PERIODIC ORBITS NEAR A HETEROCLINIC LOOP FORMED BY ONE–DIMENSIONAL ORBIT AND A 2–DIMENSIONAL MANIFOLD: APPLICATION TO THE CHARGED COLLINEAR 3–BODY PROBLEM

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Abstract. The paper is devoted to the study of a type of differential systems which appear usually in the study of some Hamiltonian systems with 2 degrees of freedom. We prove the existence of infinitely many periodic orbits on each negative energy level. All these periodic orbits pass near the total collision. Finally we apply these results to study the existence of periodic orbits in the charged collinear 3–body problem.

1. Introduction

In this paper we deal with differential systems of the form

$$\begin{array}{lllllll} & \dot r &=& rv\,, \qquad & \dot v &=& \dfrac{v^2}{2}+u^2-V(s)\,,\\ (1) & & & & & & \\ & \dot s &=& u\,, \qquad & \dot u &=& -\dfrac{1}{2}vu+V'(s)\,,\end{array}$$

where the dot denotes the derivative with respect to the time t, $V:(a,b)\longrightarrow \mathbb{R}$ is a real function and $V'(s)$ means derivation with respect to the variable s. System (1) has the first integral

$$H=\frac{1}{r}\Big[\frac{1}{2}(u^2+v^2)-V(s)\Big].$$

As we can see in [6], these type of systems appear usually in the study of many Hamiltonian systems with 2 degrees of freedom. For example the study of the collinear three body problem [7], the charged rhomboidal four body problem [5], the anisotropic Kepler problem [3], etc.

System (1) has an invariant manifold, called the *total collision manifold*

$$\Lambda=\{(r,v,s,u)\,:\,r=0,\ v^2+u^2=2V(s),\ s\in(a,b)\}.$$

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We note that Λ is empty when $V(s) < 0$ for all $s \in (a, b)$. Clearly, the shape of the collision manifold is strongly related with the shape of the function $V(s)$. System (1) is reversible, in other words is invariant under the symmetry $(r, v, s, u, t) \longrightarrow (r, -v, s, -u, -t)$ and is *gradient like* on Λ with respect to the coordinate v; i.e. all non–equilibrium solutions of system (1) on Λ are increasing with respect to the variable v.

We are interested in the situations in which the total collision manifold Λ is homeomorphic to a 2–dimensional sphere or to a 2–dimensional sphere minus one point. First we present a result which essentially is proved in [5].

More precisely, we suppose that on Λ there are two equilibrium points e^+ and e^- that are foci, and these two points, outside Λ, are connected by an orbit Γ which corresponds to an ejection–collision homothetic orbit. The unstable invariant manifold of e^+ coincides with the stable invariant manifold of e^- and it is equal to the orbit Γ. Moreover the stable invariant manifold of e^+ coincides with the unstable invariant manifold of e^- and it is equal to $\Lambda \setminus \{e^+, e^-\}$ (and consequently it is homeomorphic to a 2–dimensional sphere minus two points). Then we have a "*generalized heteroclinic loop*" formed by the stable and unstable invariant manifolds of e^+ and the points e^+ and e^-.

Theorem 1. *Suppose that Λ is a compact manifold homeomorphic to a 2–dimensional sphere and the flow generated by system (1) has exactly two equilibrium points e^+ and e^- (which correspond to the northern and southern poles of Λ, respectively) on Λ which are foci. Then, on every energy level $H(r, s, v, u) = h < 0$, the flow has infinitely many periodic orbits that pass close to Λ. All these periodic orbits intersect the plane $v = 0$ either 2, or 4, ... or $2n$ times during a period.*

We extend Theorem 1 to the case that Λ has six equilibrium points such that $e^+_{1,3}$ and $e^-_{1,3}$ are foci, and e^+_2 and e^-_2 are saddles. Each pair (e^+_i, e^-_i), for $i = 1, 2, 3$, outside Λ, is connected by an orbit Γ_i which corresponds to an ejection–collision homothetic orbit. In this case the description of the "generalized heteroclinic loop" is more complicated and it will be given on the proof.

Theorem 2. *Suppose that Λ is a compact manifold homeomorphic to a 2–dimensional sphere and the flow generated by system (1) has exactly six equilibrium points $e^+_{1,2,3}$ and $e^-_{1,2,3}$ on Λ: four are foci ($e^+_{1,3}$ and $e^-_{1,3}$) and two are saddles (e^+_2 and e^-_2). Then, on every energy level $H(r, s, v, u) = h < 0$, the flow has infinitely many periodic orbits that pass close to Λ. All these periodic orbits intersect the plane $v = 0$ either 4, or 8, ... or $4n$ times during a period.*

Also we extend Theorem 1 to the case when Λ is a *noncompact* manifold.

Theorem 3. *Suppose that Λ is a noncompact manifold homeomorphic to a 2–dimensional sphere minus one point and the flow generated by system (1) has exactly four equilibrium points $e^+_{1,2}$ and $e^-_{1,2}$ on Λ: two are foci (e^+_1 and e^-_1) and two are saddles (e^+_2 and e^-_2). Then, on every energy level $H(r, s, v, u) = h < 0$, the flow has infinitely many periodic orbits that pass close to Λ. All these periodic orbits intersect the plane $v = 0$ either 2, or 4, ... or $2n$ times during a period.*

The proofs of the theorems are based in the study of the dynamics of the flow generated by system (1) near a generalized heteroclinic loop. Finally we will apply these three theorems to study the existence of periodic orbits in the charged collinear 3–body problem.

