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QUALITATIVE STUDY OF A CHARGED RESTRICTED THREE–BODY PROBLEM

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Abstract. We characterize the global flow of the restricted three-body problem in which we have two mass points of equal masses $m_1 = m_2 > 0$ and equal charge $q_1 = q_2$ moving on a straight line under Newtonian and Coulombian forces with their centre of mass being fixed at the origin of coordinates and the third mass point with infinitesimal mass, is moving on the straight line perpendicular to the line of motion of the first two mass points and passing through their centre of mass.

1. Introduction

The classification of all the possible qualitative motions for the Newtonian three–body problem is an open and difficult problem. This leads to study several restricted subproblems, like the restricted three-body problem where one of the masses is assumed to be infinitesimally small. We consider the case when the two positive masses are equal, have the same charge and are moving under the respective Newtonian gravitational and Coulombian forces in an orbit on the x–axis while their center of mass is fixed at the origin of coordinates and the infinitesimal mass point is moving on the y–axis.

More precisely, let $m_1 = m_2$ be two mass points with the same charge $q_1 = q_2$ moving under the influence of the respective Newtonian gravitational and Coulombian forces in an orbit on the x–axis while their center of mass is fixed at the origin of coordinates. We consider a third mass point with infinitesimal mass and no charge moving on the y–axis. As usual the two bodies with masses m_1 and m_2 are called the *primaries*. Since $m_3 = 0$ the motion of the two primaries is not affected by the third body, and from the symmetry of the motion it is clear that the third mass point will remain in the y–axis. Taking the units of mass, charge, length and time conveniently we can assume that $m_1 = m_2 = 1$, $q_1 = q_2 = \sqrt{2}$, and that the gravitational constant

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and the Coulombian's constant are equal to one. The problem is to study the motion of the infinitesimal mass, and consequently we have a *charged restricted three–body problem*. This problem has no collisions because the repulsion Coulombian force between the primaries is bigger than the Newtonian attraction force when the primaries are close.

The charged three–body problem (but not restricted, i.e. with the three masses positive) has been studied recently from other points of view. Thus in the planar case the spectral stability of the equilateral equilibrium points has been studied in [7], also in the planar case the linear stability of its relative equilibria has been analyzed in [1], and in the collinear case the motion near total collision has been considered in [3].

A similar problem without charged masses was studied in [3, 4] where the authors study the qualitative behavior of the collinear restricted three–body problem with the two positive masses moving in a hyperbolic collision orbit and the infinitesimal mass point is moving on a straight line orthogonal to the motion of the primaries and passing through its center of mass. Since in our restricted three–body problem collisions between masses are not possible its dynamics is less rich than the dynamics of the restricted three–body problem studied in [3, 4].

The equations of motion of this problem in the phase space $(y, \dot{y}, t)$ are given in section 2. Every solution of this problem is defined for all time, that is, from $-\infty$ to ∞.

Initially the equations of motion of the charged three–body problem are not defined in the singularities of collision and at infinity. Then for studying the flow near and on these singularities in section 3 we extend the flow of the system to the boundaries manifold of its phase space, i.e. to its singularities of collision and infinity. For this, we use a transformation introduced by Wang [10], see also Meyer and Wang [9], based in the potential function instead of the moment of inertia as a scale factor, as it was done by McGehee [8]. By using Wang's transformation we extend analytically the flow from the initial phase space, without including the singularities of collision and infinity, to two partially compact new phase spaces including such singularities, one which corresponds to $t > 0$ and the other to $t < 0$. We perform two changes of variable to add these boundaries to the initial phase space and we give a description of the flow on them.

In section 4 we will prove that the only possible final evolutions of a solution $y(t)$ of the infinitesimal mass when $t \to \pm\infty$ either comes or escapes to infinity with the same non–zero velocity ($\mathcal{H}$ hyperbolic

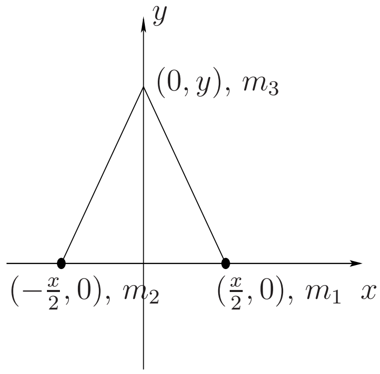

FIGURE 1

motion) or tends to a finite position with zero velocity ($\mathcal{P}$ parabolic motion).

In section 5 we study the manifold formed by parabolic orbits.

From the analytic results of sections 2–5 together with the numerical ones of section 6 we obtain the description of the global dynamics of the charged restricted three–body problem which is summarized in the conclusions section at the end of the paper.

2. Equations of motion

In this section we deduce the equations of motion of the charged restricted three–body problem.

Let x denote the distance between the primaries m_1 and m_2, so $(x/2, 0)$ denotes the position of m_1 and consequently $(-x/2, 0)$ that of m_2. If we denote by $(0, y)$ the position of m_3 (see Figure 1), then the equations of motion for x and y are

$$\begin{aligned} \ddot{x} &= \frac{2}{x^2}, \\ \ddot{y} &= -\frac{16y}{(x^2+4y^2)^{3/2}}, \end{aligned} \tag{1}$$

where the dot denotes derivative with respect to the time t. For more details on the obtention of the equations of motion (1) see for instance [1] or [7].

We note that the equations of motion (1) are invariant under the symmetry

$$(x, y, t) \to (x, -y, -t). \tag{2}$$

So if $y(t)$ is a solution for the motion of the infinitesimal mass, then $-y(-t)$ is another solution.

