BROWN REPRESENTABILITY FOLLOWS FROM ROSICKÝ

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Abstract. We prove that the dual of a well generated triangulated category satisfies Brown representability, as long as there is a combinatorial model. This settles the major open problem in [13]. We also prove that Brown representability holds for non-dualized well generated categories, but that only amounts to the fourth known proof of the fact.

The proof depends crucially on a new result of Rosický [14].

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0. Introduction

We begin by reminding the reader of Brown representability.

Definition 0.1. *Let $\mathcal{T}$ be a triangulated category. The opposite category $\mathcal{T}^{\mathrm{op}}$ is said to* satisfy Brown representability *if*

(i) *$\mathcal{T}$ is a [TR5*] triangulated category; that is we may form, in $\mathcal{T}$, any small product.*

(ii) *If $H : \mathcal{T} \longrightarrow \mathcal{A}b$ is a homological functor, and if H respects products, then H is representable.*

A Brown representability theorem *will mean a theorem asserting that some class of triangulated categories satisfies Brown representability.*

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Remark 0.2. The experts might object that I am ignoring variants; see, for example, Adams [1], or Bondal and Van den Bergh [2]. While these are very interesting and powerful, the results in this article deal only with the original, classical version of Brown representability. Explaining the variants would be a digression.

Remark 0.3. The "op" in $\mathcal{T}^{\mathrm{op}}$ is there for historical reasons; Brown's original paper [3] dealt with functors $H : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{A}b$, where $\mathcal{T}$ was the homotopy category of spectra. Thus the first example, of a Brown representability theorem, applied to the singleton; it was about the class containing only one triangulated category, the homotopy category of spectra. Over the years there have been other Brown representability theorems, which we will review next. In passing let us mention that Brown representability theorems are quite possibly the most useful structure triangulated categories can have. The list of applications, over many years, is tremendous; we will not even attempt a survey.

Let us continue the historical overview a little. In the decades following Brown [3] the theorem was confined to topology; it was widely used there, but not in other parts of mathematics. The articles [9, 10] changed this. They defined a class of triangulated categories, the compactly generated triangulated categories, for which Brown representability holds. One way to say this is that the definition of a compactly generated triangulated category is precisely tailored so that Brown's old proof goes through. The remarkable observation in [9, 10] was that the derived category of quasicoherent sheaves on a scheme is compactly generated, and that Brown representability is very applicable there. Then there followed a string of other applications, to other compactly generated categories, by many authors; but we promised not to discuss applications.

And now we come to the first puzzle. We begin with the easy observation:

Remark 0.4. Suppose $\mathcal{T}$ is a [TR5*] triangulated category. Let $\mathcal{S} \subset \mathcal{T}$ be a colocalizing subcategory; this means that $\mathcal{S}$ is closed in $\mathcal{T}$ under the formation of products. It is easy to show that the Verdier quotient $\mathcal{T}/\mathcal{S}$ is also a [TR5*] triangulated category, and that the natural projection $\pi : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ respects products.

Now suppose that $\mathcal{T}^{\mathrm{op}}$ satisfies Brown representability. If $H : \mathcal{T}/\mathcal{S} \longrightarrow \mathcal{A}b$ is a product-preserving homological functor, then so is the composite $H\pi : \mathcal{T} \longrightarrow \mathcal{A}b$. Brown representability for $\mathcal{T}^{\mathrm{op}}$ gives us that $H\pi$ is represented by an object $t \in \mathcal{T}$, and it is an easy exercise to show that $\pi(t)$ represents H in the category $\mathcal{T}/\mathcal{S}$. We conclude that Brown representability for $\mathcal{T}^{\mathrm{op}}$ implies Brown representability for $\big(\mathcal{T}/\mathcal{S}\big)^{\mathrm{op}}$.

The puzzling part was the following. Compactly generated triangulated categories were known to satisfy Brown representability, by Brown's proof. By the above remark, so do all their quotients by localizing subcategories (a localizing subcategory is a colocalizing subcategory of $\mathcal{T}^{\mathrm{op}}$). But there are many quotients of compactly generated triangulated categories which are not compactly generated. The class of categories satisfying Brown representability was clearly larger than the class of compactly generated ones. It was interesting to try to understand it.

