# ON THE BOUNDEDNESS ON INHOMOGENEOUS LIPSCHITZ SPACES OF FRACTIONAL INTEGRALS, SINGULAR INTEGRALS AND HYPERSINGULAR INTEGRALS ASSOCIATED TO NON-DOUBLING MEASURES ON METRIC SPACES

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Abstract. In this paper we prove T1 type necessary and sufficient conditions for the boundedness on inhomogeneous Lipschitz spaces of fractional integrals and singular integrals defined on a measure metric space whose measure satisfies a n-dimensional growth. We also show that hypersingular integrals are bounded on these spaces.

## 1. Introduction. Definitions and Statement of the Theorems

Let (X,d,$\mu$) be a measure metric space whose measure $\mu$ satisfies a n-dimensional growth condition, that is (X,d) is a metric space and $\mu$ a Borel measure that satisfies the following condition: there is $n > 0$ and a constant $A > 0$ such that $\mu(B_r) \leq Ar^n$,for all balls $B_r$ of radius $r$ and for all $r > 0$. Note that this condition allows the consideration of non-doubling measures as well as doubling measures.

Our results will apply to functions defined on the support of $\mu$, of course the support of $\mu$ has to be well defined, where supp($\mu$) is the smallest closed set $F$ such that for all Borel sets $E$, $E \subset F^c, \mu(E) = 0$. For example, if $X$ is separable, then the support of $\mu$ is well defined. Furthermore to avoid any confusion we will assume that $X =$supp($\mu$)

The inhomogeneous Lipschitz spaces of order $\beta, 0 < \beta < 1$, will be denoted $Lip_{[\beta]}$ and consists of all bounded functions $f$ (or class in $L^\infty$) such

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that $\sup_{x,y}\frac{|f(x)-f(y)|}{d^{\beta}(x,y)} = \|f\|_{(\beta)} < \infty$.The norm of $f$ in $Lip_{[\beta]}$is defined $\|f\|_{[\beta]} = \|f\|_{\infty} + \|f\|_{(\beta)}$ .

We will state and prove the theorems for the case $\mu(X) < \infty$.After that, we will indicate how these results can be extended to $\mu(X) = \infty$.

The letter $c$ will denote a constant not necessarily the same at each ocurrence.

Let $\Omega = X \times X\backslash\Delta$,where $\Delta = \{(x,y) : x = y\}$ .

A function $L_\alpha(x,y) : \Omega \to C$ will be called a standard fractional integral kernel of order $\alpha, 0 < \alpha < 1$,when there are constants $B_1$ and $B_2$ such that:

(L1)$|L_\alpha(x,y)| \leq \frac{B_1}{d^{n-\alpha}(x,y)}$.

(L2)$|L_\alpha(x_1,y) - L_\alpha(x_2,y)| \leq B_2\frac{d^{\gamma}(x_1,x_2)}{d^{n-\alpha+\gamma}(x_1,y)}$ ,for some $\gamma, 0 < \gamma \leq 1$, and $2d(x_1,x_2) \leq d(x_1,y)$.

The fractional integral of order $\alpha$ of a function $f$ in $Lip_{[\beta]}$ is defined by:

$$L_\alpha f(x) = \int L_\alpha(x,y)f(y)d\mu(y).$$

Note that in particular $L_\alpha(x,y) = \frac{1}{d^{n-\alpha}(x,y)}$ is a standard fractional kernel of order $\alpha$.

**Theorem 1**

Let $0 < \alpha < \gamma \leq 1$, $0 < \beta < 1$, .and $\alpha + \beta < n$ if $n \leq 1$ or $\alpha + \beta \leq 1$ if $1 < n$.The following statements are equivalent:

a)$L_\alpha 1 \in Lip_{[\alpha+\beta]}$.

b)$L_\alpha : Lip_{[\beta]} \to Lip_{[\alpha+\beta]}$ is bounded.

We will define now the singular integral kernels that we will consider in Theorem2. A function $K(x,y) : \Omega \to C$ will be called in this singular integral kernel when there are constants $C_1, C_2, C_3$ and a number $\gamma, 0 < \gamma \leq 1$,tales que:

(S1) $|K(x,y)| \leq \frac{C_1}{d^n(x,y)}$

(S2) $|K(x_1,y) - K(x_2,y)| \leq C_2\frac{d^{\gamma}(x_1,x_2)}{d^{n+\gamma}(x_1,y)}$,for $2d(x_1,x_2) \leq d(x_1,y)$

(S3) $\left|\int_{r_1<d(x,y)<r_2} K(x,y)d\mu(y)\right| \leq C_3$ for all $0 < r_1 < r_2 < \infty$.