Two times the first integral due to the energy of the primaries is

$$\dot{x}^2 + \frac{4}{x} = h. \tag{3}$$

Note that h only can take positive values.

For obtaining the motion of the primaries we introduce a new time variable τ through $dt = x d\tau$. Then, the equation of the energy integral (3) is transformed into

$$(x')^2 + 4x = hx^2, \tag{4}$$

where the prime indicates differentiation with respect to τ. The solution $x(\tau)$ of (4) such that $x(0) = 4/h$ is

$$x(\tau) = \frac{2(1 + \cosh(\sqrt{h}\,\tau))}{h}.$$

Finally, using the new time $s = \sqrt{h}\,\tau$, we have that the motion of the primaries is given by

$$\begin{aligned} x(s) &= \frac{2\,(1 + \cosh s)}{h}, \\ t(s) &= \frac{2(s + \sinh s)}{h^{3/2}}. \end{aligned} \tag{5}$$

In short, we need to study the orbits of a vector field in the three–dimensional space $(y, \dot{y}, t)$. More explicitly, the equation of motion of the *charged restricted three–body problem* is

$$\ddot{y} = -\frac{16y}{(x(t)^2 + 4y^2)^{3/2}} \tag{6}$$

where $x(t)$ is given by (5).

The phase space where the charged restricted three–body problem is defined is open. In the next section we shall extend the flow of the boundaries of this open phase space. In this way we can study better the motion of charged restricted three–body problem near and on its singularities, collision and infinity.

3. Boundary manifolds

In order to extend the flow of the system to the boundary manifold of its phase space we use a transformation introduced by Wang [10], see also Meyer and Wang [9], based in the potential function instead of the moment of inertia as a scale factor, as it was done by McGehee [8].

We do the change of coordinates

$$(x, y, \dot{x}, \dot{y}, t) \mapsto (F_1, F_2, G_1, G_2, \tau'),$$

given by

$$\begin{aligned} F_1 &= u^{-1}x, & G_1 &= u^{1/2}\dot{x}, \\ F_2 &= u^{-1}y, & G_2 &= u^{1/2}\dot{y}, \\ & & dt &= u^{3/2}d\tau', \end{aligned} \tag{7}$$

where $u^{-1} = h - 4/x = \dot{x}^2 \geq 0$.

From $u^{-1} = h - 4/x$ and (7) we get

$$\frac{d}{dt}\Big(\frac{1}{u}\Big) = \frac{4}{x^2}\frac{dx}{dt} = \frac{4G_1}{u^{\frac{5}{2}}F_1^2}.$$

From (7) and the above relation we get:

$$\begin{aligned} \frac{du}{d\tau'} &= \frac{du}{dt}u^{3/2} = -u^2\frac{d}{dt}\Big(\frac{1}{u}\Big)u^{3/2} = -\frac{4uG_1}{F_1^2}, \\ \frac{dG_2}{d\tau'} &= \frac{dG_2}{dt}u^{3/2} = \Big(\frac{1}{2}\frac{1}{\sqrt{u}}\frac{du}{dt}\frac{G_2}{\sqrt{u}} + \sqrt{u}\frac{d^2y}{dt^2}\Big)u^{e/2} \\ &= -\frac{2G_1G_2}{F_1^2} - \frac{16F_2}{(F_1^2+4F_2^2)^{3/2}}, \\ \frac{dF_1}{d\tau'} &= \frac{dF_1}{dt}u^{3/2} = \Big(\frac{dx}{dt}\frac{1}{u} + x\frac{d}{dt}\Big(\frac{1}{u}\Big)\Big)u^{3/2} \\ &= \Big(\frac{G_1}{u^{3/2}} + \frac{4uF_1G_1}{u^{5/2}F_1^2}\Big)u^{3/2} = G_1 + \frac{4G_1}{F_1}, \end{aligned}$$

$$\begin{aligned} \frac{dF_2}{d\tau'} &= \frac{dF_2}{dt}u^{3/2} = \Big(\frac{dy}{dt}\frac{1}{u} + y\frac{d}{dt}\Big(\frac{1}{u}\Big)\Big)u^{3/2} \\ &= \Big(\frac{G_2}{u^{3/2}} + \frac{4uG_1F_2}{u^{5/2}F_1^2}\Big)u^{3/2} = G_2 + \frac{4G_1F_2}{F_1^2}. \end{aligned}$$

According to the energy equation (3) for the primaries, $G_1^2 = 1$. We note that $G_1 = 1$ corresponds to the case $\dot{x} > 0$ and $G_1 = -1$ to $\dot{x} < 0$. In what follows we denote $G = G_2$ and when appears two signs together as $\pm$ or $\mp$, the upper corresponds to $G_1 = 1$ and the lower to

$G_1 = -1$. We get the following relations:

$$
(8) \qquad
\begin{aligned}
\frac{du}{d\tau'} &= \mp\frac{4u}{F_1^2},\\
\frac{dG}{d\tau'} &= \mp\frac{2G}{F_1^2} - \frac{16F_2}{(F_1^2+4F_2^2)^{3/2}},\\
\frac{dF_1}{d\tau'} &= \pm 1 + \pm\frac{4}{F_1},\\
\frac{dF_2}{d\tau'} &= G \pm \frac{4F_2}{F_1^2}.
\end{aligned}
$$

We introduce pseudo–polar coordinates through the change

$$(F_1, F_2, G, \tau') \mapsto (\varphi, G, \xi, \tau),$$

defined by

$$
(9) \qquad
\begin{aligned}
F_1 &= r\sin\varphi, & \xi &= 1 - \frac{1}{uh},\\
F_2 &= \frac{r}{2}\cos\varphi, & d\tau' &= \frac{2}{\xi}d\tau,
\end{aligned}
$$

where $0 < \varphi < \pi$, $G = G_2 \in \mathbb{R}$ and $0 < \xi \le 1$.

