

HOPF BIFURCATION FOR DEGENERATE SINGULAR POINTS OF MULTIPLICITY $2n - 1$ IN DIMENSION 3

JAUME LLIBRE¹ AND HAO WU²

ABSTRACT. The main purpose of this paper is to study the Hopf bifurcation for a class of degenerate singular points of multiplicity $2n - 1$ in dimension 3 via averaging theory. More specifically, we consider the system

$$\begin{aligned}\dot{x} &= -H_y(x, y) + P_{2n}(x, y, z) + \varepsilon P_{2n-1}(x, y), \\ \dot{y} &= H_x(x, y) + Q_{2n}(x, y, z) + \varepsilon Q_{2n-1}(x, y), \\ \dot{z} &= R_{2n}(x, y, z) + \varepsilon cz^{2n-1},\end{aligned}$$

where

$$\begin{aligned}H &= \frac{1}{2n}(x^{2l} + y^{2l})^m, & n = lm, \\ P_{2n-1} &= x(p_1x^{2n-2} + p_2x^{2n-3}y + \dots + p_{2n-1}y^{2n-2}), \\ Q_{2n-1} &= y(p_1x^{2n-2} + p_2x^{2n-3}y + \dots + p_{2n-1}y^{2n-2}),\end{aligned}$$

and P_{2n} , Q_{2n} and R_{2n} are arbitrary analytic functions starting with terms of degree $2n$, and prove that passing from $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small from the origin it can bifurcate $2n - 1$ limit cycles using averaging theory of first order, and $3n - 1$ limit cycles for $l = 1$ using averaging theory of second order.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let

$$\dot{x} = P(x, y, z), \quad \dot{y} = Q(x, y, z), \quad \dot{z} = R(x, y, z),$$

be an analytic system in \mathbb{R}^3 starting with terms in P , Q and R of order $2n - 1$. Then we say that the singular point at the origin of \mathbb{R}^3 has *multiplicity* $2n - 1$.

The main purpose of this paper is to study the Hopf bifurcation which takes place at the singular point located at the origin for a subclass analytic

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differential equations in \mathbb{R}^3 of the form

$$(1) \quad \begin{aligned} \dot{x} &= \bar{P}_{2n-1}(x, y, z) + \bar{P}_{2n}(x, y, z), \\ \dot{y} &= \bar{Q}_{2n-1}(x, y, z) + \bar{Q}_{2n}(x, y, z), \\ \dot{z} &= \bar{R}_{2n-1}(x, y, z) + \bar{R}_{2n}(x, y, z), \end{aligned}$$

where \bar{P}_{2n-1} , \bar{Q}_{2n-1} and \bar{R}_{2n-1} are homogeneous polynomials of degree $2n-1$, and \bar{P}_{2n} , \bar{Q}_{2n} and \bar{R}_{2n} are analytical functions starting with terms of order $2n$.

In general Hopf bifurcation is well studied for singular points which have an eigenvalue of the form $\alpha(\varepsilon) \pm \beta(\varepsilon)i$ with $\alpha(0) = 0$ and $\alpha'(0) \neq 0$. Also in dimension 2 the Hopf bifurcation can be obtained for the singular points having eigenvalues of the form $\pm \beta i$ using the so-called Lyapunov constants, see for instance [1, 2]. But for systems (1) with $n > 1$, the singular point located at the origin is degenerated and all its eigenvalues are zero, so the standard techniques for studying the limit cycles that bifurcate from the origin changing a parameter cannot be applied.

We shall study the Hopf bifurcation of a subclass of systems (1) using the averaging theory, see Section 2. Our main results are as follows.

Theorem 1. *We consider the differential systems in \mathbb{R}^3 given by*

$$(2) \quad \begin{aligned} \dot{x} &= -H_y(x, y) + P_{2n}(x, y, z) + \varepsilon P_{2n-1}(x, y), \\ \dot{y} &= H_x(x, y) + Q_{2n}(x, y, z) + \varepsilon Q_{2n-1}(x, y), \\ \dot{z} &= R_{2n}(x, y, z) + \varepsilon cz^{2n-1}, \end{aligned}$$

where

$$\begin{aligned} H &= \frac{1}{2n}(x^{2l} + y^{2l})^m, & n &= lm, \\ P_{2n-1} &= x(p_1x^{2n-2} + p_2x^{2n-3}y + \dots + p_{2n-1}y^{2n-2}), \\ Q_{2n-1} &= y(p_1x^{2n-2} + p_2x^{2n-3}y + \dots + p_{2n-1}y^{2n-2}), \end{aligned}$$

and P_{2n} , Q_{2n} and R_{2n} are arbitrary analytic functions starting with terms of degree $2n$, then from the origin of system (2) for $\varepsilon = 0$ can bifurcate $2n-1$ limit cycles for $\varepsilon \neq 0$ sufficiently small using averaging theory of first order.

Theorem 2. *Under the assumptions of Theorem 1 for the case $l = 1$, from the origin of system (2) for $\varepsilon = 0$ can bifurcate $3n-1$ limit cycles for $\varepsilon \neq 0$ sufficiently small using averaging theory of second order.*

Theorem 1 will be proved in Section 3. Theorem 2 will be proved in Section 4.

We remark that in the proof of Theorem 1 we only use the terms of degree $2n - 1$ and $2n$ of system (2), but in the proof of Theorem 2 we need to use the terms of degree $2n - 1$, $2n$ and $2n + 1$ of system (2).

2. LIMIT CYCLES VIA AVERAGING THEORY

Under good assumptions we can say that the averaging method [5, 6] gives a quantitative relation between the limit cycles of some non-autonomous periodic differential system and the singular points of its averaged differential system, which is an autonomous one. The next theorem provides a first order approximation in ε for the limit cycles of a periodic differential system, for a proof see Theorem 2.6.1 of Sanders and Verhulst [5] and Theorem 11.5 of Verhulst [6].

