ON MODULI OF SMOOTHNESS OF FRACTIONAL ORDER

S. TIKHONOV

Abstract. In this paper we consider the properties of moduli of smoothness of fractional order. The main result of the paper describes the equivalence of the modulus of smoothness and a function from some class.

1. Introduction

In 1977 P.L. Butzer, H. Dyckhoff, E. Goerlich, R.L. Stens (see [2]) and R.Tabersky (see [14]) introduced the modulus of smoothness of fractional order. This notion could be considered as a direct generalization of the classical modulus of smoothness, and it is more natural to use it for a number of problems of harmonic analysis (see, for example, [2], [5], [7], [10]).

The important problem of approximation theory and theory of Fourier series is the problem of description of moduli of smoothness (see [1], [4], [8], [11]). One can consider this problem from the viewpoint of description of majorant of smoothness moduli. Recently, A. Medvedev (see [6]) has proved that for any modulus of continuity on \([0, \infty)\) there exists a concave majorant that is infinitely differentiable. In this paper, we obtain the description of the modulus of smoothness of fractional order from the viewpoint of the order of decreasing to zero of the modulus of smoothness.

Let us introduce several definitions. If \(1 \leq p < \infty\), let \(L_p\) be the space of 2\(\pi\)-periodic, measurable functions \(f(x)\) such that \(\|f\|_p = \left(\frac{2\pi}{0} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty\). Similarly, let \(L_\infty\) be the space of 2\(\pi\)-periodic, continuous functions \(f(x)\) with \(\|f\|_\infty = \max_{x \in [0, 2\pi]} |f(x)|\). We will define the difference of fractional order \(\beta (\beta > 0)\) of function \(f(x)\) at the point \(x (x \in \mathbb{R})\) with increment \(h (h \in \mathbb{R})\).

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by
\[ \Delta_h^\beta f(x) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \binom{\beta}{\nu} f(x + (\beta - \nu)h), \]
where \( \binom{\beta}{\nu} = \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!} \) for \( \nu > 1 \), \( \binom{\beta}{0} = \beta \) for \( \nu = 1 \), \( \binom{\beta}{\nu} = 1 \) for \( \nu = 0 \).

The modulus of smoothness of order \( \beta (\beta > 0) \) of function \( f \in L_p \), \( 1 \leq p \leq \infty \), is given by \( \omega_\beta(f, t)_p = \sup_{|h| \leq t} \|\Delta_h^\beta f(\cdot)\|_p \) (see [2],[14]).

Let \( \Phi_\gamma (\gamma \in \mathbb{R}) \) be the set of nonnegative, bounded functions \( \varphi(\delta) \) on \((0, \infty)\) such that
\[ \begin{align*}
&\text{a): } \varphi(\delta) \to 0 \text{ as } \delta \to 0, \\
&\text{b): } \varphi(\delta) \text{ is nondecreasing,} \\
&\text{c): } \varphi(\delta)\delta^{-\gamma} \text{ is nonincreasing.}
\end{align*} \]

If for \( f \in L_p \) there exists \( g \in L_p \) such that \( \lim_{h \to 0^+} \|h^{-\beta}\Delta_h^\beta f(\cdot) - g(\cdot)\|_p = 0 \) then \( g \) is called the Liouville-Grunwald-Letnikov derivative of order \( \beta > 0 \) of a function \( f \) in the \( L_p \)-norm, denoted by \( g = D^\beta f \) (see [2], [12]). Set \( W^\beta_p := \{ f \in L_p : D^\beta f \text{ exists as element in } L_p \} \). The \( K \)-functional is given by \( K(f, t, L_p, W^\beta_p) := \inf_{g \in W^\beta_p} (\|f - g\|_p + t \|D^\beta g\|_p) \).

2. Results

Let \( f(x) \in L_p, p \in [1, \infty] \) and \( \beta > 0 \). It is clear that (see [12])
\[ \left| \binom{\beta}{\nu} \right| = \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!} \leq C(\beta) \frac{1}{\nu^{\beta+1}}, \quad \nu \in \mathbb{N} \]
implies \( C^*(\beta) := \sum_{\nu=0}^{\infty} \left| \binom{\beta}{\nu} \right| < \infty \) and the fractional difference \( \Delta_h^\beta f(x) \) is defined almost everywhere and belongs to \( L_p \):
\[ \|\Delta_h^\beta f(\cdot)\|_p \leq C^*(\beta) \|f(\cdot)\|_p. \quad (1) \]
It is easy to write the following representation for \( C^*(\beta) \) (see [14]):
\[ C^*(\beta) = \begin{cases} 
2 \sum_{\nu=0}^{k} \binom{\beta}{\nu} & \text{if } 2k < \beta \leq 2k + 1 \ (k = 0, 1, 2, \cdots), \\
2 \sum_{\nu=0}^{k} \binom{\beta}{2\nu+1} & \text{if } 2k + 1 < \beta \leq 2k + 2 \ (k = 0, 1, 2, \cdots). 
\end{cases} \quad (2) \]