2. Proof of Theorem 2 and 3

2.1. **Proof of Theorem 2.** We start to compute the equilibrium points of system (1), which are strongly related with the critical points of the function $V(s)$. Since the equilibrium points (r_i, s_i, v_i, u_i) are zeros of the vector field given by (1) and satisfy the energy relation

$$u^2 + v^2 = 2V(s) + 2hr \tag{2}$$

we obtain

$$r_i = 0, \quad u_i = 0, \quad V'(s_i) = 0, \quad v_i = \pm\sqrt{2V(s_i)}. \tag{3}$$

Note that in order to have exactly six equilibrium points we need that $V(s_i) \geq 0$ and function $V(s)$ should have exactly three critical points denoted by s_1, s_2 and s_3. The equilibrium points are on Λ and for each critical point s_i correspond two equilibrium points e^+_i and e^-_i according with $v_i > 0$ or $v_i < 0$, $i = 1, 2, 3$. (see Figure 1).

Since the last three equations of system (1) do not depend on r and the coordinate r can be obtained from the energy relation (2), in order to describe the flow of (1) on a fixed energy level $H = h$, it is sufficient to describe the flow of the system formed by the last three equations of (1)

$$\dot{s} = u, \quad \dot{v} = \frac{v^2}{2} + u^2 - V(s), \quad \dot{u} = -\frac{1}{2}vu + V'(s). \tag{4}$$

From (4) the flow on Λ is given by

$$\dot{s} = u\,, \qquad \dot{v} = \frac{u^2}{2}\,, \qquad \dot{u} = -\frac{1}{2}vu + V'(s)\,, \tag{5}$$

where we have used the energy relation (2) with $r = 0$.

Since $\dot{v} \geq 0$ and it is not identically zero on any orbit of (5) different from e^+_i and e^-_i, $i = 1, 2, 3$, the vector field given by (5) is gradient like with respect to the coordinate v.

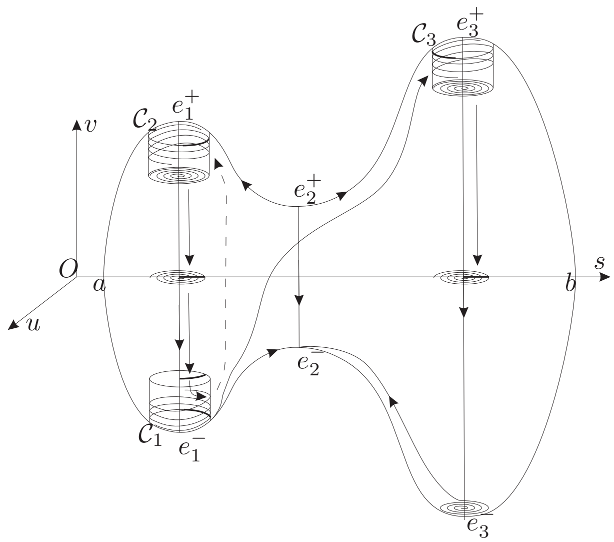

FIGURE 1

From system (1), if s_i, $i = 1, 2, 3$, is a critical point of the function $V(s)$ and $u = 0$, then $\dot{s} = 0$ and $\dot{u} = 0$. Hence the r, v–plane defined by $s = s_i$, $u = 0$ is invariant under the flow generated by system (1). The vector field on this plane is given by

$$\dot{r} = rv\,, \qquad \dot{v} = \frac{1}{2}v^2 - V(s_i) = hr\,,$$

where we used the energy relation (2) for $u = 0$. In particular, for each critical point s_i, $i = 1, 2, 3$, there is a unique *homothetic orbit* in each negative energy level which begins and ends in collision and which projects to the configuration space along the ray $s = s_i$. These homothetic orbits, denoted by Γ_i can therefore be interpreted as heteroclinic orbits connecting distinct equilibrium points e_i^+ and e_i^-. The orbit Γ_i starts at the equilibrium point e_i^+ and ends at e_i^-.

We start to describe the geometry of the problem on the energy level $E_h = H^{-1}(h)$ for $h < 0$.

Let s_1, s_2 and s_3 be the critical points of V such that $V(s_1)$ and $V(s_3)$ are local maximum, and $V(s_2)$ is local minimum of $V(s)$ and such that $e_{1,3}^+$, $e_{1,3}^-$ are foci and e_2^+, e_2^- are saddles. Let $P_1 = (0, s_1, 0) \in \Gamma_1$, $P_2 = (0, s_2, 0) \in \Gamma_2$ and $P_3 = (0, s_3, 0) \in \Gamma_3$ (see Figure 1).

By the Hartman's Theorem (see, for instance, [8]) it follows that $W^u_{e_i^+} = W^s_{e_i^-} = \{(v,s,u) : u = 0, s = s_i, -\sqrt{2V(s_i)} < v < \sqrt{2V(s_i)}\} = \Gamma_i$ where dim $W^u_{e_i^+} = 1$ and dim $W^s_{e_i^+} = 2$ for $i = 1,2,3$.

The flow on the manifold Λ is represented in the Figure 1. By hypotheses e_2^- is a saddle and we claim that one of the stable separatrices starts at e_1^- and the other one starts at e_3^-. Indeed, first due to the fact that the flow on Λ is gradient like it follows that the stable separatrices β_1 and β_2 of e_2^- must start at e_1^- or e_3^-. If we suppose that both of them started in the same point, e_1^- for example, we note that the closure of $\beta_1 \cup \beta_2$ is the boundary of a closed topological disc D. Clearly, there is an unstable separatrix β_3 of e_2^- contained in D. Then by the Poincaré-Bendixson theorem, the ω–limit set of β_3 must be: an equilibrium point, a limit cycle or a graph. Since the flow on Λ is gradient like, there are neither limit cycles, nor graphs in Λ. So the ω–limit set of β_3 must be the equilibrium point e_1^-. This is a contradiction because e_1^- is an unstable focus. The same reasons not allow the stable separatrices to start at e_3^-. In short the claim is proved.

