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ON THE EXISTENCE OF BI–PYRAMIDAL CENTRAL CONFIGURATIONS OF THE $n+2$–BODY PROBLEM WITH AN n–GON BASE

Montserrat Corbera

Departament de Tecnologies Digitals i de la Informació
Universitat de Vic
C/. Laura, 13
08500 Vic, Barcelona, Catalonia, Spain

Jaume Llibre

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Catalonia, Spain

Abstract. In this paper we prove the existence of central configurations of the $n+2$–body problem where n equal masses are located at the vertices of a regular n–gon and the remaining 2 masses, which are not necessarily equal, are located on the straight line orthogonal to the plane containing the n–gon passing through its center. Here this kind of central configurations is called *bi–pyramidal central configurations*. In particular, we prove that if the masses m_{n+1} and m_{n+2} and their positions satisfy convenient relations, then the configuration is central. We give explicitly those relations.

1. **Introduction.** We consider the spatial N–body problem

$$m_k\,\ddot{\mathbf{q}}_k = -\sum_{\substack{j=1\\ j\neq k}}^{N} G\,m_k\,m_j\,\frac{\mathbf{q}_k-\mathbf{q}_j}{|\mathbf{q}_k-\mathbf{q}_j|^3}\ ,$$

$k=1,\ldots,N$, where $\mathbf{q}_k\in\mathbb{R}^3$ is the position vector of the punctual mass m_k in an inertial coordinate system, and G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. The *configuration space* of the spatial N–body problem is

$$\mathcal{E}=\{(\mathbf{q}_1,\ldots,\mathbf{q}_N)\in\mathbb{R}^{3N}\ :\ \mathbf{q}_k\neq\mathbf{q}_j,\ \text{for } k\neq j\}.$$

Given $m_1,\ldots,m_N$ a configuration $(\mathbf{q}_1,\ldots,\mathbf{q}_N)\in\mathcal{E}$ is *central* if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_k=-\lambda\,(\mathbf{q}_k-\mathbf{c})\ ,$$

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$k = 1, \ldots, N$, where $\mathbf{c}$ is the center of mass of the system, which is given by

$$\mathbf{c} = \frac{\sum_{k=1}^{N} m_k \mathbf{q}_k}{\sum_{k=1}^{N} m_k}.$$

Thus a central configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathcal{E}$ of the N–body problem with positive masses $m_1, \ldots, m_N$ is a solution of the system of equations

$$\sum_{\substack{j=1 \\ j \neq k}}^{N} m_j \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} = \lambda \left(\mathbf{q}_k - \mathbf{c} \right), \tag{1}$$

$k = 1, \ldots, N$, for some λ.

The first known planar central configuration of the N–body problem for $N \geq 2$ consists of N equal masses located at the vertices of a regular N–gon. If we take N equal masses at the vertices of a regular polyhedron with N vertices, then we obtain a spatial central configuration of the N–body problem (see [1]). In addition to regular polyhedron central configurations, the simplest spatial central configurations of the N–body problem are the ones known as *pyramidal central configurations*. Such configurations consist of $N = n+1$ masses, n of which are coplanar and the $(n+1)$–th being off the plane (see for instance [2] and [14]). The n positions of the coplanar masses are called the base of the pyramidal central configuration.

The next simplest spatial central configurations are the ones known as *double pyramidal central configurations*. Such configurations consist of $N = n+2$ masses, n of which are coplanar and the other two being off the plane and positioned symmetrically above and below that plane. In the literature we can find some papers related with double pyramidal central configurations with different shapes of basis. For instance [16] studied for all $n \geq 4$ the double pyramidal central configurations such that the n coplanar masses are at the vertices of a regular n–gon and the $(n+1)$–th and $(n+2)$–th masses are equal, under these assumptions the $(n+1)$–th and the $(n+2)$–th mass are located symmetrically with respect to the n–gon. Liu and his coauthors have some papers related with double pyramidal central configurations of the N body problem for $N = 5, 6, 7, 9$ for different shapes of their basis (see [4, 5, 6, 7, 8, 9, 10]). Yang and Zhang in [15] also studied double pyramidal central configurations of the 6–body problem. In [12] the authors study central configurations of the $N = n+3$ body problem consisting of n masses at the vertices of a regular n-gon, 2 masses on the straight line orthogonal to the plane containing the n–gon passing through its center and positioned symmetrically above and below the n-gon, and a third mass positioned at the center of the n–gon.

In this paper we consider spatial central configurations of the N–body problem, with $N = n+2$, $n \in \mathbb{N}$ and $n \geq 2$, consisting of n equal masses $m_1 = \cdots = m_n$ at the vertices of a (regular) n–gon and 2 masses m_{n+1} and m_{n+2} on the straight line orthogonal to the plane containing the n–gon passing through its center. In contrast to what happens in the known double pyramidal central configurations, we do not impose conditions, neither on the positions, nor on the values of the masses m_{n+1} and m_{n+2}. In this paper, these configurations are called *bi–pyramidal central configurations*. Notice that the double pyramidal central configurations studied in [16] are a particular case of our bi–pyramidal central configurations when $m_{n+1} = m_{n+2}$ and m_{n+1} and m_{n+2} are positioned symmetrically above and below the plane containing the n–gon. The bi–pyramidal central configurations for $n = 3$ are studied in [3] and [11] from different points of view. In [3] the author proves that, for any pair of positive masses m_{n+1} and m_{n+2}, the number of bi-pyramidal central configurations is finite and also provides all the possible numbers of such central configurations. In [11] the authors consider the inverse problem; that is, given a bi–pyramidal configuration, they find the masses which make it central. We do not know any paper considering bi-pyramidal central configurations for $n > 3$.

