WEIGHTED HARDY SPACES FOR THE UNIT DISC: APPROXIMATION PROPERTIES

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Abstract. In this paper we study basic properties of the weighted Hardy space for the unit disc with the weight function satisfying Muckenhoupt's (A^q) condition, and study related approximation problems (expansion, moment and interpolation) with respect to two incomplete systems of holomorphic functions in this space.

1. Introduction

The weighted Hardy space with the weight function w satisfying Muckenhoupt's (A^q) condition was originally defined and studied in [7], but therein the focus of study was on the case for the upper half plane. The corresponding weighted Hardy space $H^p_w(D)$ for the unit disc D was also defined in [7] but no details were given. As pointed out in [7], it is in fact in [13] that the idea to develop a theory of weighted H^p spaces appears for the first time. In the present paper, we shall study basic properties of the space $H^p_w(D)$ and related approximation problems, which include biorthogonal expansions and moment and interpolation problems, for two systems of holomorphic functions in the space. In the classical Hardy space $H^p(D)$ (i.e. when $w \equiv 1$) for the unit disc, these kinds of approximation problems were studied, for example, in [11], [1], [2] and [14]. We shall generalize some related results to the case with general (A^q) weights.

Muckenhoupt's (A^q) condition was introduced in [12]. Assume that w is a non-negative (with $0 < w < \infty$ a.e.) and measurable function defined in $(-\infty, \infty)$. For $1 < q < \infty$, we say w satisfies Muckenhoupt's (A^q) condition or $w \in (A^q)$ (we also call w an (A^q) weight), if there is a constant C such that for every interval I,

$$\left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-\frac{1}{q-1}}dx\right)^{q-1} \leq C,$$

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where $|I|$ denotes the length of I. We say $w \in (A^1)$ if

$$\frac{1}{|I|}\int_I w(x)dx \leq C\ \|w\|_I,$$

for every interval I, where $\|w\|_I$ denotes the essential infimum of w over I.

In this paper, as in [12, Theorem 10] and [9, Theorem 1], we additionally assume that the above weight w is 2π periodic (and we shall write $w(\theta)$ rather than $w(x)$). Thus, in the above definition of (A^q) weight, "for every interval I" can be replaced by "for every interval I with $|I| \leq 2\pi$" (the value of the constant C may change). For completeness, we provide a proof of this easy fact. First assume that $1 < q < \infty$ and that there is a constant c such that for every interval I' with $|I'| \leq 2\pi$,

$$\left(\frac{1}{|I'|}\int_{I'} w(\theta)d\theta\right)\left(\frac{1}{|I'|}\int_{I'} w(\theta)^{-\frac{1}{q-1}}d\theta\right)^{q-1} \leq c. \tag{1.1}$$

We claim that there is a constant C such that for every interval I, the above inequality holds with I' and c being replaced by I and C, respectively. We need only consider the case when $|I| > 2\pi$ (if $|I| \leq 2\pi$, the conclusion is clearly true with $C = c$ by the assumption). Without loss of generality, assume that $|I| = 2k\pi + |I''|$ with $k \geq 1$ and $0 \leq |I''| < 2\pi$ (if $k = 1$ then $|I''| \neq 0$), where I'' is a sub-interval of I. Since, by (1.1), both w and $w^{-\frac{1}{q-1}}$ are integrable on $[-\pi, \pi]$, let $\int_{-\pi}^{\pi} wd\theta = A$ and $\int_{-\pi}^{\pi} w^{-\frac{1}{q-1}}d\theta = B$. Then, noting that w is 2π-periodic and nonnegative, we have

$$\begin{aligned}
&\left(\frac{1}{|I|}\int_I w(\theta)d\theta\right)\left(\frac{1}{|I|}\int_I [w(\theta)]^{-\frac{1}{q-1}}\right)^{q-1}\\
= &\left(\frac{kA}{2k\pi + |I''|} + \frac{1}{2k\pi + |I''|}\int_{I''} wd\theta\right)\\
&\cdot\left(\frac{kB}{2k\pi + |I''|} + \frac{1}{2k\pi + |I''|}\int_{I''} w^{-\frac{1}{q-1}}d\theta\right)^{q-1}\\
\leq &\left(\frac{A}{2\pi} + \frac{A}{2\pi}\right)\left(\frac{B}{2\pi} + \frac{B}{2\pi}\right)^{q-1} =: C,
\end{aligned}$$

which is independent of I.

For $q = 1$, the argument is essentially the same, noting that, for $|I| > 2\pi$, we have $\|w\|_I = \|w\|_{[-\pi,+\pi]}$.

We collect below some of the basic properties of (A^q) weights. Note that part of (i) has already been used in the proof above. It is known (see [9, p. 229, p. 231, and Lemma 5]) that (i) if $w(\theta) \in (A^q)$ with $1 < q < \infty$, then $w(\theta)$, $[w(\theta)]^{-\frac{1}{q-1}}$ and $\log w(\theta)$ are integrable on $[-\pi, \pi]$; (ii) $w \in (A^q)$

if and only if $w^{1-q'} \in (A^{q'})$ where $\frac{1}{q}+\frac{1}{q'}=1$; (iii) if $w \in (A^q)$ and $q_0 > q$, then $w \in (A^{q_0})$; (iv) if $w \in (A^q)$ with $1 < q < \infty$, then $w \in (A^{q_1})$ for some q_1 with $1 < q_1 < q$.

Given $w \in (A^q)$ for some q with $1 < q < \infty$, we denote by q_w the *critical exponent* for w, that is, the infimum of all r's such that $w \in (A^r)$. In this case, we say that w is an (A^q) weight with critical exponent q_w. We have $q_w \geq 1$, and by (iv) above, unless $q_w = 1$, w is never an (A^{q_w}) weight, but by (iii), $w \in A^r$, for all $r > q_w$.

In [9] and [12], weighted norm inequalities for the Hilbert transform, the Hardy-Littlewood maximal operator and the Poisson integral were studied. For definitions, see equations (2.14) and (2.8), respectively. The principal problem considered there is the determination of all non-negative weight functions $w(\theta)$ with period 2π such that the weighted norm inequality

$$\int_{-\pi}^{\pi} |\widetilde{f}(\theta)|^p w(\theta) d\theta \leq C \int_{-\pi}^{\pi} |f(\theta)|^p w(\theta) d\theta$$

holds, where $1 < p < \infty$, f has period 2π, C is a constant independent of f, and $\widetilde{f}$ denotes the Hilbert transform (or the Hardy-Littlewood maximal operator, or the Poisson integral) of f. The above inequality means that the operator $\widetilde{f}$ is bounded from $L^p_w(-\pi,\pi)$ into itself (the definition of the space L^p_w will be given below). The main result there is that $w(\theta)$ is such a function if and only if $w(\theta) \in (A^p)$. We shall use this important result later in this paper, namely in the proof of Lemmas 2.5 and 2.7.

For $w(\theta) \in (A^q)$, $1 \leq q < \infty$ and $0 < p < \infty$, the weighted Hardy space $H^p_w(D)$ for the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [7]) is the collection of functions $f(z)$ which are holomorphic in D and satisfy

$$\|f\|^p_{H^p_w(D)} := \sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p w(\theta) d\theta < \infty.$$

The classical Hardy space $H^p(D)$ is obtained by taking $w \equiv 1$. Similarly, we can define the space $H^p_w(D_\infty)$ where $D_\infty = \{z \in \mathbb{C} : \ |z| > 1\}$ as the collection of functions $h(z)$ which are holomorphic in D_∞ with $h(\infty) = 0$ and satisfy

$$\|h\|^p_{H^p_w(D_\infty)} := \sup_{r>1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(re^{i\theta})|^p w(\theta) d\theta < +\infty.$$

The space $L^p_w(T)$ is the collection of measurable functions $f(t)$ on $T = \{t \in \mathbb{C} : \ |t| = 1\}$ which satisfy

$$\|f\|^p_{L^p_w(T)} := \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^p w(\theta) d\theta < +\infty.$$

For $1 \le p < \infty$, $H^p_w(D), H^p_w(D_\infty)$ and $L^p_w(T)$ are Banach spaces. From now on in this paper, we assume that w is an (A^q) weight, for some $1 < q < \infty$ with critical exponent q_w, and when we study the spaces $H^p_w(D), H^p_w(D_\infty)$ and $L^p_w(T)$, in most cases, we shall assume that $q_w < p < \infty$, and hence most spaces considered in this paper are Banach spaces.

Let $\{a_k\}$ be a sequence of complex numbers with $a_k \in D$ $(k = 1, 2, \ldots)$, $a_j \ne a_k$ if $j \ne k$. Assume that $\{a_k\}$ satisfies the Blaschke condition

$$\sum_{k=1}^{\infty}(1 - |a_k|) < +\infty \tag{1.2}$$

(such a sequence is called a Blaschke sequence), and $B(z)$ is the Blaschke product of $\{a_k\}$:

$$B(z) = \prod_{k=1}^{\infty}\left[\frac{a_k - z}{1 - \overline{a}_k z} \cdot \frac{|a_k|}{a_k}\right]$$

(taking $\frac{|a_k|}{a_k} = -1$ if $a_k = 0$). Consider two systems of functions

$$e_k(z) = \frac{1}{2\pi i} \cdot \frac{1}{1 - \overline{a}_k z} \qquad (k = 1, 2, \ldots) \tag{1.3}$$

and

$$\phi_k(z) = (-i) \cdot \frac{B(z)}{(z - a_k)B'(a_k)} \qquad (k = 1, 2, \ldots). \tag{1.4}$$

We verify that all the functions in the systems (1.3) and (1.4) are in $H^p_w(D)$ (clearly, they are holomorphic in D). Indeed, for $0 \le r < 1$, we have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} |e_k(re^{i\theta})|^p w(\theta) d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{1 - \overline{a}_k re^{i\theta}}\right|^p w(\theta) d\theta \\ &\le \frac{1}{2\pi}\left(\frac{1}{1 - |\overline{a}_k|}\right)^p \int_{-\pi}^{\pi} w(\theta) d\theta < \infty, \end{aligned}$$

hence $e_k(z) \in H^p_w(D)$. To verify that $\phi_k(z) \in H^p_w(D)$, note that $|B(z)| < 1$ for $z \in D$.

It is known (see [11]) that these two systems are biorthogonal on $|z| = 1$, that is

$$\int_{|z|=1} e_k(z)\overline{\phi_j(z)}\frac{dz}{iz} = \int_{|z|=1} \overline{e_k(z)}\phi_j(z)\frac{dz}{iz} = \delta_{kj} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \ne j. \end{cases}$$

Using the residue theorem, we obtain the following integral representation for the system defined in (1.4): for $z \neq a_k$,

$$\phi_k(z) = \frac{B(z)}{2\pi} \int_{c_k} \frac{d\xi}{B(\xi)(\xi - z)}, \qquad k = 1, 2, \ldots, \tag{1.5}$$

where $c_k \subset D$ is a sufficiently small circle with center at a_k, not containing a_j $(j \neq k)$, and not containing z.

