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Abstract:

Markowitz portfolio theory (1952) has induced research into the efficiency of portfolio management. This paper studies existing nonparametric efficiency measurement approaches for single period portfolio selection from a theoretical perspective and generalises currently used efficiency measures into the full mean-variance space. Therefore, we introduce the efficiency improvement possibility function (a variation on the shortage function), study its axiomatic properties in the context of Markowitz efficient frontier, and establish a link to the indirect mean-variance utility function. This framework allows distinguishing between portfolio efficiency and allocative efficiency. Furthermore, it permits retrieving information about the revealed risk aversion of investors. The efficiency improvement possibility function thus provides a more general framework for gauging the efficiency of portfolio management using nonparametric frontier envelopment methods based on quadratic optimisation.

Keywords: shortage function, efficient frontier, risk aversion, mean-variance portfolios.

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1. Introduction

Markowitz portfolio theory (1952), based on the idea of a trade-off between portfolio risk (as measured by its variance) and portfolio expected return, is generally considered as a cornerstone of modern portfolio theory. This approach is essentially based on the efficient frontier concept, defined as the Pareto-optimal subset of portfolios, i.e., a set of portfolios such that their expected returns may not increase unless their variances increase.

In addition to its strong maintained assumptions on probability distributions and on Von Neumann-Morgenstern utility functions, the main problem with Markowitz model at the time was its computational cost.[1] Though Farrar (1962) was seemingly the first to empirically test the full-covariance Markowitz model, computing costs motivated Sharpe (1963) to formulate a simplification known as the "diagonal model". Later, Sharpe (1964) and Lintner (1965) introduced a capital asset pricing model (CAPM), an equilibrium model assuming that all agents have similar expectations about the market. Under these circumstances, it is not necessary to compute the efficient frontier.[2] Tools for gauging the efficiency of portfolios, such as the Sharpe (1966) and Treynor (1965) ratios and the Jensen (1968) alpha, have mainly been developed with reference to the above developments (in particular CAPM).[3]

Despite these developments, the static Markowitz model remains the more general framework. Our contribution focuses on integrating an efficiency measure in this single period Markowitz model and to develop a dual framework for assessing the degree of satisfaction of investors' preferences, starting from –seemingly forgotten- ideas in Farrar (1962). This leads to decomposing portfolio performance into allocative and portfolio efficiency components. In addition, duality allows revealing information about investors' risk aversion. This is an issue of great practical significance that, to the best of our knowledge, is novel.

There are both theoretical and practical motivations guiding these developments. Theoretically, this contribution brings portfolio theory in line with developments in mainly

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[1] A problem largely alleviated by today's computing power.

[2] Surveys on the history of these developments are, e.g., Constantinides and Malliaris (1995) and Philippatos (1979).

[3] General surveys on tools for measuring the performance of managed portfolios are Grinblatt and Titman (1995) or Shukla and Trzcinka (1992).

production theory, where distance functions have proven useful tools to derive efficiency measures and to develop dual relations with economic (e.g., profit) support functions (Chambers, Chung and Färe (1998)). From a practical viewpoint, one can list the following advantages. First, the integration of efficiency measures into portfolio theory responds to the needs for rating tools. Second, instead of tracing the whole efficient portfolio frontier using a critical line search method, each asset or fund is projected onto the relevant part of the frontier according to a meaningful efficiency measure. This may lead to computational gains, depending on the number of assets or funds to evaluate and the aimed fineness of the portfolio frontier representation. Third, the possibility of measuring portfolio performance using a dual approach not only allows to gauge assets or funds using given information about risk aversion, but it also allows to reveal the (shadow) risk aversion minimising portfolio inefficiency. Therefore, our contribution enriches the empirical toolbox of practitioners.

In particular, we introduce a variation of the *shortage function*, a distance function introduced in production theory by Luenberger (1995) that is dual to the profit function. This function accomplishes four goals: (i) it gauges the performance of portfolios by measuring a distance between a portfolio and an optimal portfolio projection on the Markowitz efficient frontier; (ii) it leads to a nonparametric estimation of this efficient frontier; (iii) it judges simultaneously mean return expansions and risk contractions –in fact, performance can be gauged in any direction- and thereby generalises existing approaches; and (iv) it provides a new, dual interpretation of our portfolio efficiency distance. Given the investment context, our efficiency measure is called the *Efficiency Improvement Possibility* (EIP) function.

To develop the fourth point somewhat, the paper establishes a link between the EIP function and mean-variance utility functions, thereby offering an integrated framework for assessing portfolio efficiency from the dual standpoint. To each efficient portfolio corresponds a particular utility function, whose optimal value is the indirect utility function. This approach provides a dual interpretation of the EIP function through the structure of risk preferences. Technically, this result is easily derived from Luenberger (1992, 1996). Along this line, we also establish a link to some kind of "Slutsky matrix", defined as a matrix of derivatives with respect to risk aversion (based on the structure of the mean-variance utility function).

To situate our contribution more precisely, it is possible to distinguish between several approaches to test for portfolio efficiency. It is common to develop statistical tests based on certain parametric distributional assumptions (e.g., Jobson and Korkie (1989), Gouriéroux and Jouneau (1999), Philippatos (1979a)). However, right from the start (Markowitz (1952)) there has also been attention to simple nonparametric approaches to test for portfolio efficiency. Our contribution is situated within the latter tradition.

Our work can best be contrasted with Varian (1983) and some later developments in the nonparametric test tradition using economic restrictions (Matzkin (1994)). Varian (1983) developed nonparametric nonstatistical tests checking whether the observed investment behaviour is consistent with the expected utility and the mean variance models. However, his formulation only allows to infer whether certain data are either consistent or not with the tested hypothesis. This regularity test lacks an indication about the degree of goodness of fit between data and models. In this respect, it is similar to the early nonparametric test literature in production (Diewert and Parkan (1983)) and consumption (Varian (1982)).

Sengupta (1989) is probably the first to link the Varian (1983) portfolio test approach to the nonparametric efficiency literature by explicitly introducing an efficiency measure. Färe and Grosskopf (1995) establish a link between the above literature on regularity tests in general and the growing number of efficiency contributions employing distance functions (or their inverses, efficiency measures) as an explicit (nonstatistical) indicator of goodness of fit. Recently, Morey and Morey (1999) presented a nonparametric, quadratic programming approach to measure investment fund performance focusing on radial potentials for either risk contraction or mean return expansion. By contrast, our approach gauges portfolio performance simultaneously in terms of risk contraction and mean return augmentation.

Among the obvious advantages of a nonparametric approach to production, consumption and investment one can mention: (i) it avoids the necessity to postulate specific functional forms, (ii) it is related to "revealed preference" conditions of some sort that are finite in nature and that are directly tested on any finite amount of observations, (iii) it leads to the determination of inner and outer approximations of choice sets that contain the true but unknown frontier of the set, (iv) these approximations are based on (most often piecewise linear) functions that are directly spanned by the observations in the sample at hand, (v) the computational cost is rather low (frequently limited to solving mathematical programming problems), etc. (see, e.g., Matzkin (1994), Morey and Morey (1999), Varian (1983)).

Aside from the investment setting, the problem of estimating monotone concave boundaries has recently been extensively studied and widely applied in production. Following the seminal article of Farell (1957), nonparametric efficiency methods estimate an inner bound approximation of the true, unknown production frontier using piecewise linear envelopments of the data, instead of traditional parametric, econometric estimation methods that suffer from the risk of specification error.[4] Our contribution can then be interpreted as an envelopment method for estimating an inner bound of the true but unknown Markowitz efficient frontier. While most nonparametric efficiency methods in production rely on linear programming, the portfolio context requires quadratic optimisation.[5]

The paper is organised as follows. Section 2 lays down the groundwork for our analysis. Section 3 introduces the EIP function and studies its axiomatic properties. Section 4 studies the link between the EIP function and the direct and indirect mean-variance utility functions. Section 5 presents mathematical programs to compute the EIP. A simple empirical illustration using a small sample of 26 investment funds is provided in Section 6. Conclusions and possible extensions are formulated in a final section.

