RENORMALIZATION AND α-LIMIT SET FOR EXPANDING LORENZ MAPS

YI-MING DING

ABSTRACT. We establish a one-to-one correspondence between the renormalizations and proper totally invariant closed sets (i.e., α-limit sets) of expanding Lorenz map, which enable us to distinguish periodic and non-periodic renormalizations. We describe the minimal renormalization by constructing the minimal totally invariant closed set, so that we can define the renormalization operator. Using consecutive renormalizations, we obtain complete topological characterization of α-limit sets and nonwandering set decomposition. For piecewise linear Lorenz map with slopes ≥ 1, we show that each renormalization is periodic and every proper α-limit set is countable.

1. INTRODUCTION

Lorenz equations is a system of ordinary differential equations in R^3 which has been enormous influential in Dynamics, providing inspiration for the definition of a variety of examples including the geometric models and Hénon maps [33]. The Lorenz maps we study are a simplified model for two-dimensional return maps associated to the flow of the Lorenz equations.

Numerically studies of the Lorenz equations led Lorenz to emphasize the importance of *sensitive dependence of initial conditions*—an essential factor of unpredictability in many systems. The simulations for an open neighborhood suggest that almost all points in phase space approach to a strange attractor—the Lorenz attractor. Afraimovic, Bykov and Sil'nikov [1] and Guckenheimer and Willianms [11] introduced a geometric model that is an abstraction of the numerically-observed features possessed by solution to Lorenz equations. Tucker [31] [32] proved, in a computer assistant proof, that the geometric model is valid, so the Lorenz equations define a geometric Lorenz flow.

2000 *Mathematics Subject Classification.* 37E05, 54H20.

Key words and phrases. Expanding Lorenz map, renormalization, α-limit set, nonwandering set decomposition.

Roughly speaking, a *geometric Lorenz flow* is the natural extension of a *geometric Lorenz semiflow* which is itself a suspension flow built over a roof function and a certain type of one dimensional Lorenz map.

More precisely, a *Lorenz map* on $I = [0,\ 1]$ is an interval map $f : I \to I$ such that for some $c \in (0,\ 1)$ we have

(i) f is strictly increasing on $[0,\ c)$ and on $(c,\ 1]$;

(ii) $\lim_{x\uparrow c} f(x) = 1$, $\lim_{x\downarrow c} f(x) = 0$.

If, in addition, f satisfies the topological expanding condition

(iii) The pre-images set $C = \cup_{n\geq 0} f^{-n}(c)$ of c is dense in I,

then f is said to be an *expanding Lorenz map*, [9], [10], [12].

As was mentioned in [16], maps with discontinuities are extremely natural and important, arising for example in billiards or as return maps for flows with equilibrium points, and very often in modeling and applications. Lorenz map admits a discontinuity c. It is convenient to regard c as two points, $c+$ and $c-$, $f(c+) = 0$ and $f(c-) = 1$ from the definition. So $f(c)$ is either 0 or 1, which can be identified from the context easily. Lorenz map plays an important role in the study of the global dynamics of families of vector fields near homoclinic bifurcations, see [20], [21], [28], [32], [33] and references therein. The expanding condition follows from [10], [12], [17], which is weaker than many other conditions used in[11], [28], etc. Since any Lorenz map relates to some flow in 3-dimension (cf. [15], [30], [34]), it is useful to understand the detailed dynamical behaviors of expanding Lorenz map.

Renormalization is a central concept in contemporary dynamics. The idea is to study the small-scale structure of a class of dynamical systems by means of a renormalization operator R acting on the systems in this class. This operator is constructed as a rescaled return map, where the specific definition depends essentially on the class of systems. The idea of renormalization for Lorenz map was introduced in studying simplified models of Lorenz attractor, apparently firstly in Plamer [22] and Parry [26] (cf. [14]). The renormalization operator in Lorenz map family, is the first return map of the original map to a smaller interval around the discontinuity, rescaled to the original size. Glendinning and Sparrow [10] presented a comprehensive study of the renormalization by investigating the kneading invariants of expanding Lorenz map.

Definition 1. A Lorenz map $f : I \to I$ is said to be **renormalizable** if there is a proper subinterval $[a,\ b]$ and integers $\ell, r > 1$ such that the map $g : [a,\ b] \to [a,\ b]$ defined by

$$g(x) = \begin{cases} f^{\ell}(x) & x \in [a,\ c), \\ f^{r}(x) & x \in (c,\ b], \end{cases} \tag{1. 1}$$

is itself a Lorenz map. The interval $[a,\ b]$ is called the **renormalization interval**.

If f is not renormalizable, it is said to be **prime**.

Note that we assume $\ell > 1$ and $r > 1$ in the definition of renormalization. The *trivial renormalization* ($\ell + r = 3$ in (1. 1)) appeared in [10] is not included in our definition because of two reasons: The first reason is, as we shall prove in Theorem A, renormalization of expanding Lorenz map always corresponds to a proper totally invariant closed set, while trivial renormalization does not. The second reason is that trivial renormalization is easy to characterize. In fact, as we shall see in Proposition 2, f can be trivially renormalized if and only if $\kappa > 2$, where κ is the minimal period of periodic points of f. So every Lorenz map with $\kappa > 2$ can be trivially renormalized (several times if necessary) to be a Lorenz map with $\kappa \leq 2$. As a result, the meaning of prime here is also different from which in [10].

Suppose $g = (f^\ell, f^r)$ on $[a,\ b]$ is a renormalization of f. It is easy to see that the intervals

$$f([a,\ c)),\quad f^2([a,\ c)), \cdots,\quad f^{\ell-1}([a,\ c))$$

are mutually disjoint and disjoint from $[a,\ b]$, and so are the intervals

$$f((c,\ b]),\quad f^2((c,\ b]), \cdots,\quad f^{r-1}((c,\ b])$$

(although intervals from the two collections may intersect). It follows that the renormalization map g is the first return map of f on the renormalization interval $[a,\ b]$(cf. [18]).

Let f be a renormalizable Lorenz map. f may have different renormalizations (cf. [10], [18]). A renormalization $g = (f^\ell, f^r)$ of f is said to be *minimal* if for any other renormalization $(f^{\ell'}, f^{r'})$ of f we have $\ell' \geq \ell$ and $r' \geq r$ (cf. Glendinning and Sparrow [10], Keller and Matthias [13], Martens and de Melo [18], Silva and Sousa [29], etc.).

It is not a easy problem to determine wether f is renormalizable or not. In fact, it is impossible to check if f is prime or not in finite steps, because ℓ and r in (1. 1) may be large.

In this paper we will investigate the renormalization and α-limit set of expanding Lorenz map in purely topological way. The non-expanding case is more suitable to state in terms of kneading theory, which is relegated to another paper. The key observation is that one can renormalize expanding Lorenz map via its *proper totally invariant closed set*, which turns out to be an α-limit set of some periodic points. For given expanding Lorenz map f, we establish the one-to-one correspondence between the renormalizations and the proper totally invariant closed sets of f (Theorem A). Then we characterize (Theorem B) the renormalizability of f by constructing the minimal totally invariant closed set D, which is just the α-limit set of the periodic orbit with minimal period. Since the minimal totally invariant closed set corresponds to the minimal renormalization of f, we can define the renormalization operator R on the space of (expanding) Lorenz maps: Rf is the minimal renormalization of f. Using the consecutive actions of renormalization operator, we can characterize the α-limit sets of f completely (Theorem C). To the best of the author's knowledge, Theorem C is the first full characterization of α-limit sets for nontrivial endomorphism. We also present a nonwandering set decomposition via the consecutive renormalizations (Theorem D), which is essentially the same as the decomposition proposed by Glendinning and Sparrow [10], but we emphasize that the components in the decomposition are indecomposable, and obtain the expressions of the components. At last, we use our theory to study the renormalizations and α-limit sets of a family of piecewise linear Lorenz maps with slopes ≥ 1. For any Lorenz map f in this family, each renormalization of f is periodic (Theorem E). As a result, each α-limit set of f is countable with finite depth, and one can obtain all the renormalizations of f in finite steps.

It is worthy to mention some other results which may be of interests in their own senses: the determination of the minimal period of the periodic orbits of expanding Lorenz map f (Lemma 3), the locally eventually onto property of the periodic orbit with minimal period (Lemma 5), and the construction of countable closed set with given depth (see examples in Section 5), etc.

The paper is organized as follows. We state our main results in Section 2, and establish the correspondence between proper totally invariant closed set and renormalization of expanding Lorenz map in Section 3. In Section 4, we study the renormalizability of expanding Lorenz map by constructing the minimal totally invariant closed set. We characterize the α-limit set and present the nonwandering set decomposition via consecutive renormalizations in Section 5. In the last Section, we consider the renormalizations and α-limit sets of a family of piecewise linear Lorenz maps with slopes ≥ 1.

2. Main results

In order to be self-contained, we fix some notations and terms we need in this paper. For any nonempty open interval $U \subseteq I$, put

$$N(U) = \min\{n \geq 0 : \exists z \in U \text{ such that } f^n(z) = c\}. \tag{2. 1}$$

By the definition of $N(U)$, we have $c \in f^{N(U)}(U)$, $N(U) \leq N(V)$ if $V \subseteq U$, and

$$N(f^i(U)) = N(U) - i, \qquad i = 0, 1, \cdots, N(U). \tag{2. 2}$$

In fact, $N(U)$ is the maximal integer such that $f^{N(U)}$ is continuous on U. We can regard $N(U)$ as the index of continuity for the interval U. There exists a unique $z \in U$ such that $f^{N(U)}(z) = c$ because $f^{N(U)-1}$ is continuous and strictly increasing on U. If f is expanding, $N(U) < \infty$ for all open interval U.

$A \subseteq I$, A' represents for the derived set of A, that is, the accumulation point set of $A, A'' = (A')', A^n = (A^{n-1})', n = 1, 2, \cdots$. $\overline{A}$ is the closure of A. A is perfect if $A' = A$, and A is a nowhere dense if $\overline{A}$ does not contain any interval. A is a Cantor set if A is perfect and A is nowhere dense.

2.1. **Renormalization and proper totally invariant closed set.** Recall that a subset E of I is totally invariant under f if

$$f(E) = f^{-1}(E) = E,$$

and it is proper if $E \neq I$.

Theorem A. *Let f be an expanding Lorenz map. There is a one-to-one correspondence between the renormalizations and proper totally invariant closed sets of f. More precisely, suppose E is a proper totally invariant closed set of f, put*

$$e_- = \sup\{x \in E : x < c\}, \qquad e_+ = \inf\{x \in E : x > c\}, \tag{2. 3}$$

and

$$\ell = N((e_-,\ c)), \qquad r = N((c,\ e_+)).$$

Then

$$f^\ell(e_-) = e_-, \quad f^r(e_+) = e_+ \tag{2. 4}$$

and the following map

$$R_E f(x) = \begin{cases} f^\ell(x) & x \in [f^r(c+),\ c) \\ f^r(x) & x \in (c,\ f^\ell(c-)] \end{cases} \tag{2. 5}$$

is a renormalization of f.

On the other hand, if g is a renormalization of f, then there exists a unique proper totally invariant closed set B such that $R_B f = g$.

A remarkable property of proper totally invariant closed set is illustrated by (2. 4): the two closest points to c, from the left and right, are periodic. This property is essential for us to obtain a renormalization. In their study on the renormalization theory of expanding Lorenz map via kneading invariant, Glendinning and Sparrow [10] proposed a long combinatorial proof for the existence of such two periodic points.

Definition 2. Suppose E is a proper totally invariant closed set of expanding Lorenz map f. The renormalization $R_E f$ defined by (2. 5) in Theorem A is called the *reormalization associated with* E. And E is called the *repelling set* associated to the renormalization R_E. The interval $(e_-,\ e_+)$, with endpoints e_+ and e_- defined in (2. 3), is called the *critical interval* of E and R_E.

Definition 3. A renormalization is said to be periodic if the endpoints of its *critical interval* belong to the same periodic orbit.

The periodic renormalization is interesting because β-transformation

$$T_{\beta,\alpha}(x) = \beta x + \alpha \ (\mod 1),\ 1 < \beta \leq 2,\ 0 \leq \alpha < 1$$

can only be periodically renormalized (see [8], and Section 6 for details). This kind of renormalization was studied by Alsedà and Falcò [2], Malkin [17]. It was called phase locking renormalization by Alsedà and Falcò in [2]. As we shall see in Theorem B and Theorem C, the periodic renormalization corresponds to non-perfect totally invariant closed set, and it is easy to check if the minimal renormalization is periodic or not.