Let $\eta$ be a function in $C^1[0,\infty)$ such that $\eta(s) = 0$ for $0 \leq s \leq 1/2$ and $\eta(s) = 1$ for $1 \leq s$.Let $K_\varepsilon(x,y) = \eta(\frac{d(x,y)}{\varepsilon})K(x,y)$, $\varepsilon > 0$. where K(x,y) is a standard singular kernel that satisfies (S1), (S2) and (S3). We will also denote $K_\varepsilon$ the operator $K_\varepsilon f(x) = \int K_\varepsilon(x,y)f(y)d\mu(y)$.

**Theorem 2**

Let $0 < \beta < 1$. The following two propositions are equivalent:

a) $\|K_\varepsilon 1\|_{[\beta]} \leq C_1'$, for all $\epsilon > 0$.

b) $K_\varepsilon : Lip_{[\beta]} \to Lip_{[\beta]}$ are bounded and$\|K_\varepsilon\|_{Lip_{[\beta]} \to Lip_{[\beta]}} \leq C_2'$, for all $\varepsilon > 0$.

In the next theorem we will consider principal value singular integrals. Let

(S4)$\lim_{\varepsilon\to 0} \int_{\varepsilon < d(x,y) < 1} K(x,y) d\mu(y)$ exists $\mu - a.e.$

The principal value singular integral of a function $f \in Lip_{[\beta]}$ is defined by

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x,y)} K(x,y) f(y) d\mu(y)$$

**Theorem 3**

Let $0 < \beta < 1$,and $f \in Lip_{[\beta]}$. Then $Kf(x)$ is well defined $\mu - a.e.$and the following two propositions are equivalent:

a) $K1 \in Lip_{[\beta]}$

b) $K : Lip_{[\beta]} \to Lip_{[\beta]}$ is bounded

A function $D_\alpha(x,y) : \Omega \to C$ will be called a standard hypersingular kernel of order $\alpha, 0 < \alpha < 1$,when there are constants $E$ and $E'$ such that:

(D1)$|D_\alpha(x,y)| \leq \frac{E}{d^{n+\alpha}(x,y)}$,

(D2)$|D_\alpha(x_1,y) - D_\alpha(x_2,y)| \leq E' \frac{d^\gamma(x_1,x_2)}{d^{n+\alpha+\gamma}(x_1,y)}$,for some $\gamma, 0 < \gamma \leq 1$, and $2d(x_1,x_2) \leq d(x_1,y)$.

The hypersingular integral of order $\alpha$ of a function $f \in Lip_{[\beta]}$ ,$\alpha < \beta \leq 1$is defined by:

$$D^\alpha f(x) = \int D_\alpha(x,y) \left[f(y) - f(x)\right] d\mu(y)$$

Note that in particular $D_\alpha(x,y) = \frac{1}{d^{n+\alpha}(x,y)}$ is a standard hypersingular kernel of order $\alpha$.In addition when X=R$^n$ and $\mu$ is the Lebesgue measure we have$\int \frac{1}{d^{n+\alpha}(x,y)} \left[f(y) - f(x)\right] dy = c_\alpha(\Delta^{\frac{\alpha}{2}} f)(x)$ for $f$ sufficiently smooth and $0 < \alpha < 2$.

**Theorem 4**

Let $0 < \alpha < \beta$ and $\beta \leq \gamma$, where $\gamma$ is the exponent in condition (D2).Then $D^\alpha$:$Lip_{[\beta]} \to Lip_{[\beta-\alpha]}$ is bounded.

Note that $D^\alpha 1 = 0$ by definition.

## 2. Proofs

We would like to point out that all four proofs are based in the same classical method. In a way, we are showing that this method can be extended to this modern ("T1" type theorems) formulation and to this more general context .

For carrying out the proofs we need the following known lemma about measures with the given growth condition.

**Lemma**

Let $(X, d, \mu)$ be a measure metric space such that $\mu$ satisfies the n-dimensional growth condition, $\gamma > 0, r > 0$. Then

1.$\int_{d(x,y)<r} \frac{1}{d^{n-\gamma}(x,y)} d\mu(y) \le c_1 r^{\gamma}$,

2.$\int_{r\le d(x,y)} \frac{1}{d^{n+\gamma}(x,y)} d\mu(y) \le c_2 r^{-\gamma}$

3.$\int_{r/2\le d(x,y)<r} \frac{1}{d^{n}(x,y)} d\mu(y) \le c_3$

**Proof of the Lemma**

The three parts are a consequence of the growth condition.

