This is the accepted versión of: Borges, J., Fernández-Córdoba, C. and Ten Valls, R. "Z_2-double cyclic codes" in Proceedings of IX Jornadas de Matemática Discreta y Algorítmica (Tarragona, 7-9 de juliol de 2014)

$\mathbb{Z}_2$-double cyclic codes*

Joaquim Borges Ayats
Department of Information and Communication Engineering
Universitat Autònoma de Barcelona
08193-Bellaterra, Spain
Cristina Fernández-Córdoba
Department of Information and Communication Engineering
Universitat Autònoma de Barcelona
08193-Bellaterra, Spain
and
Roger Ten-Valls
Department of Information and Communication Engineering
Universitat Autònoma de Barcelona
08193-Bellaterra, Spain

November 19, 2014

Abstract

A binary linear code C is a $\mathbb{Z}_2$-double cyclic code if the set of coordinates can be partitioned into two subsets such that any cyclic shift of the coordinates of both subsets leaves invariant the code. These codes can be identified as submodules of the $\mathbb{Z}_2[x]$-module $\mathbb{Z}_2[x]/(x^r-1) \times \mathbb{Z}_2[x]/(x^s-1)$. We determine the structure of $\mathbb{Z}_2$-double cyclic codes giving the generator polynomials of these codes. The related polynomial representation of $\mathbb{Z}_2$-double cyclic codes and its duals, and the relations between the polynomial generators of these codes are studied.

1 Introduction

Let $\mathbb{Z}_2$ be the ring of integers modulo 2. Let $\mathbb{Z}_2^n$ denote the set of all binary vectors of length n. Any non-empty subset of $\mathbb{Z}_2^n$ is a binary code and a subgroup of $\mathbb{Z}_2^n$ is called a *binary linear code*. In this paper we introduce a subfamily of binary linear codes, called $\mathbb{Z}_2$-*double cyclic codes*, with the property that the set of coordinates can be partitioned into two subsets, the first r coordinates and the last s coordinates, such that any cyclic shift of the coordinates of both subsets of a codeword is also a codeword.

Notice that if one of these sets of coordinates is empty, for example $r = 0$, then we obtain a binary cyclic code of length s. So, binary cyclic codes are a special class of $\mathbb{Z}_2$-double cyclic codes. Most of the theory of binary cyclic codes

*This work has been partially supported by the Spanish MEC grant TIN2013-40524-P and by the Catalan grant 2014SGR691.

can be found in [8]. Another special case is when $r = s$, where a $\mathbb{Z}_2$-double cyclic code is permutation equivalent to a quasi-cyclic code of index 2 and even length (see [8]).

In recent times, $\mathbb{Z}_2\mathbb{Z}_4$-additive codes have been studied (see [2], [5]). For $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, the set of coordinates is partitioned into two subsets, the first one of binary coordinates and the second one of quaternary coordinates. The simultaneous cyclic shift of the subsets of coordinates of a codeword has been defined in [1], that studies $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes and these codes can be identified as $\mathbb{Z}_4[x]$-modules of a certain ring.

The aim of this paper is the study of the algebraic structure of $\mathbb{Z}_2$-double cyclic codes and their dual codes. It is organized as follows. In Section 2, we give the definition of $\mathbb{Z}_2$-double cyclic codes, we find the relation between some canonical projections of these codes and binary cyclic codes and we present the $\mathbb{Z}_2[x]$-module $\mathbb{Z}_2[x]/(x^r-1) \times \mathbb{Z}_2[x]/(x^s-1)$, denoted by $R_{r,s}$. In Section 3, we discuss about the algebraic structure of a $\mathbb{Z}_2$-double cyclic code and we state some relations between its generators. In Section 4, we study the concept of duality and, given a $\mathbb{Z}_2$-double cyclic code, we determine the generators of the dual code in terms of the generators of the code. Finally, in Section 5, we study the relations between the generator polynomials of a $\mathbb{Z}_2$-double cyclic code and the generators of other families of cyclic codes, $\mathbb{Z}_4$-cyclic codes and $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes.

2 $\mathbb{Z}_2$-double cyclic codes

Let C be a binary code of length n. Let r and s be integers such that $n = r+s$. We consider a partition of the set of the n coordinates into two subsets of r and s coordinates, respectively, so that C is a subset of $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$.

Definition 2.1. *Let C be a binary linear code of length $n = r + s$. The code C is called* $\mathbb{Z}_2$-double cyclic *if*

$$(u_0, u_1, \ldots, u_{r-2}, u_{r-1} \mid u'_0, u'_1, \ldots, u'_{s-2}, u'_{s-1}) \in C$$

implies

$$(u_{r-1}, u_0, u_1, \ldots, u_{r-2} \mid u'_{s-1}, u'_0, u'_1, \ldots, u'_{s-2}) \in C.$$

Let $\mathbf{u} = (u_0, u_1, \ldots, u_{r-1} \mid u'_0, \ldots, u'_{s-1})$ be a codeword in C and i be an integer, then we denote by

$$\mathbf{u}^{(i)} = (u_{0+i}, u_{1+i}, \ldots, u_{r-1+i} \mid u'_{0+i}, \ldots, u'_{s-1+i})$$

the ith shift of $\mathbf{u}$, where the subscripts are read modulo r and s, respectively.

Let C_r be the canonical projection of C on the first r coordinates and C_s on the last s coordinates. The canonical projection is a linear map. Then, C_r and C_s are binary cyclic codes of length r and s, respectively. A code C is called *separable* if C is the direct product of C_r and C_s.

There is a bijective map between $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$ and $\mathbb{Z}_2[x]/(x^r-1) \times \mathbb{Z}_2[x]/(x^s-1)$ given by:

$$(u_0, u_1, \ldots, u_{r-1} \mid u'_0, \ldots, u'_{s-1}) \mapsto (u_0 + u_1 x + \cdots + u_{r-1}x^{r-1} \mid u'_0 + \cdots + u'_{s-1}x^{s-1}).$$

We denote the image of the vector $\mathbf{u}$ by $\mathbf{u}(x)$.

Definition 2.2. *Denote by $R_{r,s}$ the ring $\mathbb{Z}_2[x]/(x^r-1) \times \mathbb{Z}_2[x]/(x^s-1)$. We define the operation*

$$\star : \mathbb{Z}_2[x] \times R_{r,s} \to R_{r,s}$$

as

$$\lambda(x) \star (p(x) \mid q(x)) = (\lambda(x)p(x) \mid \lambda(x)q(x)),$$

where $\lambda(x) \in \mathbb{Z}_2[x]$ and $(p(x) \mid q(x)) \in R_{r,s}$.

The ring $R_{r,s}$ with the external operation $\star$ is a $\mathbb{Z}_2[x]$-module. Let $\mathbf{u}(x) = (u(x) \mid u'(x))$ be an element of $R_{r,s}$. Note that if we operate $\mathbf{u}(x)$ by x we get

$$\begin{aligned} x \star \mathbf{u}(x) &= x \star (u(x) \mid u'(x)) \\ &= (u_0x + \cdots + u_{r-2}x^{r-1} + u_{r-1}x^r \mid u'_0x + \cdots + u'_{s-2}x^{s-1} + u'_{s-1}x^s) \\ &= (u_{r-1} + u_0x + \cdots + u_{r-2}x^{r-1} \mid u'_{s-1} + u'_0x + \cdots + u'_{s-2}x^{s-1}). \end{aligned}$$

Hence, $x \star \mathbf{u}(x)$ is the image of the vector $\mathbf{u}^{(1)}$. Thus, the operation of $\mathbf{u}(x)$ by x in $R_{r,s}$ corresponds to a shift of $\mathbf{u}$. In general, $x^i \star \mathbf{u}(x) = \mathbf{u}^{(i)}(x)$ for all i.

3 Algebraic structure and generators

In this section, we study submodules of $R_{r,s}$. We describe the generators of such submodules and state some properties. From now on, $\langle S \rangle$ will denote the submodule generated by a subset S of $R_{r,s}$.