On the other hand, by similar arguments used in the previous claim, one of the unstable separatrices at e_2^- ends at e_1^+ and the other one ends at e_3^+. Also in the same way we can show that one of the unstable separatrices at e_2^+ ends at e_1^+ and the other one ends at e_3^+, and similarly one of the stable separatrices at e_2^+ starts at e_1^- and the other one starts at e_3^-.

By assumption the system is reversible, which means that it is invariant under the symmetry $(v,s,u,t) \longrightarrow (-v,s,-u,-t)$. That is, if $\phi(t) = (v(t),s(t),u(t))$ is an orbit, then $\psi(t) = (-v(-t),s(-t),-u(-t))$ is another orbit. This symmetry can be used in order to obtain symmetric periodic orbits in the following way. Using the symmetry and the uniqueness theorem on the solutions of the differential system (1) it is easy to see that if $v(0) = 0$ and $u(0) = 0$, then the orbits $\phi(t)$ and $\psi(t)$ must be the same. Moreover, if there exists a time $\bar{t} > 0$ such that $v(\bar{t}) = 0$, $u(\bar{t}) = 0$ and $u(\bar{t})^2 + v(\bar{t})^2 \neq 0$ for $0 < t < \bar{t}$, then the orbit must be periodic of period $2\bar{t}$. In other words, if an orbit intersects the line of symmetry $L = \{(v,s,u) : v = 0, u = 0\}$ in two different points, then it is a periodic orbit.

We give some definitions and some notations. Assume that $\varepsilon_i > 0$ take sufficiently small values for all $i = 1,\ldots,8$. We consider the segment $\gamma_1 = \{(v,s,u) \in L : s \in (s_1, s_1+\varepsilon_1)\}$, and the section $\Sigma = \{(v,s,u) \in E_h : v = 0\}$. We also consider the following small topological cylinders:

- $\mathcal{C}_1$ in a neighborhood of the equilibrium point $e_1^- = \{(-v_1, s_1, 0)\}$, where $v_1 = \sqrt{2V(s_1)} > 0$, with base on Λ and boundaries Σ_1 and Σ_2 with $\Sigma_1 = \{(v,s,u) \in E_h : v = -v_1 + \varepsilon_2, u^2 + (s-s_1)^2 \leq \varepsilon_3\}$ and $\Sigma_2 = \{(v,s,u) \in E_h : v \leq -v_1 + \varepsilon_2, u^2 + (s-s_1)^2 = \varepsilon_3\}$;

$\mathcal{C}_2$ in a neighborhood of the equilibrium point $e_1^+ = \{(v_1, s_1, 0)\}$, with base on Λ and boundaries Σ_3 and Σ_4 with $\Sigma_4 = \{(v, s, u) \in E_h : v = v_1 - \varepsilon_4, u^2 + (s - s_1)^2 \leq \varepsilon_5\}$ and $\Sigma_3 = \{(v, s, u) \in E_h : v \leq v_1 - \varepsilon_4, u^2 + (s - s_1)^2 = \varepsilon_5\}$;

$\mathcal{C}_3$ in a neighborhood of the equilibrium point $e_3^+ = \{(v_3, s_3, 0)\}$, where $v_3 = \sqrt{2V(s_3)} > 0$ with base on Λ and boundaries Σ_5 and Σ_6 with $\Sigma_6 = \{(v, s, u) \in E_h : v = v_3 - \varepsilon_6, u^2 + (s - s_3)^2 \leq \varepsilon_7\}$ and $\Sigma_5 = \{(v, s, u) \in E_h : v \leq v_3 - \varepsilon_6, u^2 + (s - s_1)^2 = \varepsilon_7\}$.

Case 1. The unstable separatrices at e_2^- does not coincide with the stable separatrices at e_2^+.

We define a map $\pi : \gamma_1 \longrightarrow \Sigma$ in the following way. We denote by $\varphi(t, q)$ the flow generated by the vector field, satisfying $\varphi(0, q) = q$. We consider the diffeomorphism $\pi_1 : \gamma_1 \to \Sigma_1$ defined by $\pi_1(q) = p$, where p is the point at which the orbit $\varphi(t, q)$ intersects the cross section Σ_1 for the first time. By the continuity of the flow φ with respect to initial conditions, if q is sufficiently close to the point P_1, then the orbit $\varphi(t, q)$ is close to the orbit Γ_1 for all t in a finite interval of time. Since the orbit Γ_1 expends a finite time for going from the point P_1 to the point $S_1 = \Sigma_1 \cap \Gamma_1$, we can guarantee that for all $q \in \gamma_1$ sufficiently close to P_1 the orbit $\varphi(t, q)$ intersects Σ_1. Consequently if ε_1 is sufficiently small, then the map π_1 is well defined. Moreover, the image by π_1 of γ_1 is an arc on Σ_1 with S_1 as one of its endpoints.