Theorem 3 (Averaging method up to first order in ε). *We consider the following two initial value problems*

$$(3) \quad \dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x, \varepsilon), \quad x(0) = x_0,$$

and

$$(4) \quad \dot{y} = \varepsilon F^0(y), \quad y(0) = x_0,$$

where $x, y, x_0 \in \Omega$ an open subset of \mathbb{R}^n , $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$, f and g are periodic of period T in the variable t , and $F^0(y)$ is the averaged function of $f(t, x)$ with respect to t , i.e.,

$$(5) \quad F^0(y) = \frac{1}{T} \int_0^T f(t, y) dt.$$

Suppose: (i) f , its Jacobian $\partial f / \partial x$, its Hessian $\partial^2 f / \partial x^2$, g and its Jacobian $\partial g / \partial x$ are defined, continuous and bounded by a constant independent on ε in $[0, \infty) \times \Omega$ and $\varepsilon \in (0, \varepsilon_0]$; (ii) T is a constant independent of ε ; and (iii) $y(t)$ belongs to Ω on the interval of time $[0, 1/\varepsilon]$. Then the following statements hold.

- (a) On the time scale $1/\varepsilon$ we have that $x(t) - y(t) = O(\varepsilon)$, as $\varepsilon \rightarrow 0$.
- (b) If p is a singular point of the averaged system (4) such that the determinant of the Jacobian matrix

$$(6) \quad \left. \frac{\partial F^0}{\partial y} \right|_{y=p}$$

is not zero, then there exists a limit cycle $\phi(t, \varepsilon)$ of period T for the system (3) which is close to p and such that $\phi(t, \varepsilon) \rightarrow p$ as $\varepsilon \rightarrow 0$.

- (c) The stability or instability of the limit cycle $\phi(t, \varepsilon)$ is given by the stability or instability of the singular point p of the averaged system

(4). *In fact, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\phi(t, \varepsilon)$.*

In [3] the authors have succeeded to weaken the hypotheses of the averaging method for the existence of periodic orbits using the Brouwer degree theory.

The next theorem provides a second order approximation for the solutions of a periodic differential system, for a proof see Llibre [4] and Theorem 3.5.1 of Sanders and Verhulst [5].

Theorem 4 (Averaging method up to second order in ε). *We consider the following two initial value problems*

$$(7) \quad \dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x) + \varepsilon^3 R(t, x, \varepsilon), \quad x(0) = x_0,$$

and

$$(8) \quad \dot{y} = \varepsilon F^0(y) + \varepsilon^2 F^{10}(y) + \varepsilon^2 G^0(y), \quad y(0) = x_0,$$

with $f, g : [0, \infty) \times D \rightarrow \mathbb{R}^n$, $R : [0, \infty) \times D \times (0, \varepsilon_0] \rightarrow \mathbb{R}^n$, D an open subset of \mathbb{R}^n , f, g and R periodic of period T in the variable t ,

$$(9) \quad F^1(t, x) = \frac{\partial f}{\partial x} y^1(t, x) - \frac{\partial y^1}{\partial x} F^0(x),$$

where

$$y^1(t, x) = \int_0^t (f(s, x) - F^0(x)) ds + z(x),$$

with $z(x)$ a C^1 function such that the averaged of y^1 is zero. Of course, F^0 , F^{10} and G^0 denote the averaged functions of f , F^1 and g , respectively, defined as in (5). Suppose: (i) $\partial f / \partial x$ is Lipschitz in x , g and R are Lipschitz in x and all these functions are continuous on their domain of definition; (ii) $|R(t, x, \varepsilon)|$ is bounded by a constant uniformly in $[0, L/\varepsilon] \times D \times (0, \varepsilon_0]$; (iii) T is independent on ε ; (iv) $y(t)$ belongs to D on the time-scale $\frac{1}{\varepsilon}$. Then the following results hold.

- (a) On the time-scale $\frac{1}{\varepsilon}$ we have that $x(t) = y(t) + \varepsilon y^1(t, y(t)) + O(\varepsilon^2)$, as $\varepsilon \rightarrow 0$.

Assume that $F^0(y) \equiv 0$ in addition.

- (b) If p is an equilibrium point of the averaged system (8) such that

$$(10) \quad \left. \frac{\partial}{\partial y} (F^{10}(y) + G^0(y)) \right|_{y=p} \neq 0,$$

then there exists a T -periodic solution $\phi(t, \varepsilon)$ of equation (7) which is close to p such that $\phi(t, \varepsilon) \rightarrow p$ as $\varepsilon \rightarrow 0$.

- (c) *The stability or instability of the limit cycle $\phi(t, \varepsilon)$ is given by the stability or instability of the singular point p of the averaged system (8). In fact, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\phi(t, \varepsilon)$.*

3. PROOF OF THEOREM 1

We assume that

$$\begin{aligned} P_{2n} &= \sum_{i+j+k=2n} A_{i,j,k} x^i y^j z^k + H.O.T., \\ Q_{2n} &= \sum_{i+j+k=2n} B_{i,j,k} x^i y^j z^k + H.O.T., \\ R_{2n} &= \sum_{i+j+k=2n} C_{i,j,k} x^i y^j z^k + H.O.T., \end{aligned}$$

where as usual H.O.T. means higher order terms. By the generalized cylinder transformation

$$x = r \, Cs\theta, \quad y = r \, Sn\theta, \quad z = z,$$

where $Cs\theta$ and $Sn\theta$ are defined by the solution of the following system

$$\dot{Cs}\theta = -Sn^{2l-1}\theta, \quad \dot{Sn}\theta = Cs^{2l-1}\theta,$$

with the initial condition $Cs(0) = 1$ and $Sn(0) = 0$. Moreover, they satisfies the following properties:

1. $Cs = \cos$ and $Sn = \sin$ if $l = 1$.
2. $Sn^{2l}\theta + Cs^{2l}\theta = 1$.
3. $T = \frac{2}{l} B(1/(2l), 1/(2l))$, where $B(\cdot, \cdot)$ is the Beta function. Let $I_{p,q} = \int_0^\theta Sn^p \varphi Cs^q \varphi d\varphi$.
4. $I_{2l-1,q} = -\frac{Cs^{q+1}\theta}{q+1}$.
5. $I_{p,2l-1} = \frac{Sn^{p+1}\theta}{p+1}$.
6. $I_{p,q} = \frac{Sn^{p+1}\theta Cs^{q-2l+1}\theta}{p+1} + \frac{q-2l+1}{p+1} I_{p+2l,q-2l}$.
7. $I_{p,q} = -\frac{Sn^{p-2l+1}\theta Cs^{q+1}\theta}{q+1} + \frac{p-2l+1}{q+1} I_{p-2l,q+2l}$.