The fractional differences and moduli of smoothness have some useful properties and we shall establish some of them in the following lemmas.

**Lemma 2.1.** ([2], [14]) Let \( f \in L_p, p \in [1, \infty], \alpha, \beta > 0; \ h \in \mathbb{R} \). Then
\[ \begin{align*}
&\text{(a): } \Delta_h^\alpha(\Delta_h^\beta f(x)) = \Delta_h^{\alpha+\beta} f(x) \text{ for almost every } x; \\
&\text{(b): } \Delta_h^\alpha(\Delta_h^\beta f(x)) = \Delta_h^{\alpha+\beta} f(x) \text{ for almost every } x.
\end{align*} \]
For a function $\omega$, \[
limit_{\delta \to 0^+} \omega(t) = 0.
\]

**Lemma 2.2.** Let $f, f_1, f_2 \in L_p$, $p \in [1, \infty]$, $\alpha, \beta > 0$; $x, h \in \mathbb{R}$. Then

(a): \[\omega_\beta(f, \delta)_p \text{ is nondecreasing nonnegative function of } \delta \text{ on } (0, \infty) \]
with \[\lim_{\delta \to 0^+} \omega_\beta(f, \delta)_p = 0;\]
(b): \[\omega_\beta(f_1 + f_2, \delta)_p \leq \omega_\beta(f_1, \delta)_p + \omega_\beta(f_2, \delta)_p;\]
(c): \[\omega_\alpha+\beta(f, \delta)_p \leq C^*(\alpha)\omega_\beta(f, \delta)_p;\]
(d): \[\text{if } \alpha \geq 1, \text{ then } \omega_\beta(f, \lambda\delta)_p \leq C(\beta)\lambda^\beta \omega_\beta(f, \delta)_p;\]
(e): \[\text{if } 0 < t \leq \delta, \text{ then } \omega_\beta(f, \delta)_p \delta^{-\beta} \leq C(\beta)\omega_\beta(f, t)_p t^{-\beta}.\]

Indeed, we immediately have (a) – (c) from Lemma 2.1, (d) was proved in [2], and (d) implies (e):

\[\omega_\beta(f, \delta)_p = \omega_\beta\left(f, \frac{\delta}{t}\right)_p \leq C(\beta)\left(\frac{\delta}{t}\right)^\beta \omega_\beta(f, t)_p.\]

**Lemma 2.3.** Let $f \in L_p$, $p \in [1, \infty]$, $\beta > 0$.

(a): \[\text{if } \beta \in \mathbb{N}, \text{ then } \|\Delta^\beta f(\cdot)\|_p \leq 2^{\frac{\beta+1}{2}}\|\Delta^\frac{\beta}{2} f(\cdot)\|_p;\]
(b): \[\text{if } \beta \not\in \mathbb{N}, \text{ then } \|\Delta^\beta f(\cdot)\|_p \leq 2^{\frac{\beta+1}{2}+1}\|\Delta^\frac{\beta}{2} f(\cdot)\|_p.\]

**Corollary 2.4.** For a function $\varphi(t) = t^\alpha$ $(0 \leq t \leq \pi)$ to be a modulus of smoothness of order $\beta$ ($\beta > 0$) of a function $f(\cdot) \in L_p$, $1 \leq p \leq \infty$ it is necessary to have $\alpha \leq \left[\frac{\beta+1}{2}\right] + 1$.

**Theorem 2.5.** Let $p \in [1, \infty]$, $\beta > 0$.

(A): \[\text{if } f(\cdot) \in L_p, \text{ then there exists a function } \varphi(\cdot) \in \Phi_\beta \text{ such that } \varphi(t) \leq \omega_\beta(f, t)_p \leq C(\beta)\varphi(t) \ (0 < t < \infty),\]
where $C(\beta)$ is a positive constant depending only on $\beta$.