Theorem 3.1. *The $\mathbb{Z}_2[x]$-module $R_{r,s}$ is a noetherian $\mathbb{Z}_2[x]$-module, and every submodule N of $R_{r,s}$ can be written as*

$$N = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle,$$

where $b(x), \ell(x) \in \mathbb{Z}_2[x]/(x^r-1)$ with $b(x) \mid (x^r-1)$ and $a(x) \in \mathbb{Z}_2[x]/(x^s-1)$ with $a(x) \mid (x^s-1)$.

Proof. Let $\pi_r : R_{r,s} \to \mathbb{Z}_2[x]/(x^r-1)$ and $\pi_s : R_{r,s} \to \mathbb{Z}_2[x]/(x^s-1)$ be the canonical projections, let N be a submodule of $R_{r,s}$.
As $\mathbb{Z}_2[x]/(x^s-1)$ is noetherian then $N_s = \pi_s(N)$ is finitely generated.
Define $N' = \{(p(x)|q(x)) \in N \mid q(x) = 0\}$. It is easy to check that $N' \cong \pi_r(N')$ by $(p(x) \mid 0) \mapsto p(x)$. Hence $\mathbb{Z}_2[x]/(x^r-1)$ is noetherian, $\pi_r(N')$ is finitely generated and so is N'.
Let $b(x)$ be a generator of $\pi_r(N')$, then $b(x) \mid (x^r-1)$ and $(b(x) \mid 0)$ is a generator of N'. Let $a(x) \in N_s$ such that $N_s = \langle a(x) \rangle$, then $a(x) \mid (x^s-1)$ and there exists $\ell(x) \in \mathbb{Z}_2[x]/(x^r-1)$ such that $(\ell(x) \mid a(x)) \in N$.
We claim that

$$N = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle.$$

Let $(p(x) \mid q(x)) \in N$, then $q(x) = \pi_s(p(x) \mid q(x)) \in N_s$. So, there exists $\lambda(x) \in \mathbb{Z}_2[x]$ such that $q(x) = \lambda(x)a(x)$. Now,

$$(p(x) \mid q(x)) - \lambda(x) \star (\ell(x) \mid a(x)) = (p(x) - \lambda(x)\ell(x) \mid 0) \in N'.$$

Then there exists $\mu(x) \in \mathbb{Z}_2[x]$ such that $(p(x) - \lambda(x)\ell(x) \mid 0) = \mu(x) \star (b(x) \mid 0)$. Thus,

$$(p(x) \mid q(x)) = \mu(x) \star (b(x) \mid 0) + \lambda(x) \star (\ell(x) \mid a(x)).$$

So, N is finitely generated by $\langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ and then $R_{r,s}$ is a noetherian $\mathbb{Z}_2[x]$-module. □

From the previous results, it is clear that we can identify $\mathbb{Z}_2$-double cyclic codes in $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$ as submodules of $R_{r,s}$. So, any submodule of $R_{r,s}$ is a $\mathbb{Z}_2$-double cyclic code. From now on, we will denote by C indistinctly both the code and the corresponding submodule.

Note that if C is a $\mathbb{Z}_2$-double cyclic code with $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$, then the canonical projections C_r and C_s are binary cyclic codes generated by $\gcd(b(x), \ell(x))$ and $a(x)$, respectively.

On the one hand, we have seen that $R_{r,s}$ is a $\mathbb{Z}_2[x]$-module, and multiply by $x \in \mathbb{Z}_2[x]$ is the right shift on the vector space $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$. On the other hand, we have that $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$ is a $\mathbb{Z}_2$-module, where the operations are addition and multiplication by elements of $\mathbb{Z}_2$.

So, our goal now is to find a set of generators for C as a $\mathbb{Z}_2$-module. We will denote the $\mathbb{Z}_2$-linear combinations of elements of a subset $S \subseteq R_{r,s}$ by $\langle S \rangle_{\mathbb{Z}_2} = \{\sum_i \lambda_i s_i \mid \lambda_i \in \mathbb{Z}_2, s_i \in S\}$, and we will call a set S a *$\mathbb{Z}_2$-linear independent* set if the relation $\sum_i \lambda_i s_i = 0$ implies that $\lambda_i s_i = 0$ for all i.

Proposition 3.2. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Define the sets*

$$S_1 = \{(b(x) \mid 0), x \star (b(x) \mid 0), \ldots, x^{r-\deg(b(x))-1} \star (b(x) \mid 0)\},$$

$$S_2 = \{(\ell(x) \mid a(x)), x \star (\ell(x) \mid a(x)), \ldots, x^{s-\deg(a(x))-1} \star (\ell(x) \mid a(x))\}.$$

Then, $S_1 \cup S_2$ forms a minimal generating set for C as a $\mathbb{Z}_2$-module.

Proof. It is easy to check that the codewords of $S_1 \cup S_2$ are $\mathbb{Z}_2$-linear independent.

Let $c(x) \in C$, such that $c(x) = p_1(x) \star (b(x) \mid 0) + p_2(x) \star (\ell(x) \mid a(x))$. We have to check that $c(x) \in \langle S_1 \cup S_2 \rangle_{\mathbb{Z}_2}$.

If $\deg(p_1(x)) < r - \deg(b(x)) - 1$, then $p_1(x) \star (b(x) \mid 0) \in \langle S_1 \rangle_{\mathbb{Z}_2}$. Otherwise, using the division algorithm, we compute $p_1(x) = q_1(x)\frac{x^r-1}{b(x)} + r_1(x)$ with $\deg(r_1(x)) < r - \deg(b(x)) - 1$, so

$$p_1(x) \star (b(x) \mid 0) = \left(q_1(x)\frac{x^r-1}{b(x)} + r_1(x)\right) \star (b(x) \mid 0) = r_1(x) \star (b(x) \mid 0) \in \langle S_1 \rangle_{\mathbb{Z}_2}.$$

So, $c(x) \in \langle S_1 \cup S_2 \rangle_{\mathbb{Z}_2}$ if $p_2(x) \star (\ell(x) \mid a(x)) \in \langle S_1 \cup S_2 \rangle_{\mathbb{Z}_2}$.

If $\deg(p_2(x)) < s - \deg(a(x)) - 1$, then $p_2(x) \star (\ell(x) \mid a(x)) \in \langle S_2 \rangle_{\mathbb{Z}_2}$. If not, using the division algorithm, consider $p_2(x) = q_2(x)\frac{x^s-1}{a(x)} + r_2(x)$ where $\deg(r_2(x)) < s - \deg(a(x)) - 1$. Then,

$$\begin{aligned} p_2(x) \star (\ell(x) \mid a(x)) &= \left(q_2(x)\frac{x^s-1}{a(x)} + r_2(x)\right) \star (\ell(x) \mid a(x)) \\ &= \left(q_2(x)\frac{x^s-1}{a(x)}\right) \star (\ell(x) \mid a(x)) + r_2(x) \star (\ell(x) \mid a(x)). \end{aligned}$$

On the one hand, $r_2(x) \star (\ell(x) \mid a(x)) \in \langle S_2 \rangle_{\mathbb{Z}_2}$. On the other hand,

$$\left(q_2(x)\frac{x^s-1}{a(x)}\right) \star (\ell(x) \mid a(x)) = (q_2(x)\frac{x^s-1}{a(x)}\ell(x) \mid 0).$$

By Proposition 3.6, $b(x)$ divides $\frac{x^s-1}{a(x)}\ell(x)$ and it follows straightforward that $(q_2(x)\frac{x^s-1}{a(x)}\ell(x) \mid 0) \in \langle S_1 \rangle_{\mathbb{Z}_2}$. Thus, $c(x) \in \langle S_1 \cup S_2 \rangle_{\mathbb{Z}_2}$. □

Proposition 3.3. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Then, C is permutation equivalent to a binary linear code with generator matrix of the form*

$$G = \left(\begin{array}{ccc|ccc} I_{r-\deg(b(x))} & A_1 & A_2 & 0 & 0 & 0 \\ 0 & B_\kappa & B & C_1 & I_\kappa & 0 \\ 0 & 0 & 0 & C_2 & R & I_{s-\deg(a(x))-\kappa} \end{array} \right),$$

where B_κ is a square matrix of full rank and $\kappa = \deg(b(x)) - \deg(\gcd(b(x), \ell(x)))$.

