three (respectively, one) limit cycles.*

Proof. For small ε, we have $G_{n+m+1}(x,y) = (x^2+y^2)^{(n+m+1)/2} + \varepsilon(x\bar{R}_{n+m}(x,y) - yQ_{n+m}(x,y)) = (x^2+y^2)^{(n+m+1)/2} + O(\varepsilon) > 0$ for $(x,y) \neq 0$. By re-parameterizing the time, we can rewrite system (9) in the form

$$x' = -y + \frac{\varepsilon \mathcal{P}(x,y)}{(x^2+y^2)^{(n+m-1)/2}}, \quad y' = x + \frac{\varepsilon \mathcal{Q}(x,y)}{(x^2+y^2)^{(n+m-1)/2}}, \tag{10}$$

which is a perturbation of the linear system

$$x' = -y, \quad y' = x. \tag{11}$$

The Abelian integral, associated to system (10), is defined as

$$I(h) = \oint_{\Gamma_h} \frac{\mathcal{P}(x,y)}{(x^2+y^2)^{(n+m-1)/2}} dy - \frac{\mathcal{Q}(x,y)}{(x^2+y^2)^{(n+m-1)/2}} dx, \tag{12}$$

where Γ_h is the the closed orbit of system (11) given by $x^2+y^2 = h^2,\ h \in (0,+\infty)$. It is well known (see for instance [17]) that the displacement function of the perturbed system (10) can be expressed in the form

$$d(h,\varepsilon) = \varepsilon I(h) + O(\varepsilon^2),$$

and the following statements hold:

(a) If there exists $h^* \in (0,+\infty)$ such that $I(h^*) = 0$ and $I'(h^*) \neq 0$, then system (10) has a unique limit cycle bifurcating from Γ_h^*, moreover, this limit cycle is hyperbolic,

(b) When $I(h) \not\equiv 0$, the total number (counting the multiplicities) of limit cycles of system (10) bifurcating from the period annulus of system (11) is bounded by the maximum number of isolated zeros (also taking into account their multiplicities) of the Abelian integral $I(h)$ for $h \in (0,+\infty)$.

The substitution $x = h\cos\theta,\ y = h\sin\theta$ and (12) gives

$$I(h) = h^{2-m}\Big(\int_0^{2\pi} f_{n+1}(\theta)d\theta + h^m \int_0^{2\pi} \bar f_{n+m+1}(\theta)d\theta + h^{2m}\int_0^{2\pi} f_{n+2m+1}(\theta)d\theta + h^{3m}\int_0^{2\pi} f_{n+3m+1}(\theta)d\theta\Big), \tag{13}$$

where $f_{k+1}(\theta)$, $k = n,\ n+2m,\ n+3m$, are defined as in (4), and

$$\bar f_{n+m+1}(\theta) = \cos\theta \bar Q_{n+m}(\cos\theta, \sin\theta) + \sin\theta \bar R_{n+m}(\cos\theta, \sin\theta).$$

Writing $P_k(x,y) = \sum_{i+j=k} a_{ij}x^iy^j$, $\bar Q_{n+m}(x,y) = \sum_{i+j=n+m} \bar a_{ij}x^iy^j$ and $R_{n+m}(x,y) = \sum_{i+j=n+m} \bar b_{ij}x^iy^j$, direct computations give that, for $k = n,\ n+2m,\ n+3m$,

$$\int_0^{2\pi} f_{k+1}(\theta)d\theta = \sum_{2i+2j=k-1} a_{2i,2j}\int_0^{2\pi}\cos^{2i}\theta\sin^{2j}\theta d\theta$$

$$\int_0^{2\pi} \bar f_{n+m+1}(\theta)d\theta = \sum_{2i+2j-1=n+m} (\bar a_{2i-1,2j} + \bar b_{2i,2j-1})\int_0^{2\pi}\cos^{2i}\theta\sin^{2j}\theta d\theta.$$

Since $h \in (0, +\infty)$, the number of the positive zeros of $I(h)$ is an upper bound of the number of limit cycles of system (9) which born from the period annulus of system (11). By Descarte's rule, this upper bound is three. Note that this result proves that even in the case where Theorem 1 does not apply, three is the maximum number of limit cycles provided by this approach.

Notice that when n is odd and m is even, then for $k = n, n+2m, n+3m$, $k-1$ is an even number. Therefore the coefficients in $I(h)$ can take arbitrary values and there exist many polynomials $\mathcal{P}(x,y)$ and $\mathcal{Q}(x,y)$ such that $I(h)$ has exactly 0, 1, 2, or 3 positive zeros counting their multiplicities. Moreover it is not difficult to find situations where $f_{n+3m+1}(\theta) = P_{n+3m-1}(\cos\theta, \sin\theta)$ does not change sign.