2.2. **Renormalizability.** By Theorem A, a possible way to characterize the renormalizability is to look for the *minimal totally invariant closed set* D of f, in the sense that $D \subset E$ for each totally invariant closed set E of f. If we can find a minimal totally invariant closed set D of f, then f is renormalizable if and only if $D \neq I$. The construction of minimal totally invariant closed set seems difficult, because we do not even know wether a Lorenz map always admits such a minimal totally invariant closed set or not.

We shall construct the minimal totally invariant closed set for expanding Lorenz map f by choosing some periodic point $p \in I$ and showing that the α-limit set of p, $\alpha(p)$, is indeed the minimal totally invariant closed set. The periodic orbit with minimal period is important in constructing the minimal totally invariant closed set. It relates naturally to the so called *primary cycle* which was used to characterize the renormalization of β-transformation [8]. We begin with the minimal period κ of periodic points of expanding Lorenz map f (see Lemma 3). Then we show (see Lemma 5) that the periodic orbit O with minimal period is unique, and the κ-periodic orbit O admits very special *locally eventually onto* (*l.e.o.*) property:

For each open interval U *with* $U \cap O \neq \emptyset$*, there exists integer* m *such that* $\bigcup_{i=0}^{m} f^i(U) = I$.

Based on the locally eventually onto property of O, it is possible to prove that the α-limit set of each κ-periodic point is the minimal totally invariant closed set.

The following Theorem B clarifies the renormalizability of expanding Lorenz map.

Theorem B. *Let f be an expanding Lorenz map with minimal period κ, $1 < \kappa < \infty$, O be the unique κ-periodic orbit, and $D = \overline{\bigcup_{n\geq 0} f^{-n}(O)}$.*

(1) *D is the minimal totally invariant closed set of f.*
(2) *f is renormalizable if and only if $D \neq I$. If f is renormalizable, then R_D, the renormalization associated to D, is the minimal renormalization of f.*
(3) *We have exactly three cases:*
   - $D = I \iff f$ *is prime;*
   - $D = O \iff R_D$ *is periodic;*
   - D *is a Cantor set* $\iff R_D$ *is not periodic.*

It is easy to see the cases $\kappa = 1$ and $\kappa = \infty$ are prime, Theorem B describes the renormalizability of expanding Lorenz map completely.

It follows from Theorem B that f is prime if and only if $D = I$. If f is prime, the locally eventually onto property of O turns out to be the locally eventually onto property of f. In this case, we say that f is *l.e.o.* (locally eventually onto).

The dynamics of prime expanding Lorenz map f is well understood: f is prime if and only if it is *l.e.o.*. This *l.e.o.* property is an ideal topological property, which is equivalent to the strong transitivity of Parry [25] (see Proposition 1 in Section 4): for each open interval $U \subset I$, there exists positive integer n such that $\bigcup_{i=0}^{n} f^i(U) = I$.

2.3. **Consecutive renormalizations.** According to Theorem B, the minimal renormalizaion of renormalizable expanding Lorenz map always exists. We can define a renormalization operator R from the set of renormalizable expanding Lorenz maps to the set expanding Lorenz maps (cf. [9], [10]). For each renormalizable expanding Lorenz map, we define Rf to be the minimal renormalization map of f. For $n > 1$, $R^n f = R(R^{n-1}f)$ if $R^{n-1}f$ is renormalizable. And f is m $(0 \leq m \leq \infty)$ *times renormalizable* if the renormalization process can proceed m times exactly. For $0 < i \leq m$, $R^i f$ is the i-th renormalization of f. Formally, we denote $R^0 f := f$ as the 0-th renormalization, whose renormalization interval is denoted by $[a_0,\ b_0] := [0,\ 1]$.

The consecutive renormalization process can be used to characterize all the α-limit sets and obtain a canonical decomposition of the nonwandering set of expanding Lorenz map.

Remember that the α-limit set $\alpha(x)$ of a point $x \in I$ under f is defined as

$$\alpha(x) = \bigcap_{n\geq 0} \overline{\bigcup_{k\geq n} \{f^{-k}(x)\}}.$$

α-limit set is important in understanding homoclinic behavior in dynamics. It is often relates to homeomorphsim because the inverse $f^{-1}(x)$ is only one point. For endomorphism f, the α-limit set is more difficult to understand than ω-limit set in general, because $f^{-k}(x)$ is more complex than $f^{k}(x)$. It seems that α-limit set is "difficult" to describe. But f may not have so many different α-limit sets because the α-limit set is "large" in some sense. We have the following unexpected result.

Theorem C. *Let f be an m ($0 \leq m \leq \infty$) renormalizable expanding Lorenz map, $[a_i,\ b_i]$ ($0 \leq i \leq m$) be the renormalization interval of the i-th renormalization $R^i f$, and $orb([a_i,\ b_i]) = \bigcup_{n\geq 0} f^n([a_i,\ b_i])$. Then we have:*

1. *f admits m proper α-limit sets which can be ordered as*
$$\emptyset = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset I.$$
2. *E_i is a Cantor set if the i-the renormalization is not periodic, and $E_i' = E_{i-1}$ if the i-the renormalization is periodic.*
3. *For $0 < i \leq m$, $\alpha(x) = E_i$ if and only if*
$$x \in orb([a_{i-1},\ b_{i-1}]) \backslash orb([a_i,\ b_i]),$$
and $\alpha(x) = I$ if and only if
$$x \in A := \bigcap_{i=0}^{m} orb([a_i,\ b_i]).$$

By Theorem C, we know that expanding Lorenz map admits a cluster of α-limit sets, and we can determine the α-limit set of each point. Note that A is the attractor of f: $A = I$ if $m = 0$, $A = orb([a_m,\ b_m])$ if $m < \infty$, and $A = \bigcap_{i=0}^{\infty} orb([a_i,\ b_i])$ is a Cantor set if $m = \infty$ (see Theorem D). So f is prime implies that $\alpha(x) = I$, $\forall x \in I$. Since I is the largest α-limit set, f admits exactly $m + 1$ different α-limit sets. To the best of the author's knowledge, Theorem C is the first full characterization for the α-limit set of a nontrivial endomorphism.

Remember that the *depth* of A is the minimal integer n such that the n-th derived set $A^{(n)}$ is empty (cf. [4], p. 33). An interesting consequence of Theorem C appears when all the renormalizations of f are periodic. In this case, Theorem C implies that, the i-th derived set of E_k is E_{k-i}:

$(E_k)^i = E_{k-i}$ for $0 \leq i \leq k \leq m < \infty$. We can construct closed sets with given depth in a dynamical way (see Section 5 for the examples).

The proof of Theorem C is based on the 1-1 correspondence between the α-limit sets and totally invariant closed sets: Each α-limit set is a totally invariant closed set (cf. Lemma 1), and each totally invariant closed set is the α-limit set for some periodic point (cf. Lemma 6).

Now we can present a global picture of the dynamics of expanding Lorenz map.

Theorem D. *Let f be an m-renormalizable ($0 \leq m \leq \infty$) expanding Lorenz map and*

$$\emptyset = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset I$$

be all the α-limit sets of f, $I_i = [a_i,\ b_i]$ be the renormalization interval of the i-th renormalization $R^i f$, and D_i is the minimal totally invariant closed set of $R^i f$.

Then there is a canonical decomposition of the nonwandring set $\Omega(f)$ of f into m-invariant closed set Ω_i ($i = 1, \cdots, m$) and an attractor A

$$\Omega(f) = \bigcup_{i=1}^{m} \Omega_i \cup A. \tag{2. 6}$$

This decomposition has the following properties:

(1) *$\Omega_i := E_i \cap orb(I_{i-1}) = orb(D_{i-1})$, $1 \leq i \leq m$ and $f|_{\Omega_i}$ is l.e.o.. Ω_i is either a periodic orbit or a Cantor set depending on wether the renormalization $R^i f$ is periodic or not.*

(2) *A is the attractor of f: $\omega(x) \subseteq A$ for $x \notin E_\infty := \bigcup_{i \geq 0} E_i$. $f|_A$ is l.e.o.. Moreover, $A = \bigcap_{i=0}^{m} orb([a_i,\ b_i])$: $A = I$ if $m = 0$, A is a finite union of closed intervals if $0 < m < \infty$, and A is a Cantor set if $m = \infty$. In the last case, $\omega(x) = A$ for $x \notin E_\infty$.*

Based on their renormalization theory on kneading invariants, Glendinning and Sparrow [10] obtained a similar nonwandering set decomposition like (2. 6). Our proof of the decomposition is independent of kneading theory. We obtain the exact expression of Ω_i, and emphasize that Ω_i is indecomposable: $f|_{\Omega_i}$ is *l.e.o.*.

We shall prove Theorem D in Section 5. We first decompose I into invariant sets:

$$I = (E_1 \backslash E_0) \cup (E_2 \backslash E_1) \cup \cdots \cup (E_m \backslash E_{m-1}) \cup (I \backslash E_m),$$

then show that $\Omega_i := \Omega(f) \cap (E_i \backslash E_{i+1}) = E_i \cap orb(I_{i-1}) = orb(D_i)$, $A := \Omega(f) \cap (I \backslash E_m)$, and obtain the detail dynamics on Ω_i and A.

2.4. **Piecewise linear Lorenz map.** A Lorenz map is said to be *piecewise linear* if it is linear on both intervals $[0, c)$ and $(c, 1]$. Such a map is of the form

$$(2.\ 7) \qquad f_{a,b,c}(x) = \begin{cases} ax + 1 - ac & x \in [0, c) \\ b(x - c) & x \in (c, 1]. \end{cases}$$

The well studied map $T_{\beta,\alpha}$ defined by

$$T_{\beta,\alpha} = \beta x + \alpha \pmod 1$$

is called a β-transformation (see [8]). When $1 < \beta \leq 2$, $T_{\beta,\alpha} = f_{\beta,\beta,c}$ with $c = (1 - \alpha)/\beta$.

Using the so called *primary cycle*, Palmer [22] proved that a β-transformation is renormalizable if and only if it admits a primary cycle (cf. [8]). In our language, every renormalization of β-transformation is periodic. When the slopes are different, we have the following Theorem E.

Theorem E. *If $a \geq 1,\ b \geq 1$, then each renormalization of the piecewise linear Lorenz map $f_{a,b,c}$ is periodic.*

Remark 1.

1. (1) Theorem E indicates the renormalization process of piecewise linear Lorenz map with slopes ≥ 1 is simple: all the renormalization are periodic. As a result, one can obtain all the renormalizations of $f_{a,b,c}$ in finite steps.
2. (2) If $a \geq 1,\ b \geq 1$, applying a Theorem of Glendinning [8], $f_{a,b,c}$ is conjugated to a β-transformation.
3. (3) If $a \geq 1,\ b \geq 1$, by Theorem C and Theorem D, the proper α-limit sets of piecewise linear Lorenz map $f_{a,b,c}$ are countable.

3. Totally invariant closed sets

Let f be an expanding Lorenz map. A set $E \subseteq I$ is said to be totally invariant under f, if

$$(3.\ 1) \qquad f(E) = E = f^{-1}(E).$$

A totally invariant set E is proper if $E \neq I$. E is totally invariant is equivalent to E is backward and forward invariant ($f^{-1}(E) \subseteq E$ and $f(E) \subseteq E$), because the forward and backward invariance of E implies

$$E \subseteq f(E) \subseteq E \quad \textit{and} \quad E \subseteq f^{-1}(E) \subseteq E.$$

Totally invariant closed set is crucial in describing the renormalization of expanding Lorenz map. There are two facts motive us to use totally invariant closed set to study the renormalization. The first one is from the Fatou-Julia-Sullivan theory. If f is a rational map then there is a dynamical decomposition of the Riemann sphere into the disjoint union of two totally invariant (i.e., both forward and backward invariant) sets $J(f)$, $F(f)$. Here, $F(f)$ is the domain of normality of the family of iterates of f, and is called the Fatou set. Its complement, which is called the Julia set of f, is a compact set, which contains all the complications of the dynamics of f. Suppose f admits a renormalization with renormalization interval $[a,\ b]$. We can obtain a decomposition:

$$I = F(f) \bigcup J(f)$$

where $F(f) = \bigcup_{n\geq 0} f^{-n}((a,\ b))$ and $J(f)$ is the complement of $F(f)$. It is easy to see that $F(f)$ is open and totally invariant, and $J(f)$ is closed and totally invariant. So renormalization induces totally invariant decomposition like rational map on Riemann sphere.

Another fact is from the combinatorial description of renormalization of expanding Lorenz map [10]. If f is renormalizable, then there exists a repel invariant closed set in the gap of the renormalization. One can check easily this invariant closed set is indeed a totally invariant closed set.

We will concentrate on the totally invariant closed set of f. Lemma 1 collects some useful facts of totally invariant closed set.