Proof. Let C be a $\mathbb{Z}_2$-double cyclic code with $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$. Then by Proposition 3.2, C is generated by the matrix whose rows are the elements of the set $S_1 \cup S_2$.

Since $r - \deg(b(x))$ and $s - \deg(a(x))$ are the dimensions of the matrices generated by the shifts of $b(x)$ and $a(x)$, respectively, the code C is permutation equivalent to a code with generator matrix of the form

$$\left(\begin{array}{cc|cc} I_{r-\deg(b(x))} & A' & 0 & 0 \\ 0 & B' & C' & I_{s-\deg(a(x))} \end{array} \right).$$

It is known that C_r is a linear cyclic code generated by $\gcd(b(x), \ell(x))$, then the submatrix B' has rank $\kappa = \deg(b(x)) - \deg(\gcd(b(x), \ell(x)))$. Moreover, C_r is permutation equivalent to a linear code generated by the matrix

$$\left(\begin{array}{ccc} I_{r-\deg(b(x))} & A_1 & A_2 \\ 0 & B_\kappa & B \\ 0 & 0 & 0 \end{array} \right),$$

with B_κ a full rank square matrix of size $\kappa \times \kappa$. Finally, applying the convenient permutations and linear combinations, we have that C is permutation equivalent to a linear code with generator matrix

$$\left(\begin{array}{ccc|ccc} I_{r-\deg(b(x))} & A_1 & A_2 & 0 & 0 & 0 \\ 0 & B_\kappa & B & C_1 & I_\kappa & 0 \\ 0 & 0 & 0 & C_2 & R & I_{s-\deg(a(x))-\kappa} \end{array} \right).$$

□

Corollary 3.4. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Then, C is a binary linear code of dimension $r + s - \deg(b(x)) - \deg(a(x))$.*

Proposition 3.5. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Then, we can assume that $\deg(\ell(x)) < \deg(b(x))$.*

Proof. Suppose that $\deg(\ell(x)) \geq \deg(b(x))$. Let $i = \deg(\ell(x)) - \deg(b(x))$ and let C' be the code generated by

$$C' = \langle (b(x) \mid 0), (\ell(x) + x^i \star b(x) \mid a(x)) \rangle.$$

On the one hand, $\deg(\ell(x) + x^i \star b(x)) < \deg(\ell(x))$ and since the generators of C' belongs to C, we have that $C' \subseteq C$. On the other hand,

$$(\ell(x) \mid a(x)) = (\ell(x) + x^i \star b(x) \mid a(x)) + x^i \star (b(x) \mid 0).$$

Then, $\langle (\ell(x) \mid a(x)) \rangle \subseteq C'$ and hence $C \subseteq C'$. Thus, $C = C'$. □

Proposition 3.6. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Then, $b(x) \mid \frac{x^s-1}{a(x)}\ell(x)$.*

Proof. Let π be the projective homomorphism of $\mathbb{Z}_2[x]$-modules defined by:

$$\begin{array}{rccc} \pi : & C & \longrightarrow & \mathbb{Z}_2[x]/(x^s-1) \\ & (p_1(x) \mid p_2(x)) & \longrightarrow & p_2(x) \end{array}$$

It can be easily checked that $\ker(\pi) = \langle (b(x) \mid 0) \rangle$.
Now, consider $\frac{x^s-1}{a(x)} \star (\ell(x) \mid a(x)) = (\frac{x^s-1}{a(x)}\ell(x) \mid 0)$. So,

$$\frac{x^s-1}{a(x)} \star (\ell(x) \mid a(x)) \in \ker(\pi) = \langle (b(x) \mid 0) \rangle.$$

Thus, $b(x) \mid \frac{x^s-1}{a(x)}\ell(x)$. □

Corollary 3.7. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Then, $b(x) \mid \frac{x^s-1}{a(x)} \gcd(\ell(x), b(x))$.*

Proposition 3.8. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a separable $\mathbb{Z}_2$-double cyclic code. Then, $\ell(x) = 0$.*

4 Duality

Let C be a $\mathbb{Z}_2$-double cyclic code and $C^\perp$ the dual code of C (see [7]). Taking a vector $\mathbf{v}$ of $C^\perp$, $\mathbf{u} \cdot \mathbf{v} = 0$ for all $\mathbf{u}$ in C. Since $\mathbf{u}$ belongs to C, we know that $\mathbf{u}^{(-1)}$ is also a codeword. So, $\mathbf{u}^{(-1)} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v}^{(1)} = 0$ for all $\mathbf{u}$ from C, therefore $\mathbf{v}^{(1)}$ is in $C^\perp$ and $C^\perp$ is also a $\mathbb{Z}_2$-double cyclic code. Consequently, we obtain the following proposition.

Proposition 4.1. *Let C be a $\mathbb{Z}_2$-double cyclic code. Then the dual code of C is also a $\mathbb{Z}_2$-double cyclic code. We denote*

$$C^\perp = \langle (\bar{b}(x) \mid 0), (\bar{\ell}(x) \mid \bar{a}(x)) \rangle,$$

where $\bar{b}(x), \bar{\ell}(x) \in \mathbb{Z}_2[x]/(x^r-1)$ with $\bar{b}(x) \mid (x^r-1)$ and $\bar{a}(x) \in \mathbb{Z}_2[x]/(x^s-1)$ with $\bar{a}(x) \mid (x^s-1)$.

The *reciprocal polynomial* of a polynomial $p(x)$ is $x^{\deg(p(x))}p(x^{-1})$ and is denoted by $p^*(x)$. As in the theory of binary cyclic codes, reciprocal polynomials have an important role in the duality (see [8]).

We denote the polynomial $\sum_{i=0}^{m-1} x^i$ by $\theta_m(x)$. Using this notation we have the following proposition.

Proposition 4.2. *Let $n, m \in \mathbb{N}$. Then, $x^{nm} - 1 = (x^n - 1)\theta_m(x^n)$.*

Proof. It is well know that $y^m - 1 = (y-1)\theta_m(y)$, replacing y by x^n the result follows. □

From now on, $\mathfrak{m}$ denotes the least common multiple of r and s.

Definition 4.3. *Let* $\mathbf{u}(x) = (u(x) \mid u'(x))$ *and* $\mathbf{v}(x) = (v(x) \mid v'(x))$ *be elements in* $R_{r,s}$*. We define the map*

$$\circ : R_{r,s} \times R_{r,s} \longrightarrow \mathbb{Z}_2[x]/(x^{\mathfrak{m}} - 1),$$

such that

$$\begin{aligned}\circ(\mathbf{u}(x), \mathbf{v}(x)) =& u(x)\theta_{\frac{\mathfrak{m}}{r}}(x^r)x^{\mathfrak{m}-1-\deg(v(x))}v^*(x) + \\ &+ u'(x)\theta_{\frac{\mathfrak{m}}{s}}(x^s)x^{\mathfrak{m}-1-\deg(v'(x))}v'^*(x) \mod (x^{\mathfrak{m}} - 1).\end{aligned}$$

The map $\circ$ is linear in each of its arguments; i.e., if we fix the first entry of the map invariant, while letting the second entry vary, then the result is a linear map. Similarly, when fixing the second entry invariant. Then, the map $\circ$ is a bilinear map between $\mathbb{Z}_2[x]$-modules.

From now on, we denote $\circ(\mathbf{u}(x), \mathbf{v}(x))$ by $\mathbf{u}(x) \circ \mathbf{v}(x)$. Note that $\mathbf{u}(x) \circ \mathbf{v}(x)$ belongs to $\mathbb{Z}_2[x]/(x^{\mathfrak{m}} - 1)$.