We want to obtain the equations of motion in the coordinates (φ, G, ξ) for $G_1 = 1$. In a similar way can be obtain the equations of motion for $G_1 = -1$.

Now using (8) and (9) we can get the equations of motion in the coordinates (φ, G, ξ):

$$\frac{d\varphi}{d\tau} = \frac{d\varphi}{d\tau'}\frac{d\tau'}{d\tau} = \frac{2}{\xi}\frac{d\varphi}{d\tau'},$$

where

$$\frac{d\varphi}{d\tau'} = \frac{d}{d\tau'}\Big(\cot^{-1}\Big(\frac{2F_2}{F_1}\Big)\Big) = -\frac{2}{r}\Big(G\sin\varphi \mp \frac{1}{2}\cos\varphi\Big),$$

and taking into account that

$$F_1 = \frac{x}{u} = \frac{4(1-\xi)}{\xi} \quad \text{and} \quad \frac{1}{r} = \frac{\xi\sin\varphi}{4(1-\xi)},$$

we get

$$
\begin{aligned}
\frac{d\varphi}{d\tau} &= \frac{\sin\varphi}{1-\xi}\Big(\pm\frac{1}{2}\cos\varphi - G\sin\varphi\Big),\\
\frac{dG}{d\tau} &= \frac{dG}{d\tau'}\frac{d\tau'}{d\tau} = \frac{4}{\xi}\Big(\mp\frac{G}{F_1^2} - \frac{8F_2}{(F_1^2+4F_2^2)^{3/2}}\Big) = -\frac{\xi}{4(1-\xi)^2}(4\sin^2\varphi\cos\varphi \pm G),\\
\frac{d\xi}{d\tau} &= \frac{d\xi}{d\tau'}\frac{d\tau'}{d\tau} = \frac{2}{\xi}\frac{1}{hu^2}\frac{du}{d\tau'} = \mp\frac{8}{\xi h u F_1^2} = \mp\frac{\xi}{2(1-\xi)}.
\end{aligned}
$$

So, finally we can write the system of equations of motion in the new coordinates as follows

$$
\begin{aligned}
\frac{d\varphi}{d\tau} &= -\frac{\sin\varphi}{1-\xi}\Big(G\sin\varphi - \frac{1}{2}\cos\varphi\Big),\\
\frac{dG}{d\tau} &= -\frac{\xi}{4(1-\xi)^2}(G + 4\sin^2\varphi\cos\varphi),\\
\frac{d\xi}{d\tau} &= -\frac{\xi}{2(1-\xi)},
\end{aligned}
\tag{10}
$$

when $G_1 = 1$. In the same way we can obtain the equations of motion in the variables (G, ξ, τ)

$$
\begin{aligned}
\frac{d\varphi}{d\tau} &= -\frac{\sin\varphi}{1-\xi}\Big(G\sin\varphi + \frac{1}{2}\cos\varphi\Big),\\
\frac{dG}{d\tau} &= \frac{\xi}{4(1-\xi)^2}(G - 4\sin^2\varphi\cos\varphi),\\
\frac{d\xi}{d\tau} &= \frac{\xi}{2(1-\xi)},
\end{aligned}
\tag{11}
$$

when $G_1 = -1$.

Now if we change $d\tau = (1-\xi)^2 d\sigma$, we get

$$
\begin{aligned}
\frac{d\varphi}{d\sigma} &= -(1-\xi)\sin\varphi\Big(G\sin\varphi - \frac{1}{2}\cos\varphi\Big),\\
\frac{dG}{d\sigma} &= -\frac{\xi}{4}\big(G + 4\sin^2\varphi\cos\varphi\big),\\
\frac{d\xi}{d\sigma} &= -\frac{\xi(1-\xi)}{2},
\end{aligned}
\tag{12}
$$

when $G_1 = 1$ and

$$
\begin{aligned}
\frac{d\varphi}{d\sigma} &= -(1-\xi)\sin\varphi\Big(G\sin\varphi + \frac{1}{2}\cos\varphi\Big),\\
\frac{dG}{d\sigma} &= \frac{\xi}{4}\big(G - 4\sin^2\varphi\cos\varphi\big),\\
\frac{d\xi}{d\sigma} &= \frac{\xi(1-\xi)}{2},
\end{aligned}
\tag{13}
$$

when $G_1 = -1$. Notice that equations (12) and (13) are naturally extended to $[0, \pi] \times \mathbb{R} \times [0, 1]$.

From (5) if $t = 0$ then $\dot{x} = 0$, if $t > 0$ then $\dot{x} > 0$, and if $t < 0$ then $\dot{x} < 0$. So, when $G_1 = 1$ we have that $t > 0$, and when $G_1 = -1$ we have that $t < 0$.

We start with a technical lemma about the asymptotical behavior of an orbit in the charged two–body problem.