2. Efficient Frontier and Portfolio Management

In developing our basic definitions, we consider n financial assets. Assets are characterised by an expected return $E(R_i)$ for i = 1...n, and, since returns of assets are correlated, by a covariance matrix $\Omega_{i,j} = Cov(R_i, R_j)$ for $i, j \in \{1,...,n\}$. A portfolio x is composed by a

4 Briec and Lesourd (2000) study mutual fund performance employing stochastic parametric frontiers. This study extends their work in an effort to avoid specification errors.

5 Nonparametric efficiency methods in production are known in operations research as Data Envelopment Analysis models. By analogy, our method could therefore be termed “Portfolio Envelopment Analysis” (PEA).

proportion of each of these n financial assets. Thus, one can define $x = (x_1, ..., x_n)$ with $\sum_{i=1...n} x_i = 1$. The condition $x_i \geq 0$ is imposed whenever short sales are excluded.

Decision-makers often face additional economic constraints (see, e.g., Pogue (1970) or Rudd and Rosenberg (1979)). For instance, the proportion of each of the n financial assets composing a portfolio can be modified by taking into account transaction costs or by imposing upper limits on any fraction invested. If these constraints are linear functions of the asset weights, then the set of admissible portfolios is defined as:

$$\Im = \left\{ x \in R^n ; \sum_{i=1...n} x_i = 1, \quad Ax \leq b, \quad x \geq 0 \right\} \tag{1}$$

where A is a $(m \times n)$ matrix and $b \in R^m$. We assume throughout the paper that $\Im \neq \varnothing$

The return of portfolio x is: $R(x) = \sum_{i=1...n} x_i R_i$. One can therefore calculate its expected return and its variance as follows:

$$E(R(x)) = \sum_{i=1...n} x_i E(R_i) \tag{2}$$

$$V(R(x)) = \sum_{i,j} x_i x_j Cov(R_i, R_j) \tag{3}$$

It is useful to define the mean-variance representation of the set $\Im$ of portfolios. From Markowitz (1952), it is straightforward to give the following definition:

$$\aleph = \{(V(R(x)), E(R(x))); \quad x \in \Im\} \tag{4}$$

However, such a representation cannot be used for quadratic programming, because the subset $\aleph$ is not convex (see, for instance, Luenberger (1998)). Thus, we extend the above set by defining a mean-variance (portfolio) representation set through:

$$\Re = \{\aleph + (R_+ \times (-R_+))\} \cap R_+^2 \tag{5}$$

This set can be rewritten as follows:

$$\Re = \{(V', E') \in R_+^2 ; \quad \exists x \in \Im, \quad (-V', E') \leq (-V(R(x)), E(R(x)))\} \tag{6}$$

The addition of the cone is necessary for the definition of a sort of "free disposal hull" of the mean variance representation of feasible portfolios. Clearly, the above definition is compatible with the definition in Markowitz (1952). To measure the degree of portfolio efficiency, it is necessary to isolate a subset of this representation set, generally known as the efficient frontier. This subset is defined as follows:

Definition 2.1. *In the mean-variance space, the weakly efficient frontier is defined as:*

$$\partial^M(\Im)=\{(V(R(x)),E(R(x)));\quad x\in\Im\wedge(-V(R(x)),E(R(x)))<(-V',E')\Rightarrow(V',E')\notin\Re\}$$

From the above definition the weakly efficient frontier is the set of all the mean-variance points that are not strictly dominated in the two dimensional space. It is also possible to define a *strongly* efficient frontier, but the above formulation simplifies most results in this contribution. Moreover, the geometric representation of the frontier (see Figure 1) is quite similar except for some rather special cases.

<FIGURE 1 ABOUT HERE>

The above definition enables us to define the set of weakly efficient portfolios:

Definition 2.2. *The set of the weakly efficient portfolios is defined, in the simplex, as:*

$$\Lambda^M(\Im)=\{x\in\Im;\quad (V(R(x)),E(R(x)))\in\partial^M(\Im)\}.$$

Markowitz (1959) defines an optimisation program to determine the portfolio corresponding to a given degree of risk aversion. This portfolio maximises a mean-variance utility function defined by:

$$U_{(\rho,\mu)}(x)=\mu E(R(x))-\rho V(R(x)) \tag{7}$$

where $\mu \geq 0$ and $\rho \geq 0$. This utility function satisfies positive marginal utility of expected return and negative marginal utility of risk. The quadratic optimisation program may simply be written as follows:

$$\begin{aligned} &max\, U_{(\rho,\mu)}(x)=\mu E(R(x))-\rho V(R(x))\\ &s.t \quad Ax\leq b\\ &\sum_{i=1\ldots n} x_i=1, x\geq 0 \end{aligned} \tag{8}$$

Traditionally, the ratio $\varphi=\rho/\mu\in[0,+\infty]$ represents the degree of absolute risk aversion.

Setting $\mu=0$ and $\rho=1$ eliminates the return information from this quadratic mathematical program and yields the efficient portfolio with minimum risk. Denoting this global minimum variance portfolio $\tilde{x}$, it can be represented in two-dimensional mean-variance space (see Figure 1) as $(\tilde{V},\tilde{R})=(V(R(\tilde{x})),E(R(\tilde{x})))$.

When shorting is allowed or there is a riskless asset with zero variance and non-zero positive return, then (from the two-fund and the one-fund theorems) the efficient frontier is determined by simple analytical solutions (e.g., Elton, Gruber and Padberg (1979) or Luenberger (1998)). Though the computational burden of the more general quadratic programming approach remains substantial, when building realistic portfolio models it is hard to avoid. The approach developed in the next section adheres to this quadratic programming tradition to maintain generality. To extend the well-known Markowitz approach, we introduce in the next section the EIP function of a portfolio as an indicator of its performance. This EIP function is similar to the shortage function (see Luenberger (1995)).

3. Efficiency Improvement Possibility Function and the Frontier of Efficient Portfolios

Intuitively stated, the shortage function in production theory measures the distance between some point of the production set and the Pareto frontier. Before we introduce this function formally, it is of interest to focus on the basic properties of the subset $\Re$ on which the shortage function is defined below:

Proposition 3.1. *The subset* $\Re$ *satisfies the following properties:*

1) $\Re$ *is a convex set.*

2) $\Re$ *is a closed set.*

3) $\forall (V,E)\in\Re,\ (-V',E')\geq 0,$ *and* $(-V',E')\leq(-V,E)\Rightarrow(V',E')\in\Re$

Proof. 1) We have immediately from equation (6), $\Re = \{(V',E')\in R_+^2;\ \ \exists x\in\Im,(-V',E')\leq(-V(R(x)),E(R(x)))\}$. Assume that $(V_1,E_1),(V_2,E_2)\in\Re$. Thus, we can deduce that there exists $x^1,x^2\in\Im$ such that $(-V_1,E_1)\leq(-V(R(x^1)),E(R(x^1)))$ and $(-V_2,E_2)\leq(-V(R(x^2)),E(R(x^2)))\in\Re$. Let us show that $\theta(V_1,E_1)+(1-\theta)(V_2,E_2)\in\Re$, $\forall\theta\in[0,1]$. Since $V(R(.))$ is a convex function, we immediately get the inequality $\theta V_1+(1-\theta)V_2\geq\theta(V(R(x^1)))+(1-\theta)(V(R(x^2)))\geq V(R(\theta x^1+(1-\theta)x^2))$. Moreover, we have $\theta E_1+(1-\theta)E_2\leq E(R(\theta x^1+(1-\theta)x^2))$. Thus, since $\left\{x\in R^n; Ax\leq b, \sum_{i=1\ldots n}x_i=1, x_i\geq 0\right\}$ is a convex set, there exists $x = \theta x^1 + (1-\theta)x^2 \in \Im$ such that $(-V(R(x)), E(R(x))) \geq \theta(-V^1, E^1) + (1-\theta)(-V^2, E^2)$. From expression (6), this implies $\theta(-V^1, E^1) + (1-\theta)(-V^2, E^2) \in \Re$ and 1) is proven. 2) The functions $V(R(.))$ and $E(R(.))$ are continuous with respect to *x*, thus $\aleph$ is a closed set. Using the result in Briec and Lesourd (1999), we get that $\{\aleph + (R_+ \times (-R_+))\}$ is closed, and obviously 2) holds. 3) From equation (6), $\forall (V, E) \in \aleph, (-V', E') \leq (-V, E) \Rightarrow (V', E') \in \Re$ and we straightforwardly deduce 3). Q.E.D.