(B): \[\text{if } \varphi(\cdot) \in \Phi_\beta, \text{ then there exist a function } f(\cdot) \in L_p \text{ and a constant } t_1 > 0 \text{ such that } C_1(\beta)\omega_\beta(f, t)_p \leq \varphi(t) \leq C_2(\beta)\omega_\beta(f, t)_p \ (0 < t < t_1),\]
where $C_1(\beta), C_2(\beta)$ are positive constants depending only on $\beta$.

**Corollary 2.6.** Let $p \in [1, \infty]$, $\beta > 0$.

(A): \[\text{if } f(\cdot) \in L_p, \text{ then there exists a function } \varphi(\cdot) \in \Phi_\beta \text{ such that } C_1(\beta)\varphi(t) \leq K(f, t^\beta, L_p, W_p^3) \leq C_2(\beta)\varphi(t).\]
(B): If \( \varphi(\cdot) \in \Phi_\beta \), then there exists a function \( f(\cdot) \in L_p \) such that (3) is true.

**Remark 2.7.**

1. We can replace the condition \( f \in L_p \) by condition \( f \in L_\infty \) in the part (B) of Theorem 2.5.

2. Note that theorem 2.5 for \( \beta \in \mathbb{N} \) was proved in [11]. Also, for \( H^p \)-spaces the analogue of Corollary 2.6 for \( \beta \in \mathbb{R}_+ \) and the analogue of theorem 2.5 for \( \beta \in \mathbb{N} \) were proved in [5].

3. **Proofs**

Proof of Lemma 2.3. The first inequality was proved in [3]. Let \( \beta > 1, \beta \notin \mathbb{N} \). We shall use the following representation (see [14])

\[
\Delta_{2h}^\beta f(x - 2\beta h) = \sum_{\nu=0}^{\infty} \left( \frac{\beta}{\nu} \right) \Delta_{\nu h}^\beta f(x - \beta h - \nu h) \quad \text{for almost every } x \quad (4)
\]

By Lemma 2.1(a) and part (a) of this Lemma, it follows that

\[
\| \Delta_{\pi}^\beta f(\cdot) \|_p = \left\| \left( \Delta_{\pi}^\beta \left( \Delta_{\pi}^{\beta - [\beta]} f \right) \right)(\cdot) \right\|_p \leq 2^{\left\lceil \frac{[\beta]+1}{2} \right\rceil} \left\| \left( \Delta_{\pi}^\beta \left( \Delta_{\pi}^{\beta - [\beta]} f \right) \right)(\cdot) \right\|_p.
\]

Here we use (4) for \( h = \frac{\pi}{2} \). We have

\[
\| \Delta_{\pi}^\beta f(\cdot) \|_p \leq 2^{\left\lceil \frac{[\beta]+1}{2} \right\rceil} \left\| \sum_{\nu=0}^{\infty} \left( \frac{\beta - [\beta]}{\nu} \right) \Delta_{\frac{\nu \pi}{2}}^\beta \left( \Delta_{\frac{\nu \pi}{2}}^{\beta - [\beta]} f \right)(\cdot) \right\|_p.
\]

Thus, by Lemma 2.1(a) and inequality (1), we get

\[
\| \Delta_{\pi}^\beta f(\cdot) \|_p \leq C^* (\beta - [\beta]) 2^{\left\lceil \frac{[\beta]+1}{2} \right\rceil} \left\| \Delta_{\pi}^\beta f(\cdot) \right\|_p.
\]

If we combine this result with \( C^* (\beta - [\beta]) = 2 \) (see (2)) and \( 2^{\left\lceil \frac{[\beta]+1}{2} \right\rceil} = 2^{[\beta]+1} \), we obtain the required inequality. If \( 0 < \beta < 1 \), then we use (1) and (4).

This completes the proof of Lemma.

We will need the following lemma.

**Lemma 3.1.** Let \( \beta > 0, n \in \mathbb{N}, \delta > 0 \).

(a): If \( f(x) = \sin x \) and \( p \in [1, \infty] \), then there exist \( t_1 > 0 \) and \( C_1(\beta), C_2(\beta) > 0 \) such that for any \( \delta \in (0, t_1) \) we have

\[
C_1(\beta) \delta^\beta \leq \omega_\beta(f, \delta)_p \leq C_2(\beta) \delta^\beta.
\]
If \( f(x) = \sin nx \) and \( p \in [1, \infty] \), then for any \( \delta \in \left(0, \frac{\pi}{2}\right) \) we have
\[
\|\Delta_{\varphi}^\beta f(\cdot)\|_p \leq (2\pi)^{\frac{\beta}{2}} (n\delta)^\beta.
\]

(c): If \( f(x) = \sin nx \), then \( \|\Delta_{x/n}^\beta f(\cdot)\|_1 = 2^{\beta+2} \).