Lemma 1. *Let f be an expanding Lorenz map.*

(1) *A totally invariant closed set E is proper if and only if $c \notin E$;*
(2) *Any proper totally invariant closed set is nowhere dense.*
(3) *The derived set of proper totally invariant closed set is also totally invariant.*
(4) *$\forall x \in I$, $\alpha(x)$ is a totally invariant closed set of f.*
(5) *If p is periodic, then $\alpha(p) = \overline{\bigcup_{n\geq 0} f^{-n}(p)}$.*
(6) *If E is a totally invariant closed set of f, then for $A \subset I$, we have*

$$f^{-1}(A \cap E) = f^{-1}(A) \cap E, \qquad f(A \cap E) = f(A) \cap E. \tag{3. 2}$$

Proof. 1, It is necessary to prove $c \in E$ implies that $E = I$. By the invariance of E under f^{-1}, $c \in E$ implies that $f^{-n}(c) \in E$. So $\bigcup_{n\geq 0} f^{-n}(c) \subset E$, which implies that $E \supseteq \overline{\bigcup_{n\geq 0} f^{-n}(c)} = I$.

2, If E contains some interval U, then $c \in f^{N(U)}(U) \subseteq E$ because E is invariant under f, we obtain a contradiction. So E contains no interval.

3, Suppose E is a proper totally invariant closed set. It follows that $c \notin E$. So both f and f^{-1} are continuous at each point of $x \in E$, which implies that E' is backward invariant and forward invariant.

4, $x \in I$, $\alpha(x) = \bigcap_{n\geq 0} \overline{\bigcup_{k\geq n}\{f^{-k}(x)\}}$. For each $n \in N$, $\bigcup_{k\geq n}\{f^{-k}(x)\}$ is invariant under f^{-1}, it follows

$$f^{-1}(\alpha(x)) \subseteq \alpha(x).$$

Remember $y \in \alpha(x)$ is equivalent to the fact that there exists a sequence $\{x_k\} \subset I$ and an increasing sequence $\{n_k\} \subset N$ such that $f^{n_k}(x_k) = x$ and $x_k \to y$ as $k \to \infty$. Assume $y \in \alpha(x)$, we have $f(y) \in \alpha(x)$ if y is not the discontinuity c. If $y = c$ we consider c as two points $c+$ and $c-$. It is easy to see $c+ \in \alpha(x)$ implies $f(c+) = 0 \in \alpha(x)$, and $c- \in \alpha(x)$ implies $f(c-) = 1 \in \alpha(x)$. So we conclude

$$f(\alpha(x)) \subseteq \alpha(x).$$

5, If p is periodic with period m, then $p \in f^{-km}(p)$ for all $k \in N$, which implies that $p \in \alpha(p)$. Since $\alpha(p)$ is totally invariant, we know that $f^{-n}(p) \subset \alpha(p)$. We have $\alpha(p) \supseteq \overline{\bigcup_{n\geq 0} f^{-n}(p)}$. The converse inclusion $\alpha(p) \subseteq \overline{\bigcup_{n\geq 0} f^{-n}(p)}$ is trivial.

6, Since E is totally invariant, it follows $f^{-1}(A\cap E) = f^{-1}(A)\cap f^{-1}(E) = f^{-1}(A) \cap E$. The first equality holds.

The inclusion $f(A \cap E) \subseteq f(A) \cap E$ is trivial. To prove the converse inclusion, suppose $x \in f(A) \cap E$. $x \in f(A)$ implies that $f(y) = x$ for some $y \in A$. $x \in E$ implies $\{f^{-1}(x)\} \subseteq E$. As a result, one gets $y \in E$. Hence, $y \in A \cap E$, which implies $f(A) \cap E \subseteq f(A \cap E)$. The second equality follows. □

For expanding Lorenz map f, Lemma 1 indicates that each totally invariant closed set containing c is trivial. It is possible that all the totally invariant closed set of f is trivial. If this is the case, f is prime because $\alpha(x) = I$ for all $x \in I$. The problem is: if f admits a proper totally invariant closed set, is it always renormalizable? The Theorem A gives a positive answer. This is why we introduce totally invariant closed set here.

It is time to prove Theorem A.

Proof. Suppose E is a proper totally invariant closed set of f. e_+, e_-, ℓ and r are defined as in the statement of Theorem A.

At first, we prove $f^\ell(e_-) = e_-$.

By the definition of ℓ, f^ℓ is continuous and monotone on $(e_-,\ c)$. Put z be the unique point in $(e_-,\ c)$ such that $f^\ell(z) = c$. Since E is totally invariant, we conclude that $f^\ell(e_-) = e_-$. In fact, if $f^\ell(e_-) > e_-$, then $e_- < f^\ell(e_-) < f^\ell(z) = c$, which contradicts to the definition of e_- because $f^\ell(e_-) \in E \bigcap(e_-,\ c)$. If $f^\ell(e_-) < e_-$, there must be some point $y \in (e_-,\ c)$

such that $f^\ell(y) = e_-$, which contradicts also the definition of e_- and the total invariance of E under f.

Similarly, we can prove $f^r(e_+) = e_+$.

Since E is totally invariant, we conclude

$$f^\ell((e_-,\ c)) = (e_-,\ f^\ell(c-)) \subseteq (e_-,\ e_+).$$

If, on the contrary, $f^\ell(c-) > e_+$, there exists $z \in (e_-,\ c)$ such that $f^\ell(z) = e_+$, which implies $z \in E$ because E is totally invariant. We arrive a contradiction.

Similarly,

$$f^r((c,\ e_+)) \subseteq (e_-,\ e_+).$$

It follows that the map $R_E f$ defined in Theorem A is a renormalization of f.

Now we prove the second statement. Suppose $g = (f^m,\ f^k)$ is a renormalization map of f with renormalization interval $[a,\ b] := [f^k(c+),\ f^m(c-)]$. Put

$$F_g = \{x \in I, orb(x) \cap (a,\ b) \neq \emptyset\},$$

$$J_g = \{x \in I, orb(x) \cap (a,\ b) = \emptyset\}.$$

Since $F_g = \bigcup_{n\geq 0} f^{-n}((a,\ b))$, F_g is a totally invariant open set. And $J_g = I \backslash F_g$ is a totally invariant closed set of f. $R_{J_g} = g$ follows from the following Lemma 2.

The proof of Theorem A is completed.

□

Lemma 2. *Let f be an expanding Lorenz map, E be a proper totally invariant closed set of f, $J_E = (e_-,\ e_+)$ be the critical interval of E. $N((e_-,\ c)) = \ell$, $N((c,\ e_+)) = r$, $[a,\ b] = [f^r(c+),\ f^\ell(c-)]$. Then*

$$I \backslash E = \bigcup_{n\geq 0} f^{-n}(J_E) = \bigcup_{n\geq 0} f^{-n}((a,\ b)). \tag{3. 3}$$

Proof. Since E is totally invariant, we have $E \cap \bigcup_{n\geq 0} f^{-n}(J_E) = \emptyset$, which indicates $\bigcup_{n\geq 0} f^{-n}(J_E) \subseteq I \backslash E$.

$x \in I \backslash E$, there exists an open interval U such that $x \in U \subset I \backslash E$ because $I \backslash E$ is open. Furthermore, we can assume that U is the maximal open interval containing x which belongs to $I \backslash E$. Since f is expanding, $N(U) < \infty$, and $c \in f^{N(U)}(U)$. It follows that $f^{N(U)}(U) \subset J_E$ because $f^{N(U)}(U) \cap E = \emptyset$. The maximality of U indicates that $f^{N(U)}(U) = J_E$. As a result, $f^{N(U)}(x) \in J_E$, i.e., $x \in f^{-N(U)}(J_E)$. Hence, $I \backslash E \subseteq \bigcup_{n\geq 0} f^{-n}(J_E)$. We have proved $I \backslash E = \bigcup_{n\geq 0} f^{-n}(J_E)$.

Since E is a totally invariant closed set, we have $(a,\ b) \subseteq J_E$. It follows that $\bigcup_{n\geq 0} f^{-n}(J_E) \supseteq \bigcup_{n\geq 0} f^{-n}((a,\ b))$.

$\forall x \in (e_-,\ c)$, put $\ell_x = N((e_-,\ x))$. We get $f^{\ell_x}(x) \in (c,\ b)$ by the total invariance of E. So we conclude $(e_-,\ c) \subset \bigcup_{n\geq 0} f^{-n}((a,\ b))$. By the same argument, we can obtain $(c,\ e_+) \subset \bigcup_{n\geq 0} f^{-n}((a,\ b))$. So $J_E = (e_-,\ e_+) \subset \bigcup_{n\geq 0} f^{-n}((a,\ b))$. Since $\bigcup_{n\geq 0} f^{-n}((a,\ b))$ is backward invariant, we have $\bigcup_{n\geq 0} f^{-n}(J_E) \subseteq \bigcup_{n\geq 0} f^{-n}((a,\ b))$.

The proof of (3. 3) is completed.

□

According to Theorem A, there is a remarkable property of totally invariant closed set E: the endpoints of the critical interval of E are periodic, which is crucial for us to construct the renormalization R_E. Compare this to the corresponding results in [10], our proof is more direct and simple.

4. Minimal totally invariant closed set and renormalizability

Applying Theorem A, the renormalizability problem of expanding Lorenz map reduces to check wether it admits a proper totally invariant closed set. In this section, we shall construct the minimal totally invariant closed set of f. We begin with the minimal period of the periodic orbits of f, and show that the periodic orbit O with minimal period of f is unique. Then we conclude that periodic orbit O has a special locally eventually onto property, which enables us to show that the α-limit set $D := \alpha(O) = \overline{\bigcup_{n\geq 0} f^{-n}(O)}$ is the minimal totally invariant closed set of f. By Theorem A, f is renormalizable if and only if $D = I$. Based on the structure of D, we can prove Theorem B. Using Theorem B, we can obtain two Propositions about the *l.e.o.* property and *trivial renormalization* of f.

4.1. Periodic orbit with minimal period.

In this subsection, we will show that the periodic orbit with minimal period is very special because it relatives to the minimal totally invariant closed set.

The period of periodic points of Lorenz map was well studied by Alsedà et al in [3]. Ding and Fan Showed that a Lorenz map is asymptotically periodic if and only if the derived set of $C(f) = \bigcup_{n\geq 0} f^{-n}(c)$ is countable [5]. The following Lemma 3 determines the minimal period κ of periodic points of expanding f via the preimages of c.

Lemma 3. *Suppose f is an expanding Lorenz map without fixed point. The minimal period of f is equal to $\kappa = m + 2$, where*

$$m = \min\{i \geq 0 : f^{-i}(c) \in [f(0), f(1)]\}. \tag{4. 1}$$

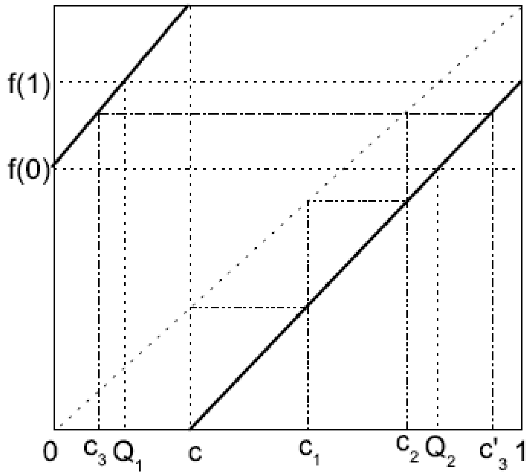

FIGURE 1. A Lorenz map with m=2

Proof. We prove the result by two steps: we first prove that f has $(m+2)$-periodic point, then we show that f has no periodic point with period less than $m+2$.

Notice that $f^{-m}(c) \in [f(0), f(1)]$ and that $[f(0), f(1)]$ consists of those points each of which has two preimages. Let c_{m+1} and c'_{m+1} with $c_{m+1} < c'_{m+1}$ be the two preimages of $f^{-m}(c)$. The set $f^{-i}(c)$ for $i = 0, 1, \cdots, m$ is a singleton. Denote $c_i := f^{-i}(c)$, $i = 0, 1, \cdots, m$. Let $Q_1 \in (0, c)$ and $Q_2 \in (c, 1)$ be the points such that $f(Q_1) = f(1)$ and $f(Q_2) = f(0)$. See Figure 1 for an intuitive picture of $m = 2$.

