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ALGORITHMIC DERIVATION OF NILPOTENT CENTERS

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Abstract. To characterize when a nilpotent singular point of an analytic differential system is a center is of particular interest, first for the problem of distinguishing between a focus and a center, and after for studying the bifurcation of limit cycles from it or from its period annulus. We give an effective algorithm in the search of necessary conditions for detecting nilpotent centers based in recent developments. Moreover we survey the last results on this problem and illustrate our approach by means of examples.

1. Introduction and statement of the main results

This work deals mainly with the distinction between a center and a focus in the case of a nilpotent singular point, the called *center problem* for nilpotent singular points. A related problem is to characterize when there exists an analytic first integral in a neighborhood of a singular point which is a center, see [10]. Let $p \in \mathbb{R}^2$ be a singular point of a differential system in $\mathbb{R}^2$. We recall that p is a *center* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ are periodic, and p is a *focus* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ spiral either in forward or in backward time to p.

Assume that p is a center that we can suppose at the origin of coordinates. After a linear change of variables and a scaling of the time variable (if necessary), the system can be written in one of the following

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three forms:

$$
\begin{array}{llll}
(1) & \dot{x} = -y + F_1(x,y), & \dot{y} = x + F_2(x,y); \\
(2) & \dot{x} = y + F_1(x,y), & \dot{y} = F_2(x,y); \\
(3) & \dot{x} = F_1(x,y), & \dot{y} = F_2(x,y);
\end{array}
$$

where F_1 and F_2 are real analytic functions without constant and linear terms, defined in a neighborhood of the origin, and the dot denotes derivative with respect to the independent variable t, usually called the time. The center is called of *linear type* (also bad called *non–degenerate*), *nilpotent* or *degenerate*, if it can be written after an affine change of variables and a scaling of time as system (1), (2) or (3), respectively.

The characterization of the linear type centers is well–known in terms of the existence of an analytic first integral, see [27, 31]. For these type of centers it is also possible to use the Poincaré return map to characterizes the existence of a center, see for instance [6]. The implementation of an algorithm for detecting the linear centers, based in the existence of an analytic first integral or using the Poincaré return map, is straightforward, but the huge amount of computations which usually are necessary becomes this problem in general computationally intractable, see [21] and references therein.

The characterization of nilpotent centers based on the existence of an analytic first integral is not possible because nilpotent and degenerate centers do not have, in general, a local analytic first integral defined in a neighborhood of the center, see [10, 11, 15, 20, 29, 30]. There are three different methods for detecting nilpotent centers.

First, the characterization of nilpotent centers is possible with the Poincaré return map using the Lyapunov generalized coordinates $x = r\mathrm{Cs}\theta$ and $y = r^n\mathrm{Sn}\theta$, see [2, 3, 6].

Second, a different method for obtaining the Poincaré–Lyapunov constants or the generalized Lyapunov constants is using normal form theory. This theory can be applied to linear type and nilpotent centers, see [1, 4, 12].

Finally, another approach is using the generalized polar coordinates $x = r\cos\theta$ and $y = r^n \sin\theta$, see [24, 25]. Using these generalized polar coordinates the generalized Lyapunov constants appear computing integrals of trigonometric functions which in general are not easy to compute. Using this technique some applications to the bifurcation of limit cycles can be found in [22]. A generalization of these polar coordinates given by $x = r^m \cos\theta$ and $y = r^n \sin\theta$ is used in [33, 34] to study nilpotent and also degenerate centers. In this case, for instance, system (3) becomes the following differential equation

$$r' = \frac{dr}{d\theta} = \frac{r(r^n F_1(r\cos\theta, r\sin\theta)\cos\theta + r^m F_2(r\cos\theta, r\sin\theta)\sin\theta)}{mr^m F_2(r\cos\theta, r\sin\theta)\cos\theta - nr^n F_1(r\cos\theta, r\sin\theta)\sin\theta}.$$