$$8. \int_0^T S n^p \varphi C s^q \varphi d\varphi = \frac{2}{l} B((p+1)/(2l), (q+1)/(2l)) \text{ if } p \text{ and } q \text{ are even.}$$

$$9. \int_0^T S n^p \varphi C s^q \varphi d\varphi = 0 \text{ if } p \text{ or } q \text{ is odd.}$$

Proposition 5. Assume \mathcal{P}_d is the homogeneous polynomial function in $\mathbb{R}[x, y]$ of degree d , if d is odd then $\int_0^{2\pi} \mathcal{P}_d(\cos \theta, \sin \theta) d\theta = 0$.

Proof. It is trivial by property 9. \square

System (2) in the variables (r, θ, z) becomes

$$(11) \quad \begin{aligned} \dot{r} &= \sum_{i=0}^{2n} K_{2n-i,i}(\theta) r^{2n-i} z^i + O_{2n+1}(r, z) + \varepsilon \Gamma(\theta) r^{2n-1}, \\ \dot{\theta} &= \frac{1}{r} \left(r^{2n-1} + \sum_{i=0}^{2n} L_{2n-i,i}(\theta) r^{2n-i} z^i \right) + O_{2n}(r, z), \\ \dot{z} &= \sum_{i=0}^{2n} M_{2n-i,i}(\theta) r^{2n-i} z^i + O_{2n+1}(r, z) + \varepsilon c z^{2n-1}, \end{aligned}$$

where

$$\begin{aligned} \Gamma(\theta) &= \sum_{i=1}^{2n-1} p_i C s^{2n-i-1} \theta S n^{i-1} \theta, \\ K_{2n-i,i}(\theta) &= \sum_{r+s=2n-i} (A_{r,s,i} C s^{r+2l-1} \theta S n^s \theta + B_{r,s,i} C s^r \theta S n^{s+2l-1} \theta), \\ L_{2n-i,i}(\theta) &= \sum_{r+s=2n-i} (B_{r,s,i} C s^{r+1} \theta S n^s \theta - A_{r,s,i} C s^r \theta S n^{s+1} \theta), \\ M_{2n-i,i}(\theta) &= \sum_{r+s=2n-i} C_{r,s,i} C s^r \theta S n^s \theta, \quad \text{in particular } G_{0,2n} = C_{0,0,2n}. \end{aligned}$$

By the coordinate transformation $r = \varepsilon R$ and $z = \varepsilon \xi$, system (11) is orbitally equivalent to the following one

$$(12) \quad \begin{aligned} \frac{dR}{d\theta} &= \varepsilon R \frac{\Gamma(\theta) R^{2n-1} + \sum_{i=0}^{2n} K_{2n-i,i}(\theta) r^{2n-i} z^i}{R^{2n-1} + \varepsilon \sum_{i=0}^{2n} L_{2n-i,i}(\theta) r^{2n-i} z^i} + O(\varepsilon^2), \\ \frac{d\xi}{d\theta} &= \varepsilon R \frac{c \xi^{2n-1} + \sum_{i=0}^{2n} M_{2n-i,i}(\theta) r^{2n-i} z^i}{R^{2n-1} + \varepsilon \sum_{i=0}^{2n} L_{2n-i,i}(\theta) r^{2n-i} z^i} + O(\varepsilon^2). \end{aligned}$$

Expanding the right part of system (12) with respect to the variable ε , we have

$$(13) \quad \begin{aligned} \frac{dR}{d\theta} &= \varepsilon f_1(\theta, R, \xi) + \varepsilon^2 g_1(\theta, R, \xi, \varepsilon), \\ \frac{d\xi}{d\theta} &= \varepsilon f_2(\theta, R, \xi) + \varepsilon^2 g_2(\theta, R, \xi, \varepsilon), \end{aligned}$$

where

$$\begin{aligned} f_1(\theta, R, \xi) &= \frac{1}{R^{2n-2}} \left(\Gamma(\theta) R^{2n-1} + \sum_{i=0}^{2n} K_{2n-i,i}(\theta) r^{2n-i} z^i \right), \\ f_2(\theta, R, \xi) &= \frac{1}{R^{2n-2}} \left(c\xi^{2n-1} + \sum_{i=0}^{2n} M_{2n-i,i}(\theta) r^{2n-i} z^i \right). \end{aligned}$$

Furthermore, $\int_0^T K_{2n-p,p}(\theta) d\theta = 0$, if p is even and $\int_0^T M_{2n-q,q}(\theta) d\theta = 0$, if q is odd by Proposition 5.

Let Ω be the open subset and ε the positive number which appear in the statement of Theorem 3. Then, it is easy to verify that system (13) satisfies the assumption of Theorem 3 if we take Ω as an open disc centered at the origin $(R, \xi) = (0, 0)$ of \mathbb{R}^2 and a sufficiently small ε_0 . Since

$$F_i^0(R, \xi) = \frac{1}{T} \int_0^T f_i(\theta, R, \xi) d\theta,$$

for $i = 1$ and 2 , then we get that

$$\begin{aligned} F_1^0(R, \xi) &= \frac{1}{R^{2n-2}} (\alpha_0 R^{2n-1} + \alpha_1 R^{2n-1} \xi + \alpha_2 R^{2n-3} \xi^3 + \dots + \\ &\quad \alpha_n R \xi^{2n-1}), \\ F_2^0(R, \xi) &= \frac{1}{R^{2n-2}} (c\xi^{2n-1} + \beta_1 R^{2n} + \beta_2 R^{2n-2} \xi^2 + \dots + \\ &\quad \beta_n R^2 \xi^{2n-2} + C_{0,0,2n} \xi^{2n}), \end{aligned}$$