Proposition 4.4. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$ *with associated polynomials* $\mathbf{u}(x) = (u(x) \mid u'(x))$ *and* $\mathbf{v}(x) = (v(x) \mid v'(x))$*, respectively. Then,* $\mathbf{u}$ *is orthogonal to* $\mathbf{v}$ *and all its shifts if and only if*

$$\mathbf{u}(x) \circ \mathbf{v}(x) = 0 \mod (x^{\mathfrak{m}} - 1).$$

Proof. Let $\mathbf{v}^{(i)} = (v_{0+i}v_{1+i}\dots v_{r-1+i} \mid v'_{0+i}\dots v'_{s-1+i})$ be the ith shift of $\mathbf{v}$. Then,

$$\mathbf{u} \cdot \mathbf{v}^{(i)} = 0 \text{ if and only if } \sum_{j=0}^{r-1} u_j v_{j+i} + \sum_{k=0}^{s-1} u'_k v'_{k+i} = 0.$$

Let $S_i = \sum_{j=0}^{r-1} u_j v_{j+i} + \sum_{k=0}^{s-1} u'_k v'_{k+i}$. One can check that

$$\begin{aligned}\mathbf{u}(x) \circ \mathbf{v}(x) &= \sum_{n=0}^{r-1} \left[\theta_{\frac{\mathfrak{m}}{r}}(x^r) \sum_{j=0}^{r-1} u_j v_{j+n} x^{\mathfrak{m}-1-n} \right] + \cdots \\ &\cdots + \sum_{t=0}^{s-1} \left[\theta_{\frac{\mathfrak{m}}{s}}(x^s) \sum_{k=0}^{s-1} u'_k v'_{k+t} x^{\mathfrak{m}-1-t} \right] \\ &= \theta_{\frac{\mathfrak{m}}{r}}(x^r) \left[\sum_{n=0}^{r-1} \sum_{j=0}^{r-1} u_j v_{j+n} x^{\mathfrak{m}-1-n} \right] + \cdots \\ &\cdots + \theta_{\frac{\mathfrak{m}}{s}}(x^s) \left[\sum_{t=0}^{s-1} \sum_{k=0}^{s-1} u'_k v'_{k+t} x^{\mathfrak{m}-1-t} \right].\end{aligned}$$

Then, arranging the terms one obtains that

$$\mathbf{u}(x) \circ \mathbf{v}(x) = \sum_{i=0}^{\mathfrak{m}-1} S_i x^{\mathfrak{m}-1-i} \mod (x^{\mathfrak{m}} - 1).$$

Thus, $\mathbf{u}(x) \circ \mathbf{v}(x) = 0$ if and only if $S_i = 0$ for $0 \leq i \leq \mathfrak{m} - 1$. □

Lemma 4.5. *Let* $\mathbf{u}(x) = (u(x) \mid u'(x))$ *and* $\mathbf{v}(x) = (v(x) \mid v'(x))$ *be elements in* $R_{r,s}$ *such that* $\mathbf{u}(x) \circ \mathbf{v}(x) = 0 \mod (x^{\mathfrak{m}} - 1)$. *If* $u'(x)$ *or* $v'(x)$ *equal* 0, *then* $u(x)v^*(x) = 0 \mod (x^r - 1)$. *Respectively, if* $u(x)$ *or* $v(x)$ *equal* 0, *then* $u'(x)v'^*(x) = 0 \mod (x^s - 1)$.

Proof. Let $u'(x)$ or $v'(x)$ equal 0, then

$$\mathbf{u}(x) \circ \mathbf{v}(x) = u(x)\theta_{\frac{\mathfrak{m}}{r}}(x^r)x^{\mathfrak{m}-1-\deg(v(x))}v^*(x) + 0 = 0 \mod (x^{\mathfrak{m}} - 1).$$

So,

$$u(x)\theta_{\frac{\mathfrak{m}}{r}}(x^r)x^{\mathfrak{m}-1-\deg(v(x))}v^*(x) = \mu'(x)(x^{\mathfrak{m}} - 1),$$

for some $\mu'(x) \in \mathbb{Z}_2[x]$. Let $\mu(x) = \mu'(x)x^{\deg(v(x))+1}$, by Proposition 4.2,

$$u(x)x^{\mathfrak{m}}v^*(x) = \mu(x)(x^r - 1),$$

$$u(x)v^*(x) = 0 \mod (x^r - 1).$$

The same argument can be used to prove the other case. □

Proposition 4.6. *Let* $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code. Then,*

$$|C_r| = 2^{r-\deg(b(x))+\kappa}, |C_s| = 2^{s-\deg(a(x))},$$
$$|(C_r)^\perp| = 2^{\deg(b(x))-\kappa}, |(C_s)^\perp| = 2^{\deg(a(x))},$$
$$|(C^\perp)_r| = 2^{\deg(b(x))}, |(C^\perp)_s| = 2^{\deg(a(x))+\kappa},$$

where $\kappa = \deg(b(x)) - \deg(\gcd(b(x), \ell(x)))$.

Corollary 4.7. *Let* $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code with dual code* $C^\perp = \langle (\bar{b}(x) \mid 0), (\bar{\ell}(x) \mid \bar{a}(x)) \rangle$. *Then,*

$$\deg(\bar{b}(x)) \quad = \quad r - \deg(\gcd(b(x), \ell(x))).$$

Proof. It is easy to prove that $(C_r)^\perp$ is a cyclic code generated by $\bar{b}(x)$, so $|(C_r)^\perp| = 2^{r-\deg(\bar{b}(x))}$. Moreover, by Proposition 4.6, $|(C_r)^\perp| = 2^{\deg(b(x))-\kappa}$.
Thus, $\deg(\bar{b}(x)) = r - \deg(\gcd(b(x), \ell(x)))$. □

Corollary 4.8. *Let* $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code with dual code* $C^\perp = \langle (\bar{b}(x) \mid 0), (\bar{\ell}(x) \mid \bar{a}(x)) \rangle$. *Then,*

$$\deg(\bar{a}(x)) \quad = \quad s - \deg(a(x)) - \deg(b(x)) + \deg(\gcd(b(x), \ell(x))).$$

Proof. Since $C^\perp$ is a $\mathbb{Z}_2$-double cyclic code, $(C^\perp)_s$ is a cyclic code generated by $\bar{a}(x)$, so $|(C^\perp)_s| = 2^{s-\deg(\bar{a}(x))}$. Moreover, by Proposition 4.6, $|(C^\perp)_s| = 2^{\deg(a(x))+\kappa}$.
Thus, $\deg(\bar{a}(x)) = s - \deg(a(x)) - \deg(b(x)) + \deg(\gcd(b(x), \ell(x)))$. □

Proposition 4.9. *Let* $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code. Then,* $\langle (0 \mid \frac{x^s-1}{a^*(x)}) \rangle \subseteq C^\perp$.

Proof. Since C_s is a binary cyclic code generated by $\langle a(x) \rangle$, then $(C_s)^\perp = \langle \frac{x^s-1}{a^*(x)} \rangle$. Let $\mathbf{v}(x) = (v(x) \mid v'(x)) \in C$. Then, $v'(x) \in C_s$ and $(0 \mid \frac{x^s-1}{a^*(x)}) \circ \mathbf{v}(x) = 0 \mod (x^{\mathfrak{m}} - 1)$. Thus, $\langle (0 \mid \frac{x^s-1}{a^*(x)}) \rangle \subseteq C^\perp$. □

Corollary 4.10. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code with $C^\perp = \langle (\bar{b}(x) \mid 0), (\bar{\ell}(x) \mid \bar{a}(x)) \rangle$. Then, $\bar{a}(x)$ divides $\frac{x^s-1}{a^*(x)}$.*

Corollary 4.11. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Let $T = \{(0 \mid p(x)) \in C^\perp\}$. Then, T is generated by $\langle (0 \mid \frac{x^s-1}{a^*(x)}) \rangle$.*

Proof. Let $T = \{(0 \mid p(x)) \in C^\perp\}$. By Proposition 4.9, we have that $\langle (0 \mid \frac{x^s-1}{a^*(x)}) \rangle \subseteq T$.
Since $T_s \subseteq (C_s)^\perp = \langle \frac{x^s-1}{a^*(x)} \rangle$, for all $(0 \mid p(x)) \in T$ we have that $p(x) \in \langle \frac{x^s-1}{a^*(x)} \rangle$. Hence, there exists $\lambda(x) \in \mathbb{Z}_2[x]$ such that $p(x) = \lambda(x)\frac{x^s-1}{a^*(x)}$. Therefore, for all $(0 \mid p(x)) \in T$ we have that

$$(0 \mid p(x)) = (0 \mid \lambda(x)\frac{x^s-1}{a^*(x)}) = \lambda(x) \star (0 \mid \frac{x^s-1}{a^*(x)}).$$

So, $T \subseteq \langle (0 \mid \frac{x^s-1}{a^*(x)}) \rangle$. □

The previous propositions and corollaries will be helpful to determine the relations between the generator polynomials of a $\mathbb{Z}_2$-double cyclic code and the generator polynomials of its dual code.