Since m is the smallest integer such that $f^{-m}(c) \in [f(0), f(1)]$, we have

$$c_{m+1} \leq Q_1 < c_i < Q_2 \leq c'_{m+1} \qquad (0 \leq i \leq m). \tag{4. 2}$$

Let c_{i_0} be the minimal point in $\{c, c_1, \cdots, c_m\}$. For interval $[c_{m+1}, c_{i_0}]$, by (4. 2), we obtain that

$$[c_{m+1},\ \ c_{i_0}] \xrightarrow{f^{i_0}} [c_{m+1-i_0},\ \ c] \xrightarrow{f} [c_{m-i_0},\ \ 1] \supseteq [c_{m-i_0},\ \ c'_{m+1}]$$

$$[c_{m-i_0},\ \ c'_{m+1}] \xrightarrow{f^{m-i_0}} [c,\ \ c_{i_0+1}] \xrightarrow{f} [0,\ \ c_{i_0}] \supseteq [c_{m+1},\ \ c_{i_0}],$$

which implies that

$$[c_{m+1},\ c_{i_0}] \subseteq f^{m+2}([c_{m+1},\ c_{i_0}]).$$

So, f has an $m+2$-periodic point in $[c_{m+1}, c_{i_0}]$.

Fix $1 < j < m+2$. We shall prove that f admits no $j-$periodic point. Put $c_\ell = \min\{c,\ c_1, \ldots,\ c_m\}$, $c_r = \max\{c,\ c_1, \ldots,\ c_m\}$.

Claim: f can not have j-periodic points in $(0,\ c_\ell)$ and $(c_r,\ 1)$.

By the selection of m, we get $N((0,\ c_\ell)) > m$. So f^j is continuous and monotone on $(0,\ c_\ell)$. It is easy to see $f^j(0) > 0$. If f^j admits a fixed point x_* in $(0,\ c_\ell)$, then

$$0 < f^j(0) < f^{2j}(0) < \cdots < f^{nj}(0) < x_*, \qquad n > 0.$$

So $\{f^{nj}(0)\}_n$ approaches to a fixed point of f^j as $n \to \infty$, which is impossible because expanding Lorenz map does not admits attractive periodic orbit.

Similarly, if f^j admits a fixed point in $(c_r,\ 1)$, then $\{f^{nj}(1)\}_n$ will converge to a fixed point of f^j, which contradicts to f is expanding.

Now, for any open interval J with both endpoints in $\{c,\ c_1, \ldots,\ c_m\}$ and $J \cap \{c,\ c_1, \ldots,\ c_m\} = \emptyset$, we know that $N(J) > m$, and at least one of the following cases hold:

- $f^j(J) \cap J = \emptyset$;
- $f^i(J) \subseteq ((0,\ c_\ell) \cup (c_r,\ 1))$ for some $1 < i \leq j$.

It follows that f admits no j-periodic point in J.

□

Remark 2. For m defined in (4. 1), it is interesting to note when $f^{-m}(c)$ is happen to be one of the endpoints of $[f(0),\ f(1)]$. If $f^{-m}(c) = f(0)$, then $c+$, as well as 0, is a periodic point with period $m+2$. If $f^{-m}(c) = f(1)$, then $f^{m+2}(c-) = c-$.

Let P_L be the largest $\kappa-$periodic point in $[0, c)$ and P_R be the smallest $\kappa-$periodic point in $(c, 1]$.

Lemma 4. *Put $L_1 = (P_L,\ c)$, $R_1 = (c, P_R)$. We have*

$$N(L_1) = N(R_1) = \kappa. \tag{4. 3}$$

Proof. We are going to show that both $N(L_1) < \kappa$ and $N(L_1) > \kappa$ are impossible.

Suppose that $N(L_1) < \kappa$. We have $f^{N(L_1)}(P_L) \neq P_L$. By the definition of $N(L_1)$, there exists $z \in L_1$ such that $f^{N(L_1)}(z) = c$. Since P_L is the largest κ-periodic point of f in $[0,\ c)$ and $f^{N(L_1)}(P_L)$ is a κ-periodic point, we must have

$$f^{N(L_1)}(P_L) < P_L.$$

$N(L_1) < \kappa$ implies that $f^{N(L_1)}$ is increasing on $[P_L, c)$. For the interval $(P_L,\ z)$, it follows that

$$f^{N(L_1)}([P_L, z)) = [f^{N(L_1)}(P_L), c) \supseteq [P_L, z).$$

So there exists $P_* \in (P_L, z)$ such that $f^{N(L_1)}(P_*) = P_*$ by the continuity of $f^{N(L_1)}$ on (P_L, z). Hence P_* is a periodic point of f with period $N(L_1) < \kappa$, which contradicts to the minimality of κ.

Assume that $N(L_1) > \kappa$. It follows from (2. 1) that f^κ is continuous and increasing on $L_1 = [P_L, c)$. We have to exclude two cases: $f^\kappa(c-) > c$ and $f^\kappa(c-) < c$, which imply that $N(L_1) > \kappa$ is also impossible.

If $f^\kappa(c-) > c$, there exists $z \in (P_L, c) = L_1$ such $f^\kappa(z) = c$, which contradicts to the minimality of $N(L_1)$.

If $f^\kappa(c-) < c$, by the monotone property of f^κ on $[P_L, c)$, we obtain a decreasing sequence $\{f^{n\kappa}(c-)\}$ with lower bound P_L. Hence,

$$f^{-n}(c) \cap [P_L, c) = \emptyset,$$

which contradicts to the fact that f is expanding.

Thus we have proved $N(L_1) = \kappa$. The equality $N(R_1) = \kappa$ can be similarly proved.

□

Lemma 5. *Suppose that f is an expanding Lorenz map, and $1 < \kappa < \infty$ is the smallest period of the periodic points of f. Then*

i) *f admits a unique $\kappa-$periodic orbit;*

ii) *We have*

$$\bigcup_{i=0}^{\kappa-1} f^i([P_L, P_R]) = I;$$

iii) *For any open interval U containing a κ periodic point, there exists positive integer n such that*

$$\bigcup_{i=0}^{n} f^i(U) = I.$$

Proof. i) Suppose that f has two distinct κ-periodic orbits $\mathrm{Orb}_f(P_L)$ and $\mathrm{Orb}_f(Q_L)$, where P_L and Q_L are the maximal points in L of these two periodic orbits respectively. Without loss of generality, we can suppose P_L is the largest κ-periodic point in L. Put $L_1 = (P_L, c)$ and $L_2 = (Q_L, c)$.

By Lemma 4 and $L_1 \subset L_2$ we know that $N(L_2) \le N(L_1) = \kappa$. If $N(L_2) = \kappa_1 < \kappa$, there exists a point $z \in (Q_L, P_L)$ such that $f^{\kappa_1}(z) = c$. Since $f^{\kappa_1}(Q_L) < c$, it follows that $f^{\kappa_1}(Q_L) < Q_L$ according to the choice of Q_L. So we have

$$f^{\kappa_1}((Q_L, z)) = (f^{\kappa_1}(Q_L), c) \supset (Q_L,\ z),$$

which implies that f admits an κ_1-periodic point in (Q_L, P_L). We obtain a contradiction because κ is the minimal period of periodic points. So we conclude that $N(L_2) = \kappa$ and f^κ is continuous on L_2.

Consider the action of f^{κ} on the interval $[Q_L,\ P_L]$, we have

$$f^{n\kappa}([Q_L,\ P_L]) = [Q_L,\ P_L],$$

which contradicts to the fact f is expanding.

So f admits a unique κ-periodic orbit.

ii) By the proof of i), we get $f^{\kappa}([P_L,\ c)) \supset [P_L,\ c)$ and $f^{\kappa}((c,\ P_R]) \supset (c,\ P_R]$. Observe that

$$f([P_L,\ c)) = [f(P_L),\ 1), \qquad f^2([P_L,\ c)) = [f^2(P_L),\ f(1))$$

$$f((c,\ P_R]) = (0,\ f(P_R)], \qquad f^2((c,\ P_R]) = (f(0),\ f^2(P_R)].$$

Since f is expanding implies $f(0) \le f(1)$, we conclude

$$f^2([P_L, P_R]) \supseteq [f^2(P_L), f^2(P_R)].$$

So

$$f^i([P_L,\ P_R]) \supseteq [f^i(P_L),\ f^i(P_R)] \quad for \quad i = 2, \cdots, \kappa. \tag{4. 4}$$

Hence,

$$\bigcup_{i=0}^{\kappa-1} f^i([P_L,\ P_R]) = I.$$

iii) Write $U = (x,\ y)$.Without loss of generality, we only consider the case $P_L \in U$ because some iterates of U contains P_L. Let $N((P_L,\ y)) = i$, $N((x,\ P_L)) = j$. Since

$$f^i([P_L,\ y)) \supseteq [P_L,\ c), \qquad f^j((x,\ P_L]) \supseteq (c,\ P_R].$$

The conclusion follows from ii).

□

The third statement in the Lemma 5 is called the *local eventually onto* (*l.e.o.*) property of the κ-periodic orbit O. The result holds trivially for each open interval V containing some point in $\bigcup_{n\ge 0} f^{-n}(O)$. This is why the α-limit set of the κ-periodic orbit $D = \overline{\bigcup_{n\ge 0} f^{-n}(O)}$ is so important in describing the renormalization of expanding Lorenz map.

4.2. **Proof of Theorem B.**

Proof. According to Lemma 5 f admits a unique periodic orbit O with period κ. We denote $D := \alpha(O) = \overline{\cup_{n\ge 0} f^{-n}(O)}$ as the α-limit set of the κ-periodic orbit of f.

(1) By Lemma 1 we know that D is a totally invariant closed set. We shall prove that D is minimal.

Suppose E is a totally invariant closed set. We have two cases:

Case 1: $E \cap (P_L,\ P_R) \neq \emptyset$.

In this case, we can suppose that $(P_L,\ c)\cap E\neq\emptyset$ without loss of generality. Assume that $y\in(P_L,\ c)\cap E$. By Lemma 4 we know that f^κ is continuous on $(P_L,\ c)$ and $f^\kappa((P_L,\ c))\supset(P_L,\ c)$. So there exists $y_1\in(P_L,\ c)\cap E$ and $y_1<y$ such that $f^\kappa(y_1)=y$. Similarly, we can obtain a decreasing sequence $\{y_n\}\subset(P_L,\ c)\cap E$ such that $f^\kappa(y_{n+1})=y_n$, $n=1,\ 2,\cdots$ and $\lim_{n\to\infty}y_n=P_L$. So $P_L\in E$ because E is closed. Hence, $\cup_{n\geq 0}f^{-n}(O)\subset E$ because E is backward invariant. E is closed implies that $D\subseteq E$.

Case 2: $E\cap(P_L,\ P_R)=\emptyset$.

By Lemma 5 ii) we know that $\bigcup_{i=0}^{\kappa-1}f^i([P_L,\ P_R])=I$. So $[P_L,\ P_R]\cap E\neq\emptyset$. The assumption $E\cap(P_L,\ P_R)=\emptyset$ indicates $[P_L,\ P_R]\cap E=\{P_L,\ P_R\}$, which implies that $D=E=O$.

The proof of the minimality of D is completed.

(2) By Theorem A, f is renormalizable if and only if f admits a proper totally invariant closed set. Since D is the minimal totally invariant closed set, we know that f is renormalizable is equivalent to $D\neq I$.

If $D\neq I$, according to Theorem A, we know that R_D is a renormalization of f, where

$$R_Df(x)=\begin{cases} f^\ell(x) & x\in[f^r(c+),c)\\ f^r(x) & x\in(c,f^\ell(c-)].\end{cases}$$

and

$$\begin{aligned} \ell&=N([d_-,c)) \qquad & d_-&=\sup\{x<c:x\in D\},\\ r&=N((c,d_+)) \qquad & d_+&=\inf\{x>c:x\in D\}.\end{aligned}$$

Assume $g=(f^{\ell'},\ f^{r'})$ is a renormalization of f with renormalization interval $(a,\ b)$. By Theorem A there exists a totally invariant closed set

$$E=\{x\in I: orb(x)\cap(a,\ b)=\emptyset\}$$

such that $g=R_E$, and

$$\begin{aligned} \ell'&=N((e_-,\ c)) \qquad & e_-&=\sup\{x<c:\ x\in E\},\\ r'&=N((c,\ e_+)) \qquad & e_+&=\inf\{x>c:\ x\in E\}.\end{aligned}$$

The minimality of D indicates $d_-\leq e_-<c<e_+\leq d_+$, which implies that $\ell\leq\ell'$ and $r\leq r'$.

So R_D is the minimal renormalization.

(3) In order to describe the structure of D, we can consider the following three cases, which cover all possible cases.

- Case A: $c\in D$,
- Case B: $c\notin D$ and $D\cap(P_L,\ P_R)=\emptyset$,
- Case C: $c\notin D$ and $D\cap(P_L,\ P_R)\neq\emptyset$.