We shall show some basic properties of the space $H_w^p(D)$ in Sections 2 and 3, then study expansions and moments in this space with respect to the system (1.3) in Section 4, and finally, the expansion and interpolation with respect to the system (1.4) in Section 5.

Before ending this section, we give a useful lemma. Assume that $\{a_k\}$ is a Blaschke sequence. For any positive integer n, denote

$$B_n(z) = \prod_{k=1}^{n} \left[\frac{a_k - z}{1 - \bar{a}_k z} \cdot \frac{|a_k|}{a_k} \right]. \tag{1.6}$$

Lemma 1.1. *For any $h(t) \in L_w^p(T)$ with $q_w < p < \infty$, as $n \to \infty$, we have*

$$\|h(t)[B_n(t) - B(t)]\|_{L_w^p(T)} \to 0, \tag{1.7}$$

where $B(z)$ is the Blaschke product of $\{a_k\}$.

Proof. Since $h(e^{i\theta}) \in L_w^p(-\pi, \pi)$, we have $h(e^{i\theta})(w(\theta))^{1/p} \in L^p(-\pi, \pi)$. Thus (see, for example, [11, p. 36]), as $n \to \infty$,

$$\|h(e^{i\theta})(w(\theta))^{1/p}[B_n(e^{i\theta}) - B(e^{i\theta})]\|_{L^p(-\pi,\pi)} \to 0$$

which is actually (1.7). □

2. Some properties of $H_w^p(D)$, Part I

The following two important lemmas are indications of the close relation that exists between the weighted and classical case.

Lemma 2.1. *Assume that $q_w < p < \infty$. Then $H_w^p(D) \subset H^{p_0}(D)$ and $L_w^p(T) \subset L^{p_0}(T)$ for some p_0 with $1 < p_0 < p$.*

Proof. Since $q_w < p < \infty$, by the definition of q_w, we have $w \in (A^{p_1})$ for some p_1 with $1 < p_1 < p$. Set $p_0 = p/p_1$. Note that $1 < p_0 < p$ and $p_1 = p/p_0$. Let $P = p/p_0$ and $1/P + 1/P' = 1$. Then $P' = p/(p - p_0)$. So,

$$w^{1-P'} = w^{-p_0/(p-p_0)} \in (A^{P'}) = (A^{p/(p-p_0)}).$$

Hence, by property (i) of the (A^q) weights mentioned above,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (w(\theta))^{-p_0/(p-p_0)} d\theta \equiv C < \infty,$$

where C is a constant. Now for any $f \in H^p_w(D)$ and any $r \in [0,1)$, we use Hölder's inequality with indices p/p_0 and $p/(p-p_0)$ to get

(2.1)

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^{p_0}d\theta\right)^{1/p_0} \leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^{p}w(\theta)d\theta\right)^{1/p} \cdot \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(w(\theta))^{-p_0/(p-p_0)}d\theta\right)^{(p-p_0)/pp_0}.$$

Taking the supremum over all r yields

$$\|f\|_{H^{p_0}(D)} \leq C'\|f\|_{H^p_w(D)} < \infty,$$

where C' is a constant, so $f \in H^{p_0}(D)$ as required. Clearly, (2.1) also holds for $r = 1$, hence $L^p_w(T) \subset L^{p_0}(T)$. The proof is complete. $\square$

Remark 2.2. If $f(z) \in H^p_w(D)$ with $q_w < p < \infty$, by Lemma 2.1, $f(z) \in H^{p_0}(D)$ for some p_0 with $1 < p_0 < p$, hence $f(z) \in H^1(D)$. By a well-known result (see [5, Theorem 2.2]), $f(z)$ has non-tangential limits $f(t)$ a.e. on T and $f(t) \in L^{p_0}(T)$. We call $f(t)$ the *boundary function* of $f(z)$.

For $w \in (A^q)$ with $1 < q < \infty$, recalling that $\log w(\theta)$ is integrable on $[-\pi,\pi]$, consider the function

$$g(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log w(t)dt\right), \qquad z \in D. \tag{2.2}$$

Clearly, we have, for $z = re^{i\theta} \in D$,

$$\begin{aligned}|g(z)| = |g(re^{i\theta})| &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\Re\left[\frac{e^{it}+z}{e^{it}-z}\right]\log w(t)dt\right)\\ &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)\log w(t)dt\right),\end{aligned} \tag{2.3}$$

where $P_r(\theta - t)$ is the Poisson kernel:

$$P_r(\theta-t) = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}.$$

It is known (see, for example, [8, p. 58]) that $g(z) \in H^1(D)$. Moreover, by (2.3), it follows that the boundary function $g(e^{i\theta})$ of $g(z)$ satisfies

$$|g(e^{i\theta})| = w(\theta) \qquad \text{a.e. in } (-\pi,\pi). \tag{2.4}$$

Lemma 2.3. *Assume that $w \in (A^q)$ with $1 < q < \infty$, and that $0 < p < \infty$. If $f(z) \in H^p_w(D)$, then $f(z)W_p(z) \in H^p(D)$, where*

$$W_p(z) = \exp\left(\frac{1}{2p\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log w(t)dt\right), \qquad z \in D.$$

Proof. Suppose $f \in H^p_w(D)$. Since f is holomorphic in $|z| < 1$ where $z = re^{i\theta}$, for $0 < \rho < 1$, writting $f_\rho(z) = f(\rho z)$, we have

$$\log|f_\rho(z)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\log|f_\rho(e^{it})|dt,$$

hence

$$|f_\rho(z)|^p \le \exp\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\log|f_\rho(e^{it})|^p dt\Big).$$

Noting that $|W_p(z)|^p = |g(z)|$ and combining the inequality above with (2.3) gives

$$|f_\rho(z)W_p(z)|^p \le \exp\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\log\big(|f_\rho(e^{it})|^p w(t)\big)dt\Big),$$

which is by Jensen's inequality

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)|f_\rho(e^{it})|^p w(t)dt.$$

Hence,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\rho re^{i\theta})W_p(re^{i\theta})|^p d\theta &\le \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_r(\theta - t)|f_\rho(e^{it})|^p w(t)dtd\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f_\rho(e^{it})|^p w(t)dt.\end{aligned}$$

Hence by Fatou's lemma,

$$\int_{-\pi}^{\pi}|f(re^{i\theta})W_p(re^{i\theta})|^p d\theta \le \liminf_{\rho\to 1}\int_{-\pi}^{\pi}|f_\rho(e^{it})|^p w(t)dt$$

$$\le \sup_{0<\rho<1}\int_{-\pi}^{\pi}|f_\rho(e^{it})|^p w(t)dt = 2\pi\|f\|^p_{H^p_w(D)}.$$

From this it now follows that $f(z)W_p(z) \in H^p(D)$. □

Remark 2.4. The proof presented here was shown to us by M. Stoll. In this lemma, the reverse is also true, that is, $f(z) \in H^p_w(D)$ if and only if $f(z)W_p(z) \in H^p(D)$. This result was mentioned in [7, p. 6] but without proof. M. Stoll also has a proof for its sufficiency. Since we shall not use the reverse part in this paper, we omit its proof here.

Lemma 2.5. *Assume that* $w \in (A^q), 1 < q < \infty$ *and that* $q_w < p < \infty$. *If* $f(z) \in H^p_w(D)$, *then its boundary function* $f(t) \in L^p_w(T)$, *and we have*

$$\lim_{r\to 1^-}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p w(\theta)d\theta = \int_{-\pi}^{\pi}|f(e^{i\theta})|^p w(\theta)d\theta \tag{2.5}$$

and

$$\|f(t)\|_{L^p_w(T)} \le \|f(z)\|_{H^p_w(D)} \le C_p \, \|f(t)\|_{L^p_w(T)}, \tag{2.6}$$

where C_p is a constant depending only on p.

Proof. Let $f(t)$ denote the boundary function of $f(z)$, which exists by Remark 2.2. Noting that $(W_p(z))^p = g(z)$, and that $g(z) \in H^1(D)$, it follows that $W_p(z) \in H^p(D)$ and hence that $W_p(z)$ has boundary function $W_p(t)$ a.e. on T. By Lemma 2.3, $f(z)W_p(z) \in H^p(D)$, and hence its boundary function, denoted $H(t)$, belongs to $L^p(T)$. From the existence of $f(t)$ and $W_p(t)$, it follows immediately that $H(t) = f(t)W_p(t)$ a.e. on T. and hence that $f(t)W_p(t) \in L^p(T)$. But by (2.4), $|W_p(e^{i\theta})|^p = w(\theta)$ a.e., and hence we get that $f(e^{i\theta}) \in L^p_w(-\pi, \pi)$.

Since $f(z) \in H^1(D)$, by [5, Theorem 3.1], $f(z)$ can be represented as the Poisson integral of its boundary function $f(e^{i\theta})$:

$$f(z) = f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})dt, \qquad |z| = r < 1. \tag{2.7}$$

Thus, by [3, p. 113],

$$|f(re^{i\theta})| \le Mf(e^{i\theta}),$$

where

$$Mf(e^{i\theta}) := \sup_{0<\phi<\pi} \frac{1}{2\phi}\int_{\theta-\phi}^{\theta+\phi} |f(e^{is})|ds \tag{2.8}$$

is the Hardy-Littlewood maximal operator. By [12], the operator Mf is bounded from $L^p_w(T)$ into itself. Therefore, (2.5) follows by applying Lebesgue's dominated convergence theorem. The first inequality in (2.6) follows immediately from (2.5), and the second one follows from (2.7) and [12, Theorem 10]. □

Remark 2.6. By the above lemma, for $q_w < p < \infty$, we see that the convergence of functions in $H^p_w(D)$ and the convergence of the corresponding boundary functions in $L^p_w(T)$ are equivalent, and the space $H^p_w(D)$ can be identified with the space of all boundary functions $f(t)$ of functions $f(z) \in H^p_w(D)$, which is a subspace of $L^p_w(T)$.