Lemma 1. *If $x(t)$ is a solution of the charged two–body problem with energy $h > 0$, then*

$$x(t) = \sqrt{h}\, t \qquad \text{when } t \to \pm\infty.$$

Proof. From equation (5) we have that $t(s) = \dfrac{2}{h^{3/2}} \sinh s$ when $s \to \pm\infty$, then $\sinh s = \dfrac{h^{3/2} t}{2}$ when $t \to \pm\infty$. So, again from (5) we have that

$$x(t) = \frac{2}{h}\left(1 + \sqrt{1 + \left(\frac{h^{3/2} t}{2}\right)^2}\right).$$

Hence $x(t) = \sqrt{h}\, t$ when $t \to \pm\infty$. □

Proposition 2. *When the variable t tends to $\pm\infty$ (i.e. the primaries tend to infinity), then the new time σ tends to $\pm\infty$.*

Proof. We know that

$$dt = u^{3/2} d\tau' = u^{3/2} \frac{2}{\xi} d\tau = \frac{2}{h^{3/2}} \frac{\sqrt{1-\xi}}{\xi} d\sigma.$$

Integrating this equality we obtain that

$$\frac{h^{3/2}}{2} \int_{t_0}^{\infty} \frac{\xi}{\sqrt{1-\xi}} dt = \int_{\sigma_0}^{\tilde{\sigma}} d\sigma = \tilde{\sigma} - \sigma_0.$$

We evaluate the integral as follows. First, from (7) and (9) we get that $\xi = \dfrac{4}{hx}$ and from Lemma 1 we have that $\xi = \dfrac{4}{h^{3/2} t}$ when $t \to \infty$. So $\xi \to 0$ when $t \to \infty$. Hence for t_0 sufficiently large we have the following approximation to our integral:

$$\int_{t_0}^{\infty} \frac{\xi}{\sqrt{1-\xi}} dt \simeq \int_{t_0}^{\infty} \xi dt \simeq \frac{4}{h^{3/2}} \int_{t_0}^{\infty} \frac{1}{t} dt = \infty.$$

So $\tilde{\sigma} \to \infty$ when $t \to \infty$.

Similarly $\tilde{\sigma} \to -\infty$ when $t \to -\infty$. □

The boundary manifolds created by equations (12) and (13) are: $\xi = 1$ which corresponds to the flow when the distance between the primaries is the smallest one; $\xi = 0$ which corresponds to the infinity manifold associated with the primaries when $t \to \pm\infty$, see Proposition 2; and $\varphi = 0$ or $\varphi = \pi$, which give us a fictitious flow because $\varphi = 0$ or

$\varphi = \pi$ means that the triangle formed by m_1, m_2 and m_3 degenerates to a line on the y–axis with m_3 at infinity.

Now we study the phase portraits on the boundaries $\xi = 0$, $\xi = 1$, $\varphi = 0$ and $\varphi = \pi$ when $G_1 = 1$. Using (12) we start by writing the equations of motion on each of these boundaries:

1. On the boundary $\xi = 0$ we have
$$\frac{d\varphi}{d\sigma} = -\sin\varphi\Big(G\sin\varphi - \frac{1}{2}\cos\varphi\Big), \qquad G = \text{constant},$$
and the phase portrait of this differential system is topologically equivalent to the one on the face $\xi = 0$ of Figure 2. The phase portrait in the closed strip $\xi = 0$ has two straight lines filled of equilibria in the boundaries of the strip, and a curve filled of equilibria in the interior of the strip. From every equilibrium point living in one of the straight lines an orbit exists and ends in one of the equilibria of the curve filled of equilibria.
2. On the boundary $\xi = 1$ we have
$$\varphi = \text{constant}, \qquad \frac{dG}{d\sigma} = -\frac{1}{4}\big(G + 4\sin^2\varphi\cos\varphi\big),$$
and the phase portrait of this differential system is topologically equivalent to the one on the face $\xi = 1$ of Figure 2. The phase portrait in the closed strip $\xi = 1$ has a curve filled of equilibria, to each of such equilibria arrives two orbits, see Figure 2.
3. On the boundaries $\varphi = 0$ and $\varphi = \pi$ we have
$$\frac{dG}{d\sigma} = -\frac{\xi G}{4}, \qquad \frac{d\xi}{d\sigma} = -\frac{\xi(1-\xi)}{2},$$
from where we get
$$\frac{dG}{d\xi} = \frac{G}{2(1-\xi)}.$$
This differential system has the first integral $H = G\sqrt{1-\xi}$. Then, the phase portrait of this differential system on the faces $\varphi = 0$ and $\varphi = \pi$ is topologically equivalent to the corresponding ones of Figure 2.

We conclude that the phase portrait of the differential system (12) on the boundaries of its domain of definition $\{(\varphi, G, \xi) \in [0,\pi]\times\mathbb{R}\times[0,1]\}$