where

$$\alpha_0 = \frac{1}{T} \int_0^T \Gamma(\theta) d\theta = \frac{\sum_{k=1}^n p_{2k-1} B\left(\frac{2n-2k+1}{2l}, \frac{2k-1}{2l}\right)}{B\left(\frac{1}{2l}, \frac{1}{2l}\right)},$$

and

$$\begin{aligned}
\alpha_i &= \\
&= \frac{1}{T} \int_0^T H_{2n-2i+1, 2i-1}(\theta) d\theta \\
&= \frac{\sum_{k=0}^{n-i} B\left(\frac{2(n+l-i-k)+1}{2l}, \frac{2k+1}{2l}\right) (A_{2(n+k-i)+1, 2k, 2i-1} + B_{2k, 2(n+k-i)+1, 2i-1})}{B\left(\frac{1}{2l}, \frac{1}{2l}\right)},
\end{aligned}$$

$$\begin{aligned}
\beta_i &= \\
&= \frac{1}{T} \int_0^T G_{2(n-i+1), 2(i-1)}(\theta) d\theta \\
&= \frac{\sum_{k=1}^{n-i+1} B\left(\frac{2(n-k-i)-1}{2l}, \frac{2k+1}{2l}\right) C_{2(n-k-i-1), 2k, 2(i+1)}}{B\left(\frac{1}{2l}, \frac{1}{2l}\right)}.
\end{aligned}$$

Moreover, we can see from the expressions of α_i and β_i that the functions $L_{2n-i, i}$ do not affect the first order averaging method and that the constants $c, C_{0,0,2n}, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ can be chosen arbitrarily.

Now we look for the solution (R, ξ) of the following system of polynomial equations with $R > 0$

$$\begin{aligned}
\alpha_0 R^{2n-2} + \alpha_1 R^{2n-2} \xi + \alpha_2 R^{2n-4} \xi^3 + \dots + \alpha_n \xi^{2n-1} &= 0, \\
c \xi^{2n-1} + \beta_1 R^{2n} + \beta_2 R^{2n-2} \xi^2 + \dots + \beta_n R^2 \xi^{2n-2} + C_{0,0,2n} \xi^{2n} &= 0,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
(14) \quad \alpha_1 R^{2n-2} \xi + \alpha_2 R^{2n-4} \xi^3 + \dots + \alpha_n \xi^{2n-1} &= -\alpha_0 R^{2n-2}, \\
(15) \quad \beta_1 R^{2n} + \beta_2 R^{2n-2} \xi^2 + \dots + \beta_n R^2 \xi^{2n-2} + C_{0,0,2n} \xi^{2n} &= -c \xi^{2n-1}.
\end{aligned}$$

So dividing equation (15) by equation (14), we have

$$\frac{\beta_1 R^{2n} + \beta_2 R^{2n-2} \xi^2 + \dots + \beta_n R^2 \xi^{2n-2} + C_{0,0,2n} \xi^{2n}}{\alpha_1 R^{2n-2} \xi + \alpha_2 R^{2n-4} \xi^3 + \dots + \alpha_n \xi^{2n-1}} = \frac{c \xi^{2n-1}}{\alpha_0 R^{2n-2}},$$

which is equivalent to

$$\frac{\beta_1 + \beta_2 \left(\frac{\xi}{R}\right)^2 + \dots + \beta_n \left(\frac{\xi}{R}\right)^{2n-2} + C_{0,0,2n} \left(\frac{\xi}{R}\right)^{2n}}{\alpha_1 \left(\frac{\xi}{R}\right)^2 + \dots + \alpha_{n-1} \left(\frac{\xi}{R}\right)^{2n-2} + \alpha_n \left(\frac{\xi}{R}\right)^{2n}} = \frac{c}{\alpha_0} \left(\frac{\xi}{R}\right)^{2n-2}.$$

That is

$$\begin{aligned}
\frac{c \alpha_n}{\alpha_0} \left(\frac{\xi}{R}\right)^{4n-2} + \frac{c \alpha_{n-1}}{\alpha_0} \left(\frac{\xi}{R}\right)^{4n-4} + \dots + \left(\frac{c \alpha_1}{\alpha_0} - C_{0,0,2n}\right) \left(\frac{\xi}{R}\right)^{2n} \\
- \beta_n \left(\frac{\xi}{R}\right)^{2n-2} - \dots - \beta_2 \left(\frac{\xi}{R}\right)^2 - \beta_1 = 0.
\end{aligned}$$

Since $c, C_{0,0,2n}, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ can be chosen arbitrarily, we can have just $2n - 1$ positive real roots for the variable $(\xi/R)^2$. Note that from equation (14), we have

$$\xi = -\frac{\alpha_0}{\alpha_1 + \dots + \alpha_{n-1} \left(\frac{\xi}{R}\right)^{2n-4} + \alpha_n \left(\frac{\xi}{R}\right)^{2n-2}}.$$

Hence, each positive real solution ξ/R corresponds to a unique solution of the polynomial system (14) and (15). Therefore, by Theorem 3, the differential system (12) can have at most $2n - 1$ limit cycles using the averaging method up to first order in ε .

Finally, since the transformation from the original system to system (12) is $r = \varepsilon R$ and $z = \varepsilon \xi$, this means that the at most $2n - 1$ limit cycles tend to zero as $\varepsilon \rightarrow 0$ in system (2). That is, all the above $2n - 1$ limit cycles bifurcate from the origin of system (2). This completes the proof of Theorem 1.