Proposition 4.12. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code and $C^\perp = \langle (\bar{b}(x) \mid 0), (\bar{\ell}(x) \mid \bar{a}(x)) \rangle$. Then,*

$$\bar{b}(x) = \frac{x^r - 1}{\gcd(b(x), \ell(x))^*}.$$

Proof. We have that $(\bar{b}(x) \mid 0)$ belongs to $C^\perp$. Then,

$$\begin{aligned} (b(x) \mid 0) \circ (\bar{b}(x) \mid 0) &= 0 \mod (x^{\mathfrak{m}} - 1), \\ (\ell(x) \mid a(x)) \circ (\bar{b}(x) \mid 0) &= 0 \mod (x^{\mathfrak{m}} - 1). \end{aligned}$$

Therefore, by Lemma 4.5,

$$\begin{aligned} b(x)\bar{b}^*(x) &= 0 \mod (x^r - 1), \\ \ell(x)\bar{b}^*(x) &= 0 \mod (x^r - 1). \end{aligned}$$

So, $\gcd(b(x), \ell(x))\bar{b}^*(x) = 0 \mod (x^r - 1)$, and there exist $\mu(x) \in \mathbb{Z}_2[x]$ such that $\gcd(b(x), \ell(x))\bar{b}^*(x) = \mu(x)(x^r - 1)$.
Moreover, since $\gcd(b(x), \ell(x))$ and $\bar{b}^*(x)$ divides $(x^r - 1)$, by Corollary 4.7, we have that $\deg(\bar{b}(x)) = r - \deg(\gcd(b(x), \ell(x)))$. Then,

$$\bar{b}^*(x) = \frac{x^r - 1}{\gcd(b(x), \ell(x))}.$$

□

Proposition 4.13. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code and $C^\perp = \langle (\bar{b}(x) \mid 0), (\bar{\ell}(x) \mid \bar{a}(x)) \rangle$. Then,*

$$\bar{a}(x) = \frac{(x^s - 1)\gcd(b(x), \ell(x))^*}{a^*(x)b^*(x)}.$$

Proof. Consider the codeword

$$\frac{b(x)}{\gcd(b(x),\ell(x))}\star(\ell(x) \mid a(x)) - \frac{\ell(x)}{\gcd(b(x),\ell(x))}\star(b(x) \mid 0) = (0 \mid \frac{b(x)}{\gcd(b(x),\ell(x))}a(x)).$$

Then,

$$(\bar{\ell}(x) \mid \bar{a}(x)) \circ (0 \mid \frac{b(x)}{\gcd(b(x),\ell(x))}a(x)) = 0 \mod (x^{\mathfrak{m}} - 1).$$

Thus, by Lemma 4.5

$$\bar{a}(x)\frac{a^*(x)b^*(x)}{\gcd(b(x),\ell(x))^*} = 0 \mod (x^s - 1),$$

and

$$\bar{a}(x)\frac{a^*(x)b^*(x)}{\gcd(b(x),\ell(x))^*} = (x^s - 1)\mu(x),$$

for some $\mu(x) \in \mathbb{Z}_2[x]$. It is known that $\bar{a}(x)$ is a divisor of $x^s - 1$ and, by Corollary 3.7, we have that $\frac{a^*(x)b^*(x)}{\gcd(b(x),\ell(x))^*}$ divides $(x^s - 1)$. By Corollary 4.8, $\deg(\bar{a}(x)) = s - \deg(a(x)) - \deg(b(x)) + \deg(\gcd(b(x),\ell(x)))$, so

$$s = \deg\left(\bar{a}(x)\frac{a^*(x)b^*(x)}{\gcd(b(x),\ell(x))^*}\right) = \deg((x^s - 1)).$$

Hence, we obtain that $\mu(x) = 1$ and

$$\bar{a}(x) = \frac{(x^s - 1)\gcd(b(x),\ell(x))^*}{a^*(x)b^*(x)}.$$

□

Proposition 4.14. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a separable $\mathbb{Z}_2$-double cyclic code. Then, $C^\perp = \langle (\frac{x^r-1}{b^*(x)} \mid 0), (0 \mid \frac{x^s-1}{a^*(x)}) \rangle$.*

Corollary 4.15. *Let C be a separable $\mathbb{Z}_2$-double cyclic code. Then, $C^\perp$ is a separable $\mathbb{Z}_2$-double cyclic code.*

Proposition 4.16. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a non separable $\mathbb{Z}_2$-double cyclic code and $C^\perp = \langle (\bar{b}(x) \mid 0), (\bar{\ell}(x) \mid \bar{a}(x)) \rangle$. Then,*

$$\bar{\ell}(x) = \frac{x^r - 1}{b^*(x)}\lambda(x),$$

for some $\lambda(x) \in \mathbb{Z}_2[x]$.

Proof. Let $\bar{c} \in C^\perp$ with $\bar{c}(x) = (\bar{b}(x) \mid 0) + (\bar{\ell}(x) \mid \bar{a}(x))$. Then

$$\begin{aligned}\bar{c}(x) \circ (b(x) \mid 0) =& ((\bar{b}(x) \mid 0)) \circ (b(x) \mid 0) + ((\bar{\ell}(x) \mid \bar{a}(x))) \circ (b(x) \mid 0) \\ =& 0 + ((\bar{\ell}(x) \mid \bar{a}(x))) \circ (b(x) \mid 0) \\ =& 0 \mod (x^{\mathfrak{m}} - 1).\end{aligned}$$

So, by Lemma 4.5

$$\bar{\ell}(x)b^*(x) = 0 \mod (x^r - 1)$$

and

$$\bar{\ell}(x) = \frac{x^r - 1}{b^*(x)}\lambda(x).$$

□

Corollary 4.17. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a non separable $\mathbb{Z}_2$-double cyclic code. Then,* $\deg(\lambda(x)) < \deg(b(x)) - \deg(\gcd(b(x), \ell(x)))$.

Proposition 4.18. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a non separable $\mathbb{Z}_2$-double cyclic code and $C^{\perp} = \langle (\bar{b}(x) \mid 0), (\bar{\ell}(x) \mid \bar{a}(x)) \rangle$. Let $\rho(x) = \frac{\ell(x)}{\gcd(b(x), \ell(x))}$ and $\bar{\ell}(x) = \frac{x^r - 1}{b^*(x)} \lambda(x)$. Then,*

$$\frac{(x^{\mathfrak{m}} - 1) \gcd^*(b(x), \ell(x))}{b^*(x)} \left(\lambda(x) x^{\mathfrak{m} - \deg(\ell(x)) - 1} \rho^*(x) + x^{\mathfrak{m} - \deg(a(x)) - 1} \right) = 0 \mod (x^{\mathfrak{m}} - 1).$$

Thus,

$$\left(\lambda(x) x^{\mathfrak{m} - \deg(\ell(x)) - 1} \rho^*(x) + x^{\mathfrak{m} - \deg(a(x)) - 1} \right) = 0 \mod \left(\frac{b^*(x)}{\gcd^*(b(x), \ell(x))} \right).$$

Proof. Let $\rho(x) = \frac{\ell(x)}{\gcd(b(x), \ell(x))}$. Computing $(\bar{\ell}(x) \mid \bar{a}(x)) \circ (\ell(x) \mid a(x))$ and arranging properly we obtain that

$$\frac{(x^{\mathfrak{m}} - 1) \gcd^*(b(x), \ell(x))}{b^*(x)} \left(\lambda(x) x^{\mathfrak{m} - \deg(\ell(x)) - 1} \rho^*(x) + x^{\mathfrak{m} - \deg(a(x)) - 1} \right),$$

that is equal $0 \mod (x^{\mathfrak{m}} - 1)$. Then,

$$\left(\lambda(x) x^{\mathfrak{m} - \deg(\ell(x)) - 1} \rho^*(x) + x^{\mathfrak{m} - \deg(a(x)) - 1} \right) = 0 \mod (x^{\mathfrak{m}} - 1), \qquad (1)$$

or

$$\left(\lambda(x) x^{\mathfrak{m} - \deg(\ell(x)) - 1} \rho^*(x) + x^{\mathfrak{m} - \deg(a(x)) - 1} \right) = 0 \mod \left(\frac{b^*(x)}{\gcd^*(b(x), \ell(x))} \right). \qquad (2)$$