Lemma 2.7. *Assume that $q_w < p < \infty$. If $f(t) \in L^p_w(T)$, then*
(1) the function

$$F^+(z) = \frac{1}{2\pi i}\int_{|t|=1} \frac{f(t)}{t - z}dt, \qquad z \in D \tag{2.9}$$

belongs to $H_w^p(D)$, and there exists a constant C_p depending only on p such that

$$\|F^+\|_{H_w^p(D)} \le C_p \|f\|_{L_w^p(T)}; \tag{2.10}$$

(2) the function

$$F^-(z) = \frac{1}{2\pi i}\int_{|t|=1} \frac{f(t)}{t-z}dt, \qquad z \in D_\infty$$

belongs to $H_w^p(D_\infty)$, and there exists a constant c_p depending only on p such that

$$\|F^-\|_{H_w^p(D_\infty)} \le c_p \|f\|_{L_w^p(T)}, \tag{2.11}$$

and $F^-(z)$ has boundary function $F^-(t)$ a.e. on T satisfying

$$F^+(t) - F^-(t) = f(t) \qquad \text{a.e. on } T. \tag{2.12}$$

Proof. By Lemma 2.1, $f(t) \in L^{p_0}(T)$ for some $1 < p_0 < p$. Consider the Poisson integral U of f: for $z = re^{i\theta}, 0 \le r < 1$,

$$U(z) = U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} \cdot f(e^{it})dt,$$

and the conjugate harmonic function $\widetilde{U}$ of U with $\widetilde{U}(0) = 0$:

$$\widetilde{U}(z) = \widetilde{U}(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2r\sin(\theta-t)}{1+r^2-2r\cos(\theta-t)} \cdot f(e^{it})dt.$$

We have

$$\begin{aligned} U(z) + i\widetilde{U}(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \cdot f(e^{it})dt \\ &= 2 \cdot \frac{1}{2\pi i}\int_{|\xi|=1} \frac{f(\xi)}{\xi - z}d\xi - \frac{1}{2\pi i}\int_{|\xi|=1} \frac{f(\xi)}{\xi}d\xi \\ &= 2F^+(z) - U(0). \end{aligned}$$

Hence,

$$F^+(z) = \frac{1}{2}U(z) + \frac{i}{2}\widetilde{U}(z) + \frac{1}{2}U(0). \tag{2.13}$$

By M. Riesz's theorem (see [10, Chapter V, Section B]), the Hilbert transform

$$Hf(e^{i\theta}) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{\varepsilon \le |\phi| \le \pi} \frac{f(e^{i(\theta-\phi)})}{2\tan\frac{\phi}{2}}d\phi \tag{2.14}$$

exists a.e. in $(-\pi, \pi)$, $Hf(e^{i\theta}) \in L^{p_0}(-\pi, \pi)$, and furthermore, for $0 \le r < 1$,

$$\widetilde{U}(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} \cdot Hf(e^{it})dt.$$

Since $w \in (A^p)$, and $f \in L^p_w(T)$, by [12, Theorem 10],

$$\int_{-\pi}^{\pi} |U(re^{i\theta})|^p w(\theta)d\theta \le C'_p \|f\|^p_{L^p_w(T)}, \tag{2.15}$$

and

$$\int_{-\pi}^{\pi} |\widetilde{U}(re^{i\theta})|^p w(\theta)d\theta \le C''_p \int_{-\pi}^{\pi} |Hf(e^{i\theta})|^p w(\theta)d\theta,$$

where C'_p, C''_p (and C'''_p, C''''_p below) are constants depending only on p. By [9, Theorem 4], the Hilbert transform Hf is a bounded operator which maps $L^p_w(T)$ into itself, that is,

$$\int_{-\pi}^{\pi} |Hf(e^{i\theta})|^p w(\theta)d\theta \le C'''_p \|f\|^p_{L^p_w(T)}.$$

Moreover,

$$U(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(e^{i\theta})(w(\theta))^{1/p}](w(\theta))^{-1/p}d\theta,$$

so by Hölder's inequality,

$$|U(0)|^p \le C''''_p \|f\|^p_{L^p_w(T)}.$$

Thus, by (2.13), we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F^+(re^{i\theta})|^p w(\theta)d\theta \le C^p_p \|f\|^p_{L^p_w(T)} < +\infty, \tag{2.16}$$

where C_p is a constant depending only on p. Here we used the inequality

$$(a + b)^p \le 2^p(a^p + b^p), \quad \text{if} \quad a \ge 0, \quad b \ge 0 \quad \text{and} \quad 1 < p < \infty.$$

Since $F^+(z)$ is holomorphic in D, so $F^+(z) \in H^p_w(D)$, and (2.10) holds. Part (1) is proved.

For $z = re^{i\theta}$ with $0 < r < 1$, we have

$$
\begin{aligned}
F^+(z) - F^-\left(\frac{1}{\overline{z}}\right) &= \frac{1}{2\pi i}\int_{|t|=1} f(t)\cdot\left[\frac{1}{t-z} - \frac{1}{t-\frac{1}{\overline{z}}}\right]dt \\
&= \frac{1}{2\pi i}\int_{|t|=1} f(t)\cdot\frac{r^2-1}{t^2\overline{z} - t(r^2+1) + z}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\cdot\frac{r^2-1}{re^{-i(\theta-\phi)} + re^{i(\theta-\phi)} - (1+r^2)}d\phi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\cdot\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}d\phi \\
&= U(re^{i\theta}) = U(z). \qquad (2.17)
\end{aligned}
$$

Thus, noting (2.16) and (2.15), we have

$$
\begin{aligned}
&\sup_{r>1}\int_{-\pi}^{\pi}|F^-(re^{i\theta})|^p w(\theta)d\theta = \sup_{r>1}\int_{-\pi}^{\pi}\left|F^+\Big(\frac{1}{r}e^{i\theta}\Big) - U\Big(\frac{1}{r}e^{i\theta}\Big)\right|^p w(\theta)d\theta \\
\leq\ & 2^p\left[\sup_{\rho<1}\int_{-\pi}^{\pi}|F^+(\rho e^{i\theta})|^p w(\theta)d\theta + \sup_{\rho<1}\int_{-\pi}^{\pi}|U(\rho e^{i\theta})|^p w(\theta)d\theta\right] \\
\leq\ & c_p^p\|f\|^p_{L^p_w(T)} < +\infty,
\end{aligned}
$$

where c_p is a constant depending only on p. Since $F^-(z)$ is holomorphic in D_∞ and by (2.17),

$$
F^-(\infty) = F^+(0) - U(0) = \frac{1}{2\pi i}\int_{|t|=1}\frac{f(t)}{t}dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})d\theta = 0,
$$

we have $F^-(z) \in H^p_w(D_\infty)$ and (2.11). By (2.17), we see that $F^-(z)$ has boundary function $F^-(t)$ a.e. on T; and letting $r \to 1^-$, we obtain (2.12). Part (2) is proved. ☐

Corollary 2.8. *Assume that $q_w < p < \infty$, and $B(z)$ is a Blaschke product. If $f(t) \in L^p_w(T)$, then*

$$
H_f(z) := \frac{B(z)}{2\pi i}\int_{|t|=1}\frac{f(t)}{B(t)(t-z)}dt \qquad (2.18)
$$

belongs to $H^p_w(D)$.

Proof. If $f(t) \in L^p_w(T)$, noting that $|B(t)| = 1$ a.e. on T, we have $\frac{f(t)}{B(t)} \in L^p_w(T)$. Thus, by Lemma 2.7,

$$
G(z) := \frac{1}{2\pi i}\int_{|t|=1}\frac{f(t)}{B(t)(t-z)}dt \qquad (2.19)
$$

belongs to $H^p_w(D)$. Hence, $H_f \in H^p_w(D)$ follows from the fact that $B(z)$ is holomorphic and $|B(z)| < 1$ for $z \in D$. ☐

Lemma 2.9. *Assume that* $q_w < p < \infty$. *If* $f(z) \in H^p_w(D)$, *then*

$$\frac{1}{2\pi i}\int_{|t|=1}\frac{f(t)}{t-z}dt = \begin{cases} f(z), & z \in D; \\ 0, & z \in D_\infty. \end{cases} \tag{2.20}$$

Proof. If $f(z) \in H^p_w(D)$, by Remark 2.2, $f(z) \in H^1(D)$. Hence, by [5, p. 40], (2.20) holds. □

Lemma 2.10. *If* $f(z) \in H^p_w(D)$ *with* $q_w < p < \infty$, *then for any compact subset* K *of* D, *we have*

$$|f(z)| \le C(K,p)\|f\|_{L^p_w(|t|=1)}, \qquad z \in K, \tag{2.21}$$

where $C(K,p)$ *is a positive constant depending only on* K *and* p.

Proof. By Lemma 2.9, and using Hölder's inequality, we have for $z \in K$,

$$\begin{aligned} |f(z)| &\le \frac{1}{2\pi}\int_{|t|=1}\frac{|f(t)|}{|t-z|}|dt| \\ &\le \frac{1}{2\pi}\Big(\int_{|t|=1}\frac{w^{-p'/p}}{|t-z|^{p'}}|dt|\Big)^{1/p'}\Big(\int_{|t|=1}|f(t)|^p w|dt|\Big)^{1/p}, \end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Noting that for $|t|=1$ and $z \in K$, we have $|t-z|>c$ where c is a constant depending only on K, and $[w(\theta)]^{-p'/p} = [w(\theta)]^{1-p'}$ is integrable in $(-\pi,\pi)$, the inequality (2.21) follows. □

Remark 2.11. By Lemma 2.10, convergence in $H^p_w(D)$ (or equivalently, in $L^p_w(T)$, by Remark 2.6) implies uniform convergence on each compact subset of D.

3. Some properties of $H^p_w(D)$, Part II

Now we deduce the representation of bounded linear functionals in $H^p_w(D)$.

Lemma 3.1. *Assume that* $q_w < p < \infty$ *and* $\frac{1}{p}+\frac{1}{p'}=1$. *Then for every bounded linear functional* $l \in (H^p_w(D))^*$, *there is a* $\Phi(z) \in H^{p'}_{w^{1-p'}}(D)$ *such that*

$$l(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{\Phi(e^{i\theta})}d\theta = \frac{1}{2\pi i}\int_{|t|=1} f(t)\frac{\overline{\Phi(t)}}{t}dt,$$

for $f(z) \in H^p_w(D)$. *Moreover*

$$\|l\| \le \|\Phi(z)\|_{H^{p'}_{w^{1-p'}}(D)} \le C\cdot\|l\|,$$

where $\|l\|$ *is the norm of* l *and* C *is a positive constant depending only on* p.