We might be tempted to ask whether every [TR5*] triangulated category satisfies Brown representability. The answer is No, a counterexample will appear in a joint article with Casacuberta.

So here was the quandry: we had Brown's proof that compactly generated triangulated categories satisfy Brown representability, and we had the trick of Remark 0.4, which constructed many other examples. We should perhaps note that these contain natural, interesting cases. For example the derived category of sheaves of abelian groups on a manifold is not compactly generated, but is easy to express as a quotient of a compactly generated category by a localizing subcategory. Hence Brown representability holds.

It was natural to look for a new proof, which would cover a larger class of triangulated categories, preferably a class closed under the formation of Verdier quotients by localizing subcategories. Up to now people have come up with three such proofs: two totally independent ones by Franke [5] and myself [13], as well as a later improvement by Krause [8]. It might be worth noting that [13] defines a class of triangulated categories called the *well generated triangulated categories.* This class contains all the compactly generated triangulated categories, and is conjecturally closed under localizing and colocalizing quotients. To put it another way, there are theorems that say that certain quotients must be well generated[1], and we know no example of a quotient which does not satisfy the hypotheses of these theorems. What is proved in both [13] and [8] is that well generated triangulated categories satisfy Brown representability. The reader is also referred to [7], for a different take on the foundations of well generated triangulated categories.

We might think that the story ends there. However the definition of well generated triangulated categories is not self-dual, and it was not clear whether Brown representability holds for $\mathcal{T}^{\mathrm{op}}$, when $\mathcal{T}$ is well generated. The article [12] was the first to prove a result in this direction: it showed that the dual of the homotopy category of spectra satisfies Brown representability. This was generalized in [13] to the dual of any compactly generated

[1]There is a technical condition on the kernel being generated by a set of objects, or else on the orthogonal of the kernel having a set of generators.

category. Krause [8] gave a second proof. The trick of Remark 0.4 still applies, and teaches us that the dual of the quotient, of a compactly generated category by a colocalizing subcategory, satisfies Brown representability. It was natural to wonder whether all duals of well generated triangulated categories satisfy Brown representability, but we were all stumped. This was the major problem left open in [13], and none of us had the foggiest clue how to proceed.

And then came Rosický's remarkable article [14]. In the remainder of the introduction we will explain Rosický's result, and show how to use it to make major progress. The way the remainder of the introduction is structured is as follows. After setting up some notation we will state a very general Brown representability theorem, Theorem 0.9. *A priori* it will not be clear that there are any categories which satisfy the hypotheses of Theorem 0.9. The relevance of Rosický's work, as we will explain at the end of the introduction, is that well generated triangulated categories and their duals satisfy these hypotheses, provided they come from combinatorial models.

There is the problem that, as the result now stands, it appeals to models. A purist might object; in fact the author has a reputation of being such a purist. In passing we note that it is quite difficult to construct triangulated categories without models. The natural ones, the ones that might actually come up in practice, all have models.

It is now time to come to results. We begin with a definition.

Definition 0.5. *Let $\mathcal{T}$ be a triangulated category satisfying [TR5*], and let $S \subset \mathcal{T}$ be a set of objects. We define, for every integer $n > 0$, a full subcategory $\mathrm{Prod}_n^{\mathcal{T}}(S) \subset \mathcal{T}$ as follows:*

(i) *The category $\mathrm{Prod}^{\mathcal{T}}(S) = \mathrm{Prod}_1^{\mathcal{T}}(S)$ is the full subcategory of $\mathcal{T}$ whose objects are all the products of objects in S.*
(ii) *Suppose $\mathrm{Prod}_n^{\mathcal{T}}(S)$ has been defined. The category $\mathrm{Prod}_{n+1}^{\mathcal{T}}(S) \subset \mathcal{T}$ is the full subcategory containing all objects $y \in \mathcal{T}$, for which there exists a triangle*
$$x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x$$
with $x \in \mathrm{Prod}_1^{\mathcal{T}}(S)$ and $z \in \mathrm{Prod}_n^{\mathcal{T}}(S)$.