Finally notice that when n is even and m is odd, then $\int_0^{2\pi} f_{n+1}(\theta)d\theta = 0$ and $\int_0^{2\pi} f_{n+2m+1}(\theta)d\theta = 0$. Therefore

$$I(h) = h^2\Big(\int_0^{2\pi} \bar f_{n+m+1}(\theta)d\theta + h^{2m}\int_0^{2\pi} f_{n+3m+1}(\theta)d\theta\Big)$$

and at most one limit cycle bifurcates from the period annulus of system (11). □

Remark 4. *In the above proposition we have proved that there are many systems of the form* (1)*, under the hypotheses of Theorem 1, having the maximum number of limit cycle that it predicts. For instance, in case a.(i), they have 3 limit cycles. In the same situation it is not difficult to construct examples having exactly k limit cycles, for* $k \in \{0,1,2\}$*. To do this it suffices to take systems of the form* (9) *with* ε *small enough and such that the function* $I(h)$ *given in* (13) *has exactly k positive simple zeros and no limit cycles bifurcate neither from the origin nor from infinity. These two bifurcations can be prevented for instance by taking perturbations satisfying that* $f_{n+1}(\theta) \neq 0$ *and* $f_{n+3m+1}(\theta) \neq 0$*, respectively. The presence of double limit cycles can as well be guaranteed by using the results of* [3, Thm. 1.3].

3. Proof of Theorem 2

To prove Theorem 2, we firstly recall a general result for Riccati equations. Consider the Riccati equation

$$\frac{dR}{d\theta} = A_0(\theta) + A_1(\theta)R + A_2(\theta)R^2, \tag{14}$$

where $A_0(t)$, $A_1(t)$, $A_2(t) : [0,2\pi] \to \mathbb{R}$ are smooth 2π-periodic functions. As for Abel equations, it is said that a solution of (14) is a *periodic solution* if it is defined in the interval $[0,2\pi]$ and $R(0) = R(2\pi)$. An isolated periodic solution, in the set of all the periodic solutions, is called a *limit cycle*. It is well known that Riccati equations can have at most two limit cycles, see for instance [7, 19, 18]. For the sake of completeness we present a proof of this fact, which is based on the approach of [7] and also takes into account the multiplicities of the limit cycles.

Proposition 5. *(a) The Riccati equation* (14) *has at most two limit cycles, taking into account their multiplicities.*

(b) The linear equation, (14) *with* $A_2(\theta) \equiv 0$*, has at most one limit cycle and when it exists it is hyperbolic.*

Proof. (a) Suppose that the equation (14) has a limit cycle $R = \rho(\theta)$. Make the change of variables

$$W(\theta) = R(\theta) - \rho(\theta). \tag{15}$$

It transforms the solution $R = \rho(\theta)$ into $W = 0$, and equation (14) into

$$\frac{dW}{d\theta} = B(\theta)W + A_2(\theta)W^2, \tag{16}$$

where $B(\theta) = A_1(\theta) + 2\rho(\theta)A_2(\theta)$. Since $R = \rho(\theta)$ is a periodic orbit, a solution of (14) is periodic if and only if the corresponding solution $W = \omega(\theta)$ is periodic. Let $W = \omega(\theta, W_0)$ be the solution of (16) with the initial condition $\omega(0, W_0) = W_0$. It is well known that the above equation can be transformed into a linear one. Direct computations give that for $\theta \in [0, \theta^*(W_0))$, where $\theta^*(W_0)$ is the first positive solution of the equation $1 - \beta(\theta)W_0 = 0$,

$$W(\theta, W_0) = \frac{\alpha(\theta)W_0}{1 - \beta(\theta)W_0},$$

where

$$\alpha(\theta) = \exp\left(\int_0^\theta B(\psi)d\psi\right), \quad \beta(\theta) = \int_0^\theta A_2(\psi)\exp\left(\int_0^\psi B(\phi)d\phi\right)d\psi.$$

Consider the displacement function

$$d(W_0) := W(2\pi, W_0) - W_0 = \frac{(\alpha(2\pi) - 1)W_0 + \beta(2\pi)W_0^2}{1 - \beta(2\pi)W_0}, \tag{17}$$

defined for $W_0 \in \mathcal{I} := \{W_0 : 1 - \beta(\theta)W_0 > 0 \text{ for } \theta \in [0,2\pi]\}$. Therefore the limit cycles of equation (14) correspond with solutions $W(\theta, W_0)$ whose initial conditions W_0 are in $\mathcal{I}$ and satisfy the quadratic equation

$$\left((\alpha(2\pi) - 1) + \beta(2\pi)W_0\right) W_0 = 0. \tag{18}$$

Moreover its multiplicity coincides with the multiplicity of W_0 as a zero of the function $d(W_0)$. Therefore, the result for Riccati equations follows. Note also that, once we know a particular periodic solution of the Riccati equation, this approach is also useful to know the total number of periodic solutions of the equation. It suffices to check how many solution has equation (18) in the interval $\mathcal{I}$. For instance, the Riccati equation has a continuum of periodic orbits if and only if $\alpha(2\pi) - 1 = \beta(2\pi) = 0$, or it has a unique limit cycle $R = \rho(\theta)$ with multiplicity two if and only if $\alpha(2\pi) = 1$ and $\beta(2\pi) \neq 0$.