We consider a second diffeomorphism $\pi_2 : \Sigma_1 \longrightarrow \Sigma_2$ defined by $\pi_2(q) = p$, where p is the point at which the orbit $\varphi(t, q)$ intersects Σ_2 for the first time. If ε_3 is sufficiently small, then the orbit $\varphi(t, q)$ intersects Σ_2 for all $q \in \Sigma_1 \setminus \{S_1\}$, because e_1^- is a hyperbolic equilibrium point with $W^s_{e_1^-} = \Gamma_1$. Moreover, since e_1^- is an unstable focus on Λ and the point $\pi_1(P_1) = S_1 \in W^s_{e_1^-}$, the image $\pi_2(\pi_1(\gamma_1))$ is a spiral on Σ_2 that approaches to Λ, when we approach to P_1, spiraling infinitely many times.

We consider on all three cylinders $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ mentioned above, the circles which represent their intersections with the manifold Λ, denoted by C_1, C_2 and C_3 respectively. The stable separatrix of e_2^- which connect e_1^- with e_2^- will intersect the circle C_1 on a point T_1. We consider now two small open arcs on C_1 denoted by γ_2 and γ_3 symmetric with respect to the point T_1 (see again Figure 1).

If γ_2 is small enough, we can assume that the image of γ_2 under the flow is a small open arc on the circle C_2, denoted by γ_4. Also if γ_3 is small enough, its image under the flow will be a small open arc on the circle C_3, denoted by γ_5. Taking into account that the image $\pi_2(\pi_1(\gamma_1))$ is a spiral on Σ_2 that approaches to Λ, when we approach to P_1, spiraling infinitely many times, we can consider on Σ_2 a sequence of small arcs of this spiral, denoted by $\{\gamma_2^n\}$, near to γ_2 as much as we want, and for the same reasons we can also consider on Σ_2 a sequence of small arcs of this spiral, denoted by $\{\gamma_3^n\}$, near to γ_3 as much as we want.

We define a third map $\pi_3^n : \gamma_2^n \longrightarrow \Sigma_3$, defined by $\pi_3(q) = p$, where p is the point at which the orbit $\varphi(t, q)$ intersects Σ_3 for the first time. Since the flow on Λ is gradient like with respect to the variable v, if γ_2^n is near enough to γ_2 then the point p is well defined and the image of γ_2^n by π_3 is an arc γ_4^n on Σ_3 near to γ_4. Since the orbits expend a finite time for going from γ_2^n to γ_4^n, π_3 is a diffeomorphism.

Consider now the fourth map $\pi_4 : \Sigma_3 \longrightarrow \Sigma_4$, defined by $\pi_4(q) = p$ where p is the point at which the orbit $\varphi(t, q)$ intersects Σ_4 for the first time. If ε_5 is sufficiently small, then this point p is well defined because e_1^+ is a stable focus on Λ and $W^u_{e_1^+} = \Gamma_1$. Moreover, the image $\pi_4(\gamma_4^n)$ is a spiral on Σ_4 that approaches to the point $S_2 = \Sigma_4 \cap \Gamma_1$, when γ_4^n approach to γ_4, spiraling finitely many times (see again Figure 1).

We define $\pi_5 : \Sigma_4 \longrightarrow \Sigma$ in the similar way than π_1^{-1}. Finally we consider the map $\pi^n : \gamma_1 \longrightarrow \Sigma$ defined by $\pi = \pi_5 \circ \pi_4 \circ \pi_3^n \circ \pi_2 \circ \pi_1$. Therefore the image $\pi^n(\gamma_1)$ contain a spiral on Σ that approaches to P_1, spiraling finitely many times. In fact we take γ_2^n sufficiently close to γ_2 in order that $\pi^n(\gamma_1)$ spirals at least a full turn around P_1. Note that all γ_2^m with $m > n$ also spirals at least a full turn around P_1.

We note that $\pi^n(\gamma_1)$ intersects the line of symmetry L finitely many times. Since the points of γ_1 belong to the line of symmetry, those intersection points correspond to orbits that cross the line of symmetry at two different points. That is, they correspond to symmetric periodic orbits. By the construction these periodic orbits cross exactly four times the plane $v = 0$. Playing with n we obtain infinitely many symmetric periodic orbits of this kind.

In the similar way we can obtain infinitely many periodic orbits using the fact that the image of γ_3 under the flow is a small open arc on the circle C_3, denoted by γ_5. Again, taking into account that the image $\pi_2(\pi_1(\gamma_1))$ is a spiral on Σ_2 that approaches to Λ, when we approach to P_1, spiraling infinitely many times, we can consider on Σ_2 a sequence of small arcs of this spiral, denoted by $\{\gamma_3^n\}$, near to γ_3 as much as we want.

We define the map $\pi_6^n : \gamma_3^n \longrightarrow \Sigma_5$, defined by $\pi_6^n(q) = p$, where p is the point at which the orbit $\varphi(t, q)$ intersects Σ_5 for the first time. Since the flow on Λ is gradient like with respect to the variable v, if γ_3^n is near enough to γ_3, then the point p is well defined and the image of γ_3^n by π_6^n is an arc γ_5^n on Σ_5 near to γ_5. Since the orbits expend a finite time for going from γ_3^n to γ_5^n, π_6^n is a diffeomorphism.

Consider now the map $\pi_7 : \Sigma_5 \longrightarrow \Sigma_6$, defined by $\pi_7(q) = p$ where p is the point at which the orbit $\varphi(t, q)$ intersects Σ_6 for the first time. If ε_7 is sufficiently small, then this point p is well defined because e_3^+ is a stable focus on Λ and $W^u_{e_3^+} = \Gamma_3$. Moreover, the image $\pi_7(\gamma_5^n)$ is a spiral on Σ_6 that approaches to the point $S_3 = \Sigma_6 \cap \Gamma_3$, when γ_5^n approach to γ_5, spiraling finitely many times (see again Figure 1). Again n must be large enough.