From the above properties of the representation set, we are now able to define the notion of an efficiency measure in the specific context of Markowitz portfolio theory. Before introducing our own approach based on the shortage function, we first briefly review existing efficiency measures in a context of portfolio benchmarking.

The first measure introduced by Morey and Morey (1999) computes the maximum expansion of the mean return while the risk is fixed at its current level.[6] From our definition of the representation set, this mean return expansion function is defined by:

$$D_{MRE}(x) = \sup\{\theta; \quad (V(R(x)), \theta\, E(R(x))) \in \Re\} \tag{9}$$

In a similar vein, the same authors define a risk contraction function as follows:

$$D_{RC}(x) = \inf\{\lambda; \quad (\lambda\, V(R(x)), E(R(x))) \in \Re\} \tag{10}$$

This function measures the maximum proportionate reduction of risk while fixing the mean return level. These authors apply these functions to measure investment fund performance.

Now, we introduce the shortage function (Luenberger (1995)) and study its properties in the context of Markowitz portfolio theory. It is shown below (see Proposition 3.2) that it encompasses the functions (9) and (10) as special cases. To achieve this objective, we introduce the efficiency improvement possibility (EIP) function defined as follows:

Definition 3.1. *The function defined as:* $S_g(x) = \sup\{\delta; (V(R(x)) - \delta g_V, E(R(x)) + \delta g_E) \in \Re\}$ *is the efficiency improvement possibility (EIP) function for the portfolio x in the direction of vector* $g = (-g_V, g_E)$.

[6] This seems also the approach taken by Sengupta (1989).

The principle of the EIP function is illustrated in Figure 2. The efficiency improvement possibility function looks for improvements in the direction of both an increased mean return and a reduced risk. For instance, the inefficient portfolio A is projected onto the efficient frontier at point B. Notice that the EIP is very similar to the directional distance function, another name for the shortage function introduced in production analysis by Chambers, Chung and Färe (1998). The directional distance function looks for simultaneous changes in the direction of reducing inputs (*x*) and expanding outputs (*y*) (i.e., $g = (-g_x, g_y)$).

<FIGURE 2 ABOUT HERE>

The pertinence of this portfolio management efficiency indicator results from some of its elementary properties, as summarised in the following proposition.

Proposition 3.2. *Let* S_g *be the EIP function defined on* $\Im$. S_g *has the following properties:*

1) $x \in \Im \Rightarrow S_g(x) < +\infty$

2) If $(g_V, g_E) > 0$*, then* $S_g(x) = 0 \Leftrightarrow x \in \partial^M(\Im)$ (weak efficiency)

3) $\forall x, y \in \Im, \quad (-V(R(y)), E(R(y))) \leq (-V(R(x)), E(R(x))) \Rightarrow S_g(x) \leq S_g(y)$ (weak monotonicity on $\aleph$)

4) S_g *is continuous on* $\Im$.

5) If $g_V = -V(R(x))$ *and* $g_R = 0$*, then* $D_{RC}(x) = 1 - S_g(x)$.

6) If $(g_V, g_E) > 0$ *and* $g_R = E(R(x))$*, then* $D_{MRE}(x) = 1 + S_g(x)$.

Proof. 1) From the definition of the representation set, if $x \in \Im$ then the subset $C(x) = \{(V', E') \in \Re; \quad (V', -E') \leq (V(R(x)), -E(R(x)))\}$ is bounded. It follows trivially that $S_g(x) < +\infty$. 2) Assume that $x \notin \Lambda^M(\Im)$. In such a case, there exists some $(V', E') \in \Re$ such that $(-V', E') \geq (-V(R(x)), E(R(x)))$. But, from Definition 2.1 it follows immediately that $S_g(x) > 0$. Consequently, we deduce that $S_g(x) = 0 \Rightarrow x \in \Lambda^M(\Im)$. To prove the converse, let $(V(R(x)) - S_g(x)g_V, E(R(x)) + S_g(x)g_E)$. Assume that $S_g(x) > 0$. Since $(g_V, g_E) > 0$, we get $(-V(R(x)) + S_g(x)g_V, E(R(x)) + S_g(x)g_E) > (-V(R(x)), E(R(x)))$. We deduce immediately that $x \notin \Lambda^M(\Im)$, and 2) holds. 3) follows from Luenberger (1995). 4) Let the function $T : \Re \rightarrow R_+$ defined by $T(V, E) = sup\{\delta; (V - \delta g_V, E + \delta g_E) \in \Re\}$. Since $\Re$ is convex and satisfies the free disposal rule, it is easy to show the continuity of *T*. Moreover, since mean and variance are continuous functions with respect to *x*, 4) holds. 5) and 6) result from making some obvious changes (see, e.g., Chambers, Chung and Färe (1998)). Q.E.D.

Briefly commenting on these properties, the use of the EIP function only guarantees weak efficiency. It does not exclude projections on vertical parts of the frontier allowing for an additional expansion in terms of expected return. Furthermore, portfolios with weakly dominated risk and return characteristics are only classified as weakly less efficient. Finally, the last two parts clearly establish a link with the Morey and Morey (1999) single dimension efficiency measurement orientations in (9) and (10). Implementing some obvious changes, a simple proof for these links is straightforwardly derived from, for instance, Chambers, Chung and Färe (1998). The next section studies the EIP function from a duality standpoint.

4. Duality, Shadow Risk Aversion and Mean-Variance Utility

Already Markowitz (1959) envisioned portfolio selection as a two step procedure, whereby the reconstruction of the efficient set of portfolios in a first step is subsequently followed by picking the optimal portfolio for a given preference structure. To provide a dual interpretation of the EIP function, we must first define the indirect mean-variance utility function (see, e.g., Farrar (1962) or Philippatos (1979a)).

Definition 4.1. *For given parameters* (ρ,μ)*, the function defined as:*

$$U^*(\mu,\rho) = sup\ \mu E(R(x)) - \rho V(R(x))$$
$$s.t.\ Ax \leq b$$
$$\sum_{i=1...n} x_i = 1, \quad x \geq 0$$

is called the indirect mean-variance utility function.

Therefore, the maximum value function for the decision maker is straightforwardly determined for a given set of parameters (ρ,μ) representing his risk-aversion. Knowledge of these parameters allows selecting a unique efficient portfolio among those on the weakly efficient frontier maximising the decision maker's direct mean-variance utility function. Furthermore, Farrar (1962) suggested to trace the set of efficient portfolios by solving this dual problem for different sets of parameters (ρ,μ).

Note that more elaborate dual frameworks exist in the literature. For instance, Varian (1983) describes nonparametric test procedures verifying whether a suitable mean-variance utility function rationalises observed portfolio choices and asset prices. Our contribution adheres to the before mentioned tradition and does not depend on asset price information.