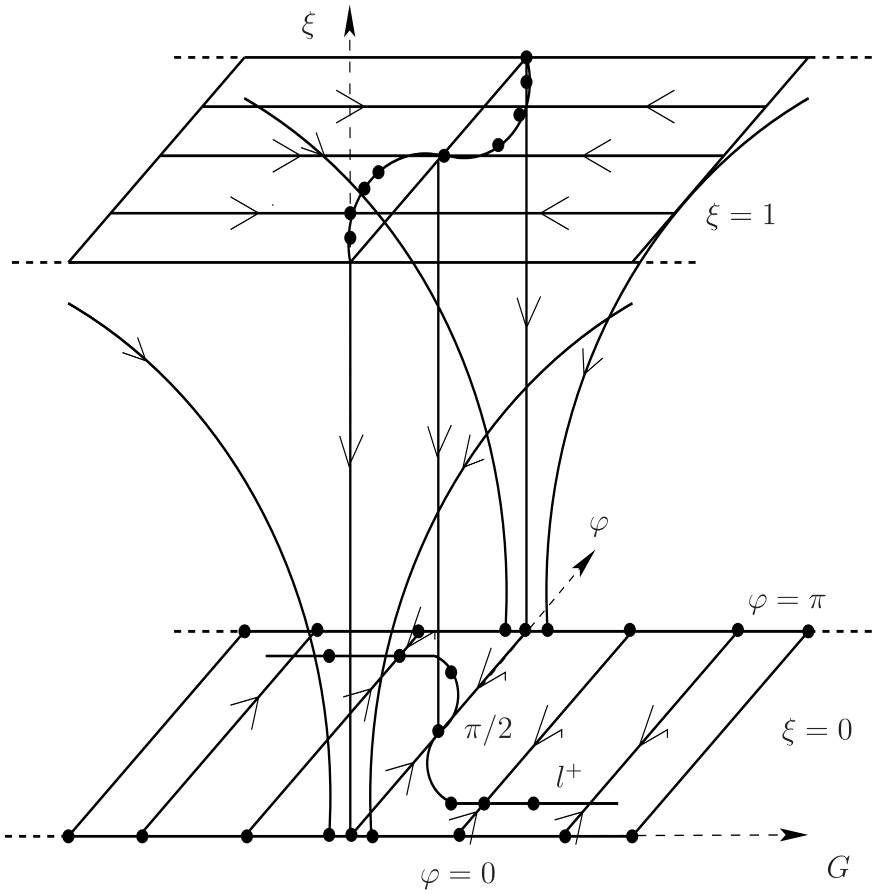

FIGURE 2. The phase portrait of system (12) on its boundaries.

is topologically equivalent to the phase portrait on the boundaries of Figure 2. A similar phase portrait can be obtained for the differential system (13).

4. Final evolutions

Let $D_{\pm} = \{(\varphi, G, \xi) \in [0,\pi] \times \mathbb{R} \times [0,1]$ with $G_1 = \pm 1\}$. Given a point $p \in D_{\pm}$, we have an initial position for the primaries at time $t = t_0$, where $t_0 > 0$ if $p \in D_+$ and $t_0 < 0$ if $p \in D_-$. We denote by $O_+(p)$ (respectively $O_-(p)$) the solution of system (12) (respectively (13)) which passes through p when $t = t_0$. We fix the new time $\sigma = 0$ when $t = t_0$. So, we can write

$$O_{\pm}(p) = \{(\varphi(p,\sigma), G(p,\sigma), \xi(p,\sigma)) : \text{ for all } \sigma \in \mathbb{R}\}.$$

We denote by $O^+_\pm(p)$ (respectively $O^-_\pm(p)$) the positive (respectively negative) semi–orbit of p, and by $\omega(p)$ the ω–limit set of $O_\pm(p)$ and by $\alpha(p)$ the α–limit set of $O_\pm(p)$. For definitions of positive or negative semi–orbit, and α– or ω–limit set, see for instance [5]. We also denote

$$
\begin{aligned}
l^+ = & \left\{(\varphi, G, 0) : G = \frac{1}{2}\cot\varphi\right\} \\
P^+ = & \left(\frac{\pi}{2}, 0, 0\right) \in \Omega^+, \\
\mathcal{L}^+ = & \; l^+ \setminus P^+.
\end{aligned}
$$

Note that l^+ is the curve formed by the equilibrium points in the boundary $\xi = 0$ of the differential system (12). The sets l^-, P^- and $\mathcal{L}^-$ are defined in a similar way. As usual we denote by $\mathrm{Int}(D_\pm)$ the interior points of $D_\pm$.

Let ϕ_t be a smooth flow on a manifold M and suppose C is a submanifold of M consisting entirely of equilibrium points for the flow. C is said to be *normally hyperbolic* if the tangent bundle to M over C splits into three subbundles TC, E^s, E^u invariant under $d\phi_t$ and satisfying

(i) $d\phi_t$ contracts E^s exponentially,
(ii) $d\phi_t$ expands E^u exponentially,
(iii) $TC =$ tangent bundle of C.

For normally hyperbolic submanifolds one has the usual existence of smooth stable and unstable manifolds together with the persistence of these invariant manifolds under small perturbations. More precisely, we have the following theorem.

Theorem 3. *Let C be a normally hyperbolic submanifold of equilibrium points for ϕ_t. Then there exist smooth stable and unstable manifolds tangent along C to $E^s \oplus TC$ and $E^u \oplus TC$, respectively. Furthermore, both C and the stable and unstable manifolds are permanent under small perturbations of the flow.*

For a proof of Theorem 3 see [6].