We define $\pi_8 : \Sigma_6 \longrightarrow \Sigma$ in the similar way than π_1^{-1}. Finally we consider the map $\Pi^n : \gamma_1 \longrightarrow \Sigma$ defined by $\Pi^n = \pi_8 \circ \pi_7 \circ \pi_6^n \circ \pi_2 \circ \pi_1$. Therefore the image $\Pi^n(\gamma_1)$ contain a spiral on Σ that approaches to P_3, spiraling finitely many times.

We note that $\Pi^n(\gamma_1)$ intersects the line of symmetry L finitely many times. Since the points of γ_1 belong to the line of symmetry, those intersection points correspond to orbits that cross the line of symmetry at two different points. That is, they correspond to symmetric periodic orbits. Its clear by the construction that these periodic orbits cross also the plane $v = 0$ exactly four times. Playing with n we obtain infinitely many symmetric periodic orbits of this kind.

Case 2. The unstable separatrices at e_2^- coincide with the stable separatrices at e_2^+.

We note that due to the symmetry of the flow with respect to the straight line L, if one stable separatrix of e_2^+ coincides with one unstable separatrix of e_2^-, then the other stable separatrix of e_2^+ also coincides with the other unstable separatrix of e_2^-.

In this case we can define the map $\pi^n : \gamma_1 \longrightarrow \Sigma$ like above. The difference is that we cannot define the map $\Pi^n : \gamma_1 \longrightarrow \Sigma$. More precisely, the image of γ_3 under the flow is a small open arc also on C_2 not on C_3, because in this case the flow on Λ started on the “left part” with respect to e_2^- is not allowed to pass to the “right part” with respect to this point and conversely. Which means that in this case the map π_6^n will be defined on γ_3^n with values on Σ_3. From now we can continue like in the previous case.

We note that when we add the coordinate r the periodic orbits of (4) give periodic orbits of (1). Clearly, by the construction, these periodic orbits cross exactly four times the plane $v = 0$.

The periodic orbits given by Theorem 2 are obtained from intersection points of $\pi^n(\gamma_1)$ and $\Pi^n(\gamma_1)$ with the line of symmetry L. If we repeat the arguments of the proof of Theorem 2 with the segment $\gamma_1' = \{(v, s, u) \in L : s \in (s_1 - \varepsilon_1, s_1)\}$ instead of γ_1 we would obtain infinitely many symmetric periodic orbits that are different from the ones obtained above.

Doing similar arguments it is not difficult to see that the periodic orbits coming from the intersection points of $(\pi^n)^2(\gamma_1)$ with the line of symmetry L provide the symmetric periodic orbits found in Theorem 2, and additionally

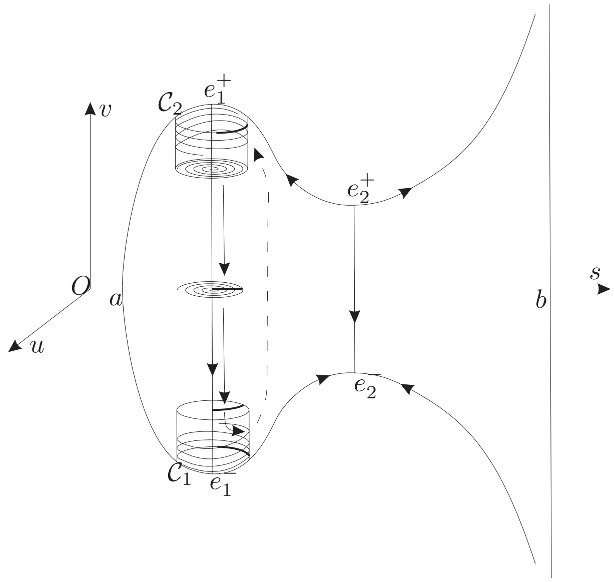

FIGURE 2

provide infinitely many symmetric periodic orbits that cross exactly eight times the plane $v = 0$ during a period; and so on. □

2.2. **Proof of Theorem 3.** We can repeat the same arguments like in the proof of Theorem 2 and we can define in a similar way the map $\pi^n : \gamma_1 \longrightarrow \Sigma$ (see Figure 2). The difference is that in this case we cannot define the map $\Pi^n : \gamma_1 \longrightarrow \Sigma$. □

3. The charged collinear 3–body problem

The charged N–body problem corresponds to the study of the dynamics of N point particles endowed with a positive mass and an electrostatic charge of any sign, moving under the influence of the respective Newtonian and Coulombian forces. These kind of problems for particular values of N have been studied among others in [1] and [5] (the charged rhomboidal four body problem), in [2] (the charged isosceles three body problem), in [4] (the restricted charged four body problem), in [9] (central configuration in the charged three body problem) etc.

Here we study the charged collinear 3–body problem. We assume that the gravitational constant and the Coulomb's constant are equal to one, and we consider three point particles with masses m_1, m_2 and m_3, charges

q_1, q_2 and q_3 and positions $x_1, x_2, x_3 \in \mathbb{R}$. The motion of the particles is described by the following system of differential equations

$$m_i \ddot{x}_i = \nabla_{x_i} U, \qquad i = 1, 2, 3, \tag{6}$$

where

$$U(x_1, x_2, x_3) = \frac{\lambda_{12}}{\|x_1 - x_2\|} + \frac{\lambda_{13}}{\|x_1 - x_3\|} + \frac{\lambda_{23}}{\|x_2 - x_3\|},$$

$\nabla_{x_i} U$ is the gradient of U with respect to x_i, and $\lambda_{ij} = m_i m_j - q_i q_j$, for $i, j = 1, 2, 3$, and $i \neq j$. If $\lambda_{ij} > 0$ the resultant force between the particles i and j is attractive, and if $\lambda_{ij} < 0$ then is repulsive. A position (x_1, x_2, x_3) of the particles will be called a *collision* if $x_i = x_j$ for some $i \neq j$.