Case A: If $c \in D$, the total invariancy of D, together with Lemma 1, implies $D = I$, which is equivalent to f is prime.

Case B: If $c \notin D$ and $D \cap (P_L,\ P_R) = \emptyset$, it follows from the proof of Claim 2 that $D = O$. In this case, one can check easily that $d_- = P_L$ and $d_+ = P_R$ in the definition of R_D. By Lemma 4 we know that $N((P_L,\ c)) = N((c,\ P_R)) = \kappa$. It follows R_D is periodic.

Conversely, assume that the minimal renormalization R_D is periodic. Follows from the definition of R_D, we know that the renormalization interval of R_D is $(f^\kappa(c+),\ f^\kappa(c-)) \subseteq (P_L,\ P_R)$ and the critcal interval of R_D is $(P_L,\ P_R)$. Consider the critical interval $(P_L,\ P_R)$, it follows $D \cap (P_L,\ P_R) = \emptyset$. So we get $D = O$ as in Case 2 of the proof of (1).

Case C: If $c \notin D$ and $D \cap (P_L,\ P_R) \neq \emptyset$, it is necessary to prove D is a Cantor set, the equivalence between D is a Cantor set and R_D is not periodic is obvious.

By Lemma 1, $c \notin D$ implies D is nowhere dense. Now we show that D is perfect, i.e., $D = D'$.

Since $D = O$ is equivalent to

$$f^\kappa((P_L,\ P_R)) = (P_L,\ P_R), \tag{4. 5}$$

$D \cap (P_L,\ P_R) \neq \emptyset$ implies (4. 5) is not true. Without loss of generality, we can suppose that $f^\kappa(c+) < P_L$. Then there exists $y_1 \in (c,\ P_R)$ such that $f^\kappa(y_1) = P_L$. And there exists $y_2 \in (c,\ P_R)$ and $y_2 > y_1$ such that $f^\kappa(y_2) = y_1$, i.e., $f^{2\kappa}(y_2) = P_L$. Repeat the above arguments, we can obtain an increasing sequence $\{y_n\}$ in $(c,\ P_R)$ such that $f^{n\kappa}(y_n) = P_L$ and $y_n \to P_R$ as $n \to \infty$. Since $\{y_n\}$ are preimages of P_L, we know that $\{y_n\} \subset D$. It follows P_R is a limit point of D, i.e. $P_R \in D'$. By Lemma 1 we know that D' is backward invariant, so $\cup_{n\geq 0} f^{-n}(P_R) \subset D'$. Therefore, $D \subset D'$, D is a perfect set.

Hence D is a Cantor set.

Theorem B is proved.

□

We say that a Lorenz map is *l.e.o.* if for any open interval U, there exists positive integer n depending on U, such that $\bigcup_{i=0}^{n} f^i(U) = I$. By Lemma 5, f is *l.e.o.* if and only if $D = I$. By Theorem B, f is prime if and only if f is *l.e.o.*, i.e., $D = I$. This result parrels to Lemma 3 in Glendinning and Sparrow [10], remember that our definition of renormalization is different from [10].

Glendinning and Sparrow [10] described the *l.e.o.* property as follows: f is said to be *locally eventually onto* if for each open interval U, there exists subintervals U_1, U_2 of U, and positive integers n_1, n_2 such that f^{n_1} and f^{n_2}

map U_1 and U_2 homeomorphically to $(0,\ c)$ and $(c,\ 1)$, respectively. The following proposition relates two different definitions of $l.e.o.$.

Proposition 1. *The two definitions of l.e.o. are coincide when* $\kappa \leq 2$.

Proof. It is necessary to show that our definition of $l.e.o.$ reduces to Glendinning and Sparrow's definition when $\kappa \leq 2$. The converse is trivial.

Now suppose f is prime and $\kappa \leq 2$. There are two cases: $\kappa = 1$ and $\kappa = 2$.

If $\kappa = 1$, at least one of the following holds:

$$f(0) = 0 \ \ and \ \ f(1) = 1.$$

Without loss of generality, we suppose $f(0) = 0$ (the case $f(1) = 1$ can be deal with similarly). For any open interval $U = (x,\ y)$, let z_0 be the point in U such that $f^{N(U)}(z_0) = c$, and z_1 be the point in $(z_0,\ y) \subset U$ such that $f^{N((z_0,\ y))}(z_1) = c$. By the definition of Lorenz map and $f(0) = 0$ we obtain

$$f^{N((z_0,\ y))+1}((z_0,\ z_1)) = (0,\ 1).$$

So there exists positive integers n and a subinterval $V \subseteq U$ such that f^n maps V to $(0,\ 1)$ homeomorphcally, which implies that f is locally eventually onto.

For the case $\kappa = 2$. Suppose f is prime, let $P_L < c < P_R$ be the $2-$periodic points. By Lemma 4, we know that $N((P_L,\ c)) = N((c,\ P_R)) = 2$, so $N((P_R,\ 1)) = 1$ because $f((P_L,\ c)) = (P_R,\ 1)$. Let x_1 be the point in $(P_R,\ 1)$ such that $f^2(x_1) = c$, y_1 be the point in $(P_R,\ 1)$ such that $f(y_1) = c$. Consider the interval $J_1 = (x_1,\ y_1)$, one can check that $f^2(J_1) = (c,\ 1) \supset J_1$. There exists an subinterval $J_2 \subset J_1$ so that $f^2(J_2) = J_1$. So we can obtain a sequence of nested intervals $\{J_n\}_n$, $J_n = (x_n,\ y_n)$ satisfy:

$$J_{n+1} \subset J_n, \quad f^2(J_{n+1}) = J_n, \quad f^{2n}(J_n) = (c,\ 1), \quad n = 1,\ 2,\ \cdots.$$

Since $\{x_n\}$ and $\{y_n\}$ are monotone and f is expanding, the length of $|J_n| \to 0$ as $n \to \infty$.

Now we prove that f is $l.e.o.$ in the sense of Glendinning and Sparrow. It is necessary to check the $l.e.o.$ conditions for intervals containing P_R, because f is prime implies that any open interval contains a subinterval which can be mapped homeomorphically to an open interval containing P_R. For any open interval F containing P_R, we can find subinterval $J_i \subset F$, which implies that f^{2i} maps J_i homeomorphically to $(c,\ 1)$ by the construction of $\{J_n\}$. Furthermore, $(c,\ 1)$ contains an interval $(c,\ y_1)$, which can be mapped by f homeomorphically to $(0,\ c)$. So J_i contains a subinterval (x_i, z_i) such that $f^{2i+1}((x_i,\ z_i)) = (0,\ c)$. Hence, f is $l.e.o.$ in the sense of Glendinning and Sparrow.

□

The exact formulation of *l.e.o.* varies in the literatures. For the definition we use, we mention the following:

(1) The *l.e.o.* property is just the strongly transitive property in Parry [25].
(2) It agrees with the one in [10], when $\kappa \leq 2$;
(3) f is prime if and only if f is *l.e.o.*;
(4) The *l.e.o.* property of expanding Lorenz map comes from the *l.e.o.* property of the periodic orbit with minimal period. According to Lemma 5, for expanding Lorenz map f, the minimal totally invariant closed set D of f admits the *l.e.o.* property: for each open interval U satisfying $U \cap D \neq \emptyset$, there exists integer $n > 0$ so that $\bigcup_{i=0}^{n} f^i(U) = I$. As a result, f is *l.e.o.* if and only if $D = I$.

Proposition 2. *Let f be an expanding Lorenz map. If $\kappa > 2$, then there exists a Lorenz map g with minimal period less than κ, such that f is renormalizable if and only if g is renormalizable. Moreover, if f is renormalizable, then the minimal renormalization of f is periodic if and only if the minimal renormalization of g is periodic.*

Proof. By Lemma 3, $\kappa > 2$ means $c \notin [f(0),\ f(1)]$. We have two cases: $c < f(0)$ or $c > f(1)$.

For the case $c < f(0)$, the following map

$$g(x) = \begin{cases} f^2(x) & x \in [0,\ c) \\ f(x) & x \in (c,\ f(1)]. \end{cases}$$

is an expanding Lorenz map with minimal period less than κ, and

$$orb(x, g) = orb(x, f) \cap [0,\ f(1)]. \tag{4. 6}$$

If $c > f(1)$, the following

$$g(x) = \begin{cases} f(x) & x \in [f(0),\ c) \\ f^2(x) & x \in (c,\ 1]. \end{cases}$$

is also an expanding Lorenz map with minimal period less than κ, and

$$orb(x, g) = orb(x, f) \cap [f(0),\ 1]. \tag{4. 7}$$

See Figure 2 (Heavy Lines) for the intuitive pictures of g.

Denote O_f and O_g as the periodic orbit with minimal period of f and g, and $D(f)$ and $D(g)$ as the minimal totally invariant closed set of f and g, respectively.

If $c < f(0)$, by (4. 6), we get $O_g = O_f \cap [0,\ f(1)]$, and $D(g) = D(f) \cap [0,\ f(1)]$. It follows that $D(f) = I$ is if and only if $D(g) = [0,\ f(1)]$, and $D(f) = O_f$ if any only if $D(g) = O_g$.

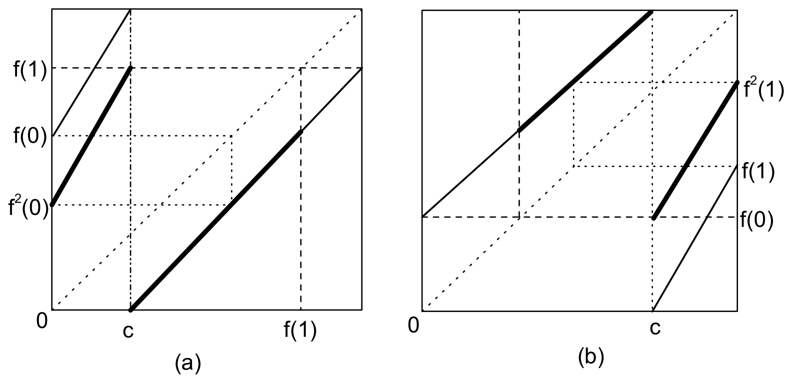

FIGURE 2. Trivial renormalizations, the pictures of g: (a) $c < f(0) < f(1)$, (b) $f(0) < f(1) < c$.

If $c > f(1)$, by (4. 7), we obtain $O_g = O_f \cap [f(0),\ 1]$, and $D(g) = D(f) \cap [f(0),\ 1]$. It follows that $D(f) = I$ is if and only if $D(g) = [f(0),\ 1]$, and $D(f) = O_f$ if any only if $D(g) = O_g$.

In both cases, according to Theorem B, we know that f is renormalizable if and only if g is renormalizable. Moreover, if f is renormalizable, the minimal renormalization of f is periodic if and only if the minimal renormalization of g is periodic. □

Remark 3. (1) In the proof of Proposition 2, the g we constructed can be regarded as some kind of renormalization with $\ell + r = 3$. This kind of renormalization is called trivial renormalization in [10]. The statement in Proposition 2 is just the the fact that f can be renormalized trivially if and only if $\kappa > 2$.

(2) Since $\kappa = 2$ if and only if $c \in [f(0),\ f(1)]$ (cf. Lemma 3), we can obtain the minimal period of f from consecutive trivial renormalizations.

5. Consecutive renormalizations: α-limit set and nonwandering set decomposition

Thanks to Theorem B, the minimal renormalizaion of renormalizable expanding Lorenz map always exists. We can define a renormalization operator R from the set of renormalizable expanding Lorenz maps to the set of expanding Lorenz maps. For each renormalizable expanding Lorenz map, $Rf := R_D f$, where D is the minimal proper totally invariant closed set of f. Obviously, Rf is also expanding. If Rf is renormalizable, we can obtain $R^2 f := R(Rf)$. In this way, we define $R^n f$ as the minimal renormalization of $R^{n-1}f$ if $R^{n-1}f$ is renormalizable. If the renormalization process can proceed m times, we say that f is m $(0 \le m \le \infty)$ *times renormalizable.* If f is m-renormalizable, then $\{R^i f\}_{i=1}^m$ are all the renormalizations of f. We call $R^i f$ the i-th renormalization of f. The process of consecutive renormalizations can be used to characterize all the α-limit sets and nonwandering set of expanding Lorenz map.

5.1. α-limit set.

Lemma 6. *Let f be an expanding Lorenz map. Each proper totally invariant closed set of f is an α-limit set.*

Proof. Suppose f is m-renormalizable $(0 \le m \le \infty)$, with renormalization intervals $[a_i,\ b_i]$, $i = 1, \cdots, m$. There are m proper totally invariant closed sets for f,

$$E_i = \{x : orb(x) \cap (a_i,\ b_i) = \emptyset\}, \quad i = 1, \cdots, m. \tag{5. 1}$$

We have

$$E_1 \subset E_2 \subset \cdots \subset E_m$$

because $[a_i,\ b_i] \supset [a_{i+1},\ b_{i+1}]$, $0 < i < m$.