Here we analyze the bi–pyramidal central configurations from the inverse problem point of view.

This paper is structured as follows. In Section 2 we give the equations of the bi–pyramidal central configurations. In Section 3 we summarize some results concerning to pyramidal central configurations. Finally, in Section 4, we analyze the 2–pyramidal central configurations. We prove that for all $n \geq 2$ we can find positions on the straight line such that convenient masses m_{n+1} and m_{n+2} placed at these positions provide central configurations. We also give the explicit relations between the masses and the positions of such central configurations. Moreover we see that for all $n < 9$ there exists a privileged position and a privileged value of the top mass, which depend on n, so that any arbitrary mass located at the same distance from all other masses provides a central configuration. The precise statement of these results is given in Theorem 4.3.

2. **Equations of the central configurations.** We consider $N = n+2$ with $n \in \mathbb{N}$, $n \geq 2$, n equal masses $m_1 = \cdots = m_n$ at the vertices of a (regular) n–gon and 2 masses m_{n+1} and m_{n+2} on the straight line orthogonal to the plane containing the n–gon passing through its center. Without loss of generality we can choose the unit of mass so that $m_1 = \cdots = m_n = 1$, and we take the unit of length in order that the radius of the circle containing the n-gon be one. By using complex coordinates in the plane that contains the regular n–gon, the positions of the vertices of the n-gon can be written as $\mathbf{q}_k = (e^{i\alpha_k}, 0) \in \mathbb{C} \times \mathbb{R}$ with $\alpha_k = 2\pi k/n$ for $k = 1, \ldots, n$. Let $m_{n+1} = \mu_1$, $m_{n+2} = \mu_2$, $\mathbf{q}_{n+1} = (0, z_1)$, and $\mathbf{q}_{n+2} = (0, z_2)$, with $z_1 > z_2$. We note that this last condition is not restrictive.

Using these notations the center of mass of the system is given by

$$\mathbf{c} = \frac{1}{n + \mu_1 + \mu_2} \left(\mu_1(0, z_1) + \mu_2(0, z_2) + \sum_{j=1}^{n} (e^{i\alpha_j}, 0) \right).$$

Since $\sum_{j=1}^{n} e^{i\alpha_j} = 0$,

$$\mathbf{c} = (c_{xy}, c_z) = \frac{1}{n + \mu_1 + \mu_2} (0, \mu_1 z_1 + \mu_2 z_2).$$

The first n equations of (1) become

$$\mu_1 \frac{\mathbf{q}_k - \mathbf{q}_{n+1}}{|\mathbf{q}_k - \mathbf{q}_{n+1}|^3} + \mu_2 \frac{\mathbf{q}_k - \mathbf{q}_{n+2}}{|\mathbf{q}_k - \mathbf{q}_{n+2}|^3} + \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} = \lambda(\mathbf{q}_k - \mathbf{c}), \tag{2}$$

for $k = 1, \ldots, n$, and the last 2 equations of (1) are

$$\begin{aligned} \mu_2 \frac{\mathbf{q}_{n+1} - \mathbf{q}_{n+2}}{|\mathbf{q}_{n+1} - \mathbf{q}_{n+2}|^3} + \sum_{j=1}^{n} \frac{\mathbf{q}_{n+1} - \mathbf{q}_j}{|\mathbf{q}_{n+1} - \mathbf{q}_j|^3} &= \lambda(\mathbf{q}_{n+1} - \mathbf{c}), \\ \mu_1 \frac{\mathbf{q}_{n+2} - \mathbf{q}_{n+1}}{|\mathbf{q}_{n+2} - \mathbf{q}_{n+1}|^3} + \sum_{j=1}^{n} \frac{\mathbf{q}_{n+2} - \mathbf{q}_j}{|\mathbf{q}_{n+2} - \mathbf{q}_j|^3} &= \lambda(\mathbf{q}_{n+2} - \mathbf{c}). \end{aligned} \tag{3}$$

It is easy to check that

$$
\begin{aligned}
&|\mathbf{q}_k - \mathbf{q_j}| = |e^{i\alpha_k} - e^{i\alpha_j}| \qquad \text{for } j \neq k, \text{ and } j,k = 1,\ldots,n,\\
&|\mathbf{q}_k - \mathbf{q}_{n+\ell}| = \sqrt{1+z_\ell^2} \qquad \text{for } k = 1,\ldots,n, \text{ and } \ell = 1,2,\\
&|\mathbf{q}_{n+1} - \mathbf{q}_{n+2}| = |z_1 - z_2|.
\end{aligned}
$$