System (6) can be written in Hamiltonian form by taking $\mathbf{q} = (x_1, x_2, x_3)^T$, $M =$ diag $\{m_1, m_2, m_3\}$, and $\mathbf{p} = M\dot{\mathbf{q}}$. In these coordinates system (6) becomes

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \tag{7}$$

where

$$H = \frac{1}{2}\mathbf{p}^T M^{-1} \mathbf{p} - U(\mathbf{q}),$$

is the Hamiltonian function.

3.1. **McGehee coordinates.** We introduce a remarkable change of coordinates due to McGehee [7] in order to analyze the behavior of the orbits in the neighborhood of the total collision. The first step is to introduce "polar" coordinates generated by the moment of inertia of the system. Let

$$r = (\mathbf{q}^T M \mathbf{q})^{1/2}, \qquad \mathbf{s} = r^{-1}\mathbf{q}.$$

The quantity r^2 is called the *moment if inertia* of the system and s is the *configuration*. Since $\mathbf{s}^T M \mathbf{s} = 1$, we think of $\mathbf{s}$ as a point in the unit sphere S in $\mathbb{R}^3$ in the metric induced by M. Also we define

$$y = \mathbf{p}^T \mathbf{s}, \qquad \mathbf{x} = \mathbf{p} - yM\mathbf{s}.$$

Then y is the *radial component* of the velocity and $\mathbf{x}$ is the *tangent component* to S. In these coordinates the equations of motion (7) become

$$\begin{aligned}
\dot{r} &= y, \\
\dot{y} &= r^{-1}\mathbf{x}^T M^{-1}\mathbf{x} - r^{-2}U(\mathbf{s}), \\
\dot{\mathbf{s}} &= r^{-1}M^{-1}\mathbf{x}, \\
\dot{\mathbf{x}} &= -r^{-1}y\mathbf{x} - r^{-1}(\mathbf{x}^T M^{-1}\mathbf{x})M\mathbf{s} + \\
&\quad + r^{-2}U(\mathbf{s})M\mathbf{s} + r^{-2}\nabla U(\mathbf{s}).
\end{aligned} \tag{8}$$

To remove the singularities at $r = 0$ (i.e. the singularity due to the total collision of the particles) we introduce new variables

$$\dot{\mathbf{u}} = r^{1/2}\mathbf{x}, \qquad v = r^{1/2}y,$$

and we scale the time variable by $dt/dt' = r^{3/2}$. After this, following Section 4 of [7], we introduce a new coordinate system. We skip the details and finally we get the following system

$$\begin{array}{rclrcl} \dot{r} & = & rv\,, & \dot{v} & = & \dfrac{v^2}{2} + u^2 - V(s)\,, \\ \dot{s} & = & \dfrac{1}{\lambda}u\,, & \dot{u} & = & -\dfrac{1}{2}vu + \dfrac{1}{\lambda}V'(s)\,, \end{array} \tag{9}$$

where

$$\begin{aligned} V(s) = \sin 2\lambda \Big[& \frac{\lambda_{12}}{(b_2 - b_1)\sin\lambda(1+s)} + \\ & + \frac{\lambda_{23}}{(a_3 - a_2)\sin\lambda(1-s)} + \\ & + \frac{\lambda_{13}}{(b_2 - b_1)\sin\lambda(1+s) + (a_3 - a_2)\sin\lambda(1-s)} \Big], \end{aligned}$$

where $\lambda \in (0, \pi/4)$ is a constant depending only on the masses m_1, m_2, and m_3, and $s \in [-1, 1]$. Now the dot denotes the derivative with respect to t'. The fixed vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are the unique points on the unit sphere S with $a_1 = a_2 < a_3$ and $b_1 < b_2 = b_3$.

The above equations define a vector field with singularities when $s = \pm 1$. We note that $r = 0$ corresponds to triple collision while $s = \pm 1$ corresponds to double collision. As we will see, since we are interested in studying the periodic orbits near the triple collision, for the values of the parameters which will be choose such that the collision manifold is homeomorphic to $\mathbb{S}^2$ or to $\mathbb{S}^2 \setminus \{\text{one point}\}$, we will be far away from the values $s = \pm 1$, and we can suppose from now that $s \in (-1, 1)$ and is not necessary to use other transformation to regularize all double collisions, like in [7].

In the new variables the energy relation $H = h$ becomes

$$\frac{1}{2}(u^2 + v^2) - V(s) = rh. \tag{10}$$

The total collision manifold Λ is characterized by

$$\Lambda = \{(r, v, s, u) : \ r = 0, \ v^2 + u^2 = 2V(s), \ s \in (-1, 1)\}. \tag{11}$$

Since $\dot{r} = 0$ when $r = 0$ in the first equation of (9), we have that Λ is invariant under the flow; from the energy relation (10) we also have that Λ is independent of the value of the constant energy h; i.e., each energy surface has the same total collision manifold Λ in its boundary.

We note that by (11) the total collision manifold Λ is not defined when $V(s) < 0$ for all $s \in (-1, 1)$. Clearly, the shape of the collision manifold is determined by the shape of the potential function $V(s)$. This function is analyzed in the following subsection.