To apprehend duality in our framework, it is useful to distinguish between overall, allocative and portfolio efficiency when evaluating the scope for improvements in portfolio management. The following definition clearly distinguishes between these concepts.

Definition 4.2. *Let S_g be the EIP function defined on $\Im$. We call:*

1) Overall Efficiency (OE) index, the quantity:

$$OE(\rho,\mu) = sup\,\{\delta\,;\mu(E(R(x)) + \delta g_E) - \rho(V(R(x)) - \delta g_V) \leq U^*(\rho,\mu)\}$$

2) *Allocative Efficiency(AE) index, the quantity:*

$$AE(x,\rho,\mu) = OE(\rho,\mu) - S_g(x)$$

3) *Portfolio Efficiency (PE) index, the quantity:*

$$PE(x) = S_g(x)$$

This definition immediately implies:

$$OE(\rho,\mu) = \frac{U^*(\rho,\mu) - U_{(\rho,\mu)}(x)}{\rho g_V + \mu g_E} \tag{11}$$

Thus, *OE* is simply the ratio between (i) the difference between (maximum) indirect mean-variance utility (Definition 4.1) and the value of the direct mean-variance utility function for the observation evaluated and (ii) the normalised value of the direction vector $g = (-g_V, g_E)$ for given parameters (ρ,μ).

Expanding on the decomposition introduced in Definition 4.2, Portfolio Efficiency only guarantees reaching a point on the portfolio frontier, not necessarily a point on the frontier maximising the investor's indirect mean-variance utility function. In this sense, it is similar to the notion of technical efficiency in production theory. Allocative Efficiency, by contrast, measures the needed portfolio reallocation, along the portfolio frontier, to achieve the maximum of the indirect mean-variance utility function. This requires adjusting an eventual Portfolio Efficient portfolio in function of relative prices, i.e., the parameters of the mean-variance utility function. Overall Efficiency ensures that both these ideals are achieved simultaneously. Obviously, the following additive decomposition identity holds:

$$OE(\rho,\mu) = AE(x,\rho,\mu) + PE(x) \tag{12}$$

Notice that changes in the risk-aversion parameters (ρ,μ) alter the slope of the indirect utility function. While the amount of *PE* is invariant to these changes, the relative importance of *AE* and *OE* normally changes.

In Figure 2, this decomposition is illustrated for a portfolio denoted by point A. For simplicity, assume that $||g|| = ||(-g_V, g_E)|| = 1$, where $||.||$, is the usual Euclidean metric. In terms of this figure, it is easy to see that $OE = ||\mathrm{C} - \mathrm{A}||$, while $PE = ||\mathrm{B} - \mathrm{A}||$ and $AE = ||\mathrm{C} - \mathrm{B}||$.

The indirect mean-variance utility function turns out to be a useful tool to characterise the representation set $\Re$. In particular, by using duality one can state the following property.

Proposition 4.1. *The representation set* $\Re$ *admits the following dual characterisation:*

$$\Re = \{(V,E) \in R^2;\quad \mu E - \rho V \le U^*(\rho,\mu)\} \cap R_+^2$$

Proof. By definition, $\Re = \{\aleph + (R_+ \times (-R_+))\} \cap R_+^2$. However, if $(\rho,\mu) \notin R_+^2$, then $sup\{U_{(\rho,\mu)}(x);\ (V(R(x)), E(R(x))) \in \aleph + (R_+ \times (-R_+))\} = +\infty$. Since for any variance-mean vector we have $(V,E) \in \aleph + (R_+ \times (-R_+))$, we deduce that $U^*(\rho,\mu) \ge \mu E - \rho V$. Now, assume that $(V,E) \notin \Re$. From Proposition 3.1, $\Re$ is convex. From the separation theorem, there exists $(\rho,\mu) \in R_+^2$ such that $\mu E - \rho V > U^*(\rho,\mu)$. Consequently, $U^*(\rho,\mu) \ge \mu E - \rho V$ implies $(V,E) \in \Re$ and Proposition 4.1 follows. Q.E.D.

This proposition allows establishing a link between the shortage function and the indirect utility function in the next proposition. In particular, the duality result in Proposition 4.2 shows that the EIP function can be derived from the indirect mean-variance utility function, and conversely. It is inspired by Luenberger (1995), who established duality between the expenditure function and the shortage function.

Proposition 4.2. *Let* S_g *be the EIP function defined on* $\Im$. S_g *has the following properties:*

1) $S_g(x) = \inf\ \{U^*(\rho,\mu) - U_{(\rho,\mu)}(x);\quad \mu g_E + \rho g_V = 1,\quad \mu \geq 0, \rho \geq 0\}$

2) $U^*(\rho,\mu) = \sup\ \{U_{(\rho,\mu)}(x) - S_g(x);\quad x \in \Im\}$

Proof. The proof is a straightforward consequence of Luenberger (1995). Q.E.D.

This result proves that the EIP function can be computed over the dual of the mean-variance space. The support function of the representation set is the indirect utility function *U**.

We are now interested in studying the properties of the EIP function that presume differentiability at the point where the function is evaluated. Therefore, we introduce the *adjusted risk aversion function:*

$$(\rho,\mu)\ (x) = \arg\min\ \{U^*(\rho,\mu) - U_{(\rho,\mu)}(x);\quad \mu g_E + \rho g_V = 1,\quad \mu \geq 0, \rho \geq 0\} \tag{13}$$

that implicitly characterises the agent's risk aversion. It could also be labelled a shadow indirect mean-variance utility function, since it adopts a reverse approach by searching for the parameters (ρ,μ) defining a shadow risk aversion that renders the current portfolio optimal for the investor. This function is similar to the adjusted price function defined by Luenberger (1995) in consumer theory, whence our naming of the *adjusted risk aversion function*.

Proposition 4.3. *Let* S_g *be the EIP function defined on* $\Im$. *At the point where* S_g *is differentiable, it has the following properties:*

1) $\dfrac{\partial S_g(x)}{\partial x} = \dfrac{\partial U_{(\rho,\mu)(x)}(x)}{\partial x} = (\mu(x)I - 2\rho(x)\Omega)R$

2) $\left.\dfrac{\partial S_g(x)}{\partial V(R(x))}\right|_{E(R(x))=Cte} = \rho(x)$ *and* $\left.\dfrac{\partial S_g(x)}{\partial E(R(x))}\right|_{V(R(x))=Cte} = -\mu(x)$

where R denotes the vector of expected asset returns and I is a unit vector of appropriate dimensions.

Proof. 1) The proof is obtained by the standard envelope theorem. We have obviously the relationship $\frac{\partial S_g(x)}{\partial x} = \frac{\partial U_{(\rho,\mu)(x)}(x)}{\partial x}$. Since $\frac{\partial U_{(\rho,\mu)(x)}(x)}{\partial x} = \mu(x)R - 2\rho(x)\Omega R$, we deduce the result. The proof for 2) is obtained in a similar way. Q.E.D.

Result 1) shows that the variations of the shortage function with respect to *x* are identical to the variation of the indirect utility function, but calculated with respect to the adjusted risk aversion function. Moreover, it can be directly linked to the return of each asset and the covariance matrix. Furthermore, result 2) shows that the shortage function decreases when the expected return increases.