To prove part1, we rewrite the integral as a series and mayorize each term using the growth condition and we add the resulting series.

In detail we have:

$$\int_{d(x,x_o)<r} \frac{1}{d^{n-\alpha}(x,x_o)} d\mu(x) = \sum_{k=0}^{\infty} \int_{2^{-k-1}r\le d(x,x_o)<2^{-k}r} \frac{1}{d^{n-\alpha}(x,x_o)} d\mu(x) \le$$
$$\sum_{k=0}^{\infty} \frac{\mu(B_{2^{-k-1}r}(x_o))}{(2^{-k-1}r)^{n-\alpha}} \le A \sum_{k=0}^{\infty} \frac{(2^{-k}r)^n}{(2^{-k-1}r)^{n-\alpha}} = Ar^{\alpha}(\frac{2^n}{2^{\alpha}-1}).$$

To prove part 2 we performe a similar estimate:

$$\int_{d(x,x_o)\ge r} \frac{1}{d^{n+\alpha}(x,x_o)} d\mu(x) = \sum_{k=0}^{\infty} \int_{2^{k}r\le d(x,x_o)<2^{k+1}r} \frac{1}{d^{n+\alpha}(x,x_o)} d\mu(x) \le$$
$$\sum_{k=0}^{\infty} \frac{\mu(B_{2^{k+1}r}(x_o))}{(2^{k}r)^{n+\alpha}} \le A \sum_{k=0}^{\infty} \frac{(2^{k+1}r)^n}{(2^{k}r)^{n+\alpha}} = Ar^{-\alpha}(\frac{2^n 2^{\alpha}}{2^{\alpha}-1})$$

Finally for part 3 we have:

$\int_{r/2\le d(x,y)<r} \frac{1}{d^{n}(x,y)} d\mu(y) \le \frac{\mu(B_r(x_o))}{(r/2)^n} \le A2^n$.

**Proof of Theorem1**

Observe first that $1 \in Lip_{[\beta]}$ and therefore condition b) implies condition a).

We will prove now that condition a) implies condition b).

We can consider the case $L_\alpha(x,y)=\frac{1}{d^{n-\alpha}(x,y)}$,because the general case is proven in the same way, and we will denote $L_\alpha = I_\alpha$.

Condition (L1) is clearly valid. To show that condition (L2) is verified we use the Mean Value Theorem, consider

$2d(x_1,x_2) \le d(x_1,y)$, and $0<\theta<1$ we have:

$$\left|\frac{1}{d^{n-\alpha}(x_1,y)} - \frac{1}{d^{n-\alpha}(x_2,y)}\right| \le$$

$$\begin{aligned} &\le \sup_\theta \left|(-n+\alpha)(\theta d(x_1,y)+(1-\theta)(d(x_2,y))^{-n+\alpha-1}\right| |d(x_1,y)-d(x_2,y)| \\ &\le B_2\frac{d(x_1,x_2)}{d^{n-\alpha+1}(x_1,y)} \end{aligned}$$

Now we will estimate $\|I_\alpha f\|_\infty$ .Let $x \in X$.We will use the Lemma to obtain

$$\begin{aligned} |I_\alpha f(x)| &\le \int_{d(x,y)<1} \frac{|f(y)|}{d^{n-\alpha}(x,y)}d\mu(y) + \int_{1\le d(x,y)} \frac{|f(y)|}{d^{n-\alpha}(x,y)}d\mu(y) \\ &\le \|f\|_\infty (c_1+\mu(X)), \end{aligned}$$

and therefore $\|I_\alpha f\|_\infty \le \|f\|_\infty (c_1+\mu(X))$.