(d): If \( f(x) = \sin nx \), then for any \( \delta \in \left(0, \frac{\pi}{2}\right) \) we have
\[
\|\Delta_{x/n}^\beta f(\cdot)\|_1 \geq 4 \left(\frac{\pi}{2}\right)^\beta (\delta n)^\beta.
\]

Proof of Lemma 3.1 Let \( T_n(x) = \sum_{\nu=-n}^n c_{\nu} e^{i\nu x} \), then
\[
\Delta_{\varphi}^\beta T_n(x - \frac{\beta \delta}{2}) = \sum_{\nu=-n}^n \left(2i \sin \frac{\nu \delta}{2}\right)^\beta c_{\nu} e^{i\nu x}.
\]
Thus, for \( f(x) = \sin nx \), \( n \in \mathbb{N} \), we get
\[
\Delta_{\varphi}^\beta f(x - \frac{\beta \delta}{2}) = \left(2 \sin \frac{n \delta}{2}\right)^\beta \sin \left(nx + \frac{\beta \pi}{2}\right).
\]
For \( n = 1 \) we obviously have \( C_1(\beta) \left(2 |\sin \frac{\delta}{2}|\right)^\beta \leq \|\Delta_{\varphi}^\beta \sin(\cdot)\|_p \leq C_2(\beta) \left(2 |\sin \frac{\delta}{2}|\right)^\beta \). If we combine this inequality with \( \sin t \leq t \ (t \geq 0) \) and \( \sin t \geq \frac{2}{\pi} \ (0 \leq t \leq \frac{\pi}{2}) \), then we obtain (5). In the same way, by (6), we shall have the proofs of (b) – (d). This completes the proof of Lemma.

Proof of Theorem 2.5. (A). Let us define \( \varphi(t) := t^\beta \inf_{0 < \xi < t} \{\xi^{-\beta} \omega_{\beta}(f, \xi)_p\} \).

We immediately have \( \varphi(t) \in \Phi_{\beta} \) from [13, §2]. It is trivial, that \( \varphi(t) \leq \omega_{\beta}(f, t)_p \). By Lemma 2.2(e), we have \( \omega_{\beta}(f, t)_p \leq C(\beta) \varphi(t) \):
\[
\omega_{\beta}(f, t)_p = t^\beta t^{-\beta} \omega_{\beta}(f, t)_p \\
\leq C(\beta) t^\beta \inf_{0 < \xi \leq t} \{\xi^{-\beta} \omega_{\beta}(f, \xi)_p\} \\
= C(\beta) \varphi(t).
\]
Therefore, for any \( t > 0 \) the following inequality \( \varphi(t) \leq \omega_{\beta}(f, t)_p \leq C(\beta) \varphi(t) \) holds and (A) follows.

(B). 1 case. Let \( \lim_{t \to 0} \varphi(t) = C \) \((0 \leq C < \infty)\). Then, by virtue of monotonicity of \( \varphi(t) \), we write
\[
(\ast) \quad \varphi(t) \leq Ct^\beta \quad \text{for} \ 0 \leq t \leq \pi;
\]
\[
(\ast\ast) \quad \text{there exists} \ t_1 > 0 \ \text{such that} \ \varphi(t) \geq \frac{C t^\beta}{2} \ \text{for} \ 0 < t \leq t_1.
\]
Indeed, \((\ast)\) is trivial like \((\ast\ast)\) for \( C = 0 \). If \( C > 0 \) and \( \lim_{t \to 0} \varphi(t) = C \), then for any \( \varepsilon > 0 \) there exists \( t_1 > 0 \) such that \( C - \frac{C t^\beta}{2} \leq \varepsilon \) for \( 0 < t \leq t_1 \).