Now we prove that E_i is an α-limit set of f for $0 < i \le m$. Put $e_-^i = \sup\{x \in E_i : x < c\}$. According to Theorem A we know that e_-^i is periodic. By Lemma 1, $\alpha(e_-^i)$ is indeed a totally invariant closed set, and $e_-^i \in \alpha(e_-^i)$. We must have $\alpha(e_-^i) = E_k$ for some $k = 1, 2, \ldots, m$, because f admits exact m proper totally invariant closed sets.

Since

$$(e_-^{i-1},\ e_+^{i-1}) \supset (a_{i-1},\ b_{i-1}) \supset (e_-^i,\ e_+^i) \supset (a_i,\ b_i) \supset (e_-^{i+1},\ e_+^{i+1}) \supset (a_{i+1},\ b_{i+1}),$$

by the definition of E_i and E_{i+1}, we know that $e_-^i \notin E_{i-1}$ and $e_-^{i+1} \in E_{i+1} \backslash E_i$.

Observe that $e_-^i \in \alpha(e_-^i)$ and $e_-^i \notin E_{i-1}$ indicate that $k \ge i$, and $e_-^{i+1} \in E_{i+1} \backslash E_i$ implies $k < i+1$, we conclude that $k = i$, *i.e.*, $\alpha(e_-^i) = E_i$. Hence, E_i is an α-limit set. □

5.1.1. *Proof of Theorem C.*

Proof. (1), By Lemma 1 we know that each α-limit set is totally invariant. And by Lemma 6 each totally invariant set is an α-limit set. So totally invariant closed set and α-limit set of f are the same thing in different names. If f is m-renormalizable, then f has exact m proper α-limit sets. Follows from the proof of Lemma 6, all the α-limit sets are $\{E_i\}_{i=1}^m$ defined in (5. 1), and

$$\emptyset = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset I.$$

(2), At first we prove that if the i-th $(0 < i \leq m < \infty)$ renormalization is periodic, then $E_i' = E_{i-1}$.

Suppose $g = R^{i-1}f$. g is an expanding Lorenz map on $[a_{i-1},\ b_{i-1}]$ with discontinuity c. Denote κ_1 as the minimal period of periodic points of g, O_1 as the κ_1-periodic orbit of g, and P_L' and P_R' are two adjacent κ_1-periodic point of g with $P_L' < c < P_R'$. By Lemma 1 and the proof of Lemma 6 we know that $E_i = \overline{\bigcup_{n\geq 0} f^{-n}(P_L')}$.

Put

$$e_-^{i-1} = \sup\{x \in E_{i-1},\ x < c\}, \qquad e_+^{i-1} = \inf\{x \in E_{i-1},\ x > c\},$$

$$\ell' = N((e_-^{i-1},\ c)), \qquad r' = N((c,\ e_+^{i-1})).$$

According to the definition of minimal renormalization, we have $e_-^{i-1} < a_{i-1} \leq P_L' < c < P_R' \leq b_{i-1} < e_+^{i-1}$.

By assumption, the minimal renormalization of g is periodic, it follows from Theorem B that the minimal totally invariant closed set of g is O_1, which implies P_L' is an isolated point of E_i. So $E_i' \neq E_i$.

Observe that $f^{\ell'}((e_-^{i-1},\ c)) = (e_-^{i-1},\ b_{i-1})$, there exists a decreasing sequence $\{x_n\}$ in $E_{i-1} \cap (e_-^{i-1},\ c)$ such that

$$f^{\ell'}(x_1) = P_R', \quad f^{\ell'}(x_{n+1}) = x_n, \quad n = 1,\ 2,\ \cdots$$

and $x_n \to e_-^{i-1}$ as $n \to \infty$. So $e_-^{i-1} \in E_i'$.

By Lemma 1 we know E_i' is also a totally invariant closed set, we have

$$E_{i-1} = \overline{\bigcup_{n\geq 0} f^{-n}(e_-^{i-1})} \subseteq E_i' \neq E_i.$$

It follows $E_i' = E_{i-1}$.

Now we show that if the i-th renormalization $R^i f$ is not periodic, then E_i a Cantor set. From the proof of first part, we know that $E_i = \overline{\bigcup_{n\geq 0} f^{-n}(P_L')}$. Since the i-th renormalization is not periodic, the minimal totally invariant closed set of $R^{i-1}f$ is a Cantor set. So E_i admits no isolated point in $[a_{i-1},\ b_{i-1}]$. $E_i' \cap [a_{i-1},\ b_{i-1}] \neq \emptyset$, which implies that $E_i' = E_i$.

(3) Now we are ready to characterize the α-limit set of every point in I. At first, we describe the set $\{x \in I, \alpha(x) = D\}$, where D is the minimal totally invariant closed set of f.

Claim: $\alpha(x) = D$ if and only if $x \notin orb([a,\ b])$, where $[a,\ b]$ is the renormalization interval of the minimal renormalization R_D.

Suppose the minimal renormalization $Rf = R_D f := (f^\ell, f^r)$. It follows that

$$orb([a,\ b]) = \bigcup_{n\geq 0} f^n([a,\ b]) = \left(\bigcup_{n=0}^{\ell-1} f^n([a,\ c])\right) \bigcup \left(\bigcup_{n=0}^{r-1} f^n([c,\ b])\right)$$

is the union of finite closed intervals, and $orb([a,\ b])$ is forward invariant under f.

Since D is the minimal totally invariant closed of f, by Lemma 6, D is also the minimal α-limit set of f. So $\alpha(x) \supset D$ for all $x \in I$.

Let D_1 be the minimal totally invariant closed set of the minimal renormalization $R_D f$. It follows that $D_1 \cap D = \emptyset$, and $D_1 \subset E_2$. If $x \notin orb([a,\ b])$, then $f^{-n}(x) \cap orb([a,\ b]) = \emptyset$ because $orb([a,\ b])$ is forward invariant under f. So $\alpha(x)$ is disjoint with the interior of $orb([a,\ b])$, which indicates $\alpha(x) \cap D_1 = \emptyset$. Hence, $\alpha(x) \neq E_2$, i.e., $\alpha(x) = D = E_1$.

On the other hand, by the minimality of D_1, $\alpha(x, R_D f) = D_1$ for all $x \in [a,\ b]$. For $x \in [a,\ b]$, since $orb(x, R_D f) = orb(x, f) \cap [a,\ b]$, we see that $\alpha(x) \supset \alpha(x, R_D f)$. So $\alpha(x) \cap D_1 \neq \emptyset$, which implies that $\alpha(x) \neq D$ for $x \in [a,\ b]$. Notice that $\alpha(x) \subseteq \alpha(f(x))$, we conclude $\alpha(x) \neq D$ for all $x \in orb([a,\ b])$.

The proof of the Claim is completed.

For $0 \leq i \leq m$, we denote $[a_i,\ b_i]$ as the renormalization interval of the i-th renormalization $R^i f$, and D_i as the minimal totally invariant closed set of $R^i f$.

By the Claim we know that $\alpha(x) = E_1$ if and only if

$$x \in I \backslash orb([a_1,\ b_1]) = orb([a_0,\ b_0]) \backslash orb([a_1,\ b_1]).$$

For the case $i = 2 \leq m$, we consider the map $Rf := R_D f$ on $[a_1,\ b_1]$. According to the Claim, we obtain that $\alpha(x,\ Rf) = D_1$ if and only if $x \notin orb([a_2,\ b_2]$. It follows that $\alpha(x) = E_2$ if and only if

$$x \in orb([a_1,\ b_1]) \backslash orb([a_2,\ b_2]).$$

Repeat the above arguments, we conclude $\alpha(x) = E_i$ if and only if

$$x \in orb([a_{i-1},\ b_{i-1}]) \backslash orb([a_i,\ b_i]) \quad for \quad 0 < i \leq m.$$

If $m < \infty$, $R^m f$ is prime on $[a_m,\ b_m]$, $\alpha(x, R^m f) = [a_m,\ b_m]$ for all $x \in [a_m,\ b_m]$. By Lemma 1, the totally invariant closed set containing $[a_m,\ b_m] \ni c$ is I, we conclude that $\alpha(x) = I$ for all $x \in orb([a_m,\ b_m])$.

For the case $m = \infty$, put $A = \cap_{i\geq 1}^{m} orb([a_i,\ b_i])$, it is known that $A := \overline{orb(c+)} = \overline{orb(c-)}$ (cf. [10], or Theorem D to be proved in this section), which is a Cantor set. Since $c \in A$, the totally invariant closed set containing A is I. As a result, $\alpha(x) = I$ for all $x \in A$.

□

5.1.2. *Example: α-limit set with given depth.*

We can use Theorem C to construct countable α-limit set with given depth.

Consider the piecewise linear symmetric Lorenz map: $1 < a \leq 2$,

$$f_a(x) = \begin{cases} ax + 1 - \frac{1}{2}a & x \in [0, \frac{1}{2}) \\ a(x-1) & x \in (\frac{1}{2}, 1]. \end{cases} \tag{5. 2}$$

According to Glendinning [8] and Palmer [22] , f_a can only be periodically renormalized finite times. Suppose $a \in [2^{2^{-(m+1)}},\ 2^{2^{-m}}]$, Parry [26] proved that f_a can be (periodically) renormalized m times. In this case, by Theorem A and Theorem C, f_a has exact m different α-limit sets. Let p_i be one of the 2^i-periodic point of f_a,

$$E_i = \overline{\bigcup_{n\geq 0} f_a^{-n}(p_i)}, \qquad i = 1,\ \cdots, m.$$

Then $\{E_i\}_{i=1}^m$ is the cluster of α-limit sets of f_a. Moreover, according to Theorem C, $E_n^{(i)} = E_{n-i}$, $i = 1,\ 2,\ \ldots,\ m$. So E_m is a countable closed set and the m-the derived set $E_m^{(m)}$ is empty. The depth of E_m is m. See Figure 3 for an intuitive picture of a cluster of α-limit sets with $m = 4$, $a = (2^{2^{-(m+1)}} + 2^{2^{-m}})/2 \approx 1.03308546\cdots$.

5.2. **Nonwandering set decomposition.**

The following Lemma 7 indicates that the dynamics on the minimal totally invariant closed D is indecomposable.

Lemma 7. *Let f be an expanding Lorenz map with $1 \leq \kappa < \infty$, D be its minimal totally invariant closed set. Then $f : D \to D$ is l.e.o., and $\Omega(f|_D) = D$.*

Proof. By Theorem B, there are three cases: $D = O$, $D = I$ and $O \subset D \subset I$, where O is the unique κ-periodic orbit. If $D = O$ or $D = I$, applying Theorem B, it is easy to see $f : D \to D$ is *l.e.o.*, and $\Omega(f|_D) = D$.

For the case $O \subset D \subset I$, we know that D is a totally invariant Cantor set of f. We shall prove that $f : D \to D$ is *l.e.o.*. Suppose A is an open set of D (in the induced topology from I), there exists an open set U of I such that $A = U \cap D$. By the *l.e.o.* property of O, there is positive integer N such that

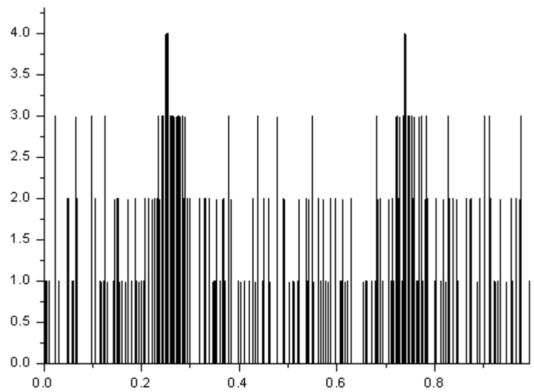

FIGURE 3. An intuitive picture of the α-limit sets of symmetric piecewise linear Lorenz map with $m = 4$, $a = (2^{2^{-(m+1)}} + 2^{2^{-m}})/2 \approx 1.03308546\cdots$. E_i is consists of points whose height equal to $(5-i)$, $i = 1, 2, 3, 4$.

$\bigcup_{n=0}^{N} f^n(U) = I$. It follows from Lemma 1 that $f^n(U \cap D) = f^n(U) \cap D$. We have

$$\bigcup_{n=0}^{n} f^n(A) = \bigcup_{n=0}^{n} f^n(D \cap U) = (\bigcup_{n=0}^{n} f^n(U)) \cap D = D,$$

which implies that $f : D \to D$ is *l.e.o.*. As a result, $\Omega(f|_D) = D$. □

5.2.1. *Proof of Theorem D.*

Now we prove the nonwandering set decomposition of expanding Lorenz map. As mentioned before, Glendinning and Sparrow [10] gave a decomposition based on kneading theory.