Proof. We claim that if $l \in (H^p_w(D))^*$ then there is a function $s(t) \in L^{p'}_{w^{1-p'}}(T)$ such that

$$\|l\| \leq C\|s(t)\|_{L^{p'}_{w^{1-p'}}(T)}, \tag{3.1}$$

and for all $f(z) \in H^p_w(D)$,

$$l(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) \cdot \overline{s(e^{i\theta})}d\theta = \frac{1}{2\pi i}\int_{|t|=1} f(t)\frac{\overline{s(t)}}{t}dt. \tag{3.2}$$

To see this, note that $H^p_w(D)$ can be identified with a subspace of $L^p_w(T)$ (see Remark 2.6), and for any $f(z) \in H^p_w(D)$,

$$\|f(e^{i\theta})\|_{L^p_w(-\pi,\pi)} = \|f(e^{i\theta})(w(\theta))^{1/p}\|_{L^p(-\pi,\pi)}.$$

Thus, by the Riesz representation theorem (see, for example, [5, Section 7.2]), to each $l \in (H^p_w(D))^*$ corresponds a function $\phi(t) \in L^{p'}(T)$, such that for all $f(z) \in H^p_w(D)$,

$$\begin{aligned} l(f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[f(e^{i\theta})(w(\theta))^{1/p}\right] \cdot \overline{\phi(e^{i\theta})}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) \cdot \overline{\phi(e^{i\theta})(w(\theta))^{1/p}}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) \cdot \overline{\phi(e^{i\theta})[(w(\theta))^{1-p'}]^{-1/p'}}d\theta, \end{aligned}$$

and $\|l\| \leq \|\phi(t)\|_{L^{p'}(T)}$. Denote

$$s(e^{i\theta}) := \phi(e^{i\theta})[(w(\theta))^{1-p'}]^{-1/p'}.$$

It is clear that

$$\|s(t)\|_{L^{p'}_{w^{1-p'}}(|t|=1)} = \|\phi(t)\|_{L^{p'}(T)},$$

hence $s(e^{i\theta}) \in L^{p'}_{w^{1-p'}}(-\pi,\pi)$ since $\phi(e^{i\theta}) \in L^{p'}(-\pi,\pi)$, and (3.1) and (3.2) hold.

Define functions $S^+(z)$ in D and $S^-(z)$ in D_∞ by the Cauchy-type integral

$$\frac{1}{2\pi i}\int_{|t|=1} \frac{\overline{s(t)}}{t(t-z)}dt.$$

Since $\frac{\overline{s(t)}}{t} \in L^{p'}_{w^{1-p'}}(T)$, by Lemmas 2.7, we have $S^+(z) \in H^{p'}_{w^{1-p'}}(D)$, $S^-(z) \in H^{p'}_{w^{1-p'}}(D_\infty)$, and

$$S^+(t) - S^-(t) = \frac{\overline{s(t)}}{t} \quad \text{a.e.} \quad . \tag{3.3}$$

For any $f(z) \in H^p_w(D)$, by Hölder's inequality, we have $f(z)S^+(z) \in H^1(D)$, hence (see [5, Theorems 3.7-3.9]),

$$\int_{|t|=1} f(t)S^+(t)dt = 0.$$

Thus, by (3.2) and (3.3), we have

$$\begin{aligned} l(f) &= \frac{1}{2\pi i}\int_{|t|=1} f(t)(-S^-(t))dt = \frac{1}{2\pi i}\int_{|t|=1} f(t)h(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})[h(e^{i\theta})e^{i\theta}]d\theta, \qquad (3.4) \end{aligned}$$

where $h(z) := -S^-(z) \in H^{p'}_{w^{1-p'}}(D_\infty)$.

Denote, for $z \in D_\infty$, $\phi_1(z) = \overline{h(z)z} = -\overline{z}\cdot\overline{S^-(z)}$. By a careful computation, we obtain

$$\phi_1(z) = \frac{1}{2\pi i}\int_{|t|=1} \frac{s(t)}{t-(\overline{z})^{-1}}dt, \qquad z \in D_\infty.$$

Setting $\xi = (\overline{z})^{-1}$, and $\Phi(\xi) := \phi_1\left((\overline{\xi})^{-1}\right)$, we have

$$\Phi(\xi) = \frac{1}{2\pi i}\int_{|t|=1} \frac{s(t)}{t-\xi}dt, \qquad \xi \in D,$$

and, by Lemma 2.7, $\Phi(\xi) \in H^{p'}_{w^{1-p'}}(D)$. Noting that $\phi_1(e^{i\theta}) = \Phi(e^{i\theta})$, by (3.4), we obtain

$$l(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{\Phi(e^{i\theta})}d\theta,$$

and, by Lemma 2.7 and (3.1),

$$\|l\| \le \|\Phi(z)\|_{H^{p'}_{w^{1-p'}}(D)} \le C\cdot\|s(t)\|_{L^{p'}_{w^{1-p'}}(T)},$$

where C is a positive constant depending only on p. The lemma is proved. □

Corollary 3.2. *There is a constant $C \ge 1$ such that, if $f(z) \in H^p_w(D)$ with $q_w < p < \infty$, and $\frac{1}{p}+\frac{1}{p'} = 1$, then*

$$\|f(z)\|_{H^p_w(D)} \le \sup_{\substack{h(z)\in H^{p'}_{w^{1-p'}}(D) \\ \|h(z)\|_{H^{p'}_{w^{1-p'}}(D)} \le C}} \left|\frac{1}{2\pi i}\int_{|t|=1} f(t)\frac{\overline{h(t)}}{t}dt\right|, \qquad (3.5)$$

and

$$\|f(z)\|_{H^p_w(D)} \le \sup_{\substack{h(z)\in H^{p'}_{w^{1-p'}}(D)\\ \|h(z)\|_{H^{p'}_{w^{1-p'}}(D)}\le C}} \left| \frac{1}{2\pi i}\int_{|t|=1} \overline{f(t)}\frac{h(t)}{t}dt \right|. \tag{3.6}$$

Proof. For $f(z) \in H^p_w(D)$, as a consequence of the Hahn-Banach theorem, there is an $l \in (H^p_w(D))^*$ such that

$$l(f) = \|f(z)\|_{H^p_w(D)} \qquad \text{and} \qquad \|l\| = 1.$$

Thus, by Lemma 3.1, there is a $\Phi(z) \in H^{p'}_{w^{1-p'}}(D)$ satisfying

$$1 \le \|\Phi(z)\|_{H^{p'}_{w^{1-p'}}(D)} \le C,$$

for some constant C, not depending on f, such that

$$\|f(z)\|_{H^p_w(D)} = \frac{1}{2\pi i}\int_{|t|=1} f(t)\frac{\overline{\Phi(t)}}{t}dt.$$

Hence, (3.5) holds, and so does (3.6). □

A system of functions is called complete in $H^p_w(D)$ if the closed linear span of elements of the system is the space $H^p_w(D)$. Otherwise, it is called incomplete.

Lemma 3.3. *Assume that $q_w < p < \infty$, and $\{a_k\}$ is a Blaschke sequence, then the system $\{e_k(z)\}$ defined by (1.3) is incomplete in $H^p_w(D)$.*

Proof. Clearly, the Blaschke product $B(z)$ of $\{a_k\}$ is in $H^{p'}_{w^{1-p'}}(D)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Consider the continuous linear functional

$$l(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{B(e^{i\theta})}d\theta, \qquad f \in H^p_w(D).$$

We have

$$\begin{aligned}
\overline{l(e_k)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} B(e^{i\theta})\overline{e_k(e^{i\theta})}d\theta \\
&= \frac{1}{2\pi}\int_{|t|=1} B(t)\left[\frac{1}{-2\pi i}\cdot\frac{1}{1-a_k\bar{t}}\right]\cdot\frac{dt}{it} \\
&= \frac{i}{2\pi}\cdot\frac{1}{2\pi i}\int_{|t|=1}\frac{B(t)}{t-a_k}dt = \frac{i}{2\pi}B(a_k) = 0.
\end{aligned}$$

Since $l(e_k) = 0$ ($k = 1, 2, \ldots$), and $B(z) \not\equiv 0$, this implies that $\{e_k(z)\}$ is incomplete in $H^p_w(D)$ by the Hahn-Banach theorem. □

Remark 3.4. By the above lemma, if $\{a_k\}$ is a Blaschke sequence, then the closed linear span of the elements of the system $\{e_k(z)\}$, namely $E_{p,w}(D)$, is a proper subspace of $H^p_w(D)$.

Lemma 3.5. *Assume that $q_w < p < \infty$. If $H_f(z)$ is defined by (2.18), with $f(z) \in E_{p,w}(D)$. Then $H_f(z) \equiv 0$ for $z \in D$.*

Proof. If $f(z) \in E_{p,w}(D)$, then there is a sequence

$$P_n(z) = \sum_{k=1}^{n} b_k^{(n)} e_k(z)$$

such that

$$\lim_{n\to\infty} \|f - P_n\|_{L^p_w(T)} = 0.$$

We claim that for $z \in D$ and all n,

$$\int_{|t|=1} \frac{P_n(t)dt}{B_n(t)(t-z)} = 0, \tag{3.7}$$

where $B_n(z)$ is defined by (1.6). Indeed, note that

$$\begin{aligned}\int_{|t|=1} \frac{P_n(t)dt}{B_n(t)(t-z)} &= \sum_{k=1}^{n} b_k^{(n)} \cdot \int_{|t|=1} \frac{e_k(t)}{B_n(t)(1-\overline{a}_k t)(t-z)} dt \\ &= \sum_{k=1}^{n} b_k^{(n)} \cdot \frac{1}{2\pi i} \int_{|t|=1} \frac{1}{B_n(t)(1-\overline{a}_k t)(t-z)} dt,\end{aligned}$$

and, for any fixed $z \in D$ and $k = 1, 2, \ldots, n$, the function

$$F(t) := \frac{1}{B_n(t)(1-\overline{a}_k t)(t-z)}$$

is holomorphic in $|t| > 1$, and as $|t| \to \infty$,

$$|F(t)| = O\Big(\frac{1}{|t|^2}\Big).$$

Thus, by the residue theorem,

$$\frac{1}{2\pi i} \int_{|t|=1} F(t)dt = -\mathrm{Res}(F, \infty) = 0,$$

hence (3.7) follows. Thus, by Hölder's inequality, noting that $|B_n(t)| = 1$ on T, for any fixed $z \in D$, we have

$$\begin{aligned}\left| \int_{|t|=1} \frac{f(t)dt}{B_n(t)(t-z)} \right| &= \left| \int_{|t|=1} \frac{[f(t) - P_n(t)]dt}{B_n(t)(t-z)} \right| \le \int_{-\pi}^{\pi} \frac{|f(e^{i\theta}) - P_n(e^{i\theta})|}{1-|z|} d\theta \\ &\le \|f - P_n\|_{L^p_w(T)} \cdot \left(\int_{-\pi}^{\pi} (w(\theta))^{1-p'} d\theta \right)^{1/p'} \cdot \frac{1}{1-|z|}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Since $w^{1-p'}\in(A^{p'})$, $w^{1-p'}$ is integrable in $(-\pi,\pi)$, we obtain, as $n\to\infty$,

$$\int_{|t|=1}\frac{f(t)dt}{B_n(t)(t-z)}\to 0,\qquad z\in D. \tag{3.8}$$

For $z\in D$, we have

$$\left|\int_{|t|=1}\frac{f(t)dt}{B(t)(t-z)}\right|\le\left|\int_{|t|=1}\frac{f(t)dt}{B_n(t)(t-z)}\right|+\left|\int_{|t|=1}\frac{f(t)}{t-z}\cdot\left[\frac{1}{B(t)}-\frac{1}{B_n(t)}\right]dt\right|$$
$$=: I_{1,n}(z)+I_{2,n}(z).$$