3.2. **The total collision manifold Λ.** We are interested for the possible shapes of $V(s)$ with respect to the parameters such that the total collision manifold is homeomorphic to $\mathbb{S}^2$ or to $\mathbb{S}^2 \setminus \{\text{one point}\}$. Introducing the following new parameters

$$\begin{aligned} A &= \frac{b_2 - b_1}{\lambda_{12}}, \qquad & B &= \frac{a_3 - a_2}{\lambda_{23}}, \\ C &= \frac{b_2 - b_1}{\lambda_{13}}, \qquad & D &= \frac{a_3 - a_2}{\lambda_{13}}, \end{aligned}$$

the potential function $V(s)$ become

$$V(s) = \sin 2\lambda \Big[\frac{1}{A \sin \lambda(1+s)} + $$
$$+ \frac{1}{B \sin \lambda(1-s)} + \frac{1}{C \sin \lambda(1+s) + D \sin \lambda(1-s)} \Big],$$

where $s \in (-1, 1)$. The derivative of the potential function $V(s)$ is

$$V'(s) = \sin 2\lambda \Big\{ - \frac{\lambda \cos \lambda(1+s)}{A \sin^2 \lambda(1+s)} + \frac{\lambda \cos \lambda(1-s)}{B \sin^2 \lambda(1-s)} -$$
$$- \frac{C\lambda \cos \lambda(1+s) - D\lambda \cos \lambda(1-s)}{\big[C \sin \lambda(1+s) + D \sin \lambda(1-s)\big]^2} \Big\}.$$

We can make the following simple remarks

- Taking into account the definition of the vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ we have $A \neq 0$, $B \neq 0$, $C \neq 0$ and $D \neq 0$;
- Taking into account that $\lambda \in (0, \pi/4)$ and $s \in (-1, 1)$, then $\sin \lambda(1 \pm s) > 0$;
- If $A < 0$ then $\lim_{s \to -1^-} V(s) = -\infty$, and if $B < 0$ then $\lim_{s \to 1^+} V(s) = -\infty$;
- By definition C and D have the same sign, i.e. $CD > 0$;
- If $A = B$ and $C = D$ then $V(s)$ is symmetric with respect to $s = 0$, $V'(0) = 0$ and

$$V(0) = \frac{\sin 2\lambda}{\sin \lambda} \Big[\frac{2}{A} + \frac{1}{2C} \Big].$$

Using these remarks and using Maple we have founded that

Example 1. If we choose $A = B = -2$, $C = D = 0.1$ and $\lambda = 0.2$, then $V(s)$ is convex, symmetric with respect to $s = 0$ and has a single critical point $s = 0$ which is a maximum and $V(0) > 0$ (see Figure 3).

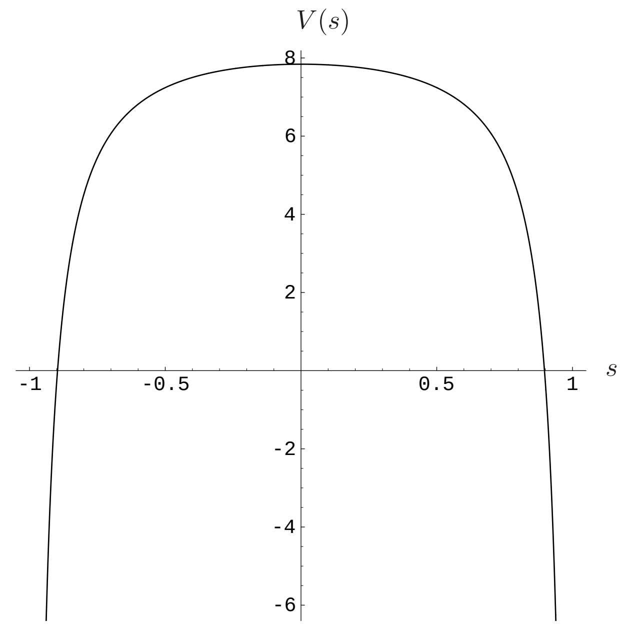

FIGURE 3. Function V for $A = B = -2$, $C = D = 0.1$ and $\lambda = 0.2$

Example 2. If we choose $A = B = -10$, $C = D = 0.1$ and $\lambda = 0.7$, then $V(s)$ is symmetric with respect to $s = 0$, and has three critical points: $s_2 = 0$ which is a local minimum, and $s_1 = -0.8150687496$ and $s_3 = 0.8150687496$ which are local maxima (see Figure 4).

Example 3. If we choose $A = -10$, $B = 1$, $C = 0.5$, $D = 0.1$ and $\lambda = 0.2$, then $V(s)$ has two critical points on the interval $(-1, 1)$: $s_1 = -0.8758139935$ which is a local maximum and $s_2 = 0,003319130892$ which is a local minimum (see Figure 5).