Proof. If $m = 0$, f is prime. By Theorem B and Theorem C, we know that f is *l.e.o.*, $A = I = \Omega(f)$, and $\alpha(x) = I$, $\forall x \in I$.

Now suppose $m > 0$, i.e., f is renormalizable. By Theorem A and Theorem C, all the totally invariant closed sets of f are:

$$\emptyset = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset E_{m+1} = I.$$

($E_{m+1} = I$ is just a notation when $m = \infty$). We can decompose $I = E_{m+1}$ as follows:

$$I = (E_1 \backslash E_0) \cup (E_2 \backslash E_1) \cup \cdots \cup (E_m \backslash E_{m-1}) \cup (E_{m+1} \backslash E_m).$$

Since E_i is totally invariant, $E_i \backslash E_{i-1}$ $(i = 1, \ldots, m)$ and $I \backslash E_m$ are invariant under f. It follows that

$$\begin{aligned} \Omega(f) &= \Omega(f) \bigcap (\bigcup_{i=1}^{m+1} (E_i \backslash E_{i-1})) \\ &= \bigcup_{i=1}^{m+1} (\Omega(f) \cap (E_i \backslash E_{i-1})) \\ &:= \bigcup_{i=1}^{m} \Omega_i \cup A \end{aligned}$$

where $\Omega_i = \Omega(f) \cap (E_i \backslash E_{i-1})$ and $A = \Omega(f) \cap (I \backslash E_m)$.

In what follows, we characterize Ω_i and A. For $0 < i \leq m$, the $(i-1)$-th renormalization of f is

$$R^{i-1} f(x) = \begin{cases} f^{\ell_{i-1}}(x) & x \in [a_{i-1}, c) \\ f^{r_{i-1}}(x) & x \in (c, b_{i-1}]. \end{cases} \tag{5. 3}$$

D_{i-1} is the minimal totally invariant closed set of $R^{i-1} f$. We claim that

$$D_{i-1} = E_i \cap [a_{i-1}, b_{i-1}].$$

In fact, it is easy to see $D_{i-1} \subseteq E_i \cap [a_{i-1}, b_{i-1}]$. On the other hand, $x \in [a_{i-1},\ b_{i-1}] \backslash E_i$ indicates $orb(R^{i-1} f, x) \cap [a_i,\ b_i] \neq \emptyset$. By Lemma 2, $x \notin D_{i-1}$. We obtain $[a_{i-1},\ b_{i-1}] \backslash D_{i-1} \subseteq [a_{i-1},\ b_{i-1}]$, which implies $D_{i-1} \supseteq E_i \cap [a_{i-1},\ b_{i-1}]$.

By Lemma 1, we obtain

$$orb(D_{i-1}) = E_i \cap orb([a_{i-1}, b_{i-1}]). \tag{5. 4}$$

Remember that

$$E_i = \{x \in I, orb(x) \cap [a_i,\ b_i] = \emptyset\},\ \ E_{i-1} = \{x \in I, orb(x) \cap [a_{i-1},\ b_{i-1}] = \emptyset\},$$

we conclude

$$D_{i-1} \subset E_i \backslash E_{i-1}$$

because $D_{i-1} \subset [a_{i-1},\ b_{i-1}]$ and $orb(x) \cap [a_i,\ b_i] = \emptyset$, $\forall x \in D_{i-1}$. It follows that $orb(D_{i-1}) \subset E_i \backslash E_{i-1}$.

By Lemma 7, we know that $R^{i-1} f|_{D_{i-1}}$ is *l.e.o.*, and $\Omega(R^{i-1} f|_{D_{i-1}}) = D_{i-1}$. According to (5. 3), $R^{i-1} f$ is the first return map of f on the renormalization interval $I_{i-1} := [a_{i-1},\ b_{i-1}]$, we have

$$orb(x, R^{i-1} f) = orb(x, f) \cap I_{i-1}, \quad \forall x \in I_{i-1}.$$

It follows that $D_{i-1} \subset \Omega(R^{i-1} f) \subset \Omega(f)$, and $orb(D_{i-1}) \subset \Omega(f)$ because $\Omega(f)$ is invariant under f. Hence, we have proved that

$$orb(D_{i-1}) \subseteq \Omega_i := \Omega(f) \cap (E_i \backslash E_{i-1}). \tag{5. 5}$$

In order to prove the converse inclusion

$$orb(D_{i-1}) \supseteq \Omega_i,$$

it is necessary to show that any point in $E_i \backslash (E_{i-1} \cup orb(D_{i-1}))$ is wandering. Remember that $x \in I$ is wandering if and only if there exists an open interval V of x such that $f^k(V) \cap V = \emptyset$ for all $k > 0$.

Fix $x \in E_i \backslash (E_{i-1} \cup orb(D_{i-1}))$.

According to (5. 4), $x \notin orb(D_i) = E_i \cap orb([a_{i-1}, \ b_{i-1}])$ implies $x \notin orb([a_{i-1}, \ b_{i-1}])$. Remember that

$$\begin{aligned} orb([a_{i-1}, \ b_{i-1}]) &= \bigcup_{n \geq 0} f^n([a_{i-1}, \ b_{i-1}]) \\ &= (\bigcup_{n=0}^{\ell_{i-1}-1} f^n([a_{i-1}, \ c])) \bigcup (\bigcup_{n=0}^{r_{i-1}-1} f^n([c, \ b_{i-1}])), \end{aligned}$$

$orb([a_{i-1}, \ b_{i-1}])$ is a closed set.

So $x \notin orb([a_{i-1}, \ b_{i-1}])$ implies there exists an open interval W containing x such that $W \cap orb([a_{i-1}, \ b_{i-1}]) = \emptyset$.

On the other hand, $x \notin E_{i-1}$ indicates $orb(x) \cap (a_{i-1}, \ b_{i-1}) \neq \emptyset$. Let $n > 0$ be the least positive integer such that $f^n(x) \in (a_{i-1}, \ b_{i-1})$. There exists an open interval U containing x such that

$$U \cap (a_{i-1}, \ b_{i-1}) = \emptyset \quad and \quad f^n(U) \subset (a_{i-1}, \ b_{i-1}),$$

because $x \notin (a_{i-1}, \ b_{i-1})$ and f^n is continuous at x. Moreover, we can choose U small enough so that

$$U \cap f^k(U) = \emptyset, \quad 0 < k < n. \tag{5. 6}$$

Put $V = U \cap W$, which is an open interval containing x. We conclude that

$$V \cap f^k(V) = \emptyset, \qquad k > 0.$$

In fact, by (5. 6), it is true for $k < n$. For $k \geq n$, since $f^n(U) \subset (a_{i-1}, \ b_{i-1})$, we have $f^n(V) \subset orb([a_{i-1}, \ b_{i-1}])$, $V \cap f^k(V) = \emptyset$ follows.

Hence, x is a wandering point of f. We finish the proof of

$$\Omega_i = orb(D_{i-1}) = E_i \cap orb([a_{i-1}, \ b_{i-1}]).$$

Lemma 7 indicates that $R^{i-1}f$ is *l.e.o.* on D_{i-1}. It follows that f is *l.e.o.* on $\Omega_i = orb(D_{i-1})$. Moreover, according to Theorem B, D_{i-1} is the periodic orbit with minimal period of $R^{i-1}f$ if the renormalization $R^i f$ is periodic, otherwise, D_{i-1} is a Cantor set. So $\Omega_i = orb(D_{i-1})$ is either a periodic orbit of f or a Cantor set depending on wether the renormalization $R^i f$ is periodic or not. The characterization of Ω_i $(1 \leq i \leq m)$ is completed.

To characterize $A = \Omega(f) \cap (I \backslash E_m)$, we consider two cases: $m < \infty$ and $m = \infty$.

Suppose $m < \infty$. In this case, $R^m f$ is prime. It is *l.e.o.* on $[a_m,\ b_m]$. $\Omega(R^m f) = [a_m,\ b_m]$. So $orb([a_m,\ b_m]) \subset \Omega(f)$. On the other hand, $x \notin E_m$ indicates that $f^n(x) \in [a_m,\ b_m]$ for some positive integer n. Since $orb([a_m,\ b_m])$ is invariant under f, we obtain

$$A = \Omega(f) \cap (I \backslash E_m) = \Omega(f) \cap orb([a_m,\ b_m]).$$

Hence, $A = orb([a_m,\ b_m])$, which is the union of finite closed intervals. It is easy to see that $\omega(x) \subset A$ for $x \notin E_m$ and f is *l.e.o.* on A.

For the case $m = \infty$. It follows that $A = \bigcap_{i=0}^{\infty} orb([a_i,\ b_i])$, where $[a_i,\ b_i]$ is the renormalization interval of the i-th renormalization. A is a Cantor set provided $\bigcap_{i=1}^{\infty} [a_i,\ b_i]$ contains just the single point c. This must be the case. In fact, by induction, one can check easily $N((a_i,\ c)) > i \to \infty$ and $N((c,\ b_i)) > i \to \infty$ as $i \to \infty$, which implies that $\cap_i [a_i,\ b_i] = c$ because f is expanding.

We conclude $A = \omega(c+) = \omega(c-)$. In fact, $\forall y \in A$, and U is an open interval containing y, there are positive integers k, m such that $f^m([a_k,\ c]) \subset U$, which implies that $orb(c+) \cap U \neq \emptyset$. So $A \subseteq \omega(c+)$. On the other hand, $\omega(c+) \subset A$ because $c+ \in A$ and A is invariant. Hence, $A = \omega(c+)$. $A = \omega(c-)$ follows from similar arguments.

$\forall x \in I \backslash E_m$ and $k > 0$, there exists positive integer $N(x)$ such that $f^n(x) \in orb([a_k,\ b_k])$ for $n \geq N(x)$. It follows that $\omega(x) \subset A$. Moreover, $c+ \in \omega(x)$ because $b_k \to c+$ as $k \to \infty$, we get $\omega(x) \supset \omega(c+) = A$. The fact that f is *l.e.o.* on A is obvious.

□

6. Piecewise linear Lorenz map

We apply our Theorems to the renormalization and α-limit set of piecewise linear Lorenz map. A Lorenz map f is said to be *piecewise linear* if it is linear on both intervals $[0, c)$ and $[c, 1)$. Such a map is of the form

$$f_{a,b,c}(x) = \begin{cases} ax + 1 - ac & x \in [0,\ c) \\ b(x - c) & x \in (c,\ 1]. \end{cases} \tag{6. 1}$$

Let $\beta > 1$ and $0 \leq \alpha < 1$. The transformation $T_{\beta,\alpha}$ defined by

$$g_{\beta,\alpha} = \beta x + \alpha \quad (\bmod 1)$$

is called a β-transformation (see [8] and [23]). When $1 < \beta \leq 2$, $g_{\beta,\alpha} = f_{\beta,\beta,c}$ with $c = (1 - \alpha)/\beta$.

The study of β-transformation goes back to Rényi. Based on his bounded distortion principe, Rényi proved that β-transformation admits an acim.

Gelfond [7] and Parry [23] [24] obtained the expression of the density of β-transformation.

In Milnor and Thurston [19] (Section 7), every piecewise monotone and continuous transformation on an interval with positive topological entropy is semi-conjugate to a piecewise linear transformation. Glendinning [8] showed that an expanding Lorenz map can be conjugated to $\beta-$transformation if its renormalizations admit some special forms. In fact, he proved , in our words, an expanding Lorenz map f is conjugated to a β-transformation if and only if f is finitely renormalizable and each renormalization is periodic.

For piecewise linear Lorenz map $f_{a,b,c}$ with $a \geq 1,\ b \geq 1$, Theorem E claims that each renormalization of $f_{a,b,c}$ is periodic. It is easy to see that $f_{a,b,c}$ can only be renormalized finite times. By Glendinning's result, we know that $f_{a,b,c}$ is conjugated to β-transformation. For the cases other than $a \geq 1,\ b \geq 1$, see [6].

It is time to prove Theorem E.

Proof. It is proved in [6] that a piecewise linear Lorenz map $f_{a,b,c}$ is expanding if and only if either $ac + b(1-c) > 1$, or $ac + b(1-c) = 1$ with irrational rotation number. Since the case $a = b = 1$ is a rotation, which is prime in our sense, independent of the rotation number is rational or not. It is necessary to consider $f_{a,b,c}$ with $a \geq 1$, $b \geq 1$ and $ac + b(1-c) > 1$, which is always expanding.