So the first components of the n vectorial equations (2) are

$$
\mu_1 \frac{e^{i\alpha_k}}{(1+z_1^2)^{3/2}} + \mu_2 \frac{e^{i\alpha_k}}{(1+z_2^2)^{3/2}} + \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{e^{i\alpha_k} - e^{i\alpha_j}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3} = \lambda\, e^{i\alpha_k}, \tag{4}
$$

for $k = 1,\ldots,n$. By dividing the k–th equation by $e^{i\alpha_k}$ we get

$$
\frac{\mu_1}{(1+z_1^2)^{3/2}} + \frac{\mu_2}{(1+z_2^2)^{3/2}} + \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1 - e^{i(\alpha_j - \alpha_k)}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3} = \lambda,
$$

for $k = 1,\ldots,n$. Defining

$$
\beta_n = \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1 - e^{i(\alpha_j - \alpha_k)}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3},
$$

after some simplifications we get

$$
\beta_n = \frac{1}{4}\sum_{j=1}^{n-1} \csc\left(\frac{\pi j}{n}\right).
$$

So the first components of the n vectorial equations (4) can be reduced to the single equation

$$
\beta_n + \frac{\mu_1}{(1+z_1^2)^{3/2}} + \frac{\mu_2}{(1+z_2^2)^{3/2}} = \lambda. \tag{5}
$$

Using the fact that $\sum_{j=1}^{n} e^{i\alpha_j} = 0$, we can see easily that the first components of the 2 vectorial equations (3) are always satisfied.

The second components of the n equations (2) become

$$
\mu_1 \frac{z_1}{(1+z_1^2)^{3/2}} + \mu_2 \frac{z_2}{(1+z_2^2)^{3/2}} = \lambda\, c_z, \tag{6}
$$

for all $k = 1,\ldots,n$. Finally, the second components of the 2 vectorial equations (3) become

$$
\begin{aligned}
n\,\frac{z_1}{(1+z_1)^{3/2}} + \mu_2\,\frac{z_1 - z_2}{|z_1 - z_2|^3} &= \lambda(z_1 - c_z),\\
n\,\frac{z_2}{(1+z_2^2)^{3/2}} + \mu_1\,\frac{z_2 - z_1}{|z_2 - z_1|^3} &= \lambda(z_2 - c_z).
\end{aligned} \tag{7}
$$

In short, system (1) can be reduced to the 4 equations (5), (6) and (7) with the 3 unknowns λ, z_1, z_2. We note that one of these equations is redundant. Indeed, if we multiply the first equation of (7) by μ_1 and the second one by μ_2 and we add the resulting equations, then we get

$$
n\left(\mu_1 \frac{z_1}{(1+z_1^2)^{3/2}} + \mu_2 \frac{z_2}{(1+z_2^2)^{3/2}}\right) = \lambda(\mu_1 z_1 + \mu_2 z_2) - \lambda(\mu_1 + \mu_2)c_z. \tag{8}
$$

¿From (6), the left hand of (8) becomes $n\,\lambda\,c_z$. On the other hand, from the definition of c_z, we have that $\mu_1\,z_1+\mu_2\,z_2=(n+\mu_1+\mu_2)c_z$. Therefore (8) becomes

$$n\,\lambda\,c_z=\lambda\,c_z\,(n+\mu_1+\mu_2)-\lambda\,c_z(\mu_1+\mu_2),$$

which is always satisfied.

We add equation (6) to each equation of (7) and we substitute the value of λ by the value given by (5) into the resulting equations. Then the system formed by equations (5), (6) and (7) reduces to a system of 2 equations that is linear in the variables μ_1 and μ_2, and its matrix form is

$$\begin{pmatrix} 0 & c_{1,2} \\ c_{2,1} & 0 \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}=\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} c_{\ell,j} &= \frac{z_\ell-z_j}{|z_\ell-z_j|^3}+\frac{z_j-z_\ell}{(1+z_j^2)^{3/2}}, \\ t_\ell &= \beta_n\,z_\ell-n\,\frac{z_\ell}{(1+z_\ell^2)^{3/2}}. \end{aligned}$$

for all $\ell,j=1,2$ and $\ell\neq j$.

We note that if in (9) we replace z_ℓ by $-z_\ell$ for all $\ell=1,2$, we obtain the same system of equations. So if $z_1=z_1^*,z_2=z_2^*$ are such that $\mu_1=\mu_1^*$, $\mu_2=\mu_2^*$ is a solution of (9), then $\mu_1=\mu_1^*,\mu_2=\mu_2^*$ is also a solution of (9) for $z_1=-z_1^*,z_2=-z_2^*$. In order to simplify our computations, in what follows we assume that $z_1>0$.

Since we have assumed that $z_1>z_2$, we have that

$$c_{1,2}=\frac{1}{(z_1-z_2)^2}+\frac{z_2-z_1}{(1+z_2^2)^{3/2}},\quad\text{and}\quad c_{2,1}=-\frac{1}{(z_1-z_2)^2}+\frac{z_1-z_2}{(1+z_1^2)^{3/2}}.$$

3. **Pyramidal central configurations.** We note that the formulation (9) allows us to find the pyramidal central configurations obtained in [14] in an easy way. Indeed, the pyramidal central configurations consists of placing only one mass on the straight line orthogonal to the plane containing the n–gon passing through its center. For these configurations equation (9) becomes the single equation

$$0\cdot\mu_1=t_1.$$

The solutions of equation $t_1=0$ are

$$z_1=0,\qquad z_1=\pm\sqrt{\left(\frac{n}{\beta_n}\right)^{2/3}-1}:=\pm z_\beta.$$

We note that the values z_β are defined only for n such that $n/\beta_n\geq 1$. In [13] it is proved that this condition is satisfied for $n<473$. In short we have proved the following known result.