3.3. **Equilibrium points.** In this subsection we will compute the equilibrium points of system (9), which are strongly related with the critical points of the potential $V(s)$. The equilibrium points (r_0, v_0, s_0, u_0) of the vector

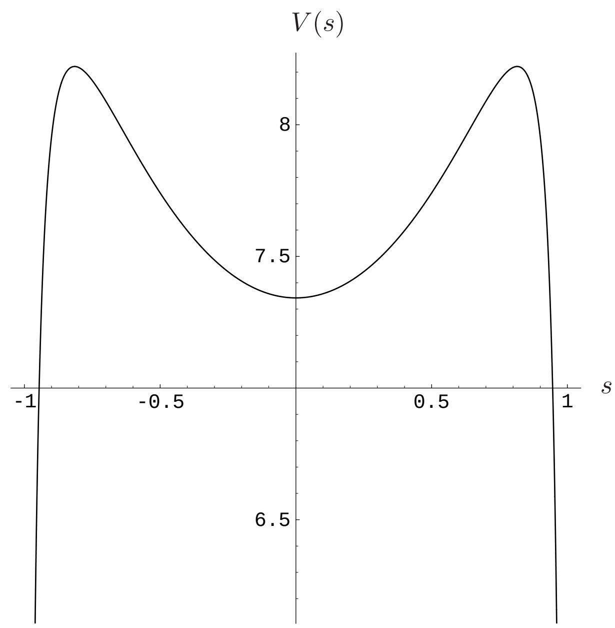

FIGURE 4. Function V for $A = B = -10$, $C = D = 0.1$ and $\lambda = 0.7$

field (9) satisfying the energy relation (10), verify

$$r_0 = 0, \quad u_0 = 0, \quad V'(s_0) = 0, \quad v_0 = \pm\sqrt{2V(s_0)}. \tag{12}$$

Note that in order to have equilibrium points we need that $V(s_0) \geq 0$. Additionally, all of them are on Λ and they are in correspondence $2:1$ with the critical points of the potential $V(s)$.

Since the last three equations of system (9) do not depend on r and the coordinate r can be obtained from the energy relation (10), in order to describe the flow of (9) on a fixed energy level $H = h$, it is sufficient to describe the flow of the system formed by the last three equations of (9)

$$\dot{v} = \frac{v^2}{2} + u^2 - V(s), \quad \dot{s} = \frac{1}{\lambda}u, \quad \dot{u} = -\frac{1}{2}vu + \frac{1}{\lambda}V'(s). \tag{13}$$

We note that the level of energy E_h of (9) with $h < 0$ is homeomorphic to a closed ball of $\mathbb{R}^3$ with boundary Λ in the first two examples and to a closed cylinder in $\mathbb{R}^3$ in the last example.

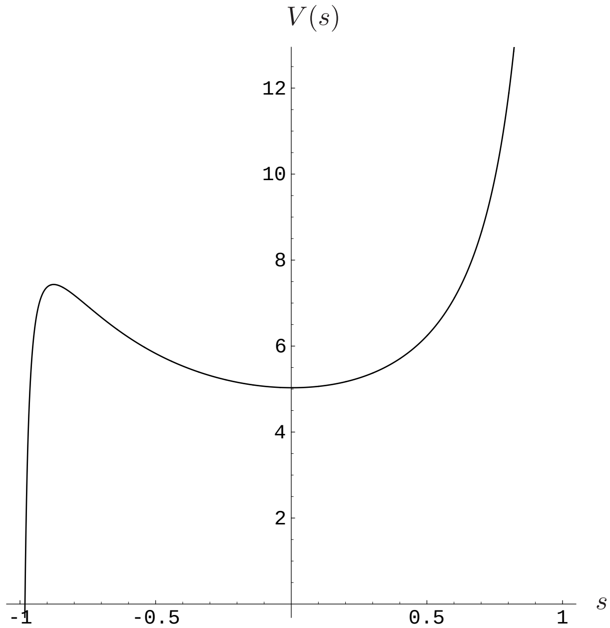

FIGURE 5. Function V for $A = -10$, $B = 1$, $C = 0.5$, $D = 0.1$ and $\lambda = 0.2$

We linearize the vector field (9) at an equilibrium point. The eigenvalues are given by

$$\mu_1 = v_0, \quad \mu_{2,3} = \frac{1}{2}\Big[-\frac{1}{2}v_0 \pm \sqrt{\frac{1}{2}V(s_0) + \frac{4}{\lambda^2}V''(s_0)}\Big],$$

with eigenvectors

$$w_1 = (1, 0, 0),$$

$$w_{2,3} = \Big(0, \frac{\lambda}{2V''(s_0)}\Big[\frac{1}{2}v_0 \pm \sqrt{\frac{1}{2}V(s_0) + \frac{4}{\lambda^2}V''(s_0)}\Big], 1\Big),$$

respectively. Note that the vectors w_2 and w_3 are tangent to the total collision manifold. We consider now the examples from the previous subsection.

Example 1. As we can see in the case of Figure 3 the function $V(s)$ has a single critical point $s_0 = 0$. The global flow (9) has two equilibrium points associated to this critical point both in Λ given by (12). Roughly speaking, they correspond to the northern and southern poles of Λ, respectively. We denote them by e^+ and e^- according with $v_0 > 0$ or $v_0 < 0$ and both of them are foci. In this case we can apply Theorem 1.

Example 2. In this case function $V(s)$ has three critical points s_1, s_2, and s_3 and corresponding to them the global flow (9) has six equilibrium points, all on Λ. We denote them by denote them by $e^+_{1,2,3}$ and $e^-_{1,2,3}$ according with $v_0 > 0$ or $v_0 < 0$. It is easy to check that four are foci ($e^+_{1,3}$ and $e^-_{1,3}$) and two are saddles (e^+_2 and e^-_2). In this case we can apply Theorem 2.

Example 3. In this case function $V(s)$ has two critical points s_1, s_2, and corresponding to them the global flow (9) has four equilibrium points, all on Λ. We denote them by $e^+_{1,2}$ and $e^-_{1,2}$ according with $v_0 > 0$ or $v_0 < 0$. It is easy to check that two are foci (e^+_1 and e^-_1) and two are saddles (e^+_2 and e^-_2). In this case we can apply Theorem 3.

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