Now it is considered the function

$$F(\theta) = \lim_{r\to 0} \frac{r'}{r}.$$

The idea is to chose n and m in order that this function $F(\theta)$ does not vanish for any value of θ. In this case the origin is monodromic and we can use the classical Bautin method to compute the focal values, see [8] and also [15, 17]. Moreover in [33, 34] the author study some families of systems which can exhibit nilpotent centers but he does not provide the necessary algebraic conditions for having a nilpotent center. He only provides numerical approximations of these conditions due to the computational complexity of the method used by the author. We study here the same systems providing necessary and in some cases sufficient algebraic conditions.

In [9] it was proved that any nilpotent center is orbitally equivalent to a time-reversible system (see also [28]). Taking into account that any nilpotent and degenerate center does not have, in general, a local analytic first integral we cannot use directly the Poincaré–Lyapunov method (seeking for analytic first integrals) for determining the center conditions in the case of nilpotent and degenerate singular points. Nevertheless in [13] it is showed that essentially the Poincaré–Lyapunov method also works to determine nilpotent centers and a subclass of analytic degenerate centers, see also [17, 18]. In fact it was proved that all the nilpotent centers are limit of linear type or non-degenerate centers. The main result of [13] is the following theorem.

Theorem 1 (Nilpotent Center Theorem). *Suppose that the origin of the real analytic differential system* (2) *is a center, then there exist analytic functions M_1 and M_2 without constants terms, such that the system*

$$(4) \quad \dot{x} = y + F_1(x,y) + \varepsilon M_1(x,y), \quad \dot{y} = -\varepsilon x + F_2(x,y) + \varepsilon M_2(x,y),$$

has a linear type center at the origin for all $\varepsilon > 0$, where $M_1 = (x + f)\ \partial f/\partial y$ and $M_2 = -(x+f)\ \partial f/\partial x - f$. Here, $f(x,y)$ is an analytic function starting with terms of degree two in x and y.

Theorem 1 was not correctly stated in [13], and later on appeared a correction to it in [14], but again the proof of [13] was not strictly correct, because in that proof it was used incorrectly an orbital equivalence. More precisely, there exists an orbital equivalence, which is an analytic transformation and a change of time, that writes any nilpotent center as a time–reversible system, i.e. as a system invariant by the symmetry $(x, y, t) \to (-x, y, -t)$. In the proof of [13] was not taken into account this change of time. Therefore we believe appropriate to give a complete proof of this useful theorem which is the base of the algorithm for detecting nilpotent centers presented in this paper. Theorem 1 is proved in section 2.

In summary Theorem 1 states that given an analytic vector field $\mathcal{X}$ defined in a neighborhood of $p \in \mathbb{R}^2$ where p is a nilpotent center of $\mathcal{X}$, there exists an one–parameter family of analytic vector fields $\mathcal{X}_\varepsilon$, with $\varepsilon \geq 0$, defined in a neighborhood of $p_\varepsilon \in \mathbb{R}^2$ having a linear type center at p_ε for $\varepsilon > 0$ and satisfying $\mathcal{X}_0 = \mathcal{X}$ and $p_0 = p$. Moreover, suppose that $H_\varepsilon(x, y)$ is a local analytic first integral at p_ε for the vector field $\mathcal{X}_\varepsilon$ with $\varepsilon > 0$. If the limit $\lim_{\varepsilon \searrow 0} H_\varepsilon(x, y)$ exists, and is a function $H(x, y)$ well defined in a neighborhood of p, then $H(x, y)$ is a local first integral (not necessary analytic) of $\mathcal{X}$ at p. Using this mechanism in [13, 19] some non–continuous first integrals for nilpotent and degenerate centers are computed.