Theorem 3.1. *The following statements hold.*

(a) *For all $\mu_1>0$ and $2\leq n<473$ the problem has three central configurations, one with μ_1 at the center of the n-gon (i.e. $z_1=0$), and two with μ_1 at the positions $z_1=z_\beta$ and $z_1=-z_\beta$ respectively.*

(b) *For all $\mu_1>0$ and $n\geq 473$ the problem has a unique central configuration, the one with μ_1 at the center or the n-gon.*

A different proof of Theorem 3.1 can be found in [14].

4. **Bi-pyramidal central configurations.** ¿From elementary algebra we get the following result.

Lemma 4.1. *System (9) has the following solutions:*

(A) a unique solution $\mu_1 = t_2/c_{2,1}$*,* $\mu_2 = t_1/c_{1,2}$ *when* $c_{1,2} \neq 0$ *and* $c_{2,1} \neq 0$*,*
(B) a continuum of solutions $\mu_1 = t_2/c_{2,1}$*,* $\mu_2 \in \mathbb{R}$ *when* $c_{1,2} = 0$*,* $t_1 = 0$*, and* $c_{2,1} \neq 0$*,*
(C) a continuum of solutions $\mu_1 \in \mathbb{R}$*,* $\mu_2 = t_1/c_{1,2}$ *when* $c_{2,1} = 0$*,* $t_2 = 0$*, and* $c_{1,2} \neq 0$*,*
(D) a continuum of solutions $\mu_1 \in \mathbb{R}$*,* $\mu_2 \in \mathbb{R}$ *when* $c_{1,2} = 0$*,* $c_{2,1} = 0$*,* $t_1 = 0$*, and* $t_2 = 0$*,*
(E) no solutions when either $c_{1,2} = 0$ *and* $t_1 \neq 0$ *or* $c_{2,1} = 0$ *and* $t_2 \neq 0$*.*

The solutions of (9) given by Lemma 4.1 correspond to bi–pyramidal cental configurations only when $\mu_1, \mu_2 > 0$.

4.1. **The curves** $c_{2,1} = 0$**,** $c_{1,2} = 0$**,** $t_1 = 0$ **and** $t_2 = 0$**.** It is easy to see that equation $c_{1,2} = 0$ has a unique solution which is given by

$$z_2 = f(z_1) = \frac{z_1^2 - 1}{2z_1},$$

and $c_{2,1} = 0$ has the unique solution

$$z_2 = g(z_1) = z_1 - \sqrt{1 + z_1^2}.$$

Analyzing the properties of the functions f and g we see that both functions are continuous for $z_1 > 0$. Since

$$f'(z_1) = \frac{z_1^2 + 1}{2z_1^2} > 0, \qquad g'(z_1) = \frac{\sqrt{z_1^2 + 1} - z_1}{\sqrt{z_1^2 + 1}} > 0,$$

they are increasing. Moreover

$$\lim_{z_1 \to 0^+} f(z_1) = -\infty, \qquad \lim_{z_1 \to +\infty} f(z_1) = +\infty,$$
$$\lim_{z_1 \to 0^+} g(z_1) = -1, \qquad \lim_{z_1 \to +\infty} g(z_1) = 0.$$

We note that

$$\lim_{z_1 \to +\infty} \frac{f(z_1)}{z_1} = \frac{1}{2},$$

this means that f tends to infinity assymptotically to the straight line $z_2 = z_1/2$ when $z_1 \to +\infty$. Finally f crosses the positive semiaxis z_1 at the point $z_1 = 1$. The plot of the curves $z_2 = f(z_1)$ and $z_2 = g(z_1)$ is given in Figure 1.

We observe that when $z_1 > 0$ the curves $z_2 = f(z_1)$ and $z_2 = g(z_1)$ intersect at the unique point $\mathbf{p}^* = (z_1, z_2) = (1/\sqrt{3}, -1/\sqrt{3})$.

On the other hand in Section 3 we have seen that equation $t_\ell = 0$ for $\ell = 1, 2$ has a unique solution $z_\ell = 0$ when $n \geq 473$ and it has three solutions $z_\ell = 0$ and $z_\ell = \pm z_\beta$ when $2 \leq n < 473$.

Finally, we analyze the sign of the functions $c_{1,2}$, $c_{2,1}$, and t_ℓ for $\ell = 1, 2$. The results that we have obtained are summarized in the next result.