4. PROOF OF THEOREM 2

We assume that

$$\begin{aligned} P_{2n} &= \sum_{i+j+k=2n}^{2n+1} A_{i,j,k} x^i y^j z^k + H.O.T., \\ Q_{2n} &= \sum_{i+j+k=2n}^{2n+1} B_{i,j,k} x^i y^j z^k + H.O.T., \\ R_{2n} &= \sum_{i+j+k=2n}^{2n+1} C_{i,j,k} x^i y^j z^k + H.O.T.. \end{aligned}$$

By the cylinder transformation

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z,$$

system (2) can be changed into

$$\begin{aligned}
 \dot{r} &= \sum_{i=0}^{2n} K_{2n-i,i}(\theta) r^{2n-i} z^i + \sum_{i=0}^{2n+1} K_{2n+1-i,i}(\theta) r^{2n+1-i} z^i + \\
 &\quad O_{2n+2}(r, z) + \varepsilon \Gamma(\theta) r^{2n-1}, \\
 \dot{\theta} &= \frac{1}{r} \left(r^{2n-1} + \sum_{i=0}^{2n} L_{2n-i,i}(\theta) r^{2n-i} z^i + \right. \\
 (16) \quad &\quad \left. \sum_{i=0}^{2n+1} L_{2n+1-i,i}(\theta) r^{2n+1-i} z^i \right) + O_{2n+1}(r, z), \\
 \dot{z} &= \sum_{i=0}^{2n} M_{2n-i,i}(\theta) r^{2n-i} z^i + \sum_{i=0}^{2n+1} M_{2n+1-i,i}(\theta) r^{2n+1-i} z^i + \\
 &\quad O_{2n+2}(r, z) + \varepsilon C z^{2n-1},
 \end{aligned}$$

where $\Gamma(\theta)$, $K_{2n-i,i}(\theta)$, $L_{2n-i,i}(\theta)$ and $M_{2n-i,i}(\theta)$ are the same defined in Section 3 and

$$\begin{aligned}
 K_{2n+1-i,i}(\theta) &= \sum_{r+s=2n+1-i} (A_{r,s,i} \cos^{r+1} \theta \sin^s \theta + B_{r,s,i} \cos^r \theta \sin^{s+1} \theta), \\
 L_{2n+1-i,i}(\theta) &= \sum_{r+s=2n+1-i} (B_{r,s,i} \cos^{r+1} \theta \sin^s \theta - A_{r,s,i} \cos^r \theta \sin^{s+1} \theta), \\
 M_{2n+1-i,i}(\theta) &= \sum_{r+s=2n+1-i} C_{r,s,i} \cos^r \theta \sin^s \theta.
 \end{aligned}$$

By the coordinate transformation $r = \varepsilon R$ and $z = \varepsilon \xi$, system (16) is orbitally equivalent to the following one

$$\begin{aligned}
 \frac{dR}{d\theta} &= \\
 &= \varepsilon R \frac{\Gamma(\theta) R^{2n-1} + \sum_{i=0}^{2n} K_{2n-i,i}(\theta) R^{2n-i} \xi^i + \varepsilon \sum_{i=0}^{2n+1} K_{2n+1-i,i}(\theta) R^{2n+1-i} \xi^i}{R^{2n-1} + \varepsilon \sum_{i=0}^{2n} L_{2n-i,i}(\theta) R^{2n-i} \xi^i + \varepsilon^2 \sum_{i=0}^{2n+1} L_{2n+1-i,i}(\theta) R^{2n+1-i} \xi^i}, \\
 \frac{d\xi}{d\theta} &= \\
 &= \varepsilon R \frac{C \xi^{2n-1} + \sum_{i=0}^{2n} M_{2n-i,i}(\theta) R^{2n-i} \xi^i + \varepsilon \sum_{i=0}^{2n+1} M_{2n+1-i,i}(\theta) R^{2n+1-i} \xi^i}{R^{2n-1} + \varepsilon \sum_{i=0}^{2n} L_{2n-i,i}(\theta) R^{2n-i} \xi^i + \varepsilon^2 \sum_{i=0}^{2n+1} L_{2n+1-i,i}(\theta) R^{2n+1-i} \xi^i},
 \end{aligned}$$

where we have omitted some terms of order 3 in ε .

Here our purpose is to apply Theorem 4 to system (16) instead of Theorem 3. So take α_i , c , β_i and $C_{0,0,2n}$ defined in Section 3 equal to zero

for $i = 1, \dots, n$ and $j = 1, \dots, n$. In this way the functions $F_i^0(R, \xi)$, for $i = 1, 2$, are identically zero, and the averaging theory of first order does not provide any information. Thus we have

$$\begin{aligned}\frac{dR}{d\theta} &= \varepsilon f_1(\theta, R, \xi) + \varepsilon^2(g_1(\theta, R, \xi) + h_1(\theta, R, \xi)) + O(\varepsilon^3), \\ \frac{d\xi}{d\theta} &= \varepsilon f_2(\theta, R, \xi) + \varepsilon^2(g_2(\theta, R, \xi) + h_2(\theta, R, \xi)) + O(\varepsilon^3),\end{aligned}$$

where

$$\begin{aligned}f_1(\theta, R, \xi) &= \Gamma(\theta)R + R^2 \sum_{i=0}^{2n} K_{2n-i,i}(\theta) \left(\frac{\xi}{R}\right)^i, \\ f_2(\theta, R, \xi) &= R^2 \sum_{i=0}^{2n-1} M_{2n-i,i}(\theta) \left(\frac{\xi}{R}\right)^i, \\ g_1(\theta, R, \xi) &= R^3 \sum_{i=0}^{2n+1} K_{2n+1-i,i}(\theta) \left(\frac{\xi}{R}\right)^i, \\ g_2(\theta, R, \xi) &= R^3 \sum_{i=0}^{2n+1} M_{2n+1-i,i}(\theta) \left(\frac{\xi}{R}\right)^i, \\ K(\theta, R, \xi) &= R \sum_{i=0}^{2n} L_{2n-i,i}(\theta) \left(\frac{\xi}{R}\right)^i,\end{aligned}$$

and $h_i = -Kf_i$ for $i = 1, 2$. Of course, the 2π -periodic averaging of f_1 and f_2 are zero by our assumptions.

By Proposition 5, the averaged of g_1 is

$$G_1^0 = R^3 \left(\lambda_0 + \lambda_1 \left(\frac{\xi}{R}\right)^2 + \dots + \lambda_n \left(\frac{\xi}{R}\right)^{2n} \right),$$

where

$$\lambda_i = \frac{1}{2\pi} \int_0^{2\pi} K_{2(n-i)+1,2i}(\theta) d\theta.$$

Similarly the averaged of g_2 is

$$G_2^0 = R^3 \left(\omega_0 \frac{\xi}{R} + \omega_1 \left(\frac{\xi}{R}\right)^3 + \dots + \omega_n \left(\frac{\xi}{R}\right)^{2n+1} \right),$$

where

$$\omega_i = \frac{1}{2\pi} \int_0^{2\pi} M_{2(n-i),2i+1}(\theta) d\theta.$$

Let \mathcal{P}_d denote a homogeneous polynomial of degree d in $\mathbb{R}[x, y]$. Then we have the following result.