Proposition 4. *For any $p \in \mathrm{Int}(D_+)$, $\omega(p)$ is one of the equilibrium points of l^+.*

Proof. Since $d\xi/d\sigma \leq 0$, from Figure 2, $\omega(p)$ is an equilibrium point either of $\{\varphi = 0\} \cap \{\xi = 0\}$, or of $\{\varphi = \pi\} \cap \{\xi = 0\}$, or of l^+. But l^+ is normally hyperbolic with a stable manifold of dimension three, because the linear part of system (12) at l^+ is given by the matrix

$$\begin{pmatrix} -\frac{1}{2} & -\sin^2\varphi & 0 \\ 0 & 0 & -\frac{1}{8}\cot\varphi(1+8\sin^3\varphi) \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

On the other hand, the lines $\varphi = 0$ or $\varphi = \pi$ are also normally hyperbolic with one stable manifold of dimension two and one unstable manifold of dimension two, because the linear part of system (12) at $\varphi = 0$ or $\varphi = \pi$ is given by the matrix

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{G}{4} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

Hence, by Theorem 3 and due to the fact that the boundaries of $\{(\varphi, G, \xi) \in [0,\pi] \times \mathbb{R} \times [0,1]\}$ are invariant by the flow of the differential system (12) we have that $\omega(p)$ is one of the equilibrium points of l^+. □

Proposition 5. *The following statements hold.*

(a) *The hyperbolic orbits when $t \to \infty$ (respectively $-\infty$) tend to infinity in position with nonzero velocity; that is, $|y| \to \infty$ and $|\dot{y}| \to$ constant $\neq 0$ when $t \to \infty$ (respectively $-\infty$).*

(b) *The parabolic orbits when $t \to \infty$ (respectively $-\infty$) tend to a finite value in position with zero velocity; that is, $|y| \to$ constant $\neq 0$ and $|\dot{y}| \to 0$ when $t \to \infty$ (respectively $-\infty$).*

Proof. We do the proof when $t \to \infty$. The proof when $t \to -\infty$ is similar.

From Lemma 1 and its proof, we know that $x(t) = \sqrt{h}\,t + O(t^\alpha)$ with $\alpha < 1$ when $t \to \infty$, and from the changes of variables (7) and (9) and Proposition 4 we have that

$$\frac{y(t)}{x(t)} = \frac{1}{2}\cot\varphi(t) \to G_0,$$

when $t \to \infty$, where G_0 is the G coordinate of the equilibrium point of l^+ which is the ω–limit of the solution $y(t)$. Therefore, if $G_0 \neq 0$ we have that

$$y(t) = G_0\sqrt{h}\,t + O(t^\alpha),$$

when $t \to \infty$.

Assume that $G_0 \neq 0$. Going back to the initial equation (6) we have

$$\begin{aligned}\ddot{y}(t) = & -\frac{16y(t)}{(x(t)^2+4y(t)^2)^{3/2}} = -\frac{16kt+O(t^\alpha)}{(ht^2+4k^2t^2+O(t^{1+\alpha}))^{3/2}} \\ = & -\frac{16kt+O(t^\alpha)}{k't^3(1+O(t^{\alpha-1}))^{3/2}} = -\frac{(16kt+O(t^\alpha))(1+O(t^{\alpha-1}))}{k't^3} \\ = & -\frac{16kt+O(t^\alpha)}{k't^3} = \frac{B}{t^2}+O(t^{\alpha-3}),\end{aligned}$$

when $t \to \infty$, where $k = G_0\sqrt{h}$, $k' = (h+4k^2)^{3/2}$, $B = -16k/k' \neq 0$ because $G_0 \neq 0$.

Integrating the last equation between t_1 and t, and if $t_1 \to \infty$ we get

$$\dot{y}(t) - \dot{y}(\infty) = -\frac{B}{t} + O(t^{\alpha-2}),$$

where $\dot{y}(\infty) = G_0\sqrt{h} = k \neq 0$ because $G_0 \neq 0$. Integrating again between t_0 and t we have

$$y(t) - y(t_0) = k(t-t_0) - B(\ln t - \ln t_0) + O(t^{\alpha-1}) - O(t_0^{\alpha-1}).$$

If we take t_0 sufficiently large and think the last equation in function of time we obtain $y(t) = kt - B\ln t + C + O(t^{\alpha-1})$, where C is a constant. So, statement (a) is proved.

If the orbit is parabolic, then $G_0 = 0$, and so $k = 0$ and $B = 0$. Hence we have that $y(t) = C + O(t^{\alpha-1})$. Then the statement (b) is also proved. □

Proposition 6. *The solution of the system* (12), $(\varphi, G, \xi) = (\frac{\pi}{2}, 0, \xi)$ *corresponds to a collinear motion of the three bodies, where the body with infinitesimal mass is in rest at the center of masses of both primaries.*

Proof. This is an immediate consequence of the changes of variables (7) and (9). □

5. The manifold of parabolic orbits

Following the ideas of Meyer and Wang [9], in order to study the flow near the equilibrium points of l^+ we introduce another transformation

$$\eta = \frac{\cos\varphi - b}{\xi}, \qquad \lambda = \frac{G - G_0}{\xi}, \tag{14}$$

where $b = \cos\varphi_0$ and the relation between φ_0 and G_0 is given by $G_0 = \frac{1}{2}\cot\varphi_0$. In a similar way we would study the flow near the equilibrium points of l^-.

From Proposition 5 and its proof the parabolic orbits when $t \to \infty$ have ω–limit the equilibrium point $(\varphi_0, G_0, 0) = (\pi/2, 0, 0)$ of the differential system (12). Therefore, the equations of motion (12) in the new variables η and λ with $(\varphi_0, G_0, 0) = (\pi/2, 0, 0)$ become

$$
\begin{aligned}
\frac{d\eta}{d\sigma} &= \frac{1}{2}(1-\xi)\left(\xi^2\eta^3 - 2\lambda(\xi\eta - 1)(\xi\eta + 1)\sqrt{1-\xi^2\eta^2}\right), \\
\frac{d\lambda}{d\sigma} &= \frac{1}{4}(2-3\xi)\lambda + \xi\eta\,(\xi^2\eta^2 - 1), \\
\frac{d\xi}{d\sigma} &= -\frac{\xi(1-\xi)}{2}.
\end{aligned} \tag{15}
$$

This differential system is defined in the space $(\eta, \lambda, \xi) \in \mathbb{R}^2 \times [0, 1]$.