Lemma 4.2. *The following conditions are satisfied:*

(a) $c_{1,2} > 0$ *when* $z_2 > f(z_1)$*,* $c_{1,2} = 0$ *when* $z_2 = f(z_1)$*, and* $c_{1,2} < 0$ *when* $z_2 < f(z_1)$*,*

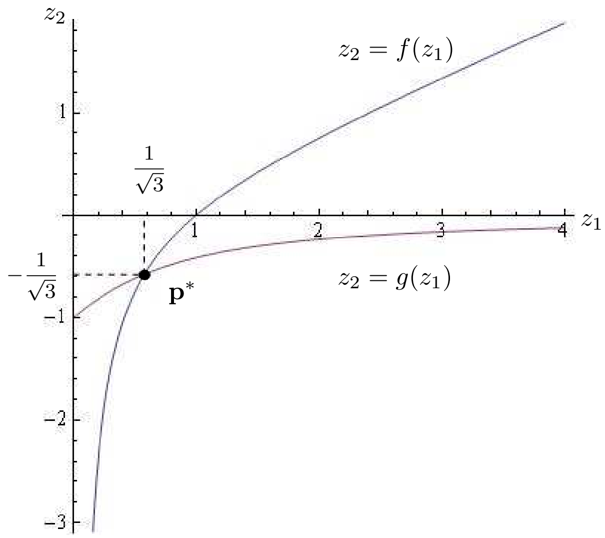


FIGURE 1. The plot of the curves $z_2 = f(z_1)$ and $z_2 = g(z_1)$.

(b) $c_{2,1} > 0$ *when* $z_2 < g(z_1)$, $c_{2,1} = 0$ *when* $z_2 = g(z_1)$, *and* $c_{2,1} < 0$ *when* $z_2 > g(z_1)$,

(c) *for* $2 \le n < 473$ *and* $\ell = 1, 2$, $t_\ell > 0$ *when* $z_\ell \in (-z_\beta, 0) \cup (z_\beta, +\infty)$, $t_\ell = 0$ *when* $z_\ell = 0, \pm z_\beta$, *and* $t_\ell < 0$ *when* $z_\ell \in (-\infty, -z_\beta) \cup (0, z_\beta)$,

(d) *for* $n \ge 473$ *and* $\ell = 1, 2$, $t_\ell > 0$ *when* $z_\ell > 0$, $t_\ell = 0$ *when* $z_\ell = 0$, *and* $t_\ell < 0$ *when* $z_\ell < 0$.

4.2. **The signs of the solutions μ_1 and μ_2 of (9).** The signs of the solutions μ_1 and μ_2 of system (9) could change at the curves $c_{2,1} = 0$, $t_2 = 0$ and $c_{1,2} = 0$, $t_1 = 0$, respectively (see Lemma 4.1). Thus, when $2 \le n < 473$ the sign of μ_1 could change at the curves $z_2 = g(z_1)$, $z_2 = 0$, $z_2 = \pm z_\beta$, and the sign of μ_2 could change at the curves $z_2 = f(z_1)$ and $z_1 = z_\beta$. When $n \ge 473$ the sign of μ_1 could change at the curves $z_2 = g(z_1)$ and $z_2 = 0$, and the sign of μ_2 could change at the curve $z_2 = f(z_1)$.

We note that the shape of the regions delimited by the curves $z_2 = g(z_1)$, $z_2 = 0$, $z_2 = \pm z_\beta$, $z_2 = f(z_1)$, and $z_1 = z_\beta$ depends on the value of z_β. More precisely, if the value of $z_1 = z_\beta$ belongs to either the interval $(0, 1/\sqrt{3})$, the interval $(1/\sqrt{3}, 1)$, or the interval $(1, +\infty)$; and if the value of $z_2 = -z_\beta$ belongs to either the interval $(-\infty, -1)$, the interval $(-1, -1/\sqrt{3})$ or the interval $(-1/\sqrt{3}, 0)$ (see Figure 1).

In [13] it is proved that the sequence $\{\beta_n/n\}_{n\in\mathbb{N}}$ is increasing, so $\{n/\beta_n\}_{n\in\mathbb{N}}$ is decreasing, and consequently the value of z_β decreases as n increases. We compute the sequence of values of z_β and we see that the first n such that $z_\beta < 1$ is $n = 9$, and that the first n such that $z_\beta < 1/\sqrt{3}$ is $n = 53$. In particular, there is no n such that either $z_\beta = 1$ or $z_\beta = 1/\sqrt{3}$. Therefore $z_\beta > 1$ for $2 \le n < 9$, $z_\beta \in (1/\sqrt{3}, 1)$ for $9 \le n < 53$ and $z_\beta \in (0, 1/\sqrt{3})$ for $53 \le n < 473$. Remember that z_β is not defined for $n \ge 473$. In short, we have four possible shapes for the regions delimited by the curves $z_2 = g(z_1)$, $z_2 = 0$, $z_2 = \pm z_\beta$, $z_2 = f(z_1)$ and $z_1 = z_\beta$, one for $2 \le n < 9$, one for $9 \le n < 53$, one for $53 \le n < 473$, and finally one for $n \ge 473$. These regions are plotted in Figures 2 and 3.