Since $\frac{b^*(x)}{\gcd^*(b(x), \ell(x))}$ divides $x^{\mathfrak{m}} - 1$, clearly (1) implies (2). □

Corollary 4.19. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a non separable $\mathbb{Z}_2$-double cyclic code and $C^{\perp} = \langle (\bar{b}(x) \mid 0), (\bar{\ell}(x) \mid \bar{a}(x)) \rangle$. Let $\rho(x) = \frac{\ell(x)}{\gcd(b(x), \ell(x))}$ and $\bar{\ell}(x) = \frac{x^r - 1}{b^*(x)} \lambda(x)$. Then,*

$$\lambda(x) = x^{\mathfrak{m} - \deg(a(x)) + \deg(\ell(x))} (\rho^*(x))^{-1} \mod \left(\frac{b^*(x)}{\gcd^*(b(x), \ell(x))} \right).$$

Proof. Let $\rho(x) = \frac{\ell(x)}{\gcd(b(x), \ell(x))}$. By Proposition 4.18,

$$\left(\lambda(x) x^{\mathfrak{m} - \deg(\ell(x)) - 1} \rho^*(x) + x^{\mathfrak{m} - \deg(a(x)) - 1} \right) = 0 \mod \left(\frac{b^*(x)}{\gcd^*(b(x), \ell(x))} \right).$$

Then,

$$\lambda(x) x^{\mathfrak{m}} \rho^*(x) = x^{\mathfrak{m} - \deg(a(x)) + \deg(\ell(x))} \mod \left(\frac{b^*(x)}{\gcd^*(b(x), \ell(x))} \right).$$

On the one hand, we have that $x^{\mathfrak{m}} = 1 \mod \left(\frac{b^*(x)}{\gcd^*(b(x), \ell(x))} \right)$. On the other hand, the great common divisor between $\rho(x)$ and $\frac{b(x)}{\gcd(b(x), \ell(x))}$ is 1, then $\rho^*(x)$ is an invertible element modulo $\left(\frac{b^*(x)}{\gcd^*(b(x), \ell(x))} \right)$. Thus,

$$\lambda(x) = x^{\mathfrak{m} - \deg(a(x)) + \deg(\ell(x))} (\rho^*(x))^{-1} \mod \left(\frac{b^*(x)}{\gcd^*(b(x), \ell(x))} \right).$$

□

We summarize the previous results in the next theorem.

Theorem 4.20. *Let $C = \langle (b(x) \mid 0), (\ell(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code and $C^\perp = \langle (\bar{b}(x) \mid 0), (\bar{\ell}(x) \mid \bar{a}(x)) \rangle$. Let $\rho(x) = \frac{\ell(x)}{\gcd(b(x),\ell(x))}$ and $\bar{\ell}(x) = \frac{x^r-1}{b^*(x)}\lambda(x)$. Then,*

1. $\bar{b}(x) = \frac{x^r-1}{\gcd(b(x),\ell(x))^*}$,
2. $\bar{a}(x) = \frac{(x^s-1)\gcd(b(x),\ell(x))^*}{a^*(x)b^*(x)}$,
3. $\bar{\ell}(x) = \frac{x^r-1}{b^*(x)}\lambda(x)$, *where*

$$\lambda(x)x^{\mathtt{m}}\rho^*(x) = x^{\mathtt{m}-\deg(a(x))+\deg(\ell(x))} \mod \left(\frac{b^*(x)}{\gcd^*(b(x),\ell(x))}\right).$$

5 Relations between $\mathbb{Z}_2$-double cyclic codes and other codes

In this section, we study how $\mathbb{Z}_2$-double cyclic codes are related with other families of cyclic codes, say $\mathbb{Z}_4$-cyclic codes and $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. Since these families of codes have part, or all the coordinates over $\mathbb{Z}_4$, then their generator polynomials also have coefficients over the ring $\mathbb{Z}_4$. From now on, the binary reduction of a polynomial $p(x) \in \mathbb{Z}_4[x]$ will be denoted by $\tilde{p}(x)$.

Let $\tilde{p}(x)$ be a divisor of $x^n - 1$ in $\mathbb{Z}_2[x]$ with n odd and let ξ be a primitive nth root of unity over $\mathbb{Z}_2$. The polynomial $(\tilde{p} \otimes \tilde{p})(x)$ is defined as the divisor of $x^n - 1$ in $\mathbb{Z}_2[x]$ whose roots are the products $\xi^i\xi^j$ such that ξ^i and ξ^j are roots of $\tilde{p}(x)$.

Let $\mathbf{u} = (u_0, \ldots, u_{n-1})$ be an element of $\mathbb{Z}_4^n$ such that $u_i = \tilde{u}_i + 2u'_i$ with $\tilde{u}_i, u'_i \in \{0, 1\}$. As in [6], the *Gray map* ϕ of $\mathbb{Z}_4^n$ to $\mathbb{Z}_2^{2n}$ is defined by

$$\phi(\mathbf{u}) = (u'_0, \ldots, u'_{n-1} \mid \tilde{u}_0 + u'_0, \ldots, \tilde{u}_{n-1} + u'_{n-1}).$$

Let $\mathbf{u}(x) = \tilde{\mathbf{u}}(x) + 2\mathbf{u}'(x)$ be the polynomial representation of $\mathbf{u} \in \mathbb{Z}_4^n$. Then, the polynomial version of the Gray map is $\phi(\mathbf{u}(x)) = (\tilde{\mathbf{u}}(x) \mid \tilde{\mathbf{u}}(x) + \mathbf{u}'(x))$. The *Nechaev permutation* is the permutation π of $\mathbb{Z}_2^{2n}$ with n odd defined by

$$\pi(v_0, v_1, \ldots, v_{2n-1}) = (v_{\tau(0)}, v_{\tau(1)}, \ldots, v_{\tau(2n-1)}),$$

where τ is the permutation on $\{0, 1, \ldots, 2n-1\}$ given by

$$(1, n+1)(3, n+3)\ldots(2i+1, n+2i+1)\ldots(n-2, 2n-2).$$

Let ψ be the map of $\mathbb{Z}_4^n$ into $\mathbb{Z}_2^{2n}$ defined by $\psi = \pi\phi$, with n odd. The map ψ is called the *Nechaev-Gray map*, [11]. We obtain the following theorem.

Theorem 5.1 ([11, Theorem 20])**.** *Let $\mathcal{C} = \langle f(x)h(x) + 2f(x) \rangle$ be a $\mathbb{Z}_4$-linear cyclic code of odd length n and where $f(x)h(x)g(x) = x^n - 1$. Let ϕ be the Gray map and let ψ the Nechaev-Gray map. The following properties are equivalent.*

1. $\gcd(\tilde{f}(x), (\tilde{g} \otimes \tilde{g})(x)) = 1$ *in* $\mathbb{Z}_2[x]$;
2. $\phi(\mathcal{C})$ *is a binary linear code of length* $2n$;
3. $\psi(\mathcal{C})$ *is a binary linear cyclic code of length* $2n$ *generated by* $\tilde{f}(x)^2\tilde{h}(x)$.

Using the last theorem, we can relate $\mathbb{Z}_2$-double cyclic codes to $\mathbb{Z}_4$-cyclic codes and $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes.

5.1 $\mathbb{Z}_2$-double cyclic codes vs $\mathbb{Z}_4$-cyclic codes

From [9] and [7], it is known that a $\mathbb{Z}_4$-cyclic code $\mathcal{C}$ of length n is generated by a single element $f(x)h(x)+2f(x) \in \mathbb{Z}_4[x]/(x^n-1)$, where $f(x)h(x)g(x) = x^n-1$ in $\mathbb{Z}_4[x]$, and $|\mathcal{C}| = 4^{\deg(g(x))}2^{\deg(h(x))}$.