As shown below, one can link the *adjusted risk aversion function* and some kind of Marshalian demand for each asset. First, let us introduce the matrix of derivatives:

$$[B]_{i,j} = \begin{bmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial \mu}{\partial x} \end{bmatrix}_{i,j} \tag{14}$$

Moreover, given a risk aversion vector (ρ, μ), we define "Marshalian" demand for assets by:

$$m(\rho, \mu) = \arg\max\{U_{(\rho,\mu)(x)}(x); \quad x \in \Im\} \tag{15}$$

One can then define some kind of Slutsky matrix:

$$[S]_{i,j} = \left[\frac{\partial m(\rho,\mu)}{\partial \rho}, \frac{\partial m(\rho,\mu)}{\partial \mu}\right]_{i,j} \tag{16}$$

As shown in the next proposition, this Slutsky matrix can be linked to the matrix *B*.

Proposition 4.4. *Let* S_g *be the EIP function defined on* $\Im$. *At the point where* S_g *is differentiable, it has the following properties:*

1) $BS = \left[\frac{1}{\rho g_V + \mu g_E} I - \frac{1}{(\rho g_V + \mu g_E)^2} \begin{pmatrix} \rho \\ \mu \end{pmatrix} \times (g_V, g_E)\right]$

2) $S^T B^T = \left[\frac{1}{\rho g_V + \mu g_E} I - \frac{1}{(\rho g_V + \mu g_E)^2} \begin{pmatrix} g_V \\ g_E \end{pmatrix} \times (\rho, \mu)\right]$

3) $BB^+ = I - \frac{1}{(\rho g_v + \mu g_E)^2} \begin{pmatrix} g_V \\ g_E \end{pmatrix} \times (g_V, g_E)$

Proof. 1) Let us consider $(\overline{\rho},\overline{\mu})=\dfrac{(\rho,\mu)}{\rho g_V+\mu g_E}$. We have the equalities:

□ $\dfrac{\partial\overline{\rho}}{\partial\rho}=\sum_{k=1\ldots n}\dfrac{\partial\overline{\rho}}{\partial x_k}\dfrac{\partial m_k}{\partial\rho}=\dfrac{1}{\rho g_V+\mu g_E}-\dfrac{\rho g_V}{(\rho g_V+\mu g_E)^2}$;

$\dfrac{\partial\overline{\mu}}{\partial\mu}=\sum_{k=1\ldots n}\dfrac{\partial\overline{\mu}}{\partial x_k}\dfrac{\partial m_k}{\partial\mu}=\dfrac{1}{\rho g_V+\mu g_E}-\dfrac{\mu g_E}{(\rho g_V+\mu g_E)^2}$;

$\dfrac{\partial\overline{\rho}}{\partial\mu}=\sum_{k=1\ldots n}\dfrac{\partial\overline{\rho}}{\partial x_k}\dfrac{\partial m_k}{\partial\rho}=-\dfrac{\rho g_E}{(\rho g_V+\mu g_E)^2}$; and $\dfrac{\partial\overline{\mu}}{\partial\rho}=\sum_{k=1\ldots n}\dfrac{\partial\overline{\rho}}{\partial x_k}\dfrac{\partial m_k}{\partial\rho}=-\dfrac{\mu g_V}{(\rho g_V+\mu g_E)^2}$. Now, since

$$BS=\begin{pmatrix}\sum_{k=1\ldots n}\dfrac{\partial\overline{\rho}}{\partial x_k}\dfrac{\partial m_k}{\partial\rho} & \sum_{k=1\ldots n}\dfrac{\partial\overline{\rho}}{\partial x_k}\dfrac{\partial m_k}{\partial\rho}\\ \sum_{k=1\ldots n}\dfrac{\partial\overline{\mu}}{\partial x_k}\dfrac{\partial m_k}{\partial\rho} & \sum_{k=1\ldots n}\dfrac{\partial\overline{\mu}}{\partial x_k}\dfrac{\partial m_k}{\partial\mu}\end{pmatrix}$$

we deduce the result. 2) is obtained by taking the transpose of 1). 3) follows by combining 1) and 2). Q.E.D.

This proof can also be derived from Luenberger (1996). This result states that the Slutstky matrix, characterizing the "Marshalian" demand for each asset, is a type of skewed pseudo-inverse of the matrix *B*.

5. Computational Aspects of the EIP Function

The representation set $\Re$, defined by expression (6), can be used directly to compute the EIP function by recourse to standard quadratic optimisations methods. Assume a sample of *m* portfolios (or investment funds) $y^1, y^2,\ldots, y^m$. Now, consider a specific portfolio y^k for $k\in\{1,\ldots,m\}$ whose performance needs to be gauged. The shortage function for this portfolio y^k under evaluation is computed by solving the following quadratic program:

$$\begin{aligned}&\max\ \delta\\ s.t.\quad & E(R(y^k))+\delta g_E\le E(R(x))\\ & V(R(y^k))-\delta g_V\ge V(R(x)) \qquad\qquad (P_1)\\ & Ax\le b\\ & \sum_{i=1\ldots n}x_i=1,\ x_i\ge 0,\ i=1\ldots n\end{aligned}$$

From equations (2) and (3), program (P_1) can be rewritten as follows:

$$\begin{aligned} & \max \delta \\ s.t.\quad & E\left(R\left(y^k\right)\right)+\delta g_E \le \sum_{i=1\ldots n} x_i E(R_i) \\ & V\left(R\left(y^k\right)\right)-\delta g_V \ge \sum_{i,j} \Omega_{i,j} x_i x_j \\ & Ax \le b \\ & \sum_{i=1\ldots n} x_i = 1,\ x_i \ge 0,\ i=1\ldots n \end{aligned} \qquad (P_2)$$

Thus, one quadratic program is solved for each portfolio to assess its performance. To obtain the entire decomposition from Definition 4.2, one only needs to compute the additional quadratic program from Definition 4.1. Then, applying expression (11) and Definition 4.2 itself, the components *OE* and *AE* follow suit.

All of the above programs can be seen as special cases of the following standard form:

$$\begin{aligned} & \min c^T z \\ s.t.\quad & L_j(z) = \alpha_j, \quad j=1\ldots q \\ & Q_k(z) \le \beta_k, \quad k=1\ldots r \\ & z \in R^p \end{aligned} \qquad (P_3)$$

where L_j is a linear map for $j = 1\ldots q$ and Q_k is a positive semi-definite quadratic form for $k = 1\ldots r$. In the case of program (P_2), $q=1$, and $r=n+3$, the latter because there are n non-negativity constraints. Program (P_3) is a standard quadratic optimisation problem (see Fiacco and McGormick (1968), Luenberger (1984)).

A novel result of some practical significance is that the adjusted risk aversion function (13) can be derived from the Kuhn-Tucker multipliers in program (P_2). This is shown in the next proposition.

Proposition 5.1. *Let* $k \in \{1,\ldots,m\}$ *such that program* (P_2) *has a regular optimal solution. Let* $\lambda_E \ge 0$ and $\lambda_V \ge 0$ *be respectively the Kuhn-Tucker multipliers of the first two constraints in program* (P_2). *If the EIP function is differentiable at point* $y^k \in \Im$, *then:*

1) *We have:*

$$\left.\frac{\partial S_g(y)}{\partial V(R(y))}\right|_{\substack{y=y^k \\ E(R(y))=E(R(y^k))}} = \lambda_V \quad \text{and} \quad \left.\frac{\partial S_g(y)}{\partial E(R(y))}\right|_{\substack{y=y^k \\ V(R(y))=V(R(y^k))}} = -\lambda_E$$

2) *The adjusted price function is identical to the Kuhn-Tucker multipliers:*

$$(\rho,\mu)\left(y^k\right)=\left(\lambda_V,\lambda_E\right)$$

Proof. 1) The proof is based on the sensitivity theorem (e.g., Luenberger (1984)). A solution of Program (P_2) is immediately obtained solving the program:

$$\begin{aligned} &\min\ -\delta \\ s.t.\quad & -\sum_{i=1\dots n} x_i E(R_i)+\delta g_E \le -E\left(R\left(y^k\right)\right) \\ & \sum_{i,j}\Omega_{i,j}x_i x_j+\delta g_V \le V\left(R\left(y^k\right)\right) \\ & Ax\le b \\ & \sum_{i=1\dots n} x_i=1,\ -x_i\le 0,\ i=1\dots n \end{aligned} \qquad (P_4)$$