We will estimate next $\|I_\alpha f\|_{(\beta)}$ . We write

$$\begin{aligned} &I_\alpha f(x_1) - I_\alpha f(x_2) = \\ &= \int_X \frac{f(y)}{d^{n-\alpha}(x_1,y)}d\mu(y) - \int_X \frac{f(y)}{d^{n-\alpha}(x_2,y)}d\mu(y) \\ &= \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_1,y)}d\mu(y) + f(x_1)\int_X \frac{1}{d^{n-\alpha}(x_1,y)}d\mu(y) \\ &\quad - \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_2,y)}d\mu(y) - f(x_1)\int_X \frac{1}{d^{n-\alpha}(x_2,y)}d\mu(y) \\ &= \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_1,y)}d\mu(y) - \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_2,y)}d\mu(y) \\ &\quad + f(x_1)\left[I_\alpha 1(x_1) - I_\alpha(x_2)\right]. \end{aligned}$$

The last term can be mayorized using the hypothesis, and we have $|f(x_1)[I_\alpha 1(x_1) - I_\alpha(x_2)]| \leq c\,\|f\|_\infty\, d^{\alpha+\beta}(x_1,x_2)$.

Let now $r = d(x_1,x_2)$ and $B_{2r}(x_1)$ the ball of radius $2r$ and center $x_1$.We write

$$\left|\int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_1,y)}d\mu(y) - \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_2,y)}d\mu(y)\right| \leq$$

$$\int_{B_{2r}(x_1)} \frac{|f(y)-f(x_1)|}{d^{n-\alpha}(x_1,y)}d\mu(y) + \int_{B_{2r}(x_1)} \frac{|f(y)-f(x_1)|}{d^{n-\alpha}(x_2,y)}d\mu(y) +$$

$$\int_{B_{2r}^c(x_1)} |f(y)-f(x_1)| \left|\frac{1}{d^{n-\alpha}(x_1,y)} - \frac{1}{d^{n-\alpha}(x_2,y)}\right| d\mu(y) = J_1 + J_2 + J_3$$

For the first term using the lemma we have

$$\begin{aligned} J_1 &\leq \|f\|_{(\beta)} \int_{B_{2r}(x_1)} \frac{d^\beta(x_1,y)}{d^{n-\alpha}(x_1,y)} d\mu(y) \leq c\,\|f\|_{(\beta)}\, r^{\alpha+\beta} \\ &= c\,\|f\|_{(\beta)}\, d^{\alpha+\beta}(x_1,x_2). \end{aligned}$$

For the second term we write

$$J_2 \leq \|f\|_{(\beta)} \int_{B_{3r}(x_2)} \frac{2r}{d^{n-\alpha}(x_2,y)} d\mu(y) \leq c\,\|f\|_{(\beta)}\, d^{\alpha+\beta}(x_1,x_2),$$

For the third term we use (L2) and the lemma to get

$$J_3 \leq \|f\|_{(\beta)} \int_{B_{2r}^c(x_1)} \frac{B_2}{d^{n-\alpha-\beta}(x_1,y)} d\mu(y) \leq c\,\|f\|_{(\beta)}\, d^{\alpha+\beta}(x_1,x_2)$$

Collecting the previous estimates, we have

$$\|I_\alpha f\|_{[\beta]} \leq C\,\|f\|_{[\beta]}$$

This concludes the proof of Theorem 1.

**Proof of Theorem 2**

Observe first that $1 \in Lip_{[\beta]}$ and therefore condition b) implies condition a).

The proof of Theorem 3 is analogous of that of Theorem 2 and we will only sketch it. Before doing the proof of the theorem and for the sake of completness, we will show that $K_\varepsilon$ satisfies conditions (S1), (S2), and (S3) with constants independent of $\varepsilon$.

Condition (S1) is true because $\eta$ is bounded.

To show condition (S2), assume that $2d(x_1, x_2) \leq d(x_1, y)$.and consider the following two cases:

Case 1: $1 < \frac{d(x_1,y)}{\varepsilon}$ and $1 < \frac{d(x_2,y)}{\varepsilon}$.In this case $K_\varepsilon(x, y) = K(x, y)$, and therefore (S2) is true with the same constant.

Case 2: $1 \geq \frac{d(x_1,y)}{\varepsilon}$ or $1 \geq \frac{d(x_2,y)}{\varepsilon}$.Assume $1 > \frac{d(x_1,y)}{\varepsilon}$.We write

$$\begin{aligned} |K_\varepsilon(x_1, y) - K_\varepsilon(x_2, y)| &\leq \left|\eta(\frac{d(x_1, y)}{\varepsilon}) - \eta(\frac{d(x_2, y)}{\varepsilon})\right| |K(x_1, y)| + \\ &\left|\eta(\frac{d(x_2, y)}{\varepsilon})\right| |K(x_1, y) - K(x_2, y)| \end{aligned}$$