(b) The result for the linear equation, *i.e.* $A_2(\theta) \equiv 0$, follows from the above reasoning noticing that in this case $\beta(\theta) \equiv 0$. □

Proof of Theorem 2. Items (a) are (b) are already proved in Theorem 1. Notice that when $P_{n+2m-1}(x, y) \equiv 0$ (respectively, $P_{n+2m-1}(x, y) \not\equiv 0$) then the differential equation (5) is a linear equation (respectively, a pure Riccati equation). The key results for proving items (c) and (d) are Proposition 5 and the following property of the equation (5), already used in the proof of Theorem 1: When n is even and m is odd, if $R = \rho(\theta)$ is one of its solutions, then so is $R = -\rho(\pi + \theta)$.

The same type of examples given in previous section can be easily adapted to this situation providing the lower bound stated in the theorem. □

Acknowledgements

The first author is partially supported by by grants MTM2005-06098-C02-1 and 2005SGR-00550. The second author is partially supported by the Spanish grant SAB-2005-0029, NSF of China (No.10571184) and SRF for ROCS, SEM. The second author also wants to express his thanks to the Departament de Matemàtiques of the Universitat Autònoma de Barcelona for the hospitality and support during the period in which this paper was started.

References

[1] M.A.M. Alwash and N.G. Lloyd, *Non-Autonomous equations related to polynomial two–dimensional systems*, Proc. Roy. Soc. Edinburgh **105A** (1987), 129–152.

[2] A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, "Theory of bifurcations of dynamic systems on a plane", John Wiley and Sons, New York-Toronto, 1967.

[3] T.R. Blows and L.M. Perko, *Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems*, SIAM Review **36** (1994), 341–376.

[4] M. Carbonell and J. Llibre, *Limit cycles of a class of polynomial systems*, Proc. Roy. Soc. Edinburgh Sect. A **109** (1988), 187–199.

[5] J. Chavarriga and J. Giné, *Integrability of cubic systems with degenerate infinity,* Differential Equations Dynam. Systems **6** (1998), 425–438.

[6] L.A. Cherkas, *Number of limit cycles of an autonomous second-order system,* Diff. Eq. **5** (1976), 666–668.

[7] A. Cima, A. Gasull and V. Mañosa, *Dynamics of some rational discrete dynamical systems via invariants*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **16** (2006),631–645.

[8] B. Coll, A. Gasull and R. Prohens, *Degenerate Hopf bifurcations in discontinuous planar systems*, J. Math. Anal. Appl. **253** (2001), 671–690.

[9] J. Devlin, N.G. Lloyd and J.M. Pearson, *Cubic systems and Abel equations,* J. Differential Equations **147** (1998), 435–454.

[10] A. Gasull and A. Guillamon, *Limit cycles for generalized Abel equations*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **16** (2006),3727–3745.

[11] A. Gasull and A. Guillamon and V. Mañosa, *An explicit expression of the first Liapunov and period constants with applications*, J. Math. Anal. Appl. **211** (1997), 190–212.

[12] A. Gasull and J. Llibre, Limit cycles for a class of Abel equations, SIAM J. Math. Anal. **21** (1990), 1235-1244.

[13] A. Gasull and R. Prohens, *Quadratic and cubic systems with degenerate infinity,* J. Math. Anal. Appl. **198** (1996), 25–34.

[14] A. Gasull, R. Prohens and J. Torregrosa, *Limit cycles for rigid cubic systems*, J. Math. Anal. Appl. **303** (2005), 391–404.

[15] J. Giné and J. Llibre, *Darboux integrability and limit cycles for a class of polynomial differential systems*, Trends in Mathematics: Differential Equations with symbolic computations, 55–65, 2005.

[16] J. Giné and J. Llibre, *Integrability, degenerate centers, and limit cycles for a class of polynomial differential systems*, Comput. Math. Appl. **51** (2006), 1453–1462.

[17] C. Li, *Abelian integrals and limit cycles*, in Limit Cycles of Differential Equations Series: Advanced Courses in Mathematics - CRM Barcelona, C. Christopher and C. Li, 2007.
[18] A. Lins Neto, *On the number of solutions of equation $dx/dt = \sum_{j=0}^{n} a_j(\theta)x^j$, $0 \le t \le 1$, for which $x(0) = x(1)$*, Invent. Math. **59** (1980), 67–76.
[19] N.G. Lloyd, *On a class of differential equations of Riccati type*, J. Lond. Math. Soc. **10** (1975), 1–10.
[20] N.G. Lloyd, C.J. Christopher, J. Devlin, J.M. Pearson and N. Yasmin *Quadratic-like cubic systems*, Planar nonlinear dynamical systems (Delft, 1995). Differential Equations Dynam. Systems **5** (1997), 329–345.
[21] H. Shariati and H.M. Mohammadi Nejad, *Relationships between Darboux integrability and limit cycles for a class of Abel equations*, J. of Sciences, Isl. Rep. of Iran **17(3)** (2006), 265–272.

Departament de Matemàtiques, Edifici C, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
E-mail address: gasull@mat.uab.cat

Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, People's Republic of China.
E-mail address: mcszyl@mail.sysu.edu.cn