Remark: all constraint functions on the left hand side in the two first inequalities are convex. Therefore, (P_4) has the standard form described in Luenberger (1984). Now, let us consider the parametric program:

$$\begin{aligned} &\min\ -\delta \\ s.t.\quad & -\sum_{i=1\dots n} x_i E(R_i)+\delta g_E \le c_E \\ & \sum_{i,j}\Omega_{i,j}x_i x_j+\delta g_V \le c_V \\ & Ax\le b \\ & \sum_{i=1\dots n} x_i=1,\quad x_i\ge 0,\quad i=1\dots n \end{aligned} \qquad (P_5)$$

Since (P_2) has a regular optimal solution, the bordered Hessian of (P_4) at the optimum is non-singular. Consequently, the sensitivity theorem applies. Let $x^*\left(c_V,c_E\right)$ be the optimal solution of the parametric program (P_5). Let us denote $-\delta^*\left(x^*\left(c_V,c_E\right)\right)$ the corresponding optimal value function. By definition, the Kuhn-Tucker multipliers of programs (P_2) and (P_4) are identical. From the sensitivity theorem, we have:

$$\left.\frac{\partial\left(-\delta^*\left(x^*\left(c_V,c_E\right)\right)\right)}{\partial c_V}\right|_{c_V=V\left(R\left(y^k\right)\right)}=-\lambda_V \quad\text{and}\quad \left.\frac{\partial\left(-\delta^*\left(x^*\left(c_V,c_E\right)\right)\right)}{\partial c_E}\right|_{c_E=-E\left(R\left(y^k\right)\right)}=-\lambda_E$$

We immediately deduce that

$$\left.\frac{\partial S_g(y)}{\partial V(R(y))}\right|_{\substack{y=y^k \\ E(R(y))=E\left(R\left(y^k\right)\right)}}=-\left.\frac{\partial\left(-\delta^*\left(x^*\left(c_V,c_E\right)\right)\right)}{\partial c_V}\right|_{c_V=V\left(R\left(y^k\right)\right)}=\lambda_V$$

Moreover:

$$\left.\frac{\partial S_g(y)}{\partial E(R(y))}\right|_{\substack{y=y^k \\ V(R(y))=V(R(y^k))}}$$

$$= -\left.\frac{\partial\left(-\delta^*\left(x^*\left(c_V,-c_E\right)\right)\right)}{\partial\left(-c_E\right)}\right|_{-c_E=E\left(R\left(y^k\right)\right)} = \left.\frac{\partial\left(-\delta^*\left(x^*\left(c_V,c_E\right)\right)\right)}{\partial c_E}\right|_{c_E=-E\left(R\left(y^k\right)\right)} = -\lambda_E .$$

This ends the proof. 2) This result is immediate from Proposition 4.3, 2). Q.E.D.

It may seem that the interest of our approach based on quadratic programming concerns essentially the original Markowitz model with short sales excluded. Our models would then simply provide a novel approach to the efficiency of portfolio management in the case of regulated investment funds, such as mutual funds in the United States, unit trusts in the UK, or EU-regulated UCITS[7], which under current regulation cannot invest in uncovered derivative instruments. Of course, if the possibility of short sales is not excluded or if there exists a riskless asset with zero variance and non-zero positive return, then the efficient frontier is straightforwardly determined by simpler, analytical solutions without recourse to quadratic optimisation (e.g., Elton, Gruber and Padberg (1979)). However, the quadratic programming approach remains valid in general. In particular, since quadratic program P_3 can be derived from P_4, it does not require a positive definite covariance matrix. Therefore, our models remain equally valid under these cases, with practical applications to measuring asset management efficiency for, e.g., regulated funds of futures and unregulated hedge funds.

Computation of these quadratic programs provides an inner bound approximation of the true, unknown portfolio frontier. This envelopment frontier is akin to the production frontiers alluded to in the introduction. This estimator is a nonparametric method, inasmuch as no functional form is specified for the Pareto frontier. Figure 3 illustrates this logic behind the performance gauging of portfolios using program (P_2). We evaluate the technically inefficient observations (V_0,E_0) to (V_3,E_3) and project them onto the portfolio frontier using the same direction vector g. By adding fictitious points or by implementing a critical line

[7] UCITS is the acronym for "Undertakings for Collective Investment in Transferable Securities", as regulated under a European Community directive of 1985 which, just as the American law of 1940 regulating mutual funds, precludes any form of leveraging, whether from borrowed funds or the use of uncovered derivatives.

search following Markowitz (1959), it is possible to refine the approximation of the efficient set of portfolios until it coincides with the Markowitz frontier.

<FIGURE 3 ABOUT HERE>

We end with two general remarks. One is concerned with the possibility of weakly efficient portfolios. The other observation focuses on the selection of a direction vector $g = (-g_V, g_E)$ in all these mathematical programs.

First, the projection of (V_0,E_0) onto a vertical segment of the set of weakly efficient portfolios illustrates the scope for further removing inefficiencies until one reaches the global minimum variance portfolio. A pragmatic solution is to substitute the global minimum variance portfolio, that provides a better expected return for the same risk, for projection points representing weakly efficient portfolios (identifiable by positive slack variables). Theoretical solutions that could be developed require sharpening the definition of the efficient frontier, or formulating doubts about the choice of direction $g = (-g_V, g_E)$ for weakly efficient portfolios (e.g., selecting a direction that guarantees at least a projection onto the global minimum variance portfolio). We deem such developments beyond the scope of this contribution. Furthermore, assuming one is interested in estimating the *OE* decomposition (*OE* implying strongly efficient portfolios), the problem of weakly efficient portfolios is limited to the *PE* component and only leads to a slight change in the relative importance of both components (*AE* versus *PE*).

Second, some remarks on the choice of the direction vector may prove useful. In principle, various alternative directions are possible (e.g., Chambers, Chung and Färe (1998)). For instance, it is possible to choose a common direction for all portfolios, as illustrated in Figure 3 above. This has a clear economic meaning in consumer theory where, for instance, utility may be measured using a type of distance function with respect to a common basket of goods (see the benefit function in Luenberger (1992)). But the economic interpretation of a common direction *g* in production and investment theory is not evident to us.

A far more straightforward choice for investment theory is to use the observation under evaluation itself (i.e., $g = (-V(R(x)), E(R(x)))$). Then, the shortage function measures the maximum percentage of risk reduction and expected return improvement. The dual formulation of the shortage function leads to a simpler interpretation:

$$\begin{aligned} S_g(x) &= inf\ \{U^*(\rho,\mu) - U_{(\rho,\mu)}(x);\quad \mu g_E + \rho g_V = 1,\quad \mu \geq 0, \rho \geq 0\} \\ &= inf\ \{U^*(\rho,\mu) - U_{(\rho,\mu)}(x);\quad -\mu E(R(x)) + \rho V(R(x)) = 1,\quad \mu \geq 0, \rho \geq 0\} \\ &= inf\ \{U^*(\rho,\mu) - U_{(\rho,\mu)}(x);\quad U_{(\rho,\mu)}(x) = 1,\quad \mu \geq 0, \rho \geq 0\} \end{aligned} \tag{17}$$

Now, by a simple normalization scheme (see Chambers, Chung and Färe (1998)), we can equivalently write:

$$S_g(x) = inf\ \left\{\frac{U^*(\rho',\mu') - U_{(\rho',\mu')}(x)}{U_{(\rho',\mu')}(x)};\quad \mu' \geq 0, \rho' \geq 0\right\} \tag{18}$$

Thus, the shortage function is now interpreted as the minimum percentage improvement in the direction to reach the maximum of the utility function (i.e., the indirect utility function). Since we work in mean-variance space, the shadow risk-aversion minimising this percentage provides a general efficiency index.

