A TOPOLOGICAL PROOF THAT SURFACE RELATORS ARE TEST WORDS

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Abstract. We use topological methods to show that all the surface relators $[x_1, x_2] \cdots [x_{2n-1}, x_{2n}]$ and $x_1^2 \cdots x_m^2$ in the free groups F_{2n} and F_m are test words.

1. Introduction

How much information does one need about an endomorphism (i.e., a self-homomorphism) of a finitely generated free group to be certain that it is an automorphism? The lazy group theorist could hardly hope for more than the existence of *test words*.

Definition. *The word $w \in F$ is a* test word *if any endomorphism fixing w is necessarily an automorphism.*

It is easy to check that if w is a test word, then an endomorphism φ is an automorphism if and only if $\varphi(w) = \alpha(w)$ for some automorphism α—and this is an issue that is easy to decide in a free group by using the algorithms developed by J. H. C. Whitehead [11]. Nielsen proved [5] that the commutator of a pair of generators of a free group of rank two is a test word. This was improved by Zieschang [12], using standard techniques of combinatorial group theory, who showed (among other things) that the words $c_g = [x_1, x_2][x_3, x_4] \cdots [x_{2g-1}, x_{2g}]$ and $s_h = x_1^2 x_2^2 \cdots x_h^2$ are test words. The words c_g and s_h are referred to as the *surface relators* since they are the relators that come from the standard representations of closed compact surfaces as CW-complexes with a single 2-cell. To the mathematician trained as a topologist, this fact cries out for a proof using the special properties of surfaces—this note provides such proof.

I am indebted to Warren Dicks for pointing out to me that Dold [2] has a proof using two-dimensional cell complexes (quite different from the one presented here) and that he (Dicks) and Formanek have a rather short proof using Fox derivatives that appears in [1], p169. Algebraic proofs can be found in [7] and [10]. I am also indebted to the Centre de Recerca

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Theorem. *All surface relators are test words.*

We assume that φ is an endomorphism of the free group F_{2g} or F_h, that $w = c_g$ or s_h is a surface relator and that S is the corresponding compact surface—so if $w = c_g$, then S is the connected sum of g tori and if $w = s_h$, then S is the connected sum of h projective planes. In either case, let B be an open ball in S with closure $\overline{B}$, boundary ∂B and center $*$ so that $\pi_1(S \setminus B)$ is a free group in which ∂B represents w. (The basepoint is no problem to deal with and will not be explicitly mentioned unless necessary—it will be assumed to be on ∂B or to be $*$ as appropriate.) The *surface groups* are the groups $(x_1, \cdots, x_{2g} \mid c_g)$ and $(x_1, \cdots, x_h \mid s_h)$.

The proof of the Theorem proceeds in the following 4 steps, which are justified by the comments and the Lemmas.

Outline of proof of the Theorem

- **A)** Represent φ by a map $f : S \setminus B \to S \setminus B$ so that $f\mid_{\partial B}$ is the identity. Extend f to $\widetilde{f} : S \to S$ as the identity on B.
- **B)** Since $\widetilde{f}$ is a degree 1 map of a surface, it is a homotopy equivalence.
- **C)** Let $\widetilde{g} : S \to S$ be a homotopy inverse for $\widetilde{f}$ and $\widetilde{H} : S \times I \to S$ be a homotopy so that $\widetilde{H}\mid_{S\times 0} = \widetilde{f} \circ \widetilde{g}$ and $\widetilde{H} \mid S \times 1$ is the identity. Choose $\widetilde{g}$ so that $\widetilde{g}\mid_D$ is the identity and $\widetilde{g}^{-1}(D) = D$ and choose $\widetilde{H}$ so that $\widetilde{H}\mid_{D\times I}$ is projection on the first factor and that $\widetilde{H}^{-1}(D) = D \times I$.
- **D)** Define $g : S \setminus B \to S \setminus B$ as $\widetilde{g}\mid_{S\setminus B}$ and $H : (S \setminus B) \times I \to (S \setminus B)$ as $\widetilde{H}\mid_{(S\setminus B)\times I}$. Then g induces a right inverse for φ so φ is onto. Therefore φ is an automorphism.

Steps **A** and **D** are applications of standard techniques as follows. The existence of f in step **A** follows from the fact that φ fixes w which is represented by ∂B. Since surface groups are Hopfian (see [4]), it suffices in **D** to show that φ is onto.