While the problem of distinguishing between a center and a focus is not algebraically solvable for degenerate centers (see for instance [23, 26]), it is algebraically solvable for analytic differential systems of the form (1) and (2), see [7]. Moreover Theorem 1 provides a new proof of this fact for the nilpotent centers from the knowledge of the algebraically solvability for systems of the form (1). However depending on the method used to determine the center conditions, sometimes the method cannot go further and determine explicitly the conditions which must be algebraic, see for instance [3]. An algorithm to detect nilpotent center conditions must find all the algebraic conditions that determine a center. The algorithm does not discriminate correctly if we obtain expressions where only numerical calculations can be used to fix the parameters of the family in order to have a center, see for instance [33, 34]. In this work we present an algorithm that correctly provides the explicit algebraic conditions for having a nilpotent center of system (2). The algorithm derived from Theorem 1 is presented in section 3.

This new algorithm improves the use of Theorem 1 presented in [13, 14]. A key point in this improvement is that instead of working with the linear part $(y, -\varepsilon x)$ of system (4) we shall work with the classical linear part $(y, -x)$ after doing a convenient rotation, and after this change we can work with trigonometric polar coordinates.

2. Proof of Theorem 1

Assume that the origin of system (2) is a center. Following [9] this system is orbitally equivalent to a time–reversible system around the origin. Therefore there exists an analytic change of variables $(x, y) \mapsto (u, v)$ of the form

$$x = u + f(x, y), \qquad y = v + g(x, y), \tag{5}$$

where f and g are analytic functions. In the new variables the system (2) becomes

$$\dot{u} = v + \overline{F}_1(u, v), \quad \dot{v} = \overline{F}_2(u, v), \tag{6}$$

where $\overline{F}_1$ and $\overline{F}_2$ are analytic functions starting with terms of second degree in x and y; and according with [9] it exists a change of time $dt = (1 + h(u, v))d\tau$ such that system (6) can written as

$$\dot{u} = (v + \overline{F}_1(u, v))(1 + h(u, v)), \quad \dot{v} = \overline{F}_2(u, v)(1 + h(u, v)); \tag{7}$$

and this system is invariant by the symmetry $(u, v, t) \mapsto (-u, v, -t)$.

Now we consider the following perturbation of system (7)

$$\dot{u} = (v + \overline{F}_1(u, v))(1 + h(u, v)), \quad \dot{v} = -\varepsilon u + \overline{F}_2(u, v)(1 + h(u, v)), \tag{8}$$

with $\varepsilon > 0$. The origin of system (8) is a linear type center for all $\varepsilon > 0$ because the eigenvalues at the singular point located at the origin are $\pm\sqrt{\varepsilon}\, i$ and the differential system (8) is invariant by the symmetry $(u, v, t) \mapsto (-u, v, -t)$. Now going back to initial variables (x, y, t) we get that the differential system (8) becomes

$$\dot{x} = y + F_1(x, y) + \varepsilon M_1(x, y), \quad \dot{y} = -\varepsilon x + F_2(x, y) + \varepsilon M_2(x, y), \tag{9}$$

where $M_1 = (x + f)\ \partial f/\partial y$ and $M_2 = -(x + f)\ \partial f/\partial x - f$. Since system (8) has a linear type center at the origin for all $\varepsilon > 0$, the same holds for system (9). This completes the proof of the theorem.

It is interesting to note that the perturbation $-\varepsilon u$ in system (8) is transformed in a perturbation in system (9) which only depends on the function f but not on the function g of the change of variables (5). This fact makes the algorithm (derived from Theorem 1) that we describe in the next section for detecting nilpotent centers more straightforward because only depends on the arbitrary parameters of the function f.

3. Algorithm derived from Theorem 1

Consider that we want to determine the necessary conditions to have a center of a certain nilpotent system. In the correct coordinates this nilpotent system has the form (2) where F_1 and F_2 are analytic functions without constants and linear terms and such that the origin of (2) is a monodromic singular point. We recall here the Andreev's theorem which characterizes when a nilpotent singular point is monodromic, see [5]. We need this theorem to assure that the nilpotent system has a monodromic singular point at the origin.