Proposition 6. *If $\int_0^{2\pi} \mathcal{P}_{d_i}(\cos \varphi, \sin \varphi) d\varphi = 0$ for $i = 1, 2$ and $d_1 + d_2$ is odd, then*

$$\int_0^{2\pi} \mathcal{P}_{d_1}(\cos \theta, \sin \theta) \left(\int_0^\theta \mathcal{P}_{d_2}(\cos \varphi, \sin \varphi) d\varphi \right) d\theta = 0.$$

Proof. Since $\int_0^{2\pi} \mathcal{P}_d(\cos \varphi, \sin \varphi) d\varphi = 0$, then we know that

$$\int_0^\theta \mathcal{P}_d(\cos \varphi, \sin \varphi) d\varphi = \sum_{i=1}^d \mathcal{Q}_i(\theta) + \text{constant},$$

where $\mathcal{Q}_i(\theta)$ is a trigonometric homogeneous polynomial of degree i in the variables $\cos \theta$ and $\sin \theta$. From the properties 4, 5, 6 and 7, we know that i and d have the same parity. So, by Proposition 5, we get the result. \square

We can write h_1 as $h_1 = \mathbf{I}_{h_1} + \mathbf{II}_{h_1}$, where

$$\mathbf{I}_{h_1} = -R^2 \sum_{i=0}^{2n} \Gamma(\theta) L_{2n-i,i}(\theta) \left(\frac{\xi}{R} \right)^i,$$

and by Proposition 6 its averaged is

$$\mathbf{I}_{h_1}^0 = -R^2 \left(\eta_0 \frac{\xi}{R} + \eta_1 \left(\frac{\xi}{R} \right)^3 + \dots + \eta_{n-1} \left(\frac{\xi}{R} \right)^{2n-1} \right),$$

where

$$\eta_i = \frac{1}{2\pi} \int_0^{2\pi} \Gamma(\theta) L_{2(n-i)-1,2i+1}(\theta) d\theta.$$

Similarly, we have that

$$\mathbf{II}_{h_1} = -R^3 \left(\sum_{i=0}^{2n} K_{2n-i,i}(\theta) \left(\frac{\xi}{R} \right)^i \right) \left(\sum_{i=0}^{2n} L_{2n-i,i}(\theta) \left(\frac{\xi}{R} \right)^i \right).$$

By the fact that $\int_0^{2\pi} K_{0,2n}(\theta) L_{0,2n}(\theta) d\theta = 0$ and by Proposition 6, its averaged is

$$\mathbf{II}_{h_1}^0 = -R^3 \left(\psi_0 + \psi_1 \left(\frac{\xi}{R} \right)^2 + \dots + \psi_{2n-1} \left(\frac{\xi}{R} \right)^{4n-2} \right),$$

where

$$\psi_i = \frac{1}{2\pi} \sum_{l+m=2i} \int_0^{2\pi} K_{2n-l,l}(\theta) L_{2n-m,m}(\theta) d\theta.$$

Now we write h_2 as

$$h_2 = -Kf_2 = -R^3 \left(\sum_{i=0}^{2n-1} M_{2n-i,i}(\theta) \left(\frac{\xi}{R} \right)^i \right) \left(\sum_{i=0}^{2n} L_{2n-i,i}(\theta) \left(\frac{\xi}{R} \right)^i \right),$$

and its averaged is

$$h_2^0 = -R^3 \left(\gamma_0 \left(\frac{\xi}{R} \right) + \gamma_1 \left(\frac{\xi}{R} \right)^3 + \dots + \gamma_{2n-1} \left(\frac{\xi}{R} \right)^{4n-1} \right),$$

where

$$\gamma_i = \frac{1}{2\pi} \sum_{l+m=2i+1} \int_0^{2\pi} M_{2n-l,l}(\theta) L_{2n-m,m}(\theta) d\theta.$$

By Theorem 4, we need to calculate the averaged of the part generated by the first order terms of ε . So equation (9) here can be written as

$$F^1(t, R, \xi) = Df(R, \xi) \cdot \int_0^t f(s, R, \xi) ds,$$

where Df is the Jacobian matrix with respect to the variable (R, ξ) , and its elements are

$$\begin{aligned} \frac{\partial f_1}{\partial R} &= \Gamma(\theta) + R \sum_{i=0}^{2n} (2-i) K_{2n-i,i}(\theta) \left(\frac{\xi}{R} \right)^i, \\ \frac{\partial f_1}{\partial \xi} &= R \sum_{i=1}^{2n} i K_{2n-i,i}(\theta) \left(\frac{\xi}{R} \right)^{i-1}, \\ \frac{\partial f_2}{\partial R} &= R \sum_{i=0}^{2n-1} (2-i) M_{2n-i,i}(\theta) \left(\frac{\xi}{R} \right)^i, \\ \frac{\partial f_2}{\partial \xi} &= R \sum_{i=1}^{2n-1} i M_{2n-i,i} \left(\frac{\xi}{R} \right)^{i-1}. \end{aligned}$$

Thus,

$$F_1^1 = \frac{\partial f_1}{\partial R} \int_0^\theta f_1(\varphi, R, \xi) d\varphi + \frac{\partial f_1}{\partial \xi} \int_0^\theta f_2(\varphi, R, \xi) d\varphi = I_{y_1} + \Pi_{y_1}.$$