Lemma 1. *A degree 1 map of a surface is a homotopy equivalence.*

Lemma 2. *If $f : (S, *) \to (S, *)$ induces an isomorphism on the fundamental group, then f has an inverse g in the category of homotopy classes of pointed maps $[(S, s_0), (S, s_0)]$—in particular, there is a homotopy $\widetilde{H} : S \times I \to S$ so that $\widetilde{H}\mid_{S\times 0} = \widetilde{f} \circ \widetilde{g}$, $\widetilde{H}\mid_{S\times 0}$ is the identity and $\widetilde{H}(*, t) = *$ for all t.*

Lemma 3. *It is possible to alter $\widetilde{H}$ as required in Step C.*

Proof of Lemma 1: This follows as an application of Edmonds' homotopy characterization of maps between surfaces [3] from which it follows that a degree 1 map is actually homotopic to a homeomorphism. We give an elementary proof.

Let $G = \pi_1(S, *)$ and consider the map $\psi : G \to G$ induced by $\widetilde{f}$ with $H = \psi(G)$. Since G is Hopfian, to show that ψ is an isomorphism, it suffices to show that $H = G$. Furthermore, since S is an Eilenberg-MacLane space $K(G, 1)$, f is a homotopy equivalence if and only if ψ is an isomorphism. A degree 1 map $\widetilde{f}$ of the surface S induces a map on first cohomology that preserves the cup product pairing, and since this pairing is non-singular, the induced map is an isomorphism. Dualizing, the induced map on first homology is an isomorphism. This completes the proof in the case that S is the torus, whose fundamental group is abelian. If S is the Klein bottle, then G is the non-trivial semi-direct product of $\mathbb{Z}$ with $\mathbb{Z}$ and the only endomorphisms that abelianize to automorphisms are automorphisms, completing the proof in this case as well. For the remainder of the proof, we assume that S is a surface other than the torus or the Klein bottle; i.e., that the Euler characteristic of S is non-zero.

Recall that the rank of a group K—denoted rank(K)—is the minimum cardinality of a set of generators.

Claim: *If H and G are as above, then rank(H)= rank(G).*

Proof of claim: Since H is a quotient of G, $\text{rank}(H) \leq \text{rank}(G)$. As a subgroup of a surface group, H is the fundamental group of a (probably non-compact) surface. But a surface whose fundamental group is finitely generated is a compact surface with finitely many punctures (see [6]): thus H is either a surface group or free. In either case, $\text{rank}(H) = \text{rank}(H^{ab})$—and of course $\text{rank}(G) = \text{rank}(G^{ab})$. Now let $j : H \to G$ be the inclusion map and consider the sequence

$$G^{ab} \xrightarrow{\psi^{ab}} H^{ab} \xrightarrow{j_*^{ab}} G^{ab}.$$

Since the composition is an isomorphism, it follows that j_*^{ab} is surjective and that $\text{rank}(G^{ab}) \leq \text{rank}(H^{ab})$, completing the proof of the claim.

Now we know that $\psi : G \to H$ is a surjective map from a surface group G to a group H of the same rank which is either a surface group or a free group. It is not hard to see that the only way this can happen is that ψ is an isomorphism. For if H were free, then the images of the generators of G would be a basis contradicting the fact that the relator is mapped to the identity. If H is a surface group, it must be isomorphic to G (by rank considerations), and it corresponds to a compact covering space of S of the same Euler characteristic. Since S is neither the torus nor the Klein bottle, this implies that the index of the covering is 1, completing the proof that G and H are isomorphic. $\square$

Proof of Lemma 2: The map $\widetilde{f} : (S, *) \to (S, *)$ has a homotopy inverse $\widetilde{g} : (S, s_0) \to (S, s_0)$ and we need to know that $\widetilde{g}$ is an inverse in the category of homotopy classes of pointed maps. Applying Theorem 8.1.9 on page 427 of [9] to the map $\widetilde{f} \circ \widetilde{g}$ with the pointed spaces $X = Y = (S, s_0)$ and using the fact that $\widetilde{f} \circ \widetilde{g}$ induces the identity on $\pi_1(S, *)$ proves precisely that. $\square$

Proof of Lemma 3:
Step 1: We begin by making $\widetilde{H}$ transverse to the point $*$ so that $\widetilde{H}^{-1}(*)$ is $* \times I$ plus a number of circles.