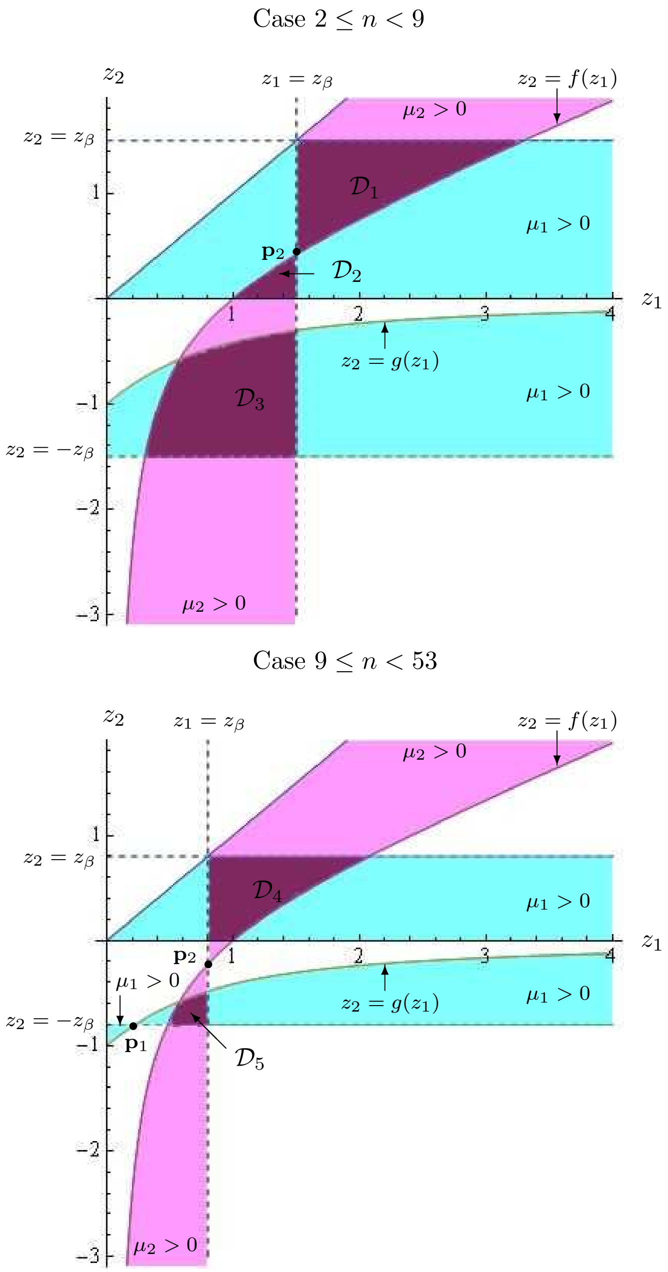
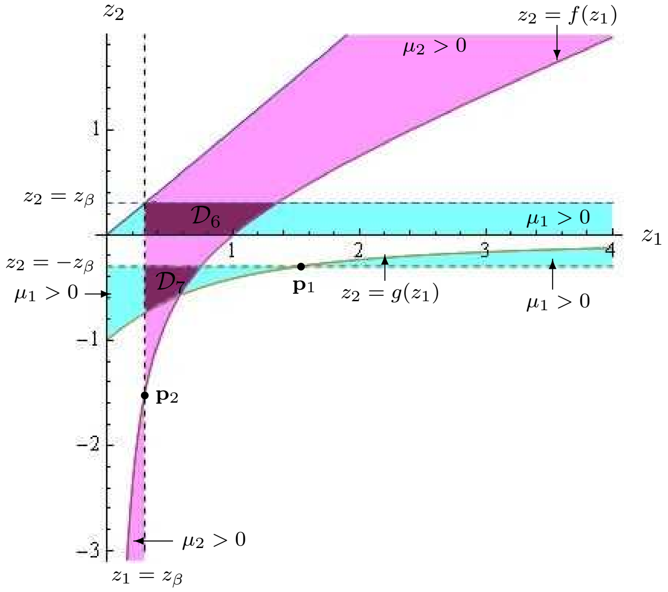


FIGURE 2. The regions delimited by the curves $z_2 = g(z_1)$, $z_2 = 0$, $z_2 = \pm z_\beta$, $z_2 = f(z_1)$, and $z_1 = z_\beta$ restricted to $\{(z_1, z_2) \in \mathbb{R}^2 : z_1 > 0, z_1 > z_2\}$, and the sign of μ_1 and μ_2 on each region for $n < 53$.

Case $53 \leq n < 473$



Case $n \geq 473$

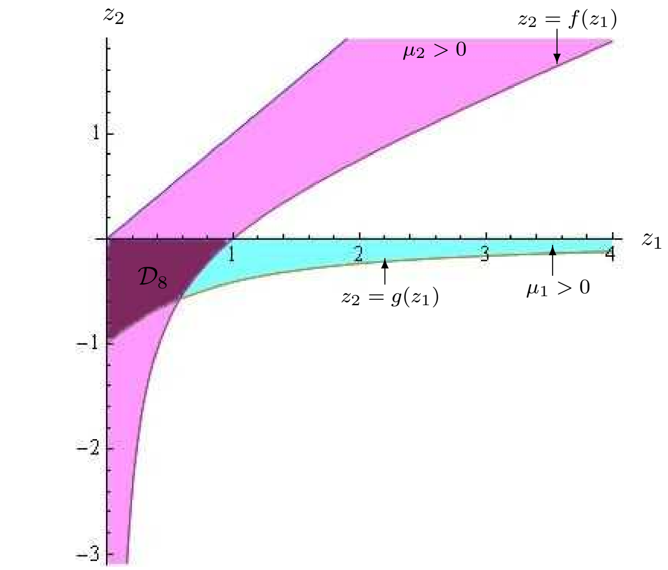


FIGURE 3. The regions delimited by the curves $z_2 = g(z_1)$, $z_2 = 0$, $z_2 = \pm z_\beta$, $z_2 = f(z_1)$, and $z_1 = z_\beta$ restricted to $\{(z_1, z_2) \in \mathbb{R}^2 : z_1 > 0, z_1 > z_2\}$, and the sign of μ_1 and μ_2 on each region for $n \geq 53$.