By (3.8), as $n\to\infty$, $I_{1,n}(z)\to 0$. For $I_{2,n}(z)$, we have

$$\begin{aligned}I_{2,n}(z)&\le\int_{|t|=1}\frac{|f(t)|}{|t-z|}\cdot\left|\frac{|B_n(t)-B(t)|}{|B(t)||B_n(t)|}\right||dt|\\&\le C_1\cdot\int_{|t|=1}|f(t)||B_n(t)-B(t)||dt|\\&\le C\cdot\|f(t)[B_n(t)-B(t)]\|_{L^p_w(T)},\end{aligned}$$

where C_1 and C are constants depending on z but independent of n, and we used Hölder's inequality. By Lemma 1.1, we obtain, as $n\to\infty$, $I_{2,n}(z)\to 0$. Thus, if $f(z)\in E_{p,w}(D)$,

$$\int_{|t|=1}\frac{f(t)dt}{B(t)(t-z)}\equiv 0,$$

hence $H_f(z)\equiv 0$ for $z\in D$. The proof is complete. □

4. Expansion and moment

In this section we study approximation problems in $H^p_w(D)$ with respect to the system (1.3). First, we give an expansion theorem under the Blaschke condition.

Theorem 4.1. *Assume that $q_w<p<\infty$, and the systems $\{e_k(z)\}$ and $\{\phi_k(z)\}$ are given by (1.3) and (1.4) with $\{a_k\}$ being a Blaschke sequence. If $f(z)\in H^p_w(D)$, then the biorthogonal expansion of $f(z)$ with respect to the system $\{e_k(z)\}$ is*

$$f(z)=P_Ef(z)+H_f(z),\qquad z\in D \tag{4.1}$$

where $H_f(z)$ is defined by (2.18), and

$$P_Ef(z)=\lim_{n\to\infty}\sum_{k=1}^{n}\alpha(f,\phi_k)\cdot\overline{\left(\frac{B(a_k)}{B_n(a_k)}\right)}\cdot e_k(z),\qquad z\in D, \tag{4.2}$$

where the limit is in the sense of $H^p_w(D)$ convergence[1],

$$\alpha(f,\phi_k) := \int_{|t|=1} f(t)\overline{\phi_k(t)}\frac{dt}{it}, \qquad k=1,2,\dots, \tag{4.3}$$

and $B(z)$ is the Blaschke product of $\{a_k\}$ and $B_n(z)$ is defined by (1.6).

Proof. For $z \in D$, let

$$l(\xi) = \frac{B(\xi)}{B_n(\xi)}\cdot\frac{1}{1-\xi\overline{z}}, \qquad \xi \in D,$$

then

$$\overline{\left(\frac{B(a_k)}{B_n(a_k)}\right)}\cdot e_k(z) = \frac{1}{2\pi i}\overline{\left(\frac{B(a_k)}{B_n(a_k)}\right)}\cdot\frac{1}{1-\overline{a}_k z}$$
$$= \frac{1}{2\pi i}\overline{\left(\frac{B(a_k)}{B_n(a_k)}\cdot\frac{1}{1-a_k\overline{z}}\right)} = \frac{1}{2\pi i}\overline{l(a_k)}.$$

Noting that $\overline{B(t)}B(t) = 1$ a.e. on T, and choosing sufficiently small c_k as in (1.5), we have

$$\begin{aligned}
S_{E,n}f(z) &:= \\
&:= \sum_{k=1}^{n}\alpha(f,\phi_k)\cdot\overline{\left(\frac{B(a_k)}{B_n(a_k)}\right)}\cdot e_k(z)\\
&= \sum_{k=1}^{n}\left(\int_{|t|=1} f(t)\overline{\left((-i)\frac{B(t)}{B'(a_k)(t-a_k)}\right)}\frac{dt}{it}\right)\cdot\frac{1}{2\pi i}\overline{l(a_k)}\\
&= \int_{|t|=1} f(t)\left(\frac{1}{2\pi i B(t)}\sum_{k=1}^{n}\overline{\left(\frac{l(a_k)}{B'(a_k)(t-a_k)}\right)}\right)\frac{dt}{t}\\
&= \int_{|t|=1} f(t)\left(\frac{1}{2\pi i B(t)}\overline{\left(\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{c_k}\frac{l(\xi)d\xi}{B(\xi)(t-\xi)}\right)}\right)\frac{dt}{t} \quad \text{(residue theorem)}\\
&= \int_{|t|=1} f(t)\left(\frac{1}{2\pi i B(t)}\overline{\left(\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{c_k}\frac{d\xi}{B_n(\xi)(t-\xi)(1-\xi\overline{z})}\right)}\right)\frac{dt}{t}\\
&=: \int_{|t|=1} f(t)G_n(z,t)dt,
\end{aligned}$$

[1]Hence, by Remark 2.11, the limit is also in the sense of uniform convergence on each compact subset of D.

where

$$G_n(z,t)=\frac{1}{2\pi i B(t)t}\overline{\left(\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{c_k}\frac{d\xi}{B_n(\xi)(t-\xi)(1-\xi\overline{z})}\right)}$$

with $|t|=1, z\in D$.

Since, as $\xi\to\infty$,

$$\left|\frac{1}{B_n(\xi)(t-\xi)(1-\xi\overline{z})}\right|=O\left(\frac{1}{|\xi|^2}\right),$$

$$\mathrm{Res}\left[\frac{1}{B_n(\xi)(t-\xi)(1-\xi\overline{z})},\xi=\infty\right]=0,$$

by the residue theorem, we have, for $|t|=1, z\in D$,

$$\begin{aligned}&\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{c_k}\frac{d\xi}{B_n(\xi)(t-\xi)(1-\xi\overline{z})}\\ =&-\mathrm{Res}\left[\frac{1}{B_n(\xi)(t-\xi)(1-\xi\overline{z})},\xi=t\right]-\mathrm{Res}\left[\frac{1}{B_n(\xi)(t-\xi)(1-\xi\overline{z})},\xi=\frac{1}{\overline{z}}\right]\\ =&-\frac{1}{\overline{z}}\left[\frac{1}{B_n(t)(t-\frac{1}{\overline{z}})}+\frac{1}{B_n(\frac{1}{\overline{z}})(\frac{1}{\overline{z}}-t)}\right].\end{aligned}$$

Hence,

$$\overline{\left(\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{c_k}\frac{d\xi}{B_n(\xi)(t-\xi)(1-\xi\overline{z})}\right)}=t\left(\frac{B_n(t)}{t-z}-\frac{B_n(z)}{t-z}\right).$$

Here we used the facts that on T, $t\overline{t}=1$,

$$B_n(t)\overline{B_n(t)}=1,$$

and for $z\in D$, $B_n(z)\overline{B_n(\frac{1}{\overline{z}})}=1$.
Thus,

$$G_n(z,t)\quad=\quad\frac{1}{2\pi i B(t)}\left(\frac{B_n(t)}{t-z}-\frac{B_n(z)}{t-z}\right),$$

and furthermore,

$$\begin{aligned}S_{E,n}f(z)\quad&=\quad\frac{1}{2\pi i}\int_{|t|=1}\frac{B_n(t)f(t)dt}{B(t)(t-z)}-\frac{B_n(z)}{2\pi i}\int_{|t|=1}\frac{f(t)dt}{B(t)(t-z)}\\ &=:\quad S^{(1)}_{E,n}(z)-S^{(2)}_{E,n}(z);\end{aligned}$$

note that this can also be obtained by taking $s=0$ in formula (16) of [11]. Noting that $|B_n(t)|=1$ on T and $|B_n(z)|<1$ for $z\in D$, by Corollary 2.8

it follows that the functions $S^{(1)}_{E,n}(z)$ and $S^{(2)}_{E,n}(z)$ belong to $H^p_w(D)$. Thus, by Lemmas 2.9, 2.7 and 1.1, we have , as $n \to \infty$,

$$\begin{aligned}\|f(z) - S^{(1)}_{E,n}(z)\|_{H^p_w(D)} &= \left\| \frac{1}{2\pi i}\int_{|t|=1}\left(1 - \frac{B_n(t)}{B(t)}\right)\frac{f(t)dt}{t-z}\right\|_{H^p_w(D)} \\ &\le C_p \left\|\left(1 - \frac{B_n(t)}{B(t)}\right) f(t)\right\|_{L^p_w(T)} \\ &= C_p\|f(t)[B(t) - B_n(t)]\|_{L^p_w(T)} \to 0;\end{aligned}$$

and by Lemmas 2.5 and 1.1, we have, as $n \to \infty$,

$$\begin{aligned}\|H_f(z) - S^{(2)}_{E,n}(z)\|_{H^p_w(D)} &= \left\| \frac{B_n(z) - B(z)}{2\pi i}\int_{|t|=1}\frac{f(t)dt}{B(t)(t-z)}\right\|_{H^p_w(D)} \\ &\le C_p\|[B_n(t) - B(t)]G(t)\|_{L^p_w(T)} \to 0,\end{aligned}$$

where $G(z)$ is defined by (2.19), $G(t) \in L^p_w(T)$. Hence, as $n \to \infty$,

$$\|S_{E,n}f(z) - [f(z) - H_f(z)]\|_{H^p_w(D)} \to 0.$$

The theorem is proved. □

Remark 4.2. In Theorem 4.1, $P_E f(z)$ is called the orthogonal projection of $f(z)$ with respect to the system $\{e_k(z)\}$ because

$$\int_{|t|=1}[f(t) - P_E f(t)]\overline{e_k(t)}\frac{dt}{it} = \int_{|t|=1} H_f(t)\overline{e_k(t)}\frac{dt}{it} = iH_f(a_k) = 0, \quad k = 1, 2, \ldots.$$

The second equality above is based on the following fact: If $h(z) \in H^p_w(D)$, by Lemma 2.9,

$$\alpha(h, e_k) := \int_{|t|=1} h(t)\overline{e_k(t)}\frac{dt}{it} = \frac{i}{2\pi i}\int_{|t|=1}\frac{h(t)}{t-a_k}dt = ih(a_k). \tag{4.4}$$

Recall that a Blaschke sequence $\{a_k\}$ is called uniformly separated [6, p. 67], if there is a positive number δ such that

$$\inf_k \prod_{j\ne k}\left|\frac{a_j - a_k}{1 - \overline{a}_j a_k}\right| \ge \delta > 0. \tag{4.5}$$

An equivalent expression of the above uniformly separated condition is (see [6, p. 158])

$$(1 - |a_k|^2)|B'(a_k)| \ge \delta > 0, \quad k = 1, 2, \ldots \tag{4.6}$$

where $B(z)$ is the Blaschke product of $\{a_k\}$.