Theorem 2 (Andreev)**.** *Let $\mathcal{X} = (y + F_1(x, y), F_2(x, y))$ be the vector field associated to system (2). Let $y = \phi(x)$ be the solution of the equation $y + F_1(x, y) = 0$ passing through the origin. Assume that the expansion of the function $F_2(x, \phi(x))$ is of the form $\xi(x) = \alpha_k x^k + O(x^{k+1})$ and $\Delta(x) = \operatorname{div}\mathcal{X}(x, \phi(x)) = \beta_n x^n + O(x^{n+1})$ with $\alpha_k \neq 0$, $k \geq 2$ and $n \geq 1$. Then, the origin is either a focus or a center if and only if k is odd, $\alpha_k < 0$, and*

- *either $k = 2n + 1$ and $\beta_n^2 + 4\alpha_k(n + 1) < 0$,*
- *or $k < 2n + 1$,*
- *or $\Delta(x) \equiv 0$,*

The first step is to perturb system (2) in such a way that it can have at the origin a linear type center system, so we consider the perturbed system

$$\dot{x} = y + F_1(x, y) + \varepsilon M_1(x, y), \quad \dot{y} = -\varepsilon x + F_2(x, y) + \varepsilon M_2(x, y), \tag{10}$$

with $M_1 = (x + f)\ \partial f/\partial y$ and $M_2 = -(x + f)\ \partial f/\partial x - f$ and where $f(x, y)$ is an analytic function without constant and linear terms, that can be expressed as

$$f(x, y) = \sum_{i,j \geq 2}^{\infty} a_{ij} x^i y^j, \tag{11}$$

where $a_{ij} \in \mathbb{R}$ are arbitrary parameters to be fixed by the algorithm. Hence we propose the function f and we try to solve using the Poincaré–Lyapunov method (i.e. looking for a first integral) the center problem for system (10).

We first simplify the linear part of system (10) performing the linear change of coordinates $(x, y) \mapsto (x/2 - y/(2\sqrt{\varepsilon}), x/2 + y/(2\sqrt{\varepsilon})$ obtaining a system of the form

$$\dot{x} = y + A(x, y, \varepsilon), \quad \dot{y} = -x + B(x, y, \varepsilon), \tag{12}$$

where A and B are analytic functions in x and y but not analytic in ε. In order to avoid to work with square roots of ε we change $\varepsilon \to \varepsilon^2$ from now on. Now the linear part of system (12) is the standard one and we can apply the classical Poincaré-Lyapunov method. We apply the method taking polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and system

(12) becomes

$$\dot{r} = \sum_{s=2}^{\infty} P_s(\theta, \varepsilon) r^s, \quad \dot{\theta} = 1 + \sum_{s=2}^{\infty} Q_s(\theta, \varepsilon) r^{s-1}, \tag{13}$$

where $P_s(\theta, \varepsilon)$ and $Q_s(\theta, \varepsilon)$ for $s = 2, 3, \ldots$ are homogeneous trigonometric polynomials respect to θ. Now we propose the Poincaré series

$$H(r, \theta, \varepsilon) = \sum_{n=2}^{\infty} H_n(\theta, \varepsilon) r^n, \tag{14}$$

where $H_2(\theta, \varepsilon) = 1/2$ and $H_n(\theta, \varepsilon)$ are trigonometric polynomials respect to θ of degree n. Imposing that this power series is a formal first integral of system (14) we obtain

$$\dot{H}(r, \theta, \varepsilon) = \sum_{k=2}^{\infty} V_{2k}(\varepsilon) r^{2k}.$$