The first term above is less than or equal to

$$\begin{aligned} &\|\eta'\|_\infty \frac{|d(x_1, y) - d(x_2, y)|}{\varepsilon} |K(x_1, y)| \leq \\ \leq\ & \|\eta'\|_\infty \frac{d(x_1, x_2)}{\varepsilon} |K(x_1, y)| \leq c(\frac{d(x_1, x_2)}{\varepsilon})^\gamma |K(x_1, y)| \\ \leq\ & c\frac{d^\gamma(x_1, x_2)}{d^{n+\gamma}(x_1, x_2)} \end{aligned}$$

On the other hand the second term is less than or equal to $c\,|K(x_1, y) - K(x_2, y)| \leq c\frac{d^\gamma(x_1,x_2)}{d^{n+\gamma}(x_1,x_2)}$.

To see condition (S3), observe that

$$\int_{\frac{1}{2}\varepsilon<d(x,y)<\varepsilon} |K_\varepsilon(x, y)|\, d\mu(y) \leq c$$

with $c$ independent of $\varepsilon$. Since $K_\varepsilon(x, y) = K(x, y)$ for $\varepsilon \leq d(x, y)$,we have $\int_{\varepsilon<d(x,y)<r_2} K_\varepsilon(x, y)d\mu(y) = \int_{\varepsilon<d(x,y)<r_2} K(x, y)d\mu(y)$ and therefore for $\varepsilon < r_2$ we also have

$$\left|\int_{\varepsilon<d(x,y)<r_2} K_\varepsilon(x, y)d\mu(y)\right| \leq C_3$$

We will estimate $\|K_\varepsilon f\|_\infty$ .Observe first that

$$|K_\varepsilon f(x)| \leq \left|\int_{d(x,y)\leq 1} K_\epsilon(x, y)f(y)d\mu(y)\right| + \left|\int_{d(x,y)>1} K_\epsilon(x, y)f(y)d\mu(y)\right| \leq$$

$$\left|\int_{d(x,y)\leq 1} |K_\epsilon(x,y)|\,|f(y)-f(x)|\,d\mu(y)\right| +$$

$$+\,|f(x)|\left|\int_{\frac{1}{2}\varepsilon\leq d(x,y)\leq 1} K_\epsilon(x,y)d\mu(y)\right| + c\,\|f\|_\infty\,\mu(X)$$

Now, by conditions (S1), (S3) and using the Lemma we can mayorize the first two terms by $\|f\|_{[\beta]}$, and therefore $\|K_\varepsilon f\|_\infty \leq c\,\|f\|_{[\beta]}$.

To estimate the $K_\varepsilon f(x_1) - K_\varepsilon f(x_2)$, we consider the following decomposition:

$$K_\varepsilon f(x_1) - K_\varepsilon f(x_2) = \int K_\varepsilon(x_1,y)f(y)d\mu(y) - \int K_\varepsilon(x_2,y)f(y)d\mu(y)$$

$$= \int K_\varepsilon(x_1,y)\left[f(y)-f(x_1)\right]d\mu(y) + f(x_1)\int K_\varepsilon(x_1,y)d\mu(y) -$$

$$\int K_\varepsilon(x_2,y)\left[f(y)-f(x_1)\right]d\mu(y) - f(x_1)\int K_\varepsilon(x_2,y)d\mu(y) =$$

$$\int K_\varepsilon(x_1,y)\left[f(y)-f(x_1)\right]d\mu(y) + \int K_\varepsilon(x_2,y)\left[f(y)-f(x_1)\right]d\mu(y) +$$

$$f(x_1)\left[K_\varepsilon 1(x_1) - K_\varepsilon 1(x_2)\right]$$

Observe now that the last term can be estimated using the hypothesis and we have

$$\left|f(x_1)\left[K_\varepsilon 1(x_1) - K_\varepsilon 1(x_2)\right]\right| \leq c\,\|f\|_\infty\, d^\beta(x_1,x_2).$$

To estimate the first two terms, let $r = d(x_1,x_2)$, we rewrite them as follows:

$$\int K_\varepsilon(x_1,y)\left[f(y)-f(x_1)\right]d\mu(y) + \int K_\varepsilon(x_2,y)\left[f(y)-f(x_1)\right]d\mu(y) =$$

$$\int_{d(x_1,y)<3r} K_\varepsilon(x_1,y)\left[f(y)-f(x_1)\right]d\mu(y) + \int_{d(x_1,y)<3r} K_\varepsilon(x_2,y)\left[f(y)-f(x_1)\right]d\mu(y) +$$

$$\int_{3r<d(x_1,y)} \left[f(y)-f(x_1)\right]\left[K_\varepsilon(x_1,y) - K_\varepsilon(x_2,y)\right]d\mu(y) = H_1 + H_2 + H_3$$