Remark 0.6. If the category $\mathcal{T}$ is clear from the context, we will drop it out of the notation; thus $\mathrm{Prod}_n(S)$ means the same as $\mathrm{Prod}_n^{\mathcal{T}}(S)$, where $\mathcal{T}$ is understood. The dual construction yields a category $\mathrm{Prod}_n^{\mathcal{T}^{\mathrm{op}}}(S)$, which we will feel free to denote by $\mathrm{Coprod}_n^{\mathcal{T}}(S)$, or more simply $\mathrm{Coprod}_n(S)$ as long as $\mathcal{T}$ is understood.

Remark 0.7. We make the following easy observations, for future reference:

(i) Because any product of distinguished triangles is distinguished, all the categories $\mathrm{Prod}_n(S)$ are closed under products. We only need to complete with respect to products once, in producing $\mathrm{Prod}_1(S)$ out of S.

(ii) A little exercise with the octahedral axiom shows that, if we know that $x \in \mathrm{Prod}_m(S)$, that $z \in \mathrm{Prod}_n(S)$ and that there is a distinguished triangle

$$x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x \ ,$$

then $y \in \mathrm{Prod}_{m+n}(S)$.

(iii) Suppose S is closed under suspension; that is S contains isomorphs of all the objects in $\Sigma S \cup \Sigma^{-1} S$. Then the same is true for all the $\mathrm{Prod}_n(S)$.

Remark 0.8. It is possible to generalize the definition of $\mathrm{Prod}_i(S)$, allowing i to be an arbitrary ordinal. This is done by transfinite induction; see [4] for more detail.

Now we come to the first theorem we will prove:

Theorem 0.9. *Let $\mathcal{T}$ be a [TR5*] triangulated category. Suppose there exists a set of objects $S \subset \mathcal{T}$, as well as an integer $n > 0$, so that $\mathcal{T} = \mathrm{Prod}_n(S)$. Then $\mathcal{T}^{\mathrm{op}}$ satisfies Brown representability.*

Remark 0.10. The theorem easily generalizes to the transfinite case of Remark 0.8: if $\mathcal{T} = \mathrm{Prod}_i(S)$, for some ordinal i, then $\mathcal{T}^{\mathrm{op}}$ satisfies Brown representablity. A minor modification of the proof works. We do not need the more general version, hence we leave it as an exercise to the reader.

So far we have seen one theorem, with very simple hypotheses which are difficult to check. To go further we need to assume that category $\mathcal{T}$ has a Rosický functor. We define our terms next. Let us begin with pre-Rosický functors.

Definition 0.11. *Let $\mathcal{T}$ be a triangulated category satisfying both [TR5] and [TR5*]; products and coproducts exist in $\mathcal{T}$. Let $\mathcal{A}$ be an abelian category satisfying both [AB3] and [AB3*]; products and coproducts exist in $\mathcal{A}$ too. A* pre-Rosický functor *$H : \mathcal{T} \longrightarrow \mathcal{A}$ is a homological functor satisfying the following properties:*

(i) *The functor H is full; that is, the natural map*

$$\mathcal{T}(x, y) \longrightarrow \mathcal{A}\big(H(x), H(y)\big)$$

is surjective.

(ii) *H reflects isomorphisms; that is, if $H(f) : H(x) \longrightarrow H(y)$ is an isomorphism then so is $f : x \longrightarrow y$.*

(iii) *H preserves both products and coproducts.*

This defines a pre-Rosický functor. Now for Rosický functors:

Definition 0.12. *A pre-Rosický functor $H : \mathcal{T} \longrightarrow \mathcal{A}$ is called a* Rosický functor *if there is a set of objects $\mathcal{P} \subset \mathcal{T}$, closed under suspension, satisfying the following properties:*

(i) *The objects $H(p)$, $p \in \mathcal{P}$ are projective in the abelian category $\mathcal{A}$, and generate it.*

(ii) *For every object $y \in \mathcal{T}$, and for every $p \in \mathcal{P}$, the natural map*

$$\mathcal{T}(p, y) \longrightarrow \mathcal{A}\big(H(p), H(y)\big)$$

is an isomorphism.