Then the averaged function of I_{y_1} is

$$\begin{aligned} I_{y_1}^0 &= R^2 \left(\mu_1 \left(\frac{\xi}{R} \right)^3 + \mu_2 \left(\frac{\xi}{R} \right)^5 + \dots + \mu_{n-1} \left(\frac{\xi}{R} \right)^{2n-1} \right) \\ &+ R^3 \left(\nu_1 \left(\frac{\xi}{R} \right)^2 + \nu_2 \left(\frac{\xi}{R} \right)^4 + \dots + \nu_{2n-1} \left(\frac{\xi}{R} \right)^{4n-2} \right), \end{aligned}$$

with

$$\begin{aligned}\mu_i &= \frac{i}{\pi} \int_0^{2\pi} \Gamma(\theta) \left(\int_0^\theta K_{2(n-i)-1, 2i+1}(\varphi) d\varphi \right) d\theta, \\ \nu_i &= \frac{1}{2\pi} \sum_{l+m=2i} \int_0^{2\pi} (2-l) K_{2n-l, l}(\theta) \left(\int_0^\theta K_{2m-m, m}(\varphi) d\varphi \right) d\theta.\end{aligned}$$

Moreover

$$\Pi_{y_1}^0 = R^3 \left(\chi_0 + \chi_1 \left(\frac{\xi}{R} \right)^2 + \dots + \chi_{2n-1} \left(\frac{\xi}{R} \right)^{4n-2} \right),$$

where

$$\chi_i = \frac{1}{2\pi} \sum_{l+m=2i+1} \int_0^{2\pi} l K_{2n-l, l}(\theta) \left(\int_0^\theta M_{2n-m, m}(\varphi) d\varphi \right) d\theta.$$

Similarly, we have

$$y_2 = \frac{\partial f_2}{\partial R} \int_0^\theta f_1(\varphi, R, \xi) d\varphi + \frac{\partial f_2}{\partial \xi} \int_0^\theta f_2(\varphi, R, \xi) d\varphi = \mathbf{I}_{y_2} + \Pi_{y_2},$$

where

$$\begin{aligned}\Gamma_{y_2}^0 &= R^2 \left(\tau_0 + \tau_1 \left(\frac{\xi}{R} \right)^2 + \dots + \tau_{n-1} \left(\frac{\xi}{R} \right)^{2n-2} \right) \\ &+ R^3 \left(\varsigma_0 \left(\frac{\xi}{R} \right) + \varsigma_1 \left(\frac{\xi}{R} \right)^3 + \dots + \varsigma_{2n-1} \left(\frac{\xi}{R} \right)^{4n-1} \right)\end{aligned}$$

with

$$\begin{aligned}\tau_i &= \frac{1-i}{\pi} \int_0^{2\pi} \Gamma(\theta) \left(\int_0^\theta M_{2(n-i), 2i}(\varphi) d\varphi \right) d\theta, \\ \varsigma_i &= \frac{1}{2\pi} \sum_{l+m=2i+1} \int_0^{2\pi} (2-l) M_{2n-l, l}(\theta) \left(\int_0^\theta K_{2n-m, m}(\varphi) d\varphi \right) d\theta.\end{aligned}$$

In particular, we have $\tau_1 = 0$, and

$$\Pi_{y_2}^0 = R^3 \left(\sigma_0 \frac{\xi}{R} + \sigma_1 \left(\frac{\xi}{R} \right)^3 + \dots + \sigma_{2n-3} \left(\frac{\xi}{R} \right)^{4n-5} \right),$$

where

$$\sigma_i = \frac{1}{2\pi} \sum_{l+m=2i+2} \int_0^{2\pi} l M_{2n-l, l}(\theta) \left(\int_0^\theta M_{2n-m, m}(\varphi) d\varphi \right) d\theta.$$

By Theorem 4, we must study the solutions of the polynomial system

$$\begin{aligned} R^2 \sum_{i=0}^{n-1} \mathcal{A}_i \left(\frac{\xi}{R} \right)^{2i+1} &= -R^3 \sum_{i=0}^{2n-1} \mathcal{B}_i \left(\frac{\xi}{R} \right)^{2i}, \\ R^2 \sum_{i=0}^{n-1} \mathcal{C}_i \left(\frac{\xi}{R} \right)^{2i} &= -R^3 \sum_{i=0}^{2n-1} \mathcal{D}_i \left(\frac{\xi}{R} \right)^{2i+1}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}_0 &= -\eta_0, \quad \mathcal{A}_i = \mu_i - \eta_i, \quad i = 1, \dots, n-1, \\ \mathcal{B}_0 &= \lambda_0 - \psi_0 + \chi_0, \quad \mathcal{B}_i = \lambda_i - \psi_i + \chi_i + \nu_i, \quad i = 1, \dots, n, \\ \mathcal{B}_j &= -\psi_j + \chi_j + \nu_j, \quad j = n+1, \dots, 2n-1, \\ \mathcal{C}_i &= \tau_i, \quad i = 0, \dots, n-1, \\ \mathcal{D}_i &= \omega_i - \gamma_i + \varsigma_i + \sigma_i, \quad i = 0, \dots, n, \\ \mathcal{D}_j &= -\gamma_j + \varsigma_j + \sigma_j, \quad j = n+1, \dots, 2n-2, \\ \mathcal{D}_{2n-1} &= -\gamma_{2n-1} + \varsigma_{2n-1}. \end{aligned}$$

Dividing the first equation by the second one, we have

$$\begin{aligned} &\frac{\mathcal{A}_0 \frac{\xi}{R} + \mathcal{A}_1 \left(\frac{\xi}{R} \right)^3 + \dots + \mathcal{A}_{n-1} \left(\frac{\xi}{R} \right)^{2n-1}}{\mathcal{C}_0 + \mathcal{C}_1 \left(\frac{\xi}{R} \right)^2 + \dots + \mathcal{C}_{n-1} \left(\frac{\xi}{R} \right)^{2n-2}} = \\ &= \frac{\mathcal{B}_0 + \mathcal{B}_1 \left(\frac{\xi}{R} \right)^2 + \dots + \mathcal{B}_{2n-1} \left(\frac{\xi}{R} \right)^{4n-2}}{\mathcal{D}_0 \frac{\xi}{R} + \mathcal{D}_1 \left(\frac{\xi}{R} \right)^3 + \dots + \mathcal{D}_{2n-1} \left(\frac{\xi}{R} \right)^{4n-1}}, \end{aligned}$$

which is equivalent to the following polynomial equation of degree $3n-1$ in the variable $z = (\xi/R)^2$

$$(17) \quad z(\mathcal{A}_{n-1}z^{n-1} + \dots + \mathcal{A}_1z + \mathcal{A}_0)(\mathcal{D}_{2n-1}z^{2n-1} + \dots + \mathcal{D}_1z + \mathcal{D}_0) - (\mathcal{B}_{2n-1}z^{2n-1} + \dots + \mathcal{B}_1z + \mathcal{B}_0)(\mathcal{C}_{n-1}z^{n-1} + \dots + \mathcal{C}_1z + \mathcal{C}_0) = 0.$$