The absolute value of $H_3$ can be estimated as follows,

$$|H_3| \leq \|f\|_{(\beta)} d^{\gamma}(x_1,x_2) \int_{3r<d(x_1,y)} \frac{d^{\beta}(x_1,y)}{d^{n+\gamma}(x_1,y)} d\mu(y) \leq c \|f\|_{(\beta)} d^{\beta}(x_1,x_2)$$

For $|H_1|$we have

$$|H_1| \leq \|f\|_{(\beta)} \int_{d(x_1,y)<3r} \frac{C_1}{d^{n-\beta}(x_1,y)} d\mu(y) \leq c \|f\|_{(\beta)} d^{\beta}(x_1,x_2)$$

Finally to estimate $H_2$ we write

$$\int_{d(x_1,y)<3r} K_{\varepsilon}(x_2,y) [f(y)-f(x_1)] \, d\mu(y) =$$

$$\int_{d(x_1,y)<3r} K_{\varepsilon}(x_2,y) [f(y)-f(x_2)] \, d\mu(y)$$

$$+ [f(x_2)-f(x_1)] \int_{\{y:\varepsilon/2<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K_{\varepsilon}(x_2,y) d\mu(y) = J_1 + J_2$$

For the first term we have

$$|J_1| \leq \int_{d(x_2,y)<4r} \frac{c \|f\|_{(\beta)}}{d^{n-\beta}(x_2,y)} d\mu(y) \leq c \|f\|_{(\beta)} d^{\beta}(x_1,x_2)$$

To estimate $J_2$ consider first

$$\int_{d(x_1,y)<3r} K_{\varepsilon}(x_2,y) d\mu(y) = \int_{d(x_2,y)<2r} K_{\varepsilon}(x_2,y) d\mu(y) +$$

$$\int_{\{y:d(x_1,y)<3r\}\setminus\{y:d(x_2,y)<2r\}} K_{\varepsilon}(x_2,y) d\mu(y)$$

Observe now that

$$\left| \int_{d(x_2,y)<2r} K_{\varepsilon}(x_2,y) d\mu(y) \right| \leq C_3$$

and using part 3 of the lemma we get

$$\left| \int_{\{y:d(x_1,y)<3r\}\setminus\{y:d(x_2,y)<2r\}} K_{\varepsilon}(x_2,y) d\mu(y) \right| \leq \int_{\{y:2r<d(x_2,y)<4r\}} |K_{\varepsilon}(x_2,y)| \, d\mu(y) \leq c$$

therefore

$$|J_2| \leq c \|f\|_{(\beta)} d^{\beta}(x_1,x_2)$$

collecting the estimates we have:

$$|K_\varepsilon f(x_1) - K_\varepsilon f(x_2)| \le c \|f\|_{[\beta]} d^\beta(x_1, x_2)$$

and finally

$$\|K_\varepsilon f\|_{[\beta]} \le c \|f\|_{[\beta]},$$

with c independent of $\varepsilon$.

**Proof of Theorem 3**

Observe first that $1 \in Lip_{[\beta]}$ and therefore condition b) implies condition a).

Let $f \in Lip_{[\beta]}$,we will show that

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x,y)} K(x,y) f(y) d\mu(y)$$

exists $\mu - a.e.$. Assume $\varepsilon < 1$,we can write

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x,y) < 1} K(x,y) [f(y) - f(x)] d\mu(y) +$$

$$f(x) \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x,y) < 1} K(x,y) d\mu(y) + \int_{1 \le d(x,y)} K(x,y) f(y) d\mu(y).$$

Note that the first integral converges absolutely, the limit of the second term exists by hypothesis and finally last integral converges absolutely because the integrand is bounded. Furthermore, we have $\|Kf\|_\infty \le c \|f\|_{[\beta]}$.