(iii) *There exists a regular cardinal α, so that each object $p \in \mathcal{P}$ is α–small. That is, any map from p, to any coproduct in $\mathcal{T}$, factors through the inclusion of a coproduct of fewer than α terms.*

Remark 0.13. The reason we split the definition in two is that Definition 0.11 is self-dual, while Definition 0.12 is not. If $H : \mathcal{T} \longrightarrow \mathcal{A}$ is a pre-Rosický functor, then so is $H^{\mathrm{op}} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{A}^{\mathrm{op}}$. When we prove some fact about categories $\mathcal{T}$ possessing pre-Rosický functors, the duals $\mathcal{T}^{\mathrm{op}}$ automatically satisfy the same property. This is not true of Rosický functors.

Rosický striking theorem asserts:

Theorem 0.14. (Rosický [14]**).** *Let $\mathcal{M}$ be a combinatorial stable model category in the sense of* [6]*, and let $\mathcal{T} = \mathrm{Ho}(\mathcal{M})$ be its homotopy category. Then*

(i) *The category $\mathcal{T}$ is well generated, in the sense of* [13].

(ii) *There is a Rosický functor $H : \mathcal{T} \longrightarrow \mathcal{A}$.*

Remark 0.15. The functor which Rosický considers is the $H : \mathcal{T} \longrightarrow \mathcal{E}x\Big(\{\mathcal{T}^\alpha\}^{op}, \mathcal{A}b\Big)$ of [13]. His remarkable discovery is that there exist arbitrarily large α for which Definition 0.11(i) holds. The other properties were known. The fact that $\mathcal{T}$ has coproducts is by definition of well generated categories. Products exist by [13, Proposition 8.4.6]. In [13, §6.1] we learn that $\mathcal{A} = \mathcal{E}x\Big(\{\mathcal{T}^\alpha\}^{op}, \mathcal{A}b\Big)$ has products and coproducts. From [13, Lemma 6.2.4] we know that H respects products, and from [13, Lemma 6.2.5] that it respects coproducts. The fact that H reflects isomorphisms is standard, as long as α is large enough so that $\mathcal{T}^\alpha$ generates $\mathcal{T}$.

For the set of objects $\mathcal{P} \subset \mathcal{T}$ we choose one representative in each isomorphism class of objects in $\mathcal{T}^\alpha$. Then every object in $p \in \mathcal{P}$ is clearly α–small, and the fact that

$$\mathcal{T}(p, y) \longrightarrow \mathcal{A}\big(H(p), H(y)\big)$$

is an isomorphism comes from Yoneda's lemma. Finally, the fact that the objects $H(p)$, $p \in \mathcal{P}$ are projective and generate may be found in [13, Lemmas 6.4.1 and 6.4.2].

Remark 0.16. Rosický's theorem goes on to identify the essential image of the functor H. This is irrelevant for us here.

Our next result says:

Theorem 0.17. *Let $\mathcal{T}$ be a triangulated category possessing a Rosický functor H. Then there exist two sets of objects $S, S' \subset \mathcal{T}$ so that*

$$\mathcal{T} = \mathrm{Coprod}_4(S) \ , \qquad\qquad \mathcal{T} = \mathrm{Prod}_{16}(S') \ .$$

Remark 0.18. Theorem 0.14 tells us that, if $\mathcal{T}$ is well generated category with a combinatorial model, then $\mathcal{T}$ has a Rosický functor. From Theorem 0.17 we learn that

$$\mathrm{Coprod}_4(S) \quad = \quad \mathcal{T} \quad = \quad \mathrm{Prod}_{16}(S') \ ,$$

and finally Theorem 0.9 tells us that Brown representability must hold both for $\mathcal{T}$ and for $\mathcal{T}^{\mathrm{op}}$. For $\mathcal{T}$ this gives the fourth proof of the fact, but for $\mathcal{T}^{\mathrm{op}}$ it is very new.