Then \( \frac{\varphi(t)}{t^\beta} \geq C - \varepsilon \), and choosing small \( \varepsilon \) we have \((\ast\ast)\).
Define \( f(x) = C \sin x \). By Lemma 3.1(a), we have
\[
\omega_{\beta}(f, \delta)_p \geq CC_1(\beta)\delta^\beta \geq C_2(\beta)\varphi(\delta) \quad \text{for} \quad 0 < \delta \leq \pi,
\]
\[
\omega_{\beta}(f, \delta)_p \leq CC_3(\beta)\delta^\beta \leq C_4(\beta)\varphi(\delta) \quad \text{for} \quad 0 < \delta \leq t_1,
\]
completing the proof in this case.

2 case. Let \( \lim_{t \to 0} \frac{\varphi(t)}{t^{\beta}} = +\infty \). Then \( \lim_{t \to 0} \varphi(t) = 0 \) and \( \lim_{t \to 0} \frac{\varphi(t)}{t^{\beta}} = 0 \). We fix \( a \geq 2 \).

Let \( \nu = 2^{\nu} \) are the numbers \( m_{\nu} \) such that
\[
m_1 = 2,
\]
\[
m_{\nu+1} = \min \left\{ m \in \mathbb{N} : \max \left( \frac{\varphi(2^{\nu})}{\varphi(2^{\nu+1})} \right) \leq \frac{1}{a} \right\} \quad (\nu \in \mathbb{N}).
\]
From the definition of \( \{n_{\nu}\}_{\nu=1}^{\infty} \) it follows that \( m_{\nu+1} > m_{\nu} \), \( n_{\nu+1} \geq 2n_{\nu} \) and for any \( \nu \in \mathbb{N} \) we have
\[
\varphi \left( \frac{1}{n_{\nu+1}} \right) \leq \frac{1}{a} \varphi \left( \frac{1}{n_{\nu}} \right); \quad (7)
\]
\[
n_{\nu}^\beta \varphi \left( \frac{1}{n_{\nu}} \right) \leq \frac{1}{a} n_{\nu+1}^\beta \varphi \left( \frac{1}{n_{\nu+1}} \right). \quad (8)
\]
Let us fix \( \kappa = 2^d (d \in \mathbb{N}) \) such that \( \kappa > 2\pi \). Note that (7) implies
\[
\sum_{\nu=1}^{\infty} \varphi \left( \frac{1}{n_{\nu}} \right) \leq \varphi \left( \frac{1}{n_1} \right) \sum_{\nu=1}^{\infty} a^{1-\nu} < \infty,
\]
and, therefore, we can define the function \( f(x) = \sum_{\nu=1}^{\infty} \varphi \left( \frac{1}{n_{\nu}} \right) \sin(\kappa n_{\nu}x) \).

First, we shall estimate \( \omega_{\beta}(f, \delta)_p \) from above. By the inequality \( \|f\|_p \leq (2\pi)^{\frac{1}{p}} \|f\|_\infty \leq 2\pi \|f\|_\infty, p \in [1, \infty) \), it is enough to prove \( \omega_{\beta}(f, \delta)_\infty \leq C(\beta)\varphi(\delta) \). Let \( \delta \in (0, \frac{1}{m_1}] \). For all \( h \in (0, \frac{1}{m_1}] \) we can find the number \( N \in \mathbb{N} \) such that \( \frac{1}{n_{N+1}} < h < \frac{1}{n_N} \). Then
\[
\left\| \Delta_h^\beta f(x) \right\|_\infty \leq \left| \sum_{\nu=1}^{N} \varphi \left( \frac{1}{n_{\nu}} \right) \Delta_h^\beta \sin(\kappa n_{\nu}x) \right|_\infty + \left| \sum_{\nu=N+1}^{\infty} \varphi \left( \frac{1}{n_{\nu}} \right) \Delta_h^\beta \sin(\kappa n_{\nu}x) \right|_\infty
\]
\[
= I_1 + I_2.
\]
Combining Lemma 3.1(b), inequality (8), and condition (c) in the definition of \( \Phi_{\beta} \), we get

\[
I_1 \leq \sum_{\nu=1}^{N} \varphi \left( \frac{1}{n_{\nu}} \right) \left\| \Delta_{h}^{\beta} \sin(\kappa n_{\nu} x) \right\|_{\infty} \\
\leq C(\beta) (\kappa h)^{\beta} \varphi \left( \frac{1}{n_{N}} \right) n_{N}^{\beta} \sum_{\nu=1}^{N} a^{- (N-\nu)} \\
\leq C(\beta) (n_{N} h)^{\beta} \varphi \left( \frac{1}{n_{N}} \right) \\
\leq C(\beta) \varphi (h).
\]