6. Empirical Illustration: Investment Funds

To show the ease of implementing the basic framework developed in this contribution, we compute the decomposition of overall efficiency for a small sample of 26 investment funds earlier analysed in Morey and Morey (1999). Return and risk are computed over a 3-year time horizon between July 1992 and June 1995 (see their Tables 1 and 8). Computing program (P_2), the quadratic program in Definition 4.1 for parameters $\mu = 1$ and $\rho = 2$, and applying the decomposition in Definition 4.2, we obtain the results summarised in Table 1. To save space, we do not report portfolio weights and slack variables. Risk aversion is based on conventional values for ρ that often range between 0.5 and 10 (e.g., Uysal, Trainer and Reis (2001)).

To underline the ease of interpretation of our performance measure, we briefly comment on the decomposition results of a single fund: “44 Wall Street Equity”. It could improve its overall efficiency by 40%, both in terms of improving its return and reducing its risk. In terms of the decomposition, 22.5% of this rather poor performance is due to portfolio inefficiency, i.e., operating below the portfolio frontier, while 17% is due to allocative inefficiency, i.e., choosing a wrong mix of return and risk given the postulated risk attitudes.

The average performance of the investment funds is rather poor. They could improve their performance by about 58%, with the majority of inefficiencies being attributed to portfolio inefficiency. Looking at individual results, none of the investment funds perfectly suits the investors' preferences. Therefore, all are to some extent overall inefficient. The last investment fund in the list comes closest to satisfying the investors' needs. Only one investment fund (number 3) is portfolio efficient and is part of the set of frontier portfolios. The residual degree of allocative efficiency, listed in the third column, is small compared to the amount of portfolio inefficiency detected. This relative importance of portfolio efficiency relative to allocative efficiency is a finding not uncommon in production analysis. Whether the same general tendency also holds in portfolio gauging remains an open question. Obviously, these efficiency measures can be easily used as a rating tool.

The same results are also depicted graphically in Figure 4. We plot the return and risk investment funds in the sample, their projections onto the portfolio frontier using the shortage function (*PE*), and the single point on the frontier maximising the investors' preferences (*OE*).

<TABLE 1 ABOUT HERE>

<FIGURE 4 ABOUT HERE>

A potential major issue is the sensitivity of our results, and in particular the decomposition, to the postulated risk aversion parameters. Using Proposition 5.1, one can easily retrieve the adjusted risk aversion function minimizing inefficiency from the Kuhn-Tucker multipliers in program (P_2). These shadow risk aversions are reported in the last column of Table 1. Note that for one fund the shadow risk aversion is zero, due to a positive slack in the risk dimension. For the sample, the shadow risk aversion is on average 0.162 with a standard deviation of 0.087. To test the sensitivity of decomposition results, we have also computed the average efficiency components for the parameter $\mu = 1$ and a rather wide range of values for ρ. The results shown in Figure 5 for values ranging from almost 0 to 10 and in the detail window for the range between 0.05 and 1 indicate that the main source of inefficiency remains portfolio efficiency, except when risk aversion approaches zero.

Allocative efficiency is minimised for the value of the above-mentioned shadow risk aversion and increases slightly for deviations on both sides of this minimum.

<FIGURE 5 ABOUT HERE>

We end with two remarks. First, the confrontation between postulated risk aversion parameters and shadow risk aversion could be instructive. For instance, it makes it possible to assess whether portfolio management strategies adhere to certain specified risk profiles. Second, while the decomposition as such depends on a specified risk aversion parameter, in case one considers risk aversion to be unknown one could equally well ignore overall efficiency to focus on portfolio efficiency as such.

7. Conclusions

The objective of this paper has been to introduce a general method for measuring the efficiency of portfolios. Portfolios are benchmarked by simultaneously looking for risk contraction and mean return augmentation using the shortage function framework (Luenberger (1995)). The virtues of our approach can be summarised as follows: (i) it does not require the complete estimation of the efficient frontier but reveals the Markowitz efficient frontier by a nonparametric envelopment method; (ii) its efficiency measure lends itself perfectly for performance gauging; (iii) it yields interesting dual interpretations (iv) it stays close to the theoretical framework of Markowitz (1959) and does not require any simplifying hypotheses (in contrast to, e.g., Sharpe (1963)). A simple empirical application on a limited sample of investment funds served to illustrate the computational feasibility of this general framework.

The general idea of looking for both risk contraction and mean return expansion may well prove useful in a wide range of both theoretical and pragmatic financial models. Just to open up some perspectives, we mention three theoretical extensions and provide a limited selection of empirical possibilities. First, right from the outset, the mean variance approach has been criticised and alternative criteria for portfolio selection based, among others, upon higher order moments have been developed (see Philippatos (1979b)). Since the shortage function is a distance (gauge) function, a perfect representation of multidimensional choice sets, we conjecture that our framework could well be extended to these multidimensional portfolio selection approaches. Second, it is obvious that statically efficient portfolios may well prove dynamically inefficient, e.g., because investors may hedge intertemporal shifts in their opportunity sets. Therefore, it may be desirable to develop a multiperiod portfolio selection generalisation of our model. This could perhaps be achieved along the lines proposed in Li and Ng (2000). Third, recent research indicates that nonparametric frontier methods are not necessarily deterministic (Simar and Wilson (2000)). Contrary to widespread opinion, these models can be conceived as measuring efficiency relative to a nonparametric, maximum likelihood estimate of an unobserved (monotone and concave) true frontier. Asymptotic sampling distributions are rarely available, but confidence bounds for efficiency scores can be obtained via smoothed bootstrapping. These developments may pave the way for introducing computer-intensive statistical testing in this nonparametric approach.

In terms of practical applications, performance measures like Jensen's alpha and the Sharpe index have recently been estimated for a sample of applied to U.S. mutual funds using a nonparametric frontier technique (Murthi, Choi and Desai (1997)) inspired by Data Envelopment Analysis. Furthermore, financial institutions have been assessed by Kestemont, Wibaut and Boussemart (1996) using nonparametric frontiers in terms of their return and the several types (transactional, market, liquidity, capital, etc.) of risks they face (see also Pastor (1999)). Finally, the performance of investment funds has been gauged over different time horizons using an approach similar to us (Morey and Morey (1999)). Instead of having one mean return dimension and one risk dimension, return and risk information pertaining to different time horizons is included (assuming investors value the information for each time horizon equally). Clearly, since these applications all use a specialised efficiency measure, a wide range of extensions appears to be possible for our more general performance index.

At the philosophical level, the question remains whether eventually detected portfolio inefficiencies reveal judgemental errors on behalf of investors, or whether these are simply the result of not accounting for additional constraints inhibiting the achievement of full mean-variance efficiency. In the latter case, additional modelling efforts are no doubt required to derive what have been called "fitted portfolios" (Gouriéroux and Jouneau (1999)).

However, in analogy with similar discussions in production theory (e.g., Førsund, Lovell and Schmidt (1980)), we are inclined to conjecture that even accounting for additional constraints does not eliminate all portfolio inefficiencies. Therefore, having an unambiguous and general efficiency measure like the one proposed in this contribution remains as useful as ever.