Remark 0.19. The sets S and S' in Theorem 0.17 are quite explicit: S is the set $\mathcal{P}$ of Definition 0.12, while S' is a set of representatives of all isomorphism classes of coproducts of $\leq \alpha$ objects in $\mathcal{P}$. The integers 4 and 16 in Theorem 0.17 are not optimal. One can show that, for the S, S' above,

$$\mathcal{T} = \mathrm{Coprod}_2(S) \ , \qquad\qquad \mathcal{T} = \mathrm{Prod}_8(S') \ .$$

The interested reader can find a hint of how to go about this in [11].

The 2 is best possible. I have made no attempt to see if the 8 is.

1. A Freyd-style representability theorem

In this section we will prove a very general representability theorem for functors on triangulated categories. The next sections will be progressively less general, but more useful. We begin with a definition.

Definition 1.1. *Let $\mathcal{T}$ be a [TR5*] triangulated category. A functor $H : \mathcal{T} \longrightarrow \mathcal{A}b$ is called* pre-representable *if*

(i) *H is homological; it takes triangles to exact sequences.*
(ii) *H respects products. That is, the natural map*

$$H\left(\prod_{\lambda \in \Lambda} t_\lambda\right) \longrightarrow \prod_{\lambda \in \Lambda} H(t_\lambda)$$

is an isomorphism.

Remark 1.2. It is immediate that any representable functor is pre-representable. In the terminology of Definition 1.1, Brown representability can be reformulated: $\mathcal{T}^{\mathrm{op}}$ satisfies Brown representability if any pre-representable functor $H : \mathcal{T} \longrightarrow \mathcal{A}b$ is representable.

Next we prove:

Theorem 1.3. *Let $\mathcal{T}$ be a [TR5*] triangulated category. Suppose that every pre-representable functor H has a solution-object. That is, for every pre-representable H there exists a representable functor $\mathcal{T}(t,-)$ and a surjective map*

$$\mathcal{T}(t,-) \longrightarrow H(-) .$$

Then $\mathcal{T}^{\mathrm{op}}$ satisfies Brown representability.

Proof. Let H be pre-representable; we need to show it representable. By hypothesis we may choose a representable functor $\mathcal{T}(s,-)$ and a surjective homomorphism $\mathcal{T}(s,-) \longrightarrow H(-)$. Complete this to a short exact sequence of functors

$$0 \longrightarrow H'(-) \longrightarrow \mathcal{T}(s,-) \longrightarrow H(-) \longrightarrow 0 .$$

Now $H(-)$ is pre-representable by assumption, and $\mathcal{T}(s,-)$ by Remark 1.2. It easily follows that H' is also pre-representable. By the hypothesis of the theorem, applied to H', we may choose a representable functor $\mathcal{T}(t,-)$ and a surjection $\mathcal{T}(t,-) \longrightarrow H'(-)$. We therefore have an exact sequence of functors

$$\mathcal{T}(t,-) \longrightarrow \mathcal{T}(s,-) \longrightarrow H(-) \longrightarrow 0 .$$

Yoneda's lemma says that the map $\mathcal{T}(t,-) \longrightarrow \mathcal{T}(s,-)$ is induced by a morphism $g : s \longrightarrow t$ in $\mathcal{T}$. Complete this to a triangle

$$r \longrightarrow s \xrightarrow{\ g\ } t \longrightarrow \Sigma r .$$

Yoneda tells us that the natural transformation $\mathcal{T}(s,-) \longrightarrow H(-)$ corresponds to an element $x \in H(s)$, and the vanishing of the composite

$$\mathcal{T}(t,-) \longrightarrow \mathcal{T}(s,-) \longrightarrow H(-)$$

guarantees that the image of $x \in H(s)$ under the homomorphism $H(g) : H(s) \longrightarrow H(t)$ vanishes. The exactness of the sequence

$$H(r) \longrightarrow H(s) \xrightarrow{H(g)} H(t)$$

says that x must be in the image of $H(r) \longrightarrow H(s)$; using Yoneda again this says that the surjection $\varphi : \mathcal{T}(s,-) \longrightarrow H(-)$ must factor as

$$\mathcal{T}(s,-) \longrightarrow \mathcal{T}(r,-) \xrightarrow{\ \psi\ } H(-) .$$