We note that the transformation (14) is a blow–up of the equilibrium point $(\pi/2, 0, 0)$ of system (12) to the straight line $\{\lambda = 0\} \cap \{\xi = 0\}$ filled of equilibrium points of system (15).

Proposition 7. *The straight line $\{\lambda = 0\} \cap \{\xi = 0\}$ is a normally hyperbolic straight line of equilibrium points for the differential system* (15)*, having a stable manifold of dimension* 2*.*

Proof. It is easy to see that the linear part of system (15) at a point of the straight line $\{\lambda = 0\} \cap \{\xi = 0\}$ is

$$
\begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & -\eta_0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.
$$

So, it has eigenvalues $\frac{1}{2}$, $-\frac{1}{2}$ and 0. Therefore, by Theorem 3 the proposition follows. □

Corollary 8. *The set of parabolic orbits when $t \to \infty$ (respectively $-\infty$) time is a two-dimensional manifold is the phase space.*

Proof. The result for the parabolic orbits when $t \to \infty$ follows directly from Proposition 7. Repeating the arguments for the differential system (13), the result follows for the parabolic orbits when $t \to -\infty$. □

6. The global flow

In this section we need the following definitions:

$\mathcal{H}^+ n \mathcal{H}^-$ with n an odd positive integer denotes the hyperbolic orbits such that $y(-\infty) = +\infty$, $\dot{y}(-\infty) < 0$ and $y(+\infty) = -\infty$, $\dot{y}(+\infty) < 0$;

$\mathcal{H}^- n \mathcal{H}^+$ with n an odd positive integer denotes the hyperbolic orbits such that $y(-\infty) = -\infty$, $\dot{y}(-\infty) = \dot{y}_0 > 0$ and $y(+\infty) = +\infty$, $\dot{y}(+\infty) = \dot{y}_1 > 0$;

$\mathcal{H}^+n\mathcal{H}^+$ with n an even positive integer denotes the hyperbolic orbits such that $y(-\infty) = +\infty$, $\dot{y}(-\infty) = \dot{y}_0 < 0$ and $y(+\infty) = \infty$, $\dot{y}(+\infty) = \dot{y}_1 > 0$;

$\mathcal{H}^-n\mathcal{H}^-$ with n an even positive integer denotes the hyperbolic orbits such that $y(-\infty) = -\infty$, $\dot{y}(-\infty) = \dot{y}_0 > 0$ and $y(+\infty) = -\infty$, $\dot{y}(+\infty) = \dot{y}_1 < 0$;

$\mathcal{H}^+n\mathcal{P}^-$ with n an odd positive integer denotes the orbits such that $y(-\infty) = +\infty$, $\dot{y}(-\infty) < 0$ and $y(+\infty) = y_0 < 0$, $\dot{y}(+\infty) = 0$;

$\mathcal{H}^-n\mathcal{P}^+$ with n an odd positive integer denotes the hyperbolic orbits such that $y(-\infty) = -\infty$, $\dot{y}(-\infty) = \dot{y}_0 > 0$ and $y(+\infty) = y_0 > 0$, $\dot{y}(+\infty) = 0$;

$\mathcal{H}^+n\mathcal{P}^+$ with n an even positive integer denotes the hyperbolic orbits such that $y(-\infty) = +\infty$, $\dot{y}(-\infty) = \dot{y}_0 < 0$ and $y(+\infty) = y_0 > 0$, $\dot{y}(+\infty) = 0$;

$\mathcal{H}^-n\mathcal{P}^-$ with n an even positive integer denotes the hyperbolic orbits such that $y(-\infty) = -\infty$, $\dot{y}(-\infty) = \dot{y}_0 > 0$ and $y(+\infty) = y_0 < 0$, $\dot{y}(+\infty) = 0$;

$\mathcal{P}^+n\mathcal{H}^-$ with n an odd positive integer denotes the hyperbolic orbits such that $y(-\infty) = y_0 > 0$, $\dot{y}(-\infty) = 0$ and $y(+\infty) = -\infty$, $\dot{y}(+\infty) < 0$;

$\mathcal{P}^-n\mathcal{H}^+$ with n an odd positive integer denotes the hyperbolic orbits such that $y(-\infty) = y_0 < 0$, $\dot{y}(-\infty) = 0$ and $y(+\infty) = +\infty$, $\dot{y}(+\infty) = \dot{y}_1 > 0$;

$\mathcal{P}^+n\mathcal{H}^+$ with n an even positive integer denotes the hyperbolic orbits such that $y(-\infty) = y_0 > 0$, $\dot{y}(-\infty) = 0$ and $y(+\infty) = \infty$, $\dot{y}(+\infty) = \dot{y}_1 > 0$;

$\mathcal{P}^-n\mathcal{H}^-$ with n an even positive integer denotes the hyperbolic orbits such that $y(-\infty) = y_0 < 0$, $\dot{y}(-\infty) = 0$ and $y(+\infty) = -\infty$, $\dot{y}(+\infty) = \dot{y}_1 < 0$;