Inequalities (1) and (7) yield that

\[
I_2 \leq \sum_{\nu=N+1}^{\infty} \varphi \left( \frac{1}{n_{\nu}} \right) \left\| \Delta_{h}^{\beta} \sin(\kappa n_{\nu} x) \right\|_{\infty} \\
\leq C(\beta) \sum_{\nu=N+1}^{\infty} \varphi \left( \frac{1}{n_{\nu}} \right) \\
\leq C(\beta) \varphi \left( \frac{1}{n_{N+1}} \right) \sum_{\nu=N+1}^{\infty} a^{N+1-\nu} \\
\leq C(\beta) \varphi \left( \frac{1}{n_{N+1}} \right) \\
\leq C(\beta) \varphi (h).
\]

Therefore, if \( h \in \left( \frac{1}{n_{N+1}}, \frac{1}{n_N} \right], N \in \mathbb{N}, \) then \( \| \Delta_{h}^{\beta} f(x) \|_{\infty} \leq C(\beta) \varphi (h) \), which implies \( \omega_{\beta}(f, \delta)_{\infty} \leq C(\beta) \varphi (\delta) \).

Now we shall obtain the inequality \( \varphi(\delta) \leq C(\beta) \omega_{\beta}(f, \delta)_{p} \). From the inequality \( \| f \|_{1} \leq 2 \pi \| f \|_{p}, \ p \in [1, \infty] \) it is sufficient to prove \( \varphi(\delta) \leq C(\beta) \omega_{\beta}(f, \delta)_{1} \). Also, we note that if the last inequality holds for \( \delta = \frac{\pi}{2k}, \ k = N, N+1, N+2, \cdots, \) where \( N \in \mathbb{N}, \) then it holds for \( \delta \in \left( \frac{\pi}{2k}, \frac{\pi}{2k+1} \right) \). Indeed, from the monotonicity of \( t^{-\beta} \varphi(t) \), we see that the estimate \( \varphi(\delta) \leq C(\beta) \varphi \left( \frac{\pi}{2k} \right) \) is true. By Lemma 2.2(a), we get

\[
\varphi(\delta) \leq C(\beta) \varphi \left( \frac{\pi}{2k} \right) \\
\leq C(\beta) \omega_{\beta} \left( f, \frac{\pi}{2k} \right)_{1} \\
\leq C(\beta) \omega_{\beta}(f, \delta)_{1}.
\]

To go further, we suppose that \( \delta = \frac{\pi}{2k} \).
Let $M$ be the integer, $M > 1$, and, let $h_1 = \frac{\pi}{\kappa n_M}$. We shall show that
\[
\| \Delta h_1 f(x) \|_1 \geq 4 \varphi \left( \frac{1}{n_M} \right) \left( 2^\beta - \frac{\pi^{\beta+1}}{a} \right).
\]  
\( (9) \)

For this purpose, we shall use the following representation of a function $f(x)$:
\[
f(x) = \sum_{\nu=1}^{M-1} \varphi \left( \frac{1}{n_\nu} \right) \sin(\kappa n_\nu x) + \varphi \left( \frac{1}{n_M} \right) \sin(\kappa n_M x) + 
\sum_{\nu=M+1}^{\infty} \varphi \left( \frac{1}{n_\nu} \right) \sin(\kappa n_\nu x)
=: f_1 + f_2 + f_3.
\]

Note, that $\sin(\kappa n_\nu x + \frac{\pi n_\nu}{n_M}) = \sin(\kappa n_\nu x)$ for $\nu > M$, and $f_3(x)$ has the period $T = h_1 = \frac{\pi}{\kappa n_M}$. We therefore obtain
\[
\Delta h_1 f_3(x) = f(x + \beta h_1) \sum_{\xi=0}^{\infty} (-1)^{\xi} \left( \frac{\beta}{\xi} \right) = 0.
\]

By Lemma 3.1(b) and (8), we have
\[
\| \Delta h_1 f_1(x) \|_1 \leq \sum_{\nu=1}^{M-1} \varphi \left( \frac{1}{n_\nu} \right) \| \Delta h_1 \sin(\kappa n_\nu x) \|_1
\leq \sum_{\nu=1}^{M-1} 2\pi (\kappa n_\nu h_1)^\beta \varphi \left( \frac{1}{n_\nu} \right)
= 2\pi \left( \frac{\pi}{n_M} \right)^\beta \sum_{\nu=1}^{M-1} \varphi \left( \frac{1}{n_\nu} \right) n_\nu^\beta
\leq 2\pi \left( \frac{\pi}{n_M} \right)^\beta \varphi \left( \frac{1}{n_{M-1}} \right) n_{M-1}^\beta \sum_{\nu=1}^{M-1} a^{-(M-1-\nu)}.
\]