Since the renormalization of piecewise linear Lorenz map is still piecewise linear, Theorem E is proved if we can show that the minimal renormalization of any renormalizable piecewise linear Lorenz map is periodic.

During the proof, we denote the piecewise linear Lorenz map $f_{a,b,c}$ by f. If $\kappa(f) > 2$, by Proposition 2, there is an expanding Lorenz map g with minimal period $\kappa(g) < \kappa(f)$, such that f is renormalizable if and only if g is renormalizable, and if f is renormalizable, then minimal renormalization of f is periodic if and only if the minimal renormalization of g is periodic. Furthermore, since f is piecewise linear with slopes ≥ 1, g is also piecewise linear with slopes ≥ 1.

Applying Proposition 2 several times if necessary, we can assume that $\kappa(f) \leq 2$. Since any expanding Lorenz map with $\kappa(f) = 1$ is prime, it is necessary to consider the case $\kappa := \kappa(f) = 2$.

Suppose that $\kappa = 2$. Let P_L and P_R are the 2-periodic points of f, $P_L < c < P_R$. $D = \alpha(P_L)$ is the minimal totally invariant closed set of f.

According to Theorem A, the minimal renormalization map of f is Rf

$$Rf(x) = \begin{cases} f^{\ell}(x) & x \in [f^r(c+), c) \\ f^r(x) & x \in (c, f^{\ell}(c-)], \end{cases}$$

where

$$\ell = N([p,c)) \qquad p = \sup\{x < c : x \in D\},$$
$$r = N((c,q)) \qquad q = \inf\{x > c : x \in D\},$$

and

$$f^{\ell}(p) = p, \qquad f^{r}(q) = q. \tag{6. 2}$$

Put $L = (p,\ c)$ and $R = (c,\ q)$. $O_p = \{p,\ f(p), \ldots, f^{\ell-1}(p)\}$ and $O_q = \{q,\ f(q), \ldots, f^{r-1}(q)\}$ are the periodic orbits of p and q.

We have

$$P_L \le p < c < q \le P_R.$$

Note that f is linear with slopes a and b on $[0,c)$ and $(c,1]$, we have

$$(f^{\ell})'(p) = \prod_{i=0}^{\ell-1} f'(f^i(p)) = a^{m_1} b^{n_1},$$

$$(f^{r})'(q) = \prod_{i=0}^{r-1} f'(f^i(q)) = a^{m_2} b^{n_2},$$

where

$$m_1 = Card\{i : f^i(p) < c,\ i = 0,\ 1,\ \cdots,\ \ell-1\}, \qquad n_1 = \ell - m_1,$$

$$m_2 = Card\{i : f^i(q) < c,\ i = 0,\ 1,\ \cdots,\ r-1\}, \qquad n_2 = r - m_2.$$

Since Rf is a Lorenz map, we must have

$$|f^{\ell}(L)| = |f^{\ell}((p,\ c))| \ = \ |(p,\ f^{\ell}(c-))| \le |L| \ + \ |R|$$
$$|f^{r}(R)| = |f^{r}((c,\ q))| \ = \ |(f^{r}(c+),\ q)| \le |L| \ + \ |R|,$$

where $|J|$ denote the length interval J.

Notice that f^{ℓ} is linear on $[p,\ c)$ and f^r is linear on $(c,\ q]$, we have

$$a^{m_1} b^{n_1} |L| \le |L| \ + \ |R| \qquad and \qquad a^{m_2} b^{m_2} |R| \le |L| \ + \ |R|,$$

which is equivalent to

$$(a^{m_1} b^{n_1} - 1)(a^{m_2} b^{m_2} - 1) \le 1. \tag{6. 3}$$

i.e.,

$$a^{m_1} b^{n_1} a^{m_2} b^{n_2} \le a^{m_1} b^{n_1} + a^{m_2} b^{n_2}. \tag{6. 4}$$

In what follows we try to find other inequalities about the length of suitable intervals, which, together with (6. 3) and (6. 4), will induce contradictions.

According to the definition of periodic renormalization, the minimal renormalization of f is periodic if and only if $\ell = r = 2$, which is equivalent to

$$(6.\ 5) \qquad f^2([P_L,\ c)) \subset [P_L,\ P_R] \quad \textit{and} \quad f^2((c,\ P_R]) \subset [P_L,\ P_R].$$

So f can not be renormalized periodically is equivalent to the fact that at least one of the inclusion in (6. 5) is not true. We have either

$$(6.\ 6) \qquad f^2([P_L,\ c)) \supset [P_L,\ P_R], \quad \textit{i.e.,} \quad f^2(c-) = f(1) > P_R$$

or

$$(6.\ 7) \qquad f^2((c,\ P_R]) \supset [P_L,\ P_R], \quad \textit{i.e.,} \quad f^2(c+) = f(0) < P_L.$$

For any renormalizable piecewise linear Lorenz map f with slopes ≥ 1, it is necessary to exclude the two inclusions in (6. 6) and (6. 7), which implies that the minimal renormalization of f is periodic. We shall consider the case (6. 6), the remain case (6. 7) can be done similarly.

Let u be the least point in $O_p \bigcup O_q$ satisfying $u > P_R$. We have to exclude two subcases: $u \in O_p$ and $u \in O_q$.

Subcase A1: $u \in O_p$.

In this case, $u = f^{\ell_*}(p)$ for some $\ell_* < \ell$. Since $N((p,\ c)) = \ell$, we have $N((f^{\ell_*}(p),\ f^{\ell_*}(c-))) = N(f^{\ell_*}((p,\ c))) = \ell - \ell_*$.

For the interval $(P_R,\ f^{\ell_*}(c-))$, we get

$$(6.\ 8) \qquad 0 < \ell'_* := N((P_R,\ f^{\ell_*}(c-)) \leq N((f^{\ell_*}(p),\ f^{\ell_*}(c-))) \leq \ell - \ell_*.$$

Let x be the point in $(P_R,\ f^{\ell_*}(c-))$ such that $f^{\ell'_*}(x) = c$. Put

$$U = (P_R,\ x).$$

We have

$$(6.\ 9) \qquad f^{\ell'_*}(U) = f^{\ell'_*}((P_R,\ x)) = (f^{\ell'_*}(P_R),\ c) = (P_L,\ c).$$

By (6. 9) and the definition of u, we obtain

$$\begin{aligned} f^{\ell'_*+2}(U) = f^2((P_L,\ c)) &= (P_L,\ f(1)) \\ &= (P_L,\ c) \cup (c,\ f(1)) \\ &\supset f^{\ell'_*}((P_R,\ x)) \cup (P_R,\ x) \\ &= f^{\ell'_*}(U) \cup U, \end{aligned}$$

i.e.,

$$(6.\ 10) \qquad |f^{\ell'_*+2}(U)| > |f^{\ell'_*}(U)| + |U|.$$

Since f is piecewise linear, $(f^{\ell'_*})'(P_R) = a^{m_*}b^{n_*}$ for some positive integers m_* and n_*. According to (6. 10), we have

$$(6.\ 11) \qquad (ab-1)a^{m_*}b^{n_*} > 1.$$

Now we consider two cases: $u < f(p)$ and $u = f(p)$.

Suppose $u < f(p)$. One can see that the orbit of p visits the left side and the right side of c at least one times before arriving u. So we get

$$a^{m_1}b^{n_1} \geq a^{m_*}b^{n_*}ab.$$

Therefore, by (6. 11), we have

$$(ab-1)a^{m_1}b^{n_1} > ab, \tag{6. 12}$$

which is equivalent to

$$a^{m_1}b^{n_1}a^{m_2}b^{n_2} > a^{m_1-1}b^{n_1-1}a^{m_2}b^{n_2} + a^{m_2}b^{n_2}. \tag{6. 13}$$

Combine (6. 4) and (6. 13) we get

$$a^{m_2}b^{n_2} < ab,$$

which is impossible for $a \geq 1$ and $b \geq 1$ because $m_2 \geq 1$ and $n_2 \geq 1$. The case $u < f(p)$ is not true.

Suppose $u = f(p)$. Since $f(c-) > P_R$ implies that $N((p,\ c)) > N((P_L,\ c)) = 2$, we have $N((u,\ 1)) > N((P_R,\ 1)) = 1$. As a result, we obtain $f(U) = (P_L,\ c)$. The inequality (6. 11) reduces to

$$(ab-1)b > 1, \qquad i.e., \qquad ab^2 > b+1.$$

In this case, it is easy to check that $\ell \geq 3$ and $a^{m_1}b^{n_1} \geq ab^2$. We have

$$(a^{m_1}b^{n_1}-1)(a^{m_2}b^{n_2}-1) \geq (ab^2-1)(a^{m_2}b^{n_2}-1) > (a^{m_2}b^{n_2}-1)b \geq (ab-1)b > 1,$$

which contradicts to (6. 3).

The case $u = f(p)$ is impossible.

So we conclude that if $f^2(c-) > P_R$, then the least point in $(P_R,\ 1) \cap \{O_p,\ O_q\}$ is not belong to O_p.

Subcase A2: $u \in O_q$.

In this subcase, $u = f^{r_*}(q)$ for some $r_* < r$. For the interval $(P_R,\ u)$, we have

$$0 < r'_* := N((P_R,\ u) \leq N((f^{r_*}(c-),\ u)) \leq r - r_*. \tag{6. 14}$$

Let y be the point in $(P_R,\ u)$ such that $f^{r'_*}(y) = c$. Put

$$V = (P_R,\ y).$$

We have

$$f^{r'_*}(V) = f^{r'_*}((P_R,\ y)) = (f^{r'_*}(P_R),\ c) = (P_L,\ c). \tag{6. 15}$$

By (6. 6) and the definition of u, we obtain

$$\begin{aligned} f^{r'_*+2}(V) = f^2((P_R,\ y)) &= (P_L,\ f(1)) \\ &= (P_L,\ c) \cup (c,\ f(1)) \\ &\supset f^{r'_*}((P_R,\ y)) \cup (P_R,\ y) \\ &= f^{r'_*}(V) \cup V, \end{aligned}$$

i.e.,

$$|f^{r'_*+2}(V)| > |f^{r'_*}(V)| + |V|. \tag{6. 16}$$

Similarly, $(f^{r'_*})'(P_R) = a^{m'_*} b^{n'_*}$ for some positive integers m'_* and n'_*.

According to (6. 16),

$$(ab-1)a^{m'_*} b^{n'_*} > 1. \tag{6. 17}$$

Since the orbit of q visits the left side and the right side of c at least one times before arriving u, we have

$$(ab-1)a^{m_2} b^{n_2} > ab.$$

By similar arguments in Subcase A1, we can obtain contradiction.

As a result, if $f^2(c-) > P_R$, then the minimal point in $(P_R,\ 1) \cap \{O_p,\ O_q\}$ does not belong to O_q.

Combine Subcases A1 and A2, we know that $f^2(c-) > P_R$ is impossible if f is renormalizable.

Similarly, one can show that $f^2(c+) < P_L$ is also impossible if f is renormalizable.

The proof is completed.

□

Assume that $f_{a,b,c}$ $(a \geq 1,\ b \geq 1)$ is m-renormalizable, $0 \leq m < \infty$. Put $f := f_{a,b,c}$, according to Theorem B, Theorem C, Theorem D and Theorem E, we have the following:

(1) f is renormalizable if and only if

$$[f^{\kappa}(c+),\ f^{\kappa}(c-)] \subseteq [P_L,\ P_R],$$

where κ is the minimal period of periodic orbits of f, P_L and P_R are two neighbor κ-periodic points satisfying $P_L < c < P_R$. So we can check wether f is prime in finite steps.

(2) If $m > 0$, the α-limit sets satisfy $E_i = (E_m)^{m-i}$, $i = 1, 2, \ldots, m$. As a result, the depth of the i-th α-limit set E_i is i.

(3) If $m > 0$, we have the nonwandering set decomposition $\Omega(f) = \bigcup_{i=1}^m \Omega_i \cup A$, where Ω_i is a periodic orbit for $0 < i < m$.

Inspired by Theorem E, we present a conjecture:

Conjecture 1. *Let f be an expanding Lorenz map satisfying $f'(x) \geq 1$ for $x \neq c$. Then each renormalization of f is periodic.*

Acknowledgements: This work is partially supported by a grant from the Spanish Ministry (No. SB2004-0149) and grants from NSFC (No.70571079, 60534080) in China. The author thanks Centre de Recerca Matemàtica for the hospitality and facilities. The author thanks Prof. Lluis Alsedà for many discussions and suggestions, and thanks Dr. Hongfei Cui for reading the manuscript.

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Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, P. R. China
E-mail address: ding@wipm.ac.cn