So it can have $3n-1$ positive real roots at most, moreover, each root with respect to the variable $(\xi/R)^2$ provides a unique solution in the variables (R, ξ) using the equality

$$\xi = -\frac{\mathcal{D}_{2n-1} \left(\frac{\xi}{R} \right)^{4n-2} + \dots + \mathcal{D}_1 \left(\frac{\xi}{R} \right)^2 + \mathcal{D}_0}{\mathcal{C}_{n-1} \left(\frac{\xi}{R} \right)^{2n-2} + \dots + \mathcal{C}_1 \left(\frac{\xi}{R} \right)^2 + \mathcal{C}_0}.$$

Now we claim that by choosing conveniently the coefficients of system (2), we can have $3n-1$ roots of (17). To prove this, we note that the

coefficients \mathcal{B}_i and \mathcal{D}_i can be chosen arbitrary for $i = 0, \dots, n$, because λ_i and ω_i can be chosen arbitrary. Then, let

$$\begin{aligned} M_{2n,0} &= C_{1,2n-1,0} \cos \theta \sin^{2n-1} \theta, \\ M_{2n-2k-2,2k+2} &= 0, \quad k = 0, \dots, n-1, \\ \Gamma &= \left(-\frac{1}{2n-3} \cos^{2n-2} \theta + \cos^{2n-4} \theta \sin^2 \theta \right) p_3. \end{aligned}$$

Thus,

$$C_0 = \frac{B((2n-3)/2, (2n+1)/2) p_3 C_{1,2n-1,0}}{(2n-1)\pi}, \quad C_i = 0, \quad i = 1, \dots, n.$$

So the coefficients of the monomials of degree from 0 to $n-1$ can be chosen arbitrarily in equation (17). Therefore, if $L_{1,2n-1} = -A_{0,1,2n-1} \sin^2 \theta$, then $K_{1,2n-1} = A_{0,1,2n-1} \cos \theta \sin \theta$, which implies

$$\mathcal{A}_{n-1} = \frac{1}{\pi} B((2n-3)/2, 3/2) p_3 A_{0,1,2n-1}.$$

So the coefficients of the monomials of degree from n to $2n$ can be chosen arbitrarily in equation (17). Finally, if we take

$$\begin{aligned} M_{2n-2i-1,2i+1} &= \sin^{2n-2i-1} \theta, \quad i = 0, \dots, n-1, \\ L_{2n-2j,2j} &= -A_{0,2n-2j,2j} \sin^{2n-2j+1} \theta, \quad j = 0, \dots, n, \end{aligned}$$

then

$$K_{2n-2j,2j} = A_{0,2n-2j,2j} \cos \theta \sin^{2n-2j} \theta.$$

So we have

$$\begin{aligned} (18) \quad & \frac{1}{2\pi} \int_0^{2\pi} M_{2n-1,1}(\theta) \left(\int_0^\theta K_{2n-2j,2j}(\varphi) d\varphi - L_{2n-2j,2j}(\theta) \right) d\theta \\ &= \frac{n-j+1}{(2n-2j+1)\pi} B((4n-2j+1)/2, 1/2) A_{0,2n-2j,2j}. \end{aligned}$$

That is, let S be a set of variables and if we denote $L(S)$ the linear combination of all variables in S , then from the expressions of γ_j , ς_j and σ_j , we have

$$(19) \quad \mathcal{D}_j = L(A_{0,2n-2k,2k}, k = 0, \dots, 2n-j-1) + \text{constant}_j$$

for $j = n+1, \dots, 2n-1$. More precisely, let $Y = (\mathcal{D}_{n+1}, \dots, \mathcal{D}_{2n-1})^T$ and $X = (A_{0,4,2n-4}, \dots, A_{0,2n,0})^T$, we can write system (19) in a matrix form $Y = SX + C$ and the coefficient matrix S is upper-triangular with non-zero elements in the main diagonal by equation (18). So the coefficients \mathcal{D}_j can be chosen arbitrarily for $j = n+1, \dots, 2n-1$, which means that the coefficients of the monomials with degree from $2n+1$ to $3n-1$ in equation (17) can also be chosen arbitrarily. By the same arguments in Section 3, we get the result.

REFERENCES

- [1] N. G. LLOYD, *The number of limit cycles of polynomial systems in the plane*, Bull. Inst. Math. Appl. **24** (1988), no. 11-12, 161- -365.
- [2] N. G. LLOYD, *Small amplitude limit cycles of p[olynomial differential equations*, Lecture Notes in Math. **1032**, Springer-Verlag, Berlin, 1983.
- [3] A. BUICĂ AND J. LLIBRE, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128** (2004), 7-22.
- [4] J. LLIBRE, *Averaging theory and limit cycles for quadratic systems*, Rad. Mat. **11** (2002), 1-14.
- [5] J.A. SANDERS AND F. VERHULST, *Averaging methods in nonlinear dynamical systems*, Applied Mathematical Sci., Vol. 59, Springer-Verlag, New York, 1985.
- [6] F. VERHULST, *Nonlinear differential equations and dynamical systems*, Universitext. Springer-Verlag, Berlin, 1996.

¹ DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, BARCELONA, SPAIN
E-mail address: jllibre@mat.uab.es

² CENTRE DE RECERCA MATEMÀTICA, UNIVERSITAT AUTÒNOMA D E BARCELONA, 08193 BELLATERRA, BARCELONA, SPAIN
E-mail address: hwu@crm.es