where $V_{2k}(\varepsilon)$ are the Poincaré-Lyapunov contants that in this case depend on ε. To determine the necessary condition to have a center at the origin we must vanish these $V_{2k}(\varepsilon)$. It is easy to see by the recursive equations that generate the $V_{2k}(\varepsilon)$ that these $V_{2k}(\varepsilon)$ are rational functions in ε, see [13]. So we take common denominator in the expression of $V_{2k}(\varepsilon)$ and we vanish the polynomial in ε that appears in the numerator. This polynomial must be vanish for all ε, hence we obtain that any coefficient of this polynomial in ε must be zero. Usually these conditions are fulfilled appropriately choosing the parameters of the analytical function f. However, in some cases this is not possible and the parameters of the differential system (2) should be used. When system (2) is polynomial, due to the Hilbert's basis theorem, we have that the set of necessary conditions to have a center will be obtained in a finite number of steps. Whenever we find a necessary condition for system (2), we must take into account if this system already has a center at the origin. For a polynomial system we need only a finite jet of the function f because the number of steps is finite. Consequently the perturbation required to detect all the necessary conditions will be polynomial, this means that the function f will be also a polynomial.

We recall the following result proved in [13].

Theorem 3. *Suppose that the origin of the real analytic differential system* (2) *is monodromy, and that this system is limit of linear type centers of the form* (4)*. Suppose also that there is no singular point of* (4) *tending to the origin when ε tends to zero. Then, system* (2) *has a center at the origin.*

We note that under the assumptions of Theorem 3 if system (10) has not a center at the origin, then it is not possible to fulfill the necessary center conditions of (10) only with the parameters of the perturbation because in that case system (10) would have a center at the origin for arbitrary values of the parameters of the family, and this is not possible.

4. Applications

In this section we illustrate how to apply the method described in the previous section to several families of polynomial differential systems for detecting nilpotent centers. Some of these families have been studied by other authors with the different methods described in the introduction of the present work. However some of these methods are unable to get the algebraic conditions and the conditions are obtained numerically. We will see that with the described algorithm we can find the algebraic conditions explicitly. There are no nilpotent centers for quadratic polynomial differential systems, see [11] and Proposition 5 in [16], so the simplest nilpotent polynomial centers must be of degree 3.

Example 1. We study when the system

$$\dot{x} = y + ax^2, \qquad \dot{y} = -x^3 + bx^2y, \tag{15}$$

has a nilpotent center.

Proposition 4. *System* (15) *has a nilpotent center at the origin if and only if* $|a| < \sqrt{2}$ *and* $b = 0$.

Proof. We separate the proof into three cases.

Case 1: $a = b = 0$. In this case system (15) becomes Hamiltonian with $H = y^2/2 + x^4/4$. So the origin is a global nilpotent center.

Case 2: $ab \neq 0$. Applying Andreev's Theorem 2 we obtain that $\phi(x) = -ax^2$, from where $\xi(x) = -x^3 - abx^4$, $\Delta(x) = 2ax + bx^2$, and then we have $k = 3$, $\alpha_3 = -1$, $n = 1$, and $\beta_1 = 2a$. Hence $k = 2n + 1$. To have monodromy we must impose $\beta_1^2 + 4\alpha_3(n+1) < 0$ which implies $4a^2 - 8 < 0$. Consequently system (15) has a monodromic singular point at the origin if and only if $0 < |a| < \sqrt{2}$.

We apply the algorithm to the perturbed system

$$\begin{aligned} \dot{x} &= y + ax^2 + \varepsilon^2(x+f)\frac{\partial f}{\partial y}, \\ \dot{y} &= -\varepsilon^2 x - x^3 + bx^2y - \varepsilon^2\left((x+f)\frac{\partial f}{\partial x} + f\right), \end{aligned} \tag{16}$$

where $\varepsilon > 0$. We obtain that the first nonzero Lyapunov constant is $V_4 = (b - 6aa_{20} - 2aa_{02}e^2)/(2\varepsilon)$. We note that in V_4 only appear the linear and quadratic terms of f. Vanishing V_4 at any order in ε we get the conditions $a_{20} = b/(6a)$ and $a_{02} = 0$ on the parameters of the perturbation. The next nonzero Lyapunov constant is