We note that the curves $z_2 = f(z_1)$ and $z_2 = g(z_1)$ intersect at the unique point $\mathbf{p}^*$. Since there is no n such that $z_\beta = 1/\sqrt{3}$, $t_1 \neq 0$ and $t_2 \neq 0$ when $(z_1, z_2) = \mathbf{p}^*$, this means that there are no solutions of system (9) of type (D) (see Lemma 4.1).

The curves $z_2 = g(z_1)$ and $z_2 = -z_\beta$ do not intersect when $2 \leq n < 9$ and they intersect at a unique point $\mathbf{p}_1 = \Big((1 - z_\beta^2)/(2\, z_\beta), -z_\beta\Big)$ when $9 \leq n < 473$ (see Figures 2 and 3). Since $z_2 = g(z_1)$ does not intersect with $z_2 = 0$ and $z_2 = z_\beta$ we have that the curves $c_{2,1} = 0$ and $t_2 = 0$ only intersect when $9 \leq n < 473$ and the intersection point is $\mathbf{p}_1$.

The curves $z_2 = f(z_1)$ and $z_1 = z_\beta$ intersect at a unique point $\mathbf{p}_2 = (z_\beta, f(z_\beta))$ for all $2 \leq n < 473$ (see again Figures 2 and 3). So the curves $c_{1,2} = 0$ and $t_1 = 0$ intersect at the point $\mathbf{p}_2$ for all $2 \leq n < 473$.

By means of Lemma 4.2, we compute the signs of μ_1 and μ_2 on all the regions delimited by the curves $z_2 = g(z_1)$, $z_2 = 0$, $z_2 = \pm z_\beta$, $z_2 = f(z_1)$ and $z_1 = z_\beta$ for each one of the four shapes depending on n. The regions on which $\mu_1 > 0$, $\mu_2 > 0$ and $\mu_1, \mu_2 > 0$ are shaded in Figures 2 and 3.

Notice that the intersection point $\mathbf{p}_1$ of the curves $c_{2,1} = 0$ and $t_2 = 0$ does not belong to the region where $\mu_2 > 0$. Therefore there are no solutions of system (9) of type (C) (see Lemma 4.1) providing central configurations.

The intersection point $\mathbf{p}_2$ of the curves $c_{1,2} = 0$ and $t_1 = 0$ belongs to the region with $\mu_1 > 0$ only when $2 \leq n < 9$. Therefore system (9) has solutions of type (B) (see Lemma 4.1) that provide central configurations only when $2 \leq n < 9$. It is easy to check that the value of μ_1 at the point $(z_1, z_2) = \mathbf{p}_2$ is

$$\mu_1 = n - 2(n\beta_n^2)^{1/3}.$$

In short, we have proved the following theorem.

Theorem 4.3. *Let*

$$\begin{aligned}
F(n, z_1, z_2) &= \left(\beta_n\, z_2 - n\,\frac{z_2}{(1+z_2^2)^{3/2}}\right) \Big/ \left(-\frac{1}{(z_1-z_2)^2} + \frac{z_1 - z_2}{(1+z_1^2)^{3/2}}\right), \\
G(n, z_1, z_2) &= \left(\beta_n\, z_1 - n\,\frac{z_1}{(1+z_1^2)^{3/2}}\right) \Big/ \left(\frac{1}{(z_1-z_2)^2} + \frac{z_2 - z_1}{(1+z_2^2)^{3/2}}\right).
\end{aligned}$$

Let $f(z_1) = (z_1^2 - 1)/(2\, z_1)$, $g(z_1) = z_1 - \sqrt{1 + z_1^2}$ *and* $z_\beta = \sqrt{(n/\beta_n)^{2/3} - 1}$,