Let $\mathcal{C}$ be a $\mathbb{Z}_4$-cyclic code of length n, and $\mathbf{w} \in \phi(\mathcal{C})$. The codeword $\mathbf{w}$ can be written as $(\tilde{u}_0, \ldots, \tilde{u}_{n-1} \,|\, \tilde{u}_0 + u'_0, \ldots, \tilde{u}_{n-1} + u'_{n-1})$, for $(u_0, \ldots, u_{n-1}) = \mathbf{u} = \phi^{-1}(\mathbf{w}) \in \mathcal{C}$. By definition of the Gray map, we have that $\mathbf{w}^{(1)}$ is $(\tilde{u}_{n-1}, \tilde{u}_0, \ldots, \tilde{u}_{n-2} \,|\, \tilde{u}_{n-1} + u'_{n-1}, \tilde{u}_0 + u'_0, \ldots, \tilde{u}_{n-2} + u'_{n-2}) = \phi(u_{n-1}, u_0, \ldots, u_{n-2})$. Therefore, since $\mathcal{C}$ is $\mathbb{Z}_4$-cyclic, we have that for $\mathbf{w} \in \phi(\mathcal{C})$, $\mathbf{w}^{(i)} \in \phi(\mathcal{C})$.

In general, the Gray map image of a $\mathbb{Z}_4$-linear code is not linear. Hence, we shall consider $\mathbb{Z}_2$-double cyclic codes as images of $\mathbb{Z}_4$-cyclic codes, $\mathcal{C} = \langle f(x)h(x)+2f(x)\rangle$, in the case that such a code $\mathcal{C}$ has linear image under the Gray map; that is, when $\gcd(\tilde{f}(x), (\tilde{g}\otimes\tilde{g})(x)) = 1$ in $\mathbb{Z}_2[x]$, by Theorem 5.1. Our goal is to stablish a relation between the generator polynomial of the $\mathbb{Z}_4$-linear cyclic code $\mathcal{C}$ and its $\mathbb{Z}_2$-double cyclic image, $\phi(\mathcal{C})$.

Let $\mathcal{C}$ be a $\mathbb{Z}_i[x]$-module with $i = 2, 4$. Let $g_1, \ldots, g_t \in \mathcal{C}$, then $\langle g_1, \ldots, g_t\rangle_i$ will denoted the $\mathbb{Z}_i[x]$-submodule of $\mathcal{C}$ generated by $g_1, \ldots, g_t$.

The following theorem is proved in [10, Theorem 8].

Theorem 5.2. *Let n be odd and let $f(x), h(x), g(x)$ be in $\mathbb{Z}_4[x]$ such that $f(x)h(x)g(x) = x^n - 1$. Then $\langle f(x)h(x)+2f(x)\rangle_4 = \langle \tilde{f}(x)\tilde{h}(x)\rangle_2 + 2\langle \tilde{f}(x)\rangle_2$ if and only if $\gcd(\tilde{f}(x), (\tilde{g}\otimes\tilde{g})(x)) = 1$ in $\mathbb{Z}_2[x]$.*

Lemma 5.3. *Let $\mathcal{C}$ be a quaternary linear code of type $2^\gamma 4^\delta$ such that $\phi(\mathcal{C})$ is a linear code. Let $\{\mathbf{u}_i\}_{i=1}^\gamma$ be codewords of order two and $\{\mathbf{v}_j\}_{j=1}^\delta$ codewords of order four such that $\mathcal{C} = \langle \{\mathbf{u}_i\}_{i=1}^\gamma, \{\mathbf{v}_j\}_{j=1}^\delta\rangle_4$. Then,*

$$\phi(\mathcal{C}) = \langle \{\phi(\mathbf{u}_i)\}_{i=1}^\gamma, \{\phi(\mathbf{v}_j)\}_{j=1}^\delta, \{\phi(2\mathbf{v}_j)\}_{j=1}^\delta\rangle_2.$$

Proof. From [4, Lemma 3], it is known that if $\mathcal{C}$ is a quaternary linear code of type $2^\gamma 4^\delta$ such that $\mathcal{C} = \langle \{\mathbf{u}_i\}_{i=1}^\gamma, \{\mathbf{v}_j\}_{j=1}^\delta\rangle_4$, then

$$\langle \phi(\mathcal{C})\rangle_2 = \langle \{\phi(\mathbf{u}_i)\}_{i=1}^\gamma, \{\phi(\mathbf{v}_j)\}_{j=1}^\delta, \{\phi(2\mathbf{v}_j)\}_{j=1}^\delta, \{\phi(2\mathbf{v}_j * \mathbf{v}_t)\}_{1\le j<t\le\delta}\rangle_2,$$

where $\mathbf{u}*\mathbf{v}$ denote the component-wise product for any $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_4^n$. We know that $\phi(\mathcal{C})$ is linear if and only if $\mathbf{u} * \mathbf{v} \in \mathcal{C}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{C}$, [6]. Since $\phi(\mathcal{C})$ is a binary linear code, then $\{2\mathbf{v}_j * \mathbf{v}_t\}_{1\le j<t\le\delta} \in \mathcal{C}$. Therefore, $\langle \{\phi(2\mathbf{v}_j * \mathbf{v}_t)\}_{1\le j<t\le\delta}\rangle_2 \subseteq \langle \{\phi(\mathbf{u}_i)\}_{i=1}^\gamma, \{\phi(\mathbf{v}_j)\}_{j=1}^\delta, \{\phi(2\mathbf{v}_j)\}_{j=1}^\delta\rangle_2$. □

Theorem 5.4. *Let $\mathcal{C} = \langle f(x)h(x)+2f(x)\rangle_4^n$ be a $\mathbb{Z}_4$-linear cyclic code of odd length n, where $f(x)h(x)g(x) = x^n - 1$ and $\gcd(\tilde{f}(x), (\tilde{g}\otimes\tilde{g})(x)) = 1$. Then,*

$$\phi(\mathcal{C}) = \langle (\tilde{f}(x)\tilde{h}(x) \mid 0), (\tilde{f}(x) \mid \tilde{f}(x))\rangle_2.$$

Proof. By Theorem 5.2, the generators of $\mathcal{C}$ are $\langle \tilde{f}(x)\tilde{h}(x)\rangle_2$ and $2\langle \tilde{f}(x)\rangle_2$, both, of order two. By Theorem 5.1, we have that $\phi(\mathcal{C})$ is linear. Hence, by Lemma 5.3, it is easy to see that the generator polynomials of $\phi(\mathcal{C})$ are $\phi(\tilde{f}(x)\tilde{h}(x))$ and $\phi(2\tilde{f}(x))$. The corresponding images of the Gray map are $\phi(\tilde{f}(x)\tilde{h}(x)) = (0 \mid \tilde{f}(x)\tilde{h}(x))$ and $\phi(2\tilde{f}(x)) = (\tilde{f}(x) \mid \tilde{f}(x))$, so $\phi(\mathcal{C}) = \langle (0 \mid \tilde{f}(x)\tilde{h}(x)), (\tilde{f}(x) \mid \tilde{f}(x))\rangle_2$.

Therefore,

$$\phi(\mathcal{C}) = \langle (\tilde{f}(x)\tilde{h}(x) \mid 0), (\tilde{f}(x) \mid \tilde{f}(x))\rangle_2.$$

□

5.2 $\mathbb{Z}_2$-double cyclic codes vs $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes

In recent times, $\mathbb{Z}_2\mathbb{Z}_4$-additive codes have been studied (see [2], [5]). A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, it is also isomorphic to a commutative structure like $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ and it has $|\mathcal{C}| = 2^{\gamma+2\delta}$ codewords.

Let X (respectively Y) be the set of $\mathbb{Z}_2$ (respectively $\mathbb{Z}_4$) coordinate positions, so $|X| = \alpha$ and $|Y| = \beta$. The set X corresponds to the first α coordinates and Y corresponds to the last β coordinates. Call $\mathcal{C}_X$ (respectively $\mathcal{C}_Y$) the punctured code of $\mathcal{C}$ by deleting the coordinates outside X (respectively Y). Notice that $\mathcal{C}_Y$ is a quaternary linear code and $\mathcal{C}_X$ is a binary linear code.