Step 2: Next, we combine all circles into one circle. If C_1 and C_2 are two circles in $\widetilde{H}^{-1}(*)$ we alter $\widetilde{H}$ to *splice* C_1 and C_2 as follows. Join C_1 to C_2 by an embedded path that meets $\widetilde{H}^{-1}(*)$ only at its endpoints and enclose the path in a box R. Do this so that $\widetilde{H}$ is constant on vertical lines in the box, the parts of C_1 and C_2 in R are vertical line segments and that cylinders about the pieces of C_1 and C_2 in B are mapped to a small disc about $*$. See Figure 1a.

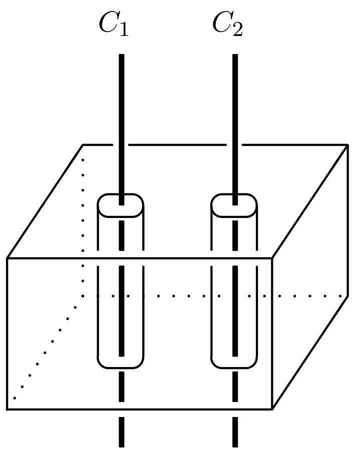

Figure 1a

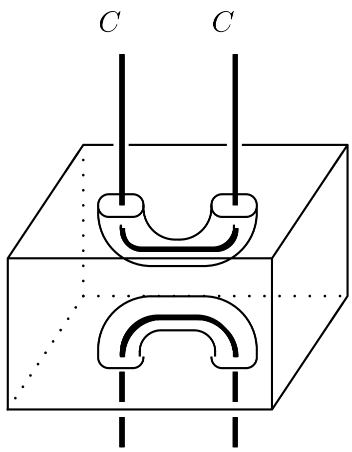

Figure 1b

The idea now is to redefine $\widetilde{H}$ in R so that the preimages of $*$ are line segments joining the same four points on the boundary of R but with the top two paired and the bottom two paired. If the interiors of the cylinders around the parts of C_1 and C_2 are removed from the box, what is left is a solid two-torus TT with boundary T—in other words, we consider R as being built up from its boundary by adding the two solid cylinders and then filling in with the solid two-torus. The map $\widetilde{H}$ on TT can be considered as a solution to the problem of extending the restriction of $\widetilde{H}$ to T over TT as a map to $S \setminus *$. (We can in fact assume that all of R is mapped to B, so that the obstructions encountered will be in $\pi_1(B \setminus *) \cong \mathbb{Z}$.) Now view R in a different way as its boundary with two cylinders added and then a solid two-torus filled in as shown in Figure 1b. The parts of C_1 and C_2 shown in Figure 1a are replaced by the cores of the cylinders in Figure 1b. One of the cylinders joins the discs on the top and the other joins the discs on the bottom. Let TT' and T' be the associated solid two-torus and its boundary. Now redefine $\widetilde{H}$ inside R by first extending over these cylinders to have the same image as the original cylinders, namely a small neighborhood of $*$. Note that the discs transverse to the pieces of C_1 and C_2 can be rotated as many times as we wish. This leaves the problem of extending the resulting map on T' over TT' (as a map to $B \setminus *$). The obstruction to this extension is the element of $H^2(TT', T', \pi_1(B \setminus *)) = Z \times Z$ whose coordinates are the homotopy classes of the maps on the boundary circles of the two two-cells added in building TT' from T'. One of these circles runs along the top side of the upper handle and then across the top face and the other runs along the bottom side of the lower handle then across the bottom face. The rotations of the transverse discs can be chosen to make these homotopy classes vanish. Note that since $S \setminus *$ is aspherical, there are no other obstructions.

The effect is to splice C_1 and C_2 together. If C_1 and C_2 are directed, this splicing can be done in a manor that preserves the directions. Repeating this process if necessary leads to a homotopy $\widetilde{H}$ for which $\widetilde{H}^{-1}(*) = * \times I \cup C$, C a single circle.