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Figure 1: Finding the Optimal Portfolio

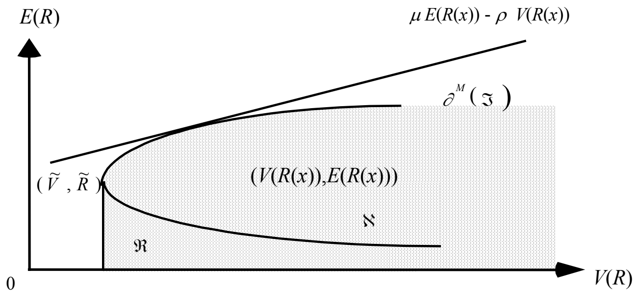

Figure 2: Efficiency Improvement Possibility Function & Decomposition

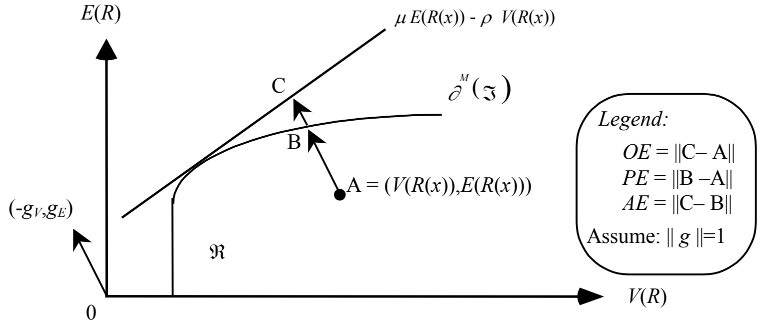

Figure 3: Portfolio Efficiency Analysis: Projections onto the Nonparametric Frontier

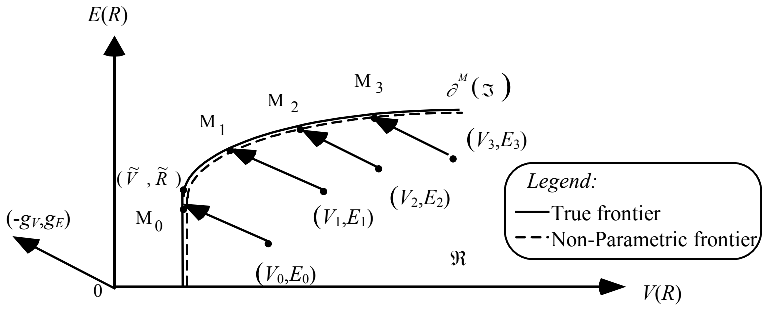

Figure 4: Portfolio Frontier: Observed Portfolios and Decomposition Results

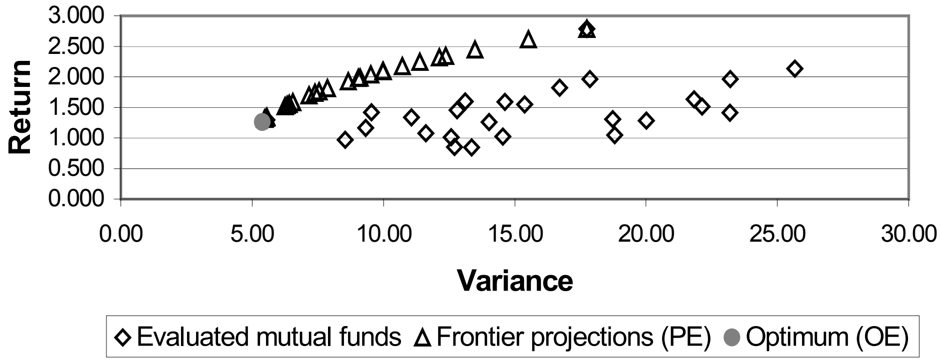

Figure 5: Sensitivity of Portfolio Efficiency Decomposition Results for ρ

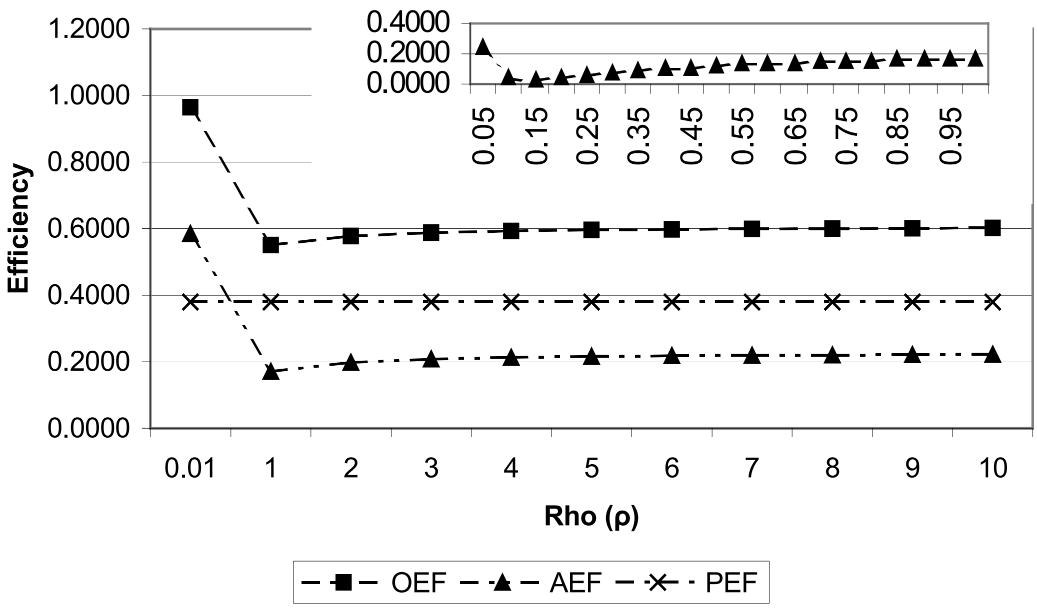

Table 1: Decomposition Results for Morey and Morey (1999) Sample

Observations	*OE*	*PE*	*AE*	φ*
20th Century Ultra Investors	0.718	0.433	0.285	0.095
44 Wall Street Equity	0.398	0.225	0.172	0.166
AIM Aggressive Growth	0.606	0.000	0.606	0.072
AIM Constellation	0.627	0.274	0.353	0.097
Alliance Quasar A	0.616	0.550	0.066	0.205
Delaware Trend A	0.610	0.351	0.259	0.116
Evergreen Aggressive Grth A	0.742	0.538	0.204	0.108
Founders Special	0.589	0.439	0.150	0.152
Fund Manager Aggressive Grth	0.366	0.357	0.009	0.330
IDS Strategy Aggressive B	0.593	0.583	0.011	0.314
Invesco Dynamics	0.543	0.274	0.269	0.122
Keystone Amer Omega A	0.521	0.448	0.073	0.213
Keystone Small Co Grth (S-4)	0.722	0.331	0.391	0.079
Oppenheimer Target A	0.402	0.320	0.082	0.219
Pacific Horizon Aggr Growth	0.700	0.619	0.081	0.175
PIMCo Adv Opportunity C	0.742	0.304	0.438	0.000
Putnam Voyager A	0.541	0.323	0.218	0.135
Security Ultra A	0.559	0.503	0.057	0.225
Seligman Capital A	0.573	0.564	0.009	0.319
Smith Barney Aggr Growth A	0.726	0.485	0.241	0.102
State St. Research Capital C	0.643	0.245	0.399	0.089
SteinRoe Capital Opport	0.588	0.317	0.272	0.116
USAA Aggressive Growth	0.708	0.545	0.162	0.128
Value Line Leveraged Gr Inv	0.481	0.319	0.163	0.161
Value Line Spec Situations	0.687	0.517	0.170	0.129
Winthrop Focus Aggr Growth	0.026	0.014	0.011	0.332
Mean	0.578	0.380	0.198	0.162
Standard deviation	0.155	0.159	0.152	0.087
Maximum	0.742	0.619	0.606	0.332

* Absolute risk aversion derived from the adjusted risk aversion function.

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