$\mathcal{P}^+n\mathcal{P}^-$ with n an odd positive integer denotes the parabolic orbits such that $y(-\infty) = y_0 > 0$, $\dot{y}(-\infty) = 0$ and $y(+\infty) = y_1 < 0$, $\dot{y}(+\infty) = 0$;

$\mathcal{P}^-n\mathcal{P}^+$ with n an odd positive integer denotes the parabolic orbits such that $y(-\infty) = y_0 < 0$, $\dot{y}(-\infty) = 0$ and $y(+\infty) = y_1 > 0$, $\dot{y}(+\infty) = 0$;

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$\mathcal{P}^-n\mathcal{P}^-$ with n an even positive integer denotes the parabolic orbits such that $y(-\infty) = y_0 < 0$, $\dot{y}(-\infty) = 0$ and $y(+\infty) = y_1 < 0$, $\dot{y}(+\infty) = 0$;

$\mathcal{P}$ denotes the special parabolic orbit for which $y(t) = 0$ for all $t \in \mathbb{R}$, i.e. the infinitesimal mass is in rest at the center of masses of both primaries.

Let $W^{s,p}(\ell^+)$ be the curve given by the p–th intersection of the local stable manifold $W^s(\ell^+)$ with $\xi = 1$ (i.e. with $\dot{x} = 0$) following the orbits of the local manifold $W^s(\ell^+)$ near ℓ^+ in backward time. Of course, these curves only can be approximated numerically for small values of p, and the orbits of them have final evolution of type $\mathcal{P}^+$ when $t \to +\infty$.

Let $W^{u,q}(\ell^-)$ be the curve given by the q–th intersection of the local unstable manifold $W^u(\ell^-)$ with $\xi = 1$ following the orbits of the local manifold $W^u(\ell^-)$ near ℓ^- in forward time. Again, these curves only can be approximated numerically for small values of q, and the orbits through the points of these curves have final evolution of type $\mathcal{P}^-$ when $t \to -\infty$.

The orbits through the points of $W^{u,q}(\ell^-) \cap W^{s,p}(\ell^+)$ with $p+q$ even are of the type $\mathcal{P}^- n\mathcal{P}^+$ with $n = p + q - 1$. Applying the symmetry (2) to these orbits we get orbits of type $\mathcal{P}^+ n\mathcal{P}^-$.

The orbits through the points of $W^{s,p}(\ell^+) \cap W^{s,q}(\ell^+)$ with $p+q$ odd are of the type $\mathcal{P}^+ n\mathcal{P}^+$ with $n = p + q - 1$. Applying the symmetry (2) to these orbits we get orbits of type $\mathcal{P}^- n\mathcal{P}^-$.

Let $\mathbb{N}$ be the set of all positive integers. The points belonging to the open regions of $\xi = 1$ which are in the complement of the curves

$$\left(\bigcup_{q \in \mathbb{N}} W^{u,q}(\ell^-) \right) \bigcup \left(\bigcup_{p \in \mathbb{N}} W^{s,p}(\ell^-) \right)$$

correspond to the hyperbolic orbits of type one of the following four types $\mathcal{H}^\pm n\mathcal{H}^\pm$.

The points belonging to the curves $\bigcup_{q \in \mathbb{N}} W^{u,q}(\ell^-)$, which are not in the curves $\bigcup_{p \in \mathbb{N}} W^{s,p}(\ell^+)$, correspond to orbits of type one of the following two types $\mathcal{P}^- n\mathcal{H}^\pm$. Applying the symmetry (2) to these orbits we get orbits of type $\mathcal{P}^+ n\mathcal{H}^\mp$.

The points belonging to the curves $\bigcup_{p \in \mathbb{N}} W^{s,p}(\ell^+)$, which are not in the curves $\bigcup_{q \in \mathbb{N}} W^{u,q}(\ell^-)$, correspond to orbits of type one of the following two types $\mathcal{H}^\pm n\mathcal{P}^+$. Applying the symmetry (2) to these orbits we get orbits of type $\mathcal{H}^\mp n\mathcal{P}^-$.

7. Conclusions

For the charged restricted three–body problem there is numerical evidence that the following statements hold:

(i) There is a two–dimensional continuum of hyperbolic solutions $y(t)$ of the infinitesimal mass of the types $\mathcal{H}^+n\mathcal{H}^-$, $\mathcal{H}^-n\mathcal{H}^+$, $\mathcal{H}^+n\mathcal{H}^+$ and $\mathcal{H}^-n\mathcal{H}^-$.

(ii) There is a one–dimensional continuum of solutions $y(t)$ of the infinitesimal mass of the types $\mathcal{P}^+n\mathcal{H}^-$, $\mathcal{P}^+n\mathcal{H}^+$, $\mathcal{P}^-n\mathcal{H}^+$, $\mathcal{P}^-n\mathcal{H}^-$, $\mathcal{H}^+n\mathcal{P}^+$, $\mathcal{H}^+n\mathcal{P}^-$, $\mathcal{H}^-n\mathcal{P}^-$ and $\mathcal{H}^-n\mathcal{P}^+$.

(iii) There are parabolic solutions $y(t)$ of the infinitesimal mass of the types $\mathcal{P}^+n\mathcal{P}^-$, $\mathcal{P}^-n\mathcal{P}^+$, $\mathcal{P}^+n\mathcal{P}^+$, $\mathcal{P}^-n\mathcal{P}^-$.

(iv) There is a unique parabolic solution $y(t)$ of the infinitesimal mass of the type $\mathcal{P}$.

(v) There are no other types of solutions than the ones of the statements (i), (ii), (iii) and (iv).

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