We will estimate now $Kf(x_1) - Kf(x_2)$ for $x_1, x_2$ two points for which $Kf(x)$ exists. We write

$$Kf(x_1) - Kf(x_2) = \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_1,y)} K(x_1,y) f(y) d\mu(y) - \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_2,y)} K(x_2,y) f(y) d\mu(y)$$

$$= \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_1,y)} K(x_1,y) [f(y) - f(x_1)] d\mu(y) + f(x_1) \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_1,y)} K(x_1,y) d\mu(y) -$$

$$\lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_2,y)} K(x_2,y) [f(y) - f(x_1)] d\mu(y) - f(x_1) \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_2,y)} K(x_2,y) d\mu(y) =$$

$$\lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_1,y)} K(x_1,y) [f(y) - f(x_1)] d\mu(y) + \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_2,y)} K(x_2,y) [f(y) - f(x_1)] d\mu(y) +$$

$$f(x_1) [K1(x_1) - K1(x_2)]$$

Observe now that the last term can be estimated using the hypothesis and we have

$$|f(x_1)\,[K1(x_1) - K1(x_2)]| \leq c\,\|f\|_\infty\, d^\beta(x_1, x_2).$$

To estimate the first two terms, let $r = d(x_1, x_2)$, and $\varepsilon < r$, we rewrite them as follows:

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_1,y)} K(x_1,y)\,[f(y)-f(x_1)]\,d\mu(y) + \lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_2,y)} K(x_2,y)\,[f(y)-f(x_1)]\,d\mu(y) =$$

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_1,y)<3r} K(x_1,y)\,[f(y)-f(x_1)]\,d\mu(y) + \lim_{\varepsilon\to 0}\int_{\{y:\varepsilon<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K(x_2,y)\,[f(y)-f(x_1)]\,d\mu(y) +$$

$$\lim_{\varepsilon\to 0}\int_{3r<d(x_1,y)} [f(y)-f(x_1)]\,[K(x_1,y)-K(x_2,y)]\,d\mu(y) = H_1 + H_2 + H_3$$

The absolute value of $H_3$ can be estimated as follows,

$$|H_3| \leq \|f\|_{(\beta)}\, d^\gamma(x_1,x_2)\int_{3r<d(x_1,y)} \frac{d^\beta(x_1,y)}{d^{n+\gamma}(x_1,y)}\,d\mu(y) \leq c\,\|f\|_{(\beta)}\, d^\beta(x_1,x_2)$$

For $|H_1|$we have

$$|H_1| \leq \|f\|_{(\beta)}\int_{d(x_1,y)<3r} \frac{C_1}{d^{n-\beta}(x_1,y)}\,d\mu(y) \leq c\,\|f\|_{(\beta)}\, d^\beta(x_1,x_2)$$

Finally to estimate $H_2$ we write

$$\lim_{\varepsilon\to 0}\int_{\{y:\varepsilon<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K(x_2,y)\,[f(y)-f(x_1)]\,d\mu(y) =$$

$$\lim_{\varepsilon\to 0}\int_{\{y:\varepsilon<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K(x_2,y)\,[f(y)-f(x_2)]\,d\mu(y)$$

$$+\,[f(x_2)-f(x_1)]\lim_{\varepsilon\to 0}\int_{\{y:\varepsilon<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K(x_2,y)d\mu(y) = J_1 + J_2$$

For the first term we have

$$|J_1| \leq \int_{d(x_2,y)<4r} \frac{c\,\|f\|_{(\beta)}}{d^{n-\beta}(x_2,y)}\,d\mu(y) \leq c\,\|f\|_{(\beta)}\, d^\beta(x_1,x_2)$$

To estimate the second $J_2$ consider first

$$\lim_{\varepsilon\to 0}\int_{\{y:\varepsilon<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K(x_2,y)d\mu(y)=\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_2,y)<2r} K(x_2,y)d\mu(y)+$$

$$\int_{\{y:d(x_1,y)<3r\}\setminus\{y:d(x_2,y)<2r\}} K(x_2,y)d\mu(y)$$

Observe now that

$$\left|\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_2,y)<2r} K(x_2,y)d\mu(y)\right|\le C_3$$

and using part 3 of the lemma we get

$$\left|\int_{\{y:d(x_1,y)<3r\}\setminus\{y:d(x_2,y)<2r\}} K(x_2,y)d\mu(y)\right|\le\int_{\{y:2r<d(x_2,y)<4r\}}|K(x_2,y)|\,d\mu(y)\le c$$

therefore

$$|J_2|\le c\,\|f\|_{(\beta)}\,d^{\beta}(x_1,x_2)$$

collecting the estimates we have:

$$|Kf(x_1)-Kf(x_2)|\le c\,\|f\|_{[\beta]}\,d^{\beta}(x_1,x_2)$$

and finally

$$\|Kf\|_{[\beta]}\le c\,\|f\|_{[\beta]}$$

This concludes the proof of Theorem 2. The rest of the proof is similar to that of Theorem 2, except for the use of the new hypothesis and we leave the details to the reader.