A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is called cyclic code if the set of coordinates can be partitioned into two subsets, the set of $\mathbb{Z}_2$ and the set of $\mathbb{Z}_4$ coordinates, such that any cyclic shift of the coordinates of both subsets leaves invariant the code. These codes can be identified as submodules of the $\mathbb{Z}_4[x]$-module $\mathbb{Z}_2[x]/(x^\alpha - 1) \times \mathbb{Z}_4[x]/(x^\beta - 1)$. From [1] and [3], we know that if $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, where β is an odd integer, then it is of the form

$$\mathcal{C} = \langle (b(x) \mid 0), (\ell(x) \mid f(x)h(x) + 2f(x)) \rangle_4$$

where $f(x)h(x)g(x) = x^\beta - 1$ in $\mathbb{Z}_4[x]$, $b(x), \ell(x) \in \mathbb{Z}_2[x]/(x^\alpha - 1)$ with $b(x)|(x^\alpha - 1)$, $deg(\ell(x)) < deg(b(x))$, and $b(x)$ divides $\frac{x^\beta - 1}{f(x)}\ell(x) \pmod 2$.

The *extended Gray map* Φ and the *extended Nechaev-Gray map* Ψ are the maps from $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ into $\mathbb{Z}_2^{\alpha+2\beta}$ given by

$$\Phi(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \phi(\mathbf{v})), \quad \Psi(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \psi(\mathbf{v})),$$

where $\mathbf{u} \in \mathbb{Z}_2^\alpha$, $\mathbf{v} \in \mathbb{Z}_4^\beta$, ϕ is the Gray map and ψ is the Nechaev-Gray map.

Notice that if $\phi(\mathcal{C}_Y)$ and $\psi(\mathcal{C}_Y)$ are binary linear codes, then $\Phi(\mathcal{C})$ and $\Psi(\mathcal{C})$ are also binary linear codes.

The following proposition can be viewed as a corollary of Theorem 5.2.

Proposition 5.5. *Let β be odd and let $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code generated by $\langle (b(x) \mid 0), (\ell(x) \mid f(x)h(x)+2f(x)) \rangle_4$ such that $f(x)h(x)g(x) = x^\beta - 1$ in $\mathbb{Z}_4[x]$. Then, $\gcd(\tilde{f}(x), (\tilde{g} \otimes \tilde{g})(x)) = 1$ in $\mathbb{Z}_2[x]$ if and only if $\mathcal{C} = \langle (b(x) \mid 0) \rangle_2 + \langle (\ell(x) \mid \tilde{f}(x)\tilde{h}(x)) \rangle_2 + 2\langle (0 \mid \tilde{f}(x)) \rangle_2$.*

Theorem 5.6. *Let $\mathcal{C} = \langle (b(x) \mid 0), (\ell(x) \mid f(x)h(x) + 2f(x)) \rangle_4 \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, where β is an odd integer and $f(x)h(x)g(x) = x^\beta - 1$. Let Ψ be the extended Nechaev-Gray map. If $\gcd(\tilde{f}(x), (\tilde{g} \otimes \tilde{g})(x)) = 1$ in $\mathbb{Z}_2[x]$, then $\Psi(\mathcal{C})$ is a $\mathbb{Z}_2$-double cyclic code of length $\alpha + 2\beta$ generated by*

$$\Psi(\mathcal{C}) = \langle (b(x) \mid 0), (\ell(x) \mid \tilde{f}(x)^2\tilde{h}(x)) \rangle_2.$$

Proof. Let $\gcd(\tilde{f}(x), (\tilde{g} \otimes \tilde{g})(x)) = 1$. By Theorem 5.1, $\psi(\mathcal{C}_Y)$ is a binary linear code, and then $\Psi(\mathcal{C})$ is also linear. By Proposition 5.5,

$$\mathcal{C} = \langle (b(x) \mid 0) \rangle_2 + \langle (\ell(x) \mid \tilde{f}(x)\tilde{h}(x)) \rangle_2 + \langle (0 \mid 2\tilde{f}(x)) \rangle_2.$$

Applying [5, Lemma 3], the extended version of Lemma 5.3, then the generator polynomials of $\Phi(\mathcal{C})$ are $\Phi((b(x) \mid 0))$, $\Phi((\ell(x) \mid \tilde{f}(x)\tilde{h}(x)))$ and $\Phi((0 \mid 2\tilde{f}(x)))$. The corresponding images of the extended Gray map are $\Phi((b(x) \mid 0)) = (b(x) \mid$

$0 \mid 0)$, $\Phi((\ell(x) \mid \tilde{f}(x)\tilde{h}(x))) = (\ell(x) \mid 0 \mid \tilde{f}(x)\tilde{h}(x))$ and $\Phi((0 \mid 2\tilde{f}(x))) = (0 \mid \tilde{f}(x) \mid \tilde{f}(x))$, so

$$\Phi(\mathcal{C}) = \langle (b(x) \mid 0 \mid 0), (\ell(x) \mid 0 \mid \tilde{f}(x)\tilde{h}(x)), (0 \mid \tilde{f}(x) \mid \tilde{f}(x)) \rangle_2 \subseteq \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_2^{\beta} \times \mathbb{Z}_2^{\beta}.$$

By [10, Corollary 7], we obtain immediately that

$$\Psi(\mathcal{C}) = \pi\Phi(\mathcal{C}) = \langle (b(x) \mid 0), (\ell(x) \mid \tilde{f}(x)^2\tilde{h}(x)) \rangle_2 \subseteq \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_2^{2\beta}.$$

□

References

[1] T. Abualrub, I. Siap, N. Aydin. $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. *IEEE Trans. Info. Theory*, vol. 60, No. 3, pp. 1508-1514, Mar. 2014.

[2] J. Borges, C. Fernández-Córdoba, J. Pujol, J. Rifà and M. Villanueva. $\mathbb{Z}_2\mathbb{Z}_4$-linear codes: generator matrices and duality. *Designs, Codes and Cryptography*, vol. 54, No. 2, pp. 167-179, 2010.

[3] J. Borges, C. Fernández-Córdoba, R. Ten-Valls. $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes, generator polynomials and dual codes. *IEEE Trans. Info. Theory*, submitted, arXiv:1406.4425.

[4] C. Fernández-Córdoba, J. Pujol and M. Villanueva. On rank and kernel of $\mathbb{Z}_4$-linear codes. *Lecture Notes in Computer Science*, n. 5228, pp. 46-55, 2008.

[5] C. Fernández-Córdoba, J. Pujol and M. Villanueva. $\mathbb{Z}_2\mathbb{Z}_4$-linear codes: rank and kernel. *Designs, Codes and Cryptography*, vol. 56, pp. 43 - 59, 2010.

[6] A.R. Hammons, P.V. Kumar, A.R. Calderbank, N.J.A. Sloane, P. Solé. The $\mathbb{Z}_4$-linearity of kerdock, preparata, goethals and related codes. *IEEE Trans. Info. Theory*, vol. 40, pp. 301-319, 1994.

[7] W.C. Huffman, V. Pless. *Fundamentals of Error-Correcting Codes.* Cambridge University Press, Cambridge, 2003.

[8] F.J. MacWilliams, N.J.A. Sloane. *The Theory of Error-Correcting Codes.* North-Holland Publishing Company, Amsterdam, New York, Oxford, 1975.

[9] V.S. Pless and Z. Qian. Cyclic codes and quadratic residue codes over $\mathbb{Z}_4$. *IEEE Trans. Info. Theory*, vol. 42, No. 5, pp. 1594-1600, 1996.

[10] G. Vega, J. Wolfmann. Some families of $\mathbb{Z}_4$-cyclic codes. *Finite Fields Their Appl.*, vol. 10, pp. 530-539, 2004.

[11] J. Wolfmann. Binary Images of Cyclic Codes over Z_4. *IEEE Trans. Info. Theory*, vol. 47, No. 5, pp. 1773-1779, Jul. 2001.