Step 3: Next we arrange that the circle C does not link the arc $* \times I$. Consider first the following example of a homotopy $G : D^2 \times I \to D^2$ from the identity to itself so that $G(s,t) = s$ for all $s \in S^1$ and that $G^{-1}(*)$ is $* \times I \cup C'$ where C' is a circle that links $* \times I$ once. Start with the Hopf fibration map $\eta : S^3 \to S^2$, for which the preimage of any two distinct points is a pair of linked circles. By following this with a map of S^2 to itself that identifies two points, we get a map $\eta' : S^3 \to S^2$ so that the preimage of a point is two linked circles. Now take a small box about a point on one of the circles and parameterize the map so that its restriction to the box is projection on one side—the desired homotopy is the restriction of this map to the closure of the complement of the box.

The homotopy G can be extended over $S \times I$ as the identity to produce $\widetilde{G} : S \times I \to S$. Now if C links $* \times I$ k times, we can stack k copies of $\widetilde{G}$ on top of $\widetilde{H}$ and then use the splicing procedure of Step 1 to splice these circles onto C but in the opposite direction. The resulting circle does not link $* \times I$.

Step 4: Next we remove C from $\widetilde{H}^{-1}(*)$. Since $\widetilde{H}$ is a homotopy equivalence and $\widetilde{H}(C)$ is null homotopic, C bounds an immersed disc in $S \times I$. We may as well assume that C lies in $B \times I$. If D were actually embedded, then since C does not link $* \times I$, (and there are no other circles) we can pull D off of $* \times I$ so that D lies in the complement of $* \times I$. Then the boundary of a small neighborhood of D is a 2-sphere in $S \times I$ whose image lies in $(S \ \setminus *) \times I$, which is aspherical. In this case, $\widetilde{H}$ can be redefined inside the 2-sphere to miss $* \times I$, thus removing the circle C from $\widetilde{H}^{-1}(*)$ and completing the argument.

The problem of finding an embedded disc bounded by C is the problem of unknotting C. Any circle can be unknotted by a sequence of moves corresponding to the replacement of an overcrossing by an undercrossing or vice versa in a knot projection. We do this with C as follows.

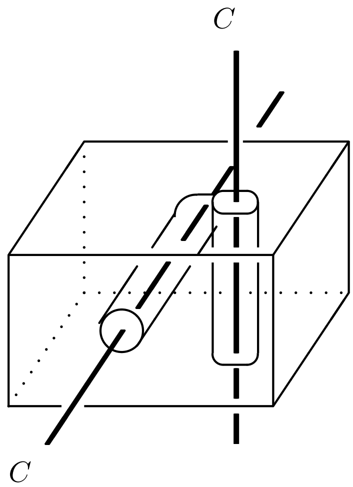

Figure 2a

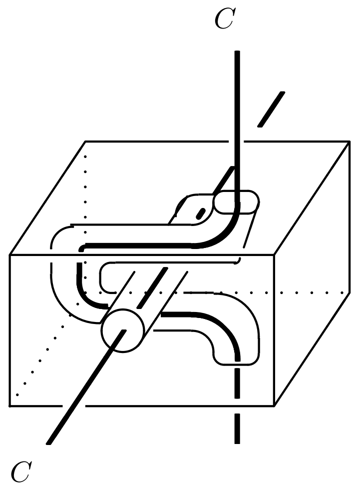

Figure 2b

Assume that a path from one point on C to another is enclosed in a box R with the potions of C in R as in Figure 2a with surrounding cylinders—so again R is viewed as its boundary with two handles adjoined and filed in with a solid two-handled torus, but this time one cylinder goes form top to bottom and the other from front to back. Now view R differently with the handles of Figure 2b adjoined and filled in with the corresponding solid two-torus TT' with boundary $T\ $. Again we have freedom to rotate the discs transverse to the pieces of C when defining the maps on the cylinders.

Again, we can choose the rotations to guarantee that the obstruction to the extension over TT' vanish, completing the argument. In this case, one of the two boundary circles runs from front to back on the right side of the front-back tube and then along the back, right and front faces at constant height to close up. The other runs from top to bottom along the left side of the redirected tube and then along the bottom, left and top faces to close up.

Once C is unknotted, we find an embedded disc and remove C as indicated above.

Step 5: Finally, "blow up" $* \times I$ to $\overline{B} \times I$ to produce the desired $\tilde{\tilde{H}}$. □

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