Remark 4.3. An important result about uniformly separated sequences is (see [6, Chapter 6, p. 157], or [5, p. 152]): If $\{a_k\} \subset D$ is uniformly separated, then there exists a constant C such that for every $f(z) \in H^p(D)$ $(0 < p < \infty)$,

$$\sum_{k=1}^{\infty}(1-|a_k|^2)|f(a_k)|^p \leq C\|f\|^p_{L^p(T)}.$$

Later, we shall use this result in the proof of Theorems 4.5 and 5.3.

Assume that $q_w < p < \infty$, and define the function

$$v(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}[\log(w(t))^{1-p'}]dt\right), \quad z \in D, \tag{4.7}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Comparing with the function $g(z)$ defined in (2.2), we see that

$$v(z) = [g(z)]^{1-p'}, \quad |v(z)|^{-\frac{p}{p'}} = |g(z)|, \quad z \in D, \tag{4.8}$$

and by (2.4),

$$|v(e^{i\theta})| = [w(\theta)]^{-p'/p} = [w(\theta)]^{1-p'} \quad \text{a.e.} \tag{4.9}$$

Remark 4.4. Since $w \in (A^p)$, we have $w^{1-p'} \in (A^{p'})$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Thus, by Lemma 2.3, if $f(z) \in H^{p'}_{w^{1-p'}}(D)$ then $f(z)V_{p'}(z) \in H^{p'}(D)$, where

$$V_{p'}(z) = \exp\left(\frac{1}{2p'\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}[\log(w(t))^{1-p'}]dt\right), \quad z \in D.$$

Now we are ready to give a moment theorem.

Theorem 4.5. *Assume that (a)* $q_w < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$*; (b) the sequence* $\{a_k\}$ *is uniformly separated; (c) the systems* $\{e_k(z)\}$ *and* $\{\phi_k(z)\}$ *are defined by (1.3) and (1.4), and (d)* $v(z)$ *is defined by (4.7). If a sequence of complex numbers* $\{b_k\}$ *satisfies*

$$\sum_{k=1}^{\infty}|b_k|^p\Big[(1-|a_k|^2)|v(a_k)|\Big]^{-\frac{p}{p'}} < +\infty, \tag{4.10}$$

then

$$J(z) = \sum_{k=1}^{\infty} b_k e_k(z), \quad z \in D, \tag{4.11}$$

belongs to $E_{p,w}(D)$*, where the series converges in the sense of* $H^p_w(D)$*, and*

$$\alpha(J,\phi_k) := \int_{|t|=1} J(t)\overline{\phi_k(t)}\frac{dt}{it} = b_k, \quad k = 1, 2, \ldots. \tag{4.12}$$

Proof. By Corollary 3.2 (3.6), we have

$$\|S_m - S_n\|_{H^p_w(D)} = \left\| \sum_{k=n+1}^{m} b_k e_k(z) \right\|_{H^p_w(D)}$$

$$\leq \sup_{\substack{h(z)\in H^{p'}_{w^{1-p'}}(D) \\ \|h(z)\|_{H^{p'}_{w^{1-p'}}(D)} \leq C}} \left| \sum_{k=n+1}^{m} \bar{b}_k \left(\frac{1}{2\pi} \int_{|t|=1} h(t)\overline{e_k(t)} \frac{dt}{it} \right) \right|,$$

where $S_n(z)$ is the partial sum of the series (4.11). For the above h, we have

$$\frac{1}{2\pi} \int_{|t|=1} h(t)\overline{e_k(t)} \frac{dt}{it} = \frac{i}{2\pi} \frac{1}{2\pi i} \int_{|t|=1} \frac{h(t)}{t - a_k} dt = H(a_k),$$

where

$$H(z) = \frac{1}{4\pi^2} \int_{|t|=1} \frac{h(t)}{t - z} dt \in H^{p'}_{w^{1-p'}}(D),$$

and, if we require $\|h(z)\|_{H^{p'}_{w^{1-p'}}(D)} \leq C$,

$$\|H(z)\|_{H^{p'}_{w^{1-p'}}(D)} \leq C_1 \|h\|_{L^{p'}_{w^{1-p'}}(T)} \leq C_1 C := C_2,$$

where C_2 is a constant depending only on p. Thus,

$$\left\| \sum_{k=n+1}^{m} b_k e_k(z) \right\|_{H^p_w(D)} \leq \sup_{\substack{H(z)\in H^{p'}_{w^{1-p'}}(D) \\ \|H(z)\|_{H^{p'}_{w^{1-p'}}(D} \leq C_2}} \sum_{k=n+1}^{m} |\bar{b}_k H(a_k)|. \tag{4.13}$$

Moreover, by Hölder's inequality, we have

$$\sum_{k=n+1}^{m} |\bar{b}_k H(a_k)| \leq \left(\sum_{k=n+1}^{m} (1 - |a_k|^2)|H(a_k)|^{p'} |v(a_k)| \right)^{\frac{1}{p'}} \cdot \left(\sum_{k=n+1}^{m} |b_k|^p \left[(1 - |a_k|^2)|v(a_k)| \right]^{-\frac{p}{p'}} \right)^{\frac{1}{p}}. \tag{4.14}$$

By Remark 4.4, $H(z)V_{p'}(z) \in H^{p'}(D)$, and since $|V_{p'}(z)|^{p'} = |v(z)|$ and $\{a_k\}$ is uniformly separated, hence by Remark 4.3, and by (4.9), we have

$$\sum_{k=1}^{\infty} (1 - |a_k|^2)|H(a_k)|^{p'} |v(a_k)| < C_3 \|HV_{p'}\|^{p'}_{L^{p'}(T)} = C_3 \|H\|^{p'}_{L^{p'}_{w^{1-p'}}(T)} < C_4. \tag{4.15}$$

Thus, by (4.10), (4.13), (4.14) and (4.15), it follows that the series (4.11) is Cauchy, hence converges in $H^p_w(D)$ to a function $J(z) \in E_{p,w}(D)$. And since $J \to \alpha(J, \phi_k)$ is a continuous linear functional on $L^p_w(T)$, by the biorthogonality of the two systems, (4.12) holds. The proof is complete. $\square$

As an application of the above moment theorem, now we give the biorthogonal expansion in $H^p_w(D)$ under the uniformly separated condition. We need a lemma:

Lemma 4.6. *Assume that $q_w < p < \infty$; $\{a_k\}$ is a Blaschke sequence, $\{e_k(z)\}$ and $\{\phi_k(z)\}$ are defined by (1.3) and (1.4). If $h(t) \in L^p_w(T)$, then*

$$\int_{|t|=1} h(t)\overline{e_k(t)}\frac{dt}{it} = 0 \qquad (k = 1, 2, \ldots)$$

and

$$\int_{|t|=1} h(t)\overline{\phi_k(t)}\frac{dt}{it} = 0 \qquad (k = 1, 2, \ldots)$$

are equivalent.

Proof. Since, by Lemma 2.1, $h(t) \in L^p_w(T)$ implies $h(t) \in L^{p_0}(T)$ for some p_0 with $1 < p_0 < p$, this lemma follows immediately from [14, Theorem 1]. $\square$

Theorem 4.7. [2] *In Theorem 4.1, if $\{a_k\}$ is uniformly separated, then the expansion (4.1) becomes*

$$f(z) = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha(f, \phi_k) e_k(z) + H_f(z), \qquad z \in D, \tag{4.16}$$

where the limit is in the sense of $H^p_w(D)$ convergence.

Proof. We claim that

$$\sum_{k=1}^{\infty} |\alpha(f, \phi_k)|^p \Big[(1 - |a_k|^2)|v(a_k)|\Big]^{-\frac{p}{p'}} < \infty, \tag{4.17}$$

where $v(z)$ is defined by (4.7). Noting that, under the uniformly separated condition, by (4.6),

$$\frac{1}{|B'(a_k)|} \le \frac{1 - |a_k|^2}{\delta}, \qquad k = 1, 2, \ldots.$$

[2]This result for the classical case was obtained in [1].

Thus, we have

$$
\begin{aligned}
|\alpha(f,\phi_k)|^p &= \left|\int_{|t|=1} f(t)\overline{\phi_k(t)}\frac{dt}{it}\right|^p = \left|\int_{|t|=1} \overline{f(t)}\phi_k(t)\frac{dt}{it}\right|^p \\
&= \left|\frac{1}{|B'(a_k)|}\int_{|t|=1}\frac{\overline{f(t)}B(t)}{t(t-a_k)}dt\right|^p \leq C_1(1-|a_k|^2)^p|G(a_k)|^p,
\end{aligned}
$$

where C_1 is a constant and

$$G(z) = \frac{1}{2\pi i}\int_{|t|=1}\frac{\overline{f(t)}B(t)}{t}\cdot\frac{1}{t-z}dt.$$

Since $\frac{\overline{f(t)}B(t)}{t} \in L^p_w(T)$, by Lemmas 2.7 and 2.5, $G(z) \in H^p_w(D)$, and

$$\|G\|_{L^p_w(T)} \leq C\left\|\frac{\overline{f(t)}B(t)}{t}\right\|_{L^p_w(T)} = C\|f\|_{L^p_w(T)},$$

where C is a constant depending only on p. Noting that $p - \frac{p}{p'} = 1$, by (4.8), we have

$$
\begin{aligned}
& \sum_{k=1}^{\infty}(1-|a_k|^2)^p|G(a_k)|^p\Big[(1-|a_k|^2)|v(a_k)|\Big]^{-\frac{p}{p'}} \\
= & \sum_{k=1}^{\infty}(1-|a_k|^2)|G(a_k)|^p|v(a_k)|^{-\frac{p}{p'}} \\
= & \sum_{k=1}^{\infty}(1-|a_k|^2)|G(a_k)|^p|g(a_k)| \\
= & \sum_{k=1}^{\infty}(1-|a_k|^2)|G(a_k)|^p|W_p(a_k)|^p \\
\leq & C_2\|GW_p\|^p_{L^p(T)} = C_2\|G\|^p_{L^p_w(T)} \leq C_3\|f\|^p_{L^p_w(T)} < \infty,
\end{aligned}
$$

where C_2 and C_3 are constants. Here we used (4.8), the fact that (see Lemma 2.3) $G(z)W_p(z) \in H^p(D)$, and a result in [6, Chapter 6, p. 157] (see Remark 4.3). Hence (4.17) holds. Thus, by Theorem 4.5,

$$J(z) = \sum_{k=1}^{\infty}\alpha(f,\phi_k)e_k(z) \in E_{p,w}(D),$$

and

$$\alpha(J,\phi_k) = \alpha(f,\phi_k), \qquad k = 1,2,\ldots.$$

By Theorem 4.1, $f(z) - H_f(z) \in E_{p,w}(D)$. Let $h(z) = f(z) - H_f(z) - J(z)$. We have $h(z) \in E_{p,w}(D)$, and

$$\alpha(h, \phi_k) = \alpha(f, \phi_k) - \alpha(H_f, \phi_k) - \alpha(J, \phi_k) = -\alpha(H_f, \phi_k), \qquad k = 1, 2, \dots.$$