**Proof of Theorem 4**

We will prove the theorem for $D_\alpha(x,y)=\frac{1}{d^{n+\alpha}(x,y)}$,the general case is identical.

We will estimate first $\|D^\alpha f\|_\infty$ for $f\in Lip_{[\beta]}$.We have

$$\begin{aligned}|D^\alpha f(x)| &\le \int_{d(x,y)\le 1}\frac{|f(y)-f(x)|}{d^{n+\alpha}(x,y)}d\mu(y)+c\,\|f\|_\infty\,\mu(X)\\ &\le \|f\|_{(\beta)}\int_{d(x,y)\le 1}\frac{1}{d^{n+\alpha-\beta}(x,y)}d\mu(y)+c\,\|f\|_\infty\,\mu(X)\end{aligned}$$

Since $0 < \alpha < \beta \leq 1$, we use the lemma to estimate the integral and we obtain that $D^\alpha f(x)$ is well defined everywhere and

$$\|D^\alpha f\|_\infty \leq c \|f\|_{[\beta]} .$$

To estimate the Lipschitz norm of $D^\alpha f$, we consider $r = d(x, y)$ and write

$$\begin{aligned}
&D^\alpha f(x_1) - D^\alpha f(x_2) = \\
&= \int_{d(x_1,y)\leq 2r} \frac{f(y) - f(x_1)}{d^{n+\alpha}(x_1, y)} d\mu(y) - \int_{d(x_1,y)\leq 2r} \frac{f(y) - f(x_2)}{d^{n+\alpha}(x_2, y)} d\mu(y) + \\
&\quad + \int_{d(x_1,y)>2r} [f(y) - f(x_1)] \left[\frac{1}{d^{n+\alpha}(x_1, y)} - \frac{1}{d^{n+\alpha}(x_2, y)}\right] d\mu(y) - \\
&\quad \int_{d(x_1,y)>2r} \frac{f(x_1) - f(x_2)}{d^{n+\alpha}(x_2, y)} d\mu(y)
\end{aligned}$$

Using part1 of the Lemma and the fact that $f$ is $Lip(\beta)$ we can mayorize the absolute value of each of the first two terms by $c \|f\|_{(\beta)} d^{\beta-\alpha}(x, y)$. Using part 2 of the Lemma we can also mayorize the absolute value of the fourth term by $c \|f\|_{(\beta)} d^{\beta-\alpha}(x, y)$.

To estimate the third term observe first that for $2d(x_1, x_2) \leq d(x_1, y)$,

$$\begin{aligned}
&\left|\frac{1}{d^{n+\alpha}(x_1, y)} - \frac{1}{d^{n+\alpha}(x_2, y)}\right| \leq \\
&\leq \sup_\theta \left|(-n-\alpha)(\theta d(x_1, y) + (1-\theta)(d(x_2, y))^{-n-\alpha-1}\right| |d(x_1, y) - d(x_2, y)| \\
&\leq c \frac{d(x_1, x_2)}{d^{n+\alpha+1}(x_1, y)}.
\end{aligned}$$

Therefore using this estimate, the fact that $f \epsilon Lip(\beta)$ and the Lemma we obtain that the third term is less than or equal to $c \|f\|_{(\beta)} d^{\beta-\alpha}(x, y)$and consequently $\|D^\alpha f\|_{(\beta)} \leq c \|f\|_{[\beta]}$.

Finally combinig the estimates above we get $\|D^\alpha f\|_{[\beta]} \leq c \|f\|_{[\beta]}$ .

To conclude we would like to mention that these results can be extended to spaces with $\mu(X) = \infty$. Theorem 4 extends without changes. To extend the other theorems fractional integrals and singular integrals have to be redefined so they converge for $d(x, y) > 1$.The operator's norm in each result will depend on the normalization.

We will denote with $'$ the normalizations. Let $x_o \in X$ be a fixed point and define:

$$L'_\alpha f(x) = \int \left[L'_\alpha(x,y) - L'_\alpha(x_o,y)\right] f(y)d\mu(y)$$

$$K'f(x) = \lim_{\varepsilon\to 0} \int \left[K(x,y) - K(x_o,y)\right] f(y)d\mu(y)$$

where in addition $x_o$ is such that $\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_o,y)<1} K(x_o,y)d\mu(y)$ exists $\mu - a.e.$

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