$$\begin{aligned}
\mathcal{D}_1 &= \{(z_1, z_2) \in \mathbb{R}^2 \,:\, z_1 \in \big(z_\beta, f^{-1}(z_\beta)\big)\,,\; z_2 \in (f(z_1), z_\beta)\}, \\
\mathcal{D}_2 &= \{(z_1, z_2) \in \mathbb{R}^2 \,:\, z_1 \in (1, z_\beta)\,,\; z_2 \in (0, f(z_1))\}, \\
\mathcal{D}_3 &= \{(z_1, z_2) \in \mathbb{R}^2 \,:\, z_1 \in \Big(f^{-1}(-z_\beta), 1/\sqrt{3}\Big)\,,\; z_2 \in (-z_\beta, f(z_1))\} \cup \\
&\quad\; \{(z_1, z_2) \in \mathbb{R}^2 \,:\, z_1 \in \Big(1/\sqrt{3}, z_\beta\Big)\,,\; z_2 \in (-z_\beta, g(z_1))\}, \\
\mathcal{D}_4 &= \{(z_1, z_2) \in \mathbb{R}^2 \,:\, z_1 \in \big(z_\beta, f^{-1}(z_\beta)\big)\,,\; z_2 \in (0, z_\beta)\}, \\
\mathcal{D}_5 &= \{(z_1, z_2) \in \mathbb{R}^2 \,:\, z_1 \in \Big(f^{-1}(-z_\beta), 1/\sqrt{3}\Big)\,,\; z_2 \in (-z_\beta, f(z_1))\} \cup \\
&\quad\; \{(z_1, z_2) \in \mathbb{R}^2 \,:\, z_1 \in \Big(1/\sqrt{3}, z_\beta\Big)\,,\; z_2 \in (-z_\beta, g(z_1))\ \ ,
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_6 &= \{(z_1,z_2)\in\mathbb{R}^2 \,:\, z_1\in\left(z_\beta, f^{-1}(z_\beta)\right),\ z_2\in(0,z_\beta)\},\\
\mathcal{D}_7 &= \{(z_1,z_2)\in\mathbb{R}^2 \,:\, z_1\in\left(z_\beta, 1/\sqrt{3}\right),\ z_2\in(g(z_1),-z_\beta)\}\cup\\
&\quad\ \{(z_1,z_2)\in\mathbb{R}^2 \,:\, z_1\in\left(1/\sqrt{3}, f^{-1}(-z_\beta)\right),\ z_2\in(f(z_1),-z_\beta)\},\\
\mathcal{D}_8 &= \{(z_1,z_2)\in\mathbb{R}^2 \,:\, z_1\in\left(0, 1/\sqrt{3}\right),\ z_2\in(g(z_1),0)\}\cup\\
&\quad\ \{(z_1,z_2)\in\mathbb{R}^2 \,:\, z_1\in\left(1/\sqrt{3}, 1\right),\ z_2\in(f(z_1),0)\},
\end{aligned}$$

(see Figures 2 and 3).

The configuration $(q_1,\dots,q_{n+2}) = \left((e^{i\,\alpha_1},0),\dots,(e^{i\,\alpha_n},0),(0,z_1),(0,z_2)\right)$ *is central for the* $n+2$*–body problem in the following cases:*

- *(a) If* $2\le n<9$*,*
  - *(i)* $(z_1,z_2)\in\mathcal{D}_1\cup\mathcal{D}_2\cup\mathcal{D}_3$ *(see Figure 2),* $\mu_1=F(n,z_1,z_2)$ *and* $\mu_2=G(n,z_1,z_2)$*.*
  - *(ii)* $z_1=z_\beta$, $z_2=f(z_\beta)$, $\mu_1=n-2(n\beta_n^2)^{1/3}$ *and* $\mu_2\in(0,+\infty)$*.*
- *(b) If* $9\le n<53$, $(z_1,z_2)\in\mathcal{D}_4\cup\mathcal{D}_5$ *(see Figure 2),* $\mu_1=F(n,z_1,z_2)$ *and* $\mu_2=G(n,z_1,z_2)$*.*
- *(c) If* $53\le n<473$, $(z_1,z_2)\in\mathcal{D}_6\cup\mathcal{D}_7$ *(see Figure 3),* $\mu_1=F(n,z_1,z_2)$ *and* $\mu_2=G(n,z_1,z_2)$*.*
- *(d) If* $n\ge 473$, $(z_1,z_2)\in\mathcal{D}_8$ *(see Figure 3),* $\mu_1=F(n,z_1,z_2)$ *and* $\mu_2=G(n,z_1,z_2)$*.*

Of course there are the symmetric central configurations obtained changing z_1 *and* z_2 *to* $-z_1$ *and* $-z_2$ *respectively.*

We note that the positions of the masses $m_i=1$ for $i=1,\dots,n$, $m_{n+1}=\mu_1$ and $m_{n+2}=\mu_2$ at the configurations given by Theorem 4.3 (a)(ii) are

$$\begin{aligned}
&\mathbf{q}_k=(\cos(2\pi\,k/n),\sin(2\pi\,k/n),0)\qquad \text{for } k=1,\dots,n,\\
&\mathbf{q}_{n+1}=(0,0,z_\beta),\qquad \mathbf{q}_{n+2}=(0,0,f(z_\beta)).
\end{aligned}$$

It is not difficult to check that

$$|\mathbf{q}_{n+2}-\mathbf{q}_k| = \frac{z_\beta^2+1}{2\,z_\beta}\quad \text{for } k=1,\dots,n+1.$$

We note that in this configuration the arbitrary mass $m_{n+2}=\mu_2$ is located at the same distance from all other masses.

We have done some numerical computations in order to see how many classes of bi-pyramidal central configurations there are for given values of the two masses μ_1 and μ_2. As usual we denote by z_i the z–coordinate of the mass μ_i for $i=1,2$, and we assume that $z_1>z_2$ with $z_1>0$. Our numerical results provide evidence for the following claims:

- If $3<n\le 473$ then it can be 1, 2 or 3 classes of bi-pyramidal central configurations, one always having negative z–coordinate and the others if exist have positive z–coordinate.
- If $n>473$ then there is only 1 class of bi-pyramidal central configurations.

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E-mail address: montserrat.corbera@uvic.cat
E-mail address: jllibre@mat.uab.cat