Note that, by Corollary 2.8, $H_f(z) \in H^p_w(D)$. Thus, by (4.4), $\alpha(H_f, e_k) = iH_f(a_k)$. Hence $\alpha(H_f, e_k) = 0, k=1, 2, \dots$, and by Lemma 4.6, $\alpha(H_f, \phi_k) = 0, k = 1, 2, \dots$. So, $\alpha(h, \phi_k) = 0,\ k = 1, 2, \dots$, and by Theorem 4.1, $h(z) = H_h(z),\ z \in D$. But, $H_h(z) \equiv 0$ by Lemma 3.5 due to $h(z) \in E_{p,w}(D)$, hence $h(z) \equiv 0$. Thus, $f(z) = J(z) + H_f(z)$, this is (4.16). The theorem is proved. □

5. Expansion and interpolation

In this section, we give results corresponding to Theorems 4.1, 4.5 and 4.7 but with respect to the system $\{\phi_k(z)\}$.

Denote by $\Phi_{p,w}$ the closed linear span of the elements of the system $\{\phi_k(z)\}$ in $H^p_w(D)$. By the next Lemma and Lemma 3.3, we see that $\Phi_{p,w}$ is also a proper subspace of $H^p_w(D)$, and the system $\{\phi_k(z)\}$ is also incomplete in $H^p_w(D)$.

Lemma 5.1. *Assume that $q_w < p < \infty$, and $\{a_k\} \subset D$ is a Blaschke sequence. Then*

$$\Phi_{p,w}(D) \subset E_{p,w}(D).$$

Proof. If there were a function, say $\phi_j(z)$, in the system $\{\phi_k(z)\}$, which did not belong to $E_{p,w}(D)$, then by the Hahn-Banach theorem, there would be a linear functional $l \in (H^p_w(D))^*$ such that

$$l(e_k) = 0 \qquad (k = 1, 2, \dots) \tag{5.1}$$

but

$$l(\phi_j) \neq 0. \tag{5.2}$$

By Lemma 3.1, there exists $h \in H^{p'}_{w^{1-p'}}(D)$ such that

$$l(f) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(e^{i\theta}) \overline{h(e^{i\theta})} d\theta = \frac{1}{2\pi i} \int_{|t|=1} f(t) \overline{h(t)} \frac{dt}{t}$$

with $f \in H^p_w(D)$ and $\frac{1}{p} + \frac{1}{p'} = 1$. So, (5.2) becomes

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_j(e^{i\theta}) \overline{h(e^{i\theta})} d\theta \neq 0, \tag{5.3}$$

and by (5.1), it follows that $\overline{l(e_k)} = 0$. Hence

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{i\theta})\left[\frac{1}{-2\pi i}\cdot\frac{1}{1-a_k e^{-i\theta}}\right]d\theta \\ &= \frac{1}{-2\pi i}\cdot\frac{1}{2\pi i}\int_{|t|=1}\frac{h(t)}{(1-a_k t^{-1})t}dt = \frac{1}{-2\pi i}\cdot\frac{1}{2\pi i}\int_{|t|=1}\frac{h(t)dt}{t-a_k}. \end{aligned}$$

So

$$\frac{1}{2\pi i}\int_{|t|=1}\frac{h(t)}{t-a_k}dt = 0 \qquad (k = 1, 2, \ldots),$$

that is, by Lemma 2.9,

$$h(a_k) = 0 \qquad (k = 1, 2, \ldots). \tag{5.4}$$

Since, by (1.5),

$$\phi_j(z) = \frac{B(z)}{2\pi}\int_{c_j}\frac{d\xi}{B(\xi)(\xi - z)},$$

we have, by (5.3), and Fubini's theorem,

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{B(e^{i\theta})}{2\pi}\int_{c_j}\frac{d\xi}{B(\xi)(\xi - e^{i\theta})}\right]\overline{h(e^{i\theta})}d\theta \\ &= \frac{1}{2\pi}\int_{c_j}\frac{1}{B(\xi)}\cdot\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{B(e^{i\theta})\overline{h(e^{i\theta})}}{\xi - e^{i\theta}}d\theta\right]d\xi \neq 0. \end{aligned} \tag{5.5}$$

But

$$\begin{aligned} \overline{\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{B(e^{i\theta})\overline{h(e^{i\theta})}}{\xi - e^{i\theta}}d\theta\right]} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h(e^{i\theta})}{B(e^{i\theta})(\overline{\xi} - e^{-i\theta})}d\theta \\ &= \frac{1}{\overline{\xi}}\frac{1}{2\pi i}\int_{|t|=1}\frac{h(t)}{B(t)(t-(\overline{\xi})^{-1})}dt, \end{aligned} \tag{5.6}$$

where we used the fact that $\overline{B(t)} = 1/B(t)$ for almost every $t \in T$. Noting that (i) since $h \in H^{p'}_{w^{1-p'}}(D)$, $w^{1-p'} \in (A^{p'})$, and thus by Lemma 2.1, we have $h \in H^{p_0}(D)$ for some $1 < p_0 < p'$, and since, by (5.4), $h(a_k) = 0 = B(a_k)$, $k = 1, 2, \ldots$, it follows by [5, Theorem 2.5] that

$$\frac{h(z)}{B(z)} \in H^{p_0}(D);$$

(ii) since $\xi \in c_j$, we have $(\overline{\xi})^{-1} \in D_\infty$, and thus, by Lemma 2.9, the integral in (5.6) must be zero, a contradiction with (5.5). The proof is complete. □

Now, we give an expansion theorem which is corresponding to Theorem 4.1.

Theorem 5.2. *Under the assumptions of Theorem 4.1, if $f(z) \in H^p_w(D)$, then the biorthogonal expansion of $f(z)$ with respect to the system $\{\phi_k(z)\}$ is*

$$f(z) = P_\Phi f(z) + H_f(z), \qquad z \in D, \tag{5.7}$$

where $H_f(z)$ is defined by (2.18), and

$$P_\Phi f(z) = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha(f, e_k) \cdot \frac{B(a_k)}{B_n(a_k)} \cdot \phi_k(z), \qquad z \in D,$$

where the limit is in the sense of $H^p_w(D)$ convergence[3]*, and*

$$\alpha(f, e_k) := \int_{|t|=1} f(t)\overline{e_k(t)}\frac{dt}{it}, \qquad k = 1, 2, \ldots.$$

Proof. First, let us find the representation for the partial sum

$$S_{\Phi,n} f(z) := \sum_{k=1}^{n} \alpha(f, e_k) \cdot \frac{B(a_k)}{B_n(a_k)} \cdot \phi_k(z).$$

For $z = a_j$ ($j = 1, 2, \ldots$), (5.7) must hold since $\phi_k(a_j) = -i\delta_{kj}$ and $\alpha(f, e_j) = if(a_j)$ (see 4.4). So, we assume that $|z| < 1$ and $z \neq a_j$ ($j =$

[3]Hence, by Remark 2.11, the limit is also in the sense of uniform convergence on each compact subset of D.

$1, 2, \ldots$). Choose sufficiently small circles c_k as in (1.5). We have

$$
\begin{aligned}
S_{\Phi,n}f(z) = & \\
= & \sum_{k=1}^{n} [if(a_k)] \cdot \frac{B(a_k)}{B_n(a_k)} \cdot \Big[(-i) \cdot \frac{B(z)}{B'(a_k)(z-a_k)}\Big] \\
= & \sum_{k=1}^{n} B(z) \cdot \frac{f(a_k)B(a_k)}{B_n(a_k)(z-a_k)} \cdot \frac{1}{B'(a_k)} \\
= & \sum_{k=1}^{n} \frac{-B(z)}{2\pi i} \int_{c_k} \frac{B(\xi)f(\xi)}{B_n(\xi)(\xi-z)} \cdot \frac{1}{B(\xi)} d\xi \quad \text{(residue theorem)} \\
= & \sum_{k=1}^{n} \frac{-B(z)}{2\pi i} \int_{c_k} \frac{f(\xi)}{B_n(\xi)(\xi-z)} d\xi \\
= & \sum_{k=1}^{n} \frac{-B(z)}{2\pi i} \int_{c_k} \Big[\frac{1}{2\pi i} \int_{|t|=1} \frac{f(t)dt}{t-\xi}\Big] \frac{d\xi}{B_n(\xi)(\xi-z)} \quad \text{(Lemma 2.9)} \\
= & \int_{|t|=1} f(t) \Big[\sum_{k=1}^{n} \frac{-B(z)}{(2\pi i)^2} \int_{c_k} \frac{d\xi}{B_n(\xi)(\xi-z)(t-\xi)}\Big] dt \\
=: & \int_{|t|=1} f(t) G_n(z,t) dt,
\end{aligned}
$$

where

$$G_n(z,t) = \sum_{k=1}^{n} \frac{-B(z)}{(2\pi i)^2} \int_{c_k} \frac{d\xi}{B_n(\xi)(\xi-z)(t-\xi)}$$

with $|t| = 1,\ z \in D$ and $z \neq a_k\ (k = 1, 2, \ldots)$.