$$\begin{aligned} V_6 &= \frac{1}{108a^2\varepsilon^3}\Big[-18a^2(6a^2-5)b - 108a^3(a_{04} - 2a_{03}a_{11})\varepsilon^6 \\ &\quad +18a^2(14a^2a_{03} + 24aa_{11}a_{21} - 6aa_{22} - 4a_{11}^2 b + a_{12}b)\varepsilon^4 \\ &\quad +(324a^4a_{21} - 540a^3a_{40} - 234a^3a_{11}b + 270a^2a_{30}b + 5b^3)\varepsilon^2\Big]. \end{aligned}$$

We remark that in V_6 only appear terms of f up to degree 4. Vanishing V_6 at any order in ε we get the first necessary center condition $a^2(6a^2 - 5)b = 0$ for system (15) and for an arbitrary perturbation. We obtain also other conditions on the parameters of the perturbation. The next nonzero Lyapunov constant has the form

$$V_8 = -\frac{1}{5184a^4\varepsilon^5}\Big[-18a^4(6a^2-5)(135+688a^2)b + O(\varepsilon^2)\Big].$$

Therefore, as $ab \neq 0$, a necessary conditions is $6a^2 - 5 = 0$ which implies $a = \sqrt{5/6} \approx 0.912871$ or $a = -\sqrt{5/6} \approx -0.912871$. It seems that the necessary condition $6a^2 - 5 = 0$ is also sufficient. However going further with the algorithmic method appears the condition $b = 0$ in contradiction with the assumptions of this case. This condition can be also detected using the theory of normal forms, used for instance in [1]. Hence in this case there are no nilpotent centers.

Case 3: $a = 0$ and $b \neq 0$. Applying Andreev's Theorem 2 we obtain that $\phi(x) = 0$, from where $\xi(x) = -x^3$, $\Delta(x) = bx^2$, and then we have $k = 3$ and $n = 2$. Since $k < 2n + 1$ then the origin is monodromic. Applying the algorithm described in section 3 to system (15) we can obtain $V_4 = b/(2\varepsilon)$. Hence implies $b = 0$ in contradiction with the assumptions of this case. Hence in this case there are no nilpotent centers.

Case 4: $a \neq 0$ and $b = 0$. Applying Andreev's Theorem 2 we obtain that $\phi(x) = -ax^2$, from where $\xi(x) = -x^3$, $\Delta(x) = 2ax$, and then we have $k = 3$, $\alpha_3 = -1$, $n = 1$, and $\beta_1 = 2a$. Hence it is verified $k = 2n+1$. To have monodromy we must impose $\beta_1^2 + 4\alpha_3(n+1) < 0$ which implies $4a^2 - 8 < 0$. Consequently system (15) has a monodromic singular point at the origin if and only if $|a| < \sqrt{2}$. Moreover, if $b = 0$ system (15) is invariant by the symmetry $(x, y, t) \mapsto (-x, y, -t)$ and it has a center at the origin. □

However, in general, we are far to solve completely the center problem given any necessary condition which seems sufficient. System (15) is studied in [34] using generalized polar coordinates and computing some generalized Lyapunov constants. The method developed in [34] is not useful to give the algebraic condition $6a^2 - 5 = 0$, although the author claims that using the Lyapunov generalized coordinates it is possible to arrive to such condition.

Example 2. We now consider the system

$$\dot{x} = y + ax^2 + 5xy^2, \qquad \dot{y} = -2x^3 + 3xy^2 - 4y^3. \tag{17}$$

For system (17) we can establish the following result.

Proposition 5. *A necessary condition in order that system* (17) *has a nilpotent center at the origin is that* $-98 + 47a^2 + 20a^4 = 0$.