Using $\sum_{\nu=1}^{M-1} a^{-(M-1-\nu)} \leq 2$ and (8), we obtain
\[
\| \Delta h_1 f_1(x) \|_1 \leq \frac{4\pi^\beta+1}{a} \varphi \left( \frac{1}{n_M} \right).
\]

By Lemma 3.1(c),
\[
\| \Delta h_1 f_2(x) \|_1 = \varphi \left( \frac{1}{n_M} \right) \| \Delta h_1 \sin(\kappa n_M x) \|_1 = 2^\beta + 2 \varphi \left( \frac{1}{n_M} \right).
\]
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Therefore, for $h_1 = \frac{\pi}{2m_{i+1}}$, the inequality $|f| \geq |f_2| - |f_1| - |f_3|$ implies

$$\left\| \triangle^\beta h_{i+1} f(x) \right\|_1 \geq \left\| \triangle^\beta h_{i+1} f_2(x) \right\|_1 - \left\| \triangle^\beta h_{i+1} f_1(x) \right\|_1 - \left\| \triangle^\beta h_{i+1} f_3(x) \right\|_1$$

$$\geq 4\varphi \left( \frac{1}{n_{i+1}} \right) \left( 2^\beta - \frac{\pi^{\beta+1}}{a} \right),$$

i.e. we obtain (9).

Further, we choose the integer $i$ such that

$$\frac{1}{n_{i+1}} = \frac{1}{2^{m_{i+1}}} < \delta \leq \frac{1}{2^{m_i}} = \frac{1}{n_i}.$$ 

Note, that, by definition of $m_i$, at the least one of the following inequalities is true:

$$2^{\beta(m_{i+1}-1)} \varphi \left( \frac{1}{2^{m_{i+1}-1}} \right) < a \ 2^{\beta m_i} \varphi \left( \frac{1}{2^{m_i}} \right), \quad (10)$$

$$\varphi \left( \frac{1}{2^{m_{i+1}-1}} \right) > \frac{1}{a} \varphi \left( \frac{1}{2^{m_i}} \right) \quad (11)$$

Case 2(a). Let (10) be true. Using the monotonicity of $\varphi(t)$ and (10), we get

$$n_{i+1}^\beta \varphi \left( \frac{1}{n_{i+1}} \right) \leq 2^{\beta} 2^{\beta(m_{i+1}-1)} \varphi \left( \frac{1}{2^{m_{i+1}-1}} \right)$$

$$< a \ 2^\beta n_i^\beta \varphi \left( \frac{1}{n_i} \right). \quad (12)$$

We write

$$f(x) = \sum_{\nu=1}^{i-1} \varphi \left( \frac{1}{n_\nu} \right) \sin(\kappa n_\nu x) + \varphi \left( \frac{1}{n_i} \right) \sin(\kappa n_i x)$$

$$+ \sum_{\nu=i+1}^{\infty} \varphi \left( \frac{1}{n_\nu} \right) \sin(\kappa n_\nu x)$$

$$=: f_1 + f_2 + f_3.$$

It is clear, that the function $f_3$ has a period $T = \frac{2\pi}{2^{m_{i+1}}}$. Then, for $\kappa = 2^d > 2\pi$ we have $\delta = \frac{\pi}{\kappa} > \frac{1}{n_{i+1}} > T$, therefore, $f_3$ has a period $\delta$ and