Since, as $\xi \to \infty$,

$$\left| \frac{1}{B_n(\xi)(\xi-z)(\xi-t)} \right| = O\left(\frac{1}{|\xi|^2}\right),$$

we have, for $|t| = 1, z \in D$,

$$\text{Res}\left[\frac{1}{B_n(\xi)(\xi-z)(\xi-t)}, \xi = \infty\right] = 0,$$

hence, by the residue theorem,

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{1}{2\pi i} \int_{c_k} \frac{d\xi}{B_n(\xi)(\xi - z)(\xi - t)} \\
= & -\mathrm{Res}\left[\frac{1}{B_n(\xi)(\xi - z)(\xi - t)}, \xi = z\right] - \mathrm{Res}\left[\frac{1}{B_n(\xi)(\xi - z)(\xi - t)}, \xi = t\right] \\
= & \frac{1}{B_n(z)(t - z)} - \frac{1}{B_n(t)(t - z)}.
\end{aligned}$$

Thus, we obtain

$$G_n(z,t) = \frac{B(z)}{2\pi i}\left[\frac{1}{B_n(z)(t - z)} - \frac{1}{B_n(t)(t - z)}\right].$$

So, by Lemma 2.9, we have

$$S_{\Phi,n} f(z) = \frac{B(z)}{B_n(z)} f(z) - \frac{B(z)}{2\pi i} \int_{|t|=1} \frac{f(t)dt}{B_n(t)(t - z)}. \tag{5.8}$$

Thus, as in the proof of Theorem 4.1, we obtain, as $n \to \infty$,

$$\begin{aligned}
\|S_{\Phi,n} f(z) - f(z) + H_f(z)\|_{H^p_w(D)} &\leq \left\|\frac{B(z)}{B_n(z)} f(z) - f(z)\right\|_{H^p_w(D)} \\
&+ \left\|\frac{B(z)}{2\pi i} \int_{|\xi|=1} \frac{[B(\xi) - B_n(\xi)]f(\xi)}{B_n(\xi)B(\xi)} \cdot \frac{d\xi}{\xi - z}\right\|_{H^p_w(D)} \\
&\leq 2C_p \|f(t)[B(t) - B_n(t)]\|_{L^p_w(T)} \to 0.
\end{aligned}$$

The proof is complete. □

Next, we give an interpolation theorem.

Theorem 5.3. *Under the assumptions of Theorem 4.5, if a sequence of complex numbers* $\{b_k\}$ *satisfies*

$$\sum_{k=1}^{\infty} |b_k|^p (1 - |a_k|^2) |v(a_k)|^{-\frac{p}{p'}} < +\infty, \tag{5.9}$$

then the series

$$\sum_{k=1}^{\infty} i b_k \phi_k(z) \tag{5.10}$$

converges in the sense of $H^p_w(D)$, *and its sum function* $I(z)$ *belongs to* $\Phi_{p,w}(D)$ *and satisfies*

$$I(a_k) = b_k, \quad k = 1, 2, \ldots.$$

Proof. For any positive integer n, let $S_n(z)$ be the partial sum of the series (5.10). By Corollary 3.2 (3.5), we have

$$\|S_m - S_n(z)\|_{H^p_w(D)} = \left\| \sum_{k=n+1}^{m} b_k \phi_k(z) \right\|_{H^p_w(D)}$$

$$\leq \sup_{\substack{h(z)\in H^{p'}_{w^{1-p'}}(D) \\ \|h(z)\|_{H^{p'}_{w^{1-p'}}(D)} \leq C}} \left| \frac{1}{2\pi} \int_{|t|=1} \left(\sum_{k=n+1}^{m} b_k \phi_k(t) \right) \overline{h(t)} \frac{dt}{it} \right|$$

$$= \sup_{\substack{h(z)\in H^{p'}_{w^{1-p'}}(D) \\ \|h(z)\|_{H^{p'}_{w^{1-p'}}(D)} \leq C}} \left| \sum_{k=n+1}^{m} b_k \left(\frac{1}{2\pi} \int_{|t|=1} \overline{h(t)} \phi_k(t) \frac{dt}{it} \right) \right|$$

$$= \sup_{\substack{h(z)\in H^{p'}_{w^{1-p'}}(D) \\ \|h(z)\|_{H^{p'}_{w^{1-p'}}(D)} \leq C}} \left| \sum_{k=n+1}^{m} \frac{b_k}{B'(a_k)} \left(\frac{1}{2\pi i} \int_{|t|=1} \frac{\overline{h(t)} B(t) dt}{t(t-a_k)} \right) \right|,$$

where C is a constant depending only on p with $C \geq 1$. Let

$$H(z) = \frac{1}{2\pi i} \int_{|t|=1} \frac{\overline{h(t)} B(t)}{t(t-z)} dt.$$

Since

$$\frac{\overline{h(t)} B(t)}{t} \in L^{p'}_{w^{1-p'}}(T),$$

by Lemma 2.7, we have $H(z) \in H^{p'}_{w^{1-p'}}(D)$, and, if we require $\|h(z)\|_{H^{p'}_{w^{1-p'}}(D} \leq C$,

$$\|H(z)\|_{H^{p'}_{w^{1-p'}}(D)} \leq C_1 \left\| \frac{\overline{h(t)} B(t)}{t} \right\|_{L^{p'}_{w^{1-p'}}(T)} = C_1 \|h\|_{L^{p'}_{w^{1-p'}}(T)} \leq C_1 \cdot C := C_2$$

where C_2 is a constant depending only on p. Thus, by (4.6), we have

(5.11)
$$\left\| \sum_{k=n+1}^{m} b_k \phi_k(z) \right\|_{H^p_w(D)} \leq \sup_{\substack{H(z)\in H^{p'}_{w^{1-p'}}(D) \\ \|H(z)\|_{H^{p'}_{w^{1-p'}}(D)} \leq C_2}} \sum_{k=n+1}^{m} \left| \frac{(1-|a_k|^2)}{\delta} b_k H(a_k) \right|.$$

By Hölder's inequality,

(5.12)
$$\sum_{k=n+1}^{m} |b_k|(1-|a_k|^2)H(a_k)| \leq$$
$$\left(\sum_{k=n+1}^{m}(1-|a_k|^2)|H(a_k)|^{p'}|v(a_k)|\right)^{\frac{1}{p'}}\left(\sum_{k=n+1}^{m}(1-|a_k|^2)|b_k|^p|v(a_k)|^{-\frac{p}{p'}}\right)^{\frac{1}{p}}.$$

Since $H(z)V_{p'}(z) \in H^{p'}(D)$ and $|V_{p'}(z)|^{p'} = |v(z)|$ (see Remark 4.4), and since $\{a_k\}$ is uniformly separated, as in the proof of Theorem 4.5, by Remark 4.3 and by (4.9), we have

$$\begin{aligned}\sum_{k=1}^{\infty}(1-|a_k|^2)|H(a_k)|^{p'}|v(a_k)| &\leq C_3\|HV_{p'}\|_{L^{p'}(T)} \\ &= C_3\|H\|_{L^{p'}_{w^{1-p'}}(T)} < C_2C_3 := C_4.\end{aligned} \tag{5.13}$$

Thus, by (5.9), (5.11), (5.12) and (5.13), it follows that the series in (5.10) is Cauchy, hence converges in $H^p_w(D)$ to a function $I(z) \in \Phi_{p,w}$. And, by Remark 2.11, the series uniformly converges on each compact subset of D, so it is pointwise convergent in D. Noting that $\phi_k(a_j) = -i\delta_{kj}$, we get $I(a_k) = b_k$ $(k = 1, 2, \ldots)$. The proof is complete. □

Finally, we give an expansion theorem which is corresponding to Theorem 4.7.

Theorem 5.4. *In Theorem 5.2, if $\{a_k\}$ is uniformly separated, then the expansion (5.7) becomes*

$$f(z) = \sum_{k=1}^{\infty}\alpha(f, e_k)\phi_k(z) + H_f(z), \qquad z \in D, \tag{5.14}$$

where the limit is in the sense of $H^p_w(D)$ convergence.

Proof. By Lemma 2.3, $f(z)W_p(z) \in H^p(D)$, and since $|W_p(z)|^p = |g(z)|$, hence, by the uniformly separated condition, we have (see Remark 4.3)

$$\sum_{k=1}^{\infty}(1-|a_k|^2)|f(a_k)|^p|g(a_k)| < \infty.$$

Thus, by (4.8),

$$\sum_{k=1}^{\infty}|f(a_k)|^p(1-|a_k|^2)|v(a_k)|^{-\frac{p}{p'}} < \infty.$$

Hence by Theorem 5.3,

$$I(z) = \sum_{k=1}^{\infty} if(a_k)\phi_k(z) \in \Phi_{p,w}(D), \qquad z \in D,$$

and since $\phi_k(a_j) = -i\delta_{kj}$, we get $I(a_k) = f(a_k)$ $(k = 1, 2, \ldots)$. But, by Theorem 5.2, $f(z) - H_f(z) \in \Phi_{p,w}(D)$. Let $G(z) = f(z) - H_f(z) - I(z)$. Clearly, $G(z) \in \Phi_{p,w}(D)$, hence, by Lemma 5.1, $G(z) \in E_{p,w}(D)$. Thus, by Lemma 3.5, $H_G(z) \equiv 0$ for $z \in D$. Meanwhile, $G(a_k) = f(a_k) - H_f(a_k) - I(a_k) = 0$, hence, by (4.4), $\alpha(G, e_k) = 0$ $(k = 1, 2, \ldots)$. Thus, by Theorem 5.2, $G(z) = H_G(z),\ z \in D$. So, $G(z) \equiv 0$ for $z \in D$, this is (5.14) due to $if(a_k) = \alpha(f, e_k)$. The proof is complete. □

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References

[1] G. M. Airapetian, *On the bases of rational functions in subspaces of Hardy spaces H_p $(1 < p < \infty)$*, Izv. Akad. Nauk Armjan SSR, Ser. Mat., **6** (1973), 429-450 (In Russian).

[2] G. M. Airapetian, *Multple interpolation and bases of certain biorthogonal systems of rational functions in Hardy classes H_p*, Izv. Akad. Nauk Armjan SSR, Ser. Mat., **12** (1977), 262-277 (In Russian).

[3] S. Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Springer-Verlag, 1992.

[4] J. A. Cima, A. L. Matheson and W. T. Ross, *The Cauchy Transform*, Amer. Math. Soc., 2006.

[5] P. L. Duren, *Theory of H^p Spaces*, Academic Press, 1970; Dover Publications, 2000.

[6] P. L. Duren and A. Schuster, *Bergman Spaces*, Amer. Math. Soc., 2004.

[7] J. García-Cuerva, *Weighted H^p spaces*, Dissertationes Mathematicae, **162** (1979), 1-63.

[8] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, 1962.

[9] R. Hunt, B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc., **176** (1973), 227-251.

[10] P. Koosis, *Introduction to H_p Spaces*, Cambridge Univ. Press, 2nd Ed., 1998.

[11] M. S. Martirosian and V.Kh. Musoyan, *Summation of biorthogonal expansion in non-complete system of rational functions in the spaces H^p* $(1 < p < \infty)$, J. Contemp. Math. Anal., Armen. Acad. Sci., **32** (1997), no. 5, 27-38; translation from Izv. Nats. Akad. Nauk Armen., Mat. **32**, no. 5, 32-44 (1997).

[12] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165** (1972), 207-226.

[13] M. Rosenblum, *Summability of Fourier series in $L^p(d\mu)$*, Trans. Amer. Math. Soc., **105** (1962), 32-42.

[14] X. Shen, *On the problem of incompleteness of the biorthogonal systems of functions*, Approx. Theory and its Appl., **4** (1988), 1-7.

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