Proof. Applying Andreev's Theorem 2 we obtain that $\phi(x) = -ax^2 + O(x^3)$, from where $\xi(x) = -2x^3 + O(x^4)$, $\Delta(x) = 2ax + O(x^2)$, and then we have $k = 3$, $\alpha_3 = -2$, $n = 1$, and $\beta_1 = 2a$ if $a \neq 0$. Hence it is verified $k = 2n+1$. To have monodromy we must impose $\beta_1^2 + 4\alpha_3(n+1) < 0$ which implies $4a^2 - 16 < 0$. Consequently system (15) has a monodromic singular point at the origin if and only if $|a| < 2$. For the case $a = 0$ the origin is always monodromic because we have $\phi(x) = 0$, from where $\xi(x) = -2x^3$ and $\Delta(x) = 0$. Now we apply the algorithm described in section 3 to system (17). For the case $a = 0$ it is easy to see using the algorithm that system (17) has always a focus at the origin. For $a \neq 0$, we apply the algorithm to the perturbed system

$$\begin{array}{rcl} \dot{x} & = & y + ax^2 + 5xy^2 + \varepsilon^2 M_1(x,y), \\ \dot{y} & = & -\varepsilon^2 x - 2x^3 + 3xy^2 - 4y^3 + \varepsilon^2 M_2(x,y), \end{array} \tag{18}$$

where M_1 and M_2 are given in Theorem 1 and $\varepsilon > 0$. We obtain that the first nonzero Lyapunov constant is $V_4 = -(6aa_{20} + 7\varepsilon^2 + 2aa_{02}\varepsilon^2)/(2\varepsilon)$. Vanishing V_4 we get the conditions $a_{20} = 0$ and $a_{02} = -7/(2a)$ on the parameters of the perturbation. Next we compute the second nonzero Lyapunov constant given by

$$\begin{aligned} V_6 \;=\; & \frac{1}{6a\varepsilon}\Big[2a(-21 + 28a^2 + 9a^2 a_{21} - 15aa_{40}) + a(-156 + 14a^2 a_{03} \\ & +121aa_{11} - 18a_{21} + 24aa_{11}a_{21} - 6aa_{22} - 63a_{30})\varepsilon^2 \\ & +3(-6aa_{03} - 2a^2 a_{04} - 42a_{11} + 4a^2 a_{03}a_{11} + 14aa_{11}^2 - 7aa_{12})\varepsilon^4\Big]. \end{aligned}$$

Vanishing V_6 at any order in ε we obtain conditions on the parameters of the perturbation. We have isolated the parameters

$$a_{21} = \frac{1}{9a^2}(21 - 28a^2 + 15aa_{40}),$$

$$a_{30} = \frac{1}{63}(-156 + 14a^2 a_{03} + 121aa_{11} - 18a_{21} + 24aa_{11}a_{21} - 6aa_{22}),$$

$$a_{12} = \frac{2}{7a}(-3aa_{03} - a^2 a_{04} - 21a_{11} + 2a^2 a_{03} a_{11} + 7aa_{11}^2).$$

The next nonzero Lyapunov constant has the form

$$V_8 = \frac{1}{54432a^4\varepsilon^3}\Big[13608\, a^4(-98 + 47a^2 + 20a^4) + O(\varepsilon^2)\Big].$$

Therefore, a necessary condition to have a center is $-98+47a^2+20a^4 = 0$ which implies $a \approx 1.153741$ or $a \approx -1.153741$. □

System (17) is studied in [33] using generalized polar coordinates and computing some generalized Lyapunov constants. The method developed in [33] is not useful to give the algebraic condition $-98 + 47a^2 + 20a^4 = 0$. In some cases the values of the parameters of the original system that vanish the Lyapunov constants are not roots of algebraic equations but it is also difficult to prove that we have a sufficient condition as the following example shows.

Example 3. Consider the system

$$\dot{x} = y, \qquad \dot{y} = -x^3 + ay^2 + bx^3 y + cy^3. \tag{19}$$

For system (19) we have the following result.