$$\triangle^\beta \delta f_3(x) = 0.$$
For $0 < \delta \leq \frac{\pi}{\kappa n_i}$, by Lemma 3.1(d), we have
\[
\left\| \Delta^\beta f_2(x) \right\|_1 = \varphi \left( \frac{1}{n_i} \right) \left\| \Delta^\beta \sin(\kappa n_i x) \right\|_1 
\geq 4 \left( \frac{2}{\pi} \right)^\beta \varphi \left( \frac{1}{n_i} \right) (\kappa n_i \delta)^\beta.
\]
Using Lemma 3.1(b) and inequality (8), we estimate $f_1$:
\[
\left\| \Delta^\beta f_1(x) \right\|_1 \leq \sum_{\nu=1}^{i-1} \varphi \left( \frac{1}{n_\nu} \right) \left\| \Delta^\beta \sin(\kappa n_\nu x) \right\|_1 
\leq \sum_{\nu=1}^{i-1} 2\pi (\kappa n_\nu \delta)^\beta \varphi \left( \frac{1}{n_\nu} \right) 
\leq \frac{4\pi (\kappa n_{i-1} \delta)^\beta \varphi \left( \frac{1}{n_{i-1}} \right)}{\frac{1}{a}} 
\leq \frac{4\pi (\kappa n_i \delta)^\beta \frac{1}{a} \varphi \left( \frac{1}{n_i} \right)}{\frac{1}{a}}.
\]
For $\frac{1}{n_{i+1}} < \delta \leq \frac{\pi}{\kappa n_i}$ we obtain
\[
\left\| \Delta^\beta f(x) \right\|_1 \geq \left\| \Delta^\beta f_2(x) \right\|_1 - \left\| \Delta^\beta f_1(x) \right\|_1 
\geq \varphi \left( \frac{1}{n_i} \right) (\kappa n_i \delta)^\beta \left\{ 4 \left( \frac{2}{\pi} \right)^\beta - \frac{4\pi}{a} \right\}.
\]
Now we choose $a$ such that $2^\beta - \frac{\pi^{n+1}}{\alpha} = \gamma_i > 0$ (then $4 \left( \frac{3}{\pi} \right)^\beta - \frac{4\pi}{a} = \gamma_2 > 0$).

From (12) and the condition (c) in the definition of $\Phi_\beta$, we have
\[
(\delta n_i)^\beta \varphi \left( \frac{1}{n_i} \right) \geq \left( \frac{\delta n_{i+1}}{2} \right)^\beta \frac{1}{a} \varphi \left( \frac{1}{n_{i+1}} \right) 
\geq 2^{-\beta} \frac{1}{a} \varphi (\delta).
\]
Thus, the inequality $\omega_\beta(f, \delta)_p \geq C(\beta)\varphi(\delta)$ holds for $\frac{1}{n_{i+1}} < \delta \leq \frac{\pi}{\kappa n_i}$. If $\frac{\pi}{\kappa n_i} < \delta \leq \frac{1}{n_i}$, then (9) implies
\[
\omega_\beta(f, \delta)_p \geq \omega_\beta \left( f, \frac{\pi}{\kappa n_i} \right)_p 
\geq C(\beta)\varphi \left( \frac{1}{n_i} \right) 
\geq C(\beta)\varphi (\delta).
\]
The theorem has been proved in case 2(a).

Case 2(b). Let (11) be true. By virtue of monotonicity of \( \frac{\varphi(t)}{t^\beta} \), we write \( \varphi \left( \frac{1}{2^{m_i+1-\varepsilon}} \right) \leq 2^\beta \varphi \left( \frac{1}{2^{m_i+1-\varepsilon}} \right) \).

Hence,

\[
\varphi \left( \frac{1}{n_i+1} \right) = \varphi \left( \frac{1}{2^{m_i+1}} \right) \\
\geq 2^{-\beta} \varphi \left( \frac{1}{2^{m_i+1-1}} \right) \\
> 2^{-\beta} \frac{\varphi}{a} \left( \frac{1}{2^{m_i}} \right) \\
= \frac{2^{-\beta}}{a} \varphi \left( \frac{1}{n_i} \right). 
\]  

(13)

It follows from (9) and (13) that

\[
\omega_{\beta} (f, \delta)_1 \geq \omega_{\beta} \left( f, \frac{1}{n_i+1} \right)_1 \\
\geq \omega_{\beta} \left( f, \frac{\pi}{2n_i+1} \right)_1 \\
\geq C(\beta) \varphi \left( \frac{1}{n_i+1} \right) \\
\geq C(\beta) \varphi \left( \frac{1}{n_i} \right) \\
\geq C(\beta) \varphi (\delta). 
\]

This completes the proof of case 2(b) and Theorem 2.5.

Proof of Corollary 2.6 follows from the following estimates (see [2]):

\( C_1(\beta) \omega_{\beta} (f, t)_{p} \leq K(f, t^\beta, L_{p}, W_{p}^\beta) \leq C_2(\beta) \omega_{\beta} (f, t)_{p} \).

References


S. Tikhonov

Centre de Recerca Matemàtica (CRM)
Bellaterra (Barcelona) E-08193, Spain

e-mail: stikhonov@crm.es