Proposition 6. *A necessary condition in order that system* (19) *has a nilpotent center at the origin is that* $ab(ab + 3c) = 0$. *Moreover the conditions* $a = c = 0$ *or* $a = b = 0$ *are also sufficient.*

Proof. Applying Andreev's Theorem 2 we can see that the origin of system (19) is monodromic. Actually we obtain that $\phi(x) = 0$, from where $\xi(x) = -x^3$, $\Delta(x) = bx^3$, and then we have $k = 3$, $\alpha_3 = -2$, $n = 3$, and $\beta_3 = b$. Hence it is verified $k > 2n + 1$ with k odd which implies that the origin is always monodromic. Now we apply the algorithm described in section 3 to the perturbed system

$$\dot{x} = y+\varepsilon^2 M_1(x, y), \quad \dot{y} = -\varepsilon^2 x - x^3 + ay^2 + bx^3 y + cy^3 + \varepsilon^2 M_2(x, y), \tag{20}$$

where M_1 and M_2 are given in Theorem 1 and $\varepsilon > 0$. We obtain that the first nonzero Lyapunov constant is $V_4 = (3c - 2aa_{11})\varepsilon/2$. Vanishing V_4 we get the condition $a_{11} = 3c/(2a)$ on the parameter of the perturbation. Now, we compute the second nonzero Lyapunov constant which is

$$\begin{aligned} V_6 \;=\; -\frac{1}{6\varepsilon}\Big[& -7ab + 15a_{20}b - 9c + (14a^2 a_{21} - 24aa_{20}a_{21} + 6aa_{31} \\ & +3a_{02}b + 9a^2 c - 51aa_{20}c + 72a_{20}^2 c - 27a_{30}c)\varepsilon^2 + (18a^2 a_{03} + 6aa_{13} \\ & -12aa_{03}a_{20} - 12aa_{02}a_{21} - 15aa_{02}c - 9a_{12}c + 36a_{02}a_{20}c)\varepsilon^4\Big]. \end{aligned}$$

Vanishing V_6 at any order in ε we obtain conditions on the parameters of the perturbation. We have isolated the parameters. In fact we isolate a_{20} from the term without ε, a_{02} form the term with ε^2 and a_{13} form the term with ε^4. The next nonzero Lyapunov constant has the form

$$V_8 = -\frac{1}{540000a^2b^4\varepsilon^3}\Big[472500a^2b^4(ab+3c)+O(\varepsilon^2)\Big].$$

Therefore a necessary condition is $ab(ab+3c)=0$. If $a=0$, then $V_4 = 3c/2$ which implies $c=0$. If $b=0$, then V_6 takes the form

$$V_6 \;=\; -\frac{1}{6\varepsilon}\big[-9c+O(\varepsilon^2)\big],$$

which implies $c=0$. Now we prove that these two conditions are sufficient. If $a=c=0$ or $b=c=0$, we have that system (19) has the symmetry $(x,y,t)\mapsto(-x,y,-t)$ or $(x,y,t)\mapsto(x,-y,-t)$, respectively. Therefore, since the origin is monodromic, it is a center. □

The difficulty in the implementation of the algorithm developed in this work is the same as the classical Poincaré–Liapunov method for linear type centers. For obtaining the necessary conditions a big amount of computations are necessary owing to the number of parameters of the perturbation that we must add to the initial system. However the amount of computations that appear in the application of our algorithm is the usual that appear when we apply the classical Poincaré–Liapunov method to a system with several parameters. In general, the algorithm proposed in this work to detect nilpotent centers is easier than the algorithm that we can deduce from the results of [9]. In our algorithm we have only one arbitrary function f. However in the algorithm based in the results of [9] three arbitrary functions must be used. One which appear in the normal form of the nilpotent center and the two coming from the change of variables. Furthermore the arbitrary function f that appear in Theorem 3 is always polynomial for polynomial systems contrary to what happens in the algorithm based on the results of [9].

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