Now consider the diagram with exact rows

$$\begin{tikzcd}
\mathcal{T}(t,-) \arrow[r] \arrow[d, "1"'] & \mathcal{T}(s,-) \arrow[r, "\varphi"] \arrow[d, "1"] & H(-) \arrow[r] & 0 \\
\mathcal{T}(t,-) \arrow[r] & \mathcal{T}(s,-) \arrow[r] & \mathcal{T}(r,-) &
\end{tikzcd} ;$$

it permits us to factor $\mathcal{T}(s,-) \longrightarrow \mathcal{T}(r,-)$ as

$$\mathcal{T}(s,-) \xrightarrow{\ \varphi\ } H(-) \xrightarrow{\ \theta\ } \mathcal{T}(r,-) .$$

We conclude that $\psi\theta\varphi = \varphi$; since φ is surjective it follows that $\psi\theta = 1$.

This makes $\theta\psi : \mathcal{T}(r,-) \longrightarrow \mathcal{T}(r,-)$ an idempotent, whose image is $H(-)$. But the idempotent $\theta\psi$ must be induced by an idempotent $e : r \longrightarrow r$ in $\mathcal{T}$. The category $\mathcal{T}$ is a [TR5*] triangulated category; applying [13, Proposition 1.6.8] to the dual category $\mathcal{T}^{\mathrm{op}}$, we conclude that the idempotent e must split in $\mathcal{T}$. Hence we obtain an object $h \in \mathcal{T}$ with $H(-) \cong \mathcal{T}(h,-)$. □

2. If it doesn't take many products

This section is about proving Theorem 0.9. Perhaps we should remind the reader.

Theorem 0.9. *Let $\mathcal{T}$ be a [TR5*] triangulated category. Suppose there exists a set of objects $S \subset \mathcal{T}$, as well as an integer $n > 0$, so that $\mathcal{T} = \mathrm{Prod}_n(S)$. Then $\mathcal{T}^{\mathrm{op}}$ satisfies Brown representability.*

The entire section will be devoted to the proof, and so we will adopt throughout the notation of Theorem 0.9; the category $\mathcal{T}$ will be fixed in the entire section, as will the set of objects $S \subset \mathcal{T}$.

Discussion 2.1. We will prove that Theorem 0.9 is a consequence of Theorem 1.3. For every pre-representable functor H we will produce a surjection $\mathcal{T}(t,-) \longrightarrow H(-)$. Fix a pre-representable functor H, and let us remind ourselves what it means to find such a surjection.

A natural transformation $\varphi : \mathcal{T}(t,-) \longrightarrow H(-)$ corresponds, under Yoneda, to an element $x \in H(t)$. To say that φ is surjective is to assert that, for every $t' \in \mathcal{T}$, the map

$$\mathcal{T}(t,t') \longrightarrow H(t')$$

is an epimorphism. In other words: φ will be an epimorphism as long as, for any $x' \in H(t')$, we can produce a morphism $f : t \longrightarrow t'$ so that $H(f) : H(t) \longrightarrow H(t')$ takes $x \in H(t)$ to $x' \in H(t')$.

We define therefore a category $\mathcal{C}$. The objects are pairs (t,x), with t an object of $\mathcal{T}$ and x an element in the abelian group $H(t)$. A morphism in $\mathcal{C}$, from the object (t,x) to the object (t',x'), is a morphism $f : t \longrightarrow t'$, with $H(f) : H(t) \longrightarrow H(t')$ taking $x \in H(t)$ to $x' \in H(t')$. By the previous paragraph an object (t,x) corresponds to a natural transformation $\varphi : \mathcal{T}(t,-) \longrightarrow H(-)$, and φ will be surjective if and only if (x,t) is weakly initial in the category $\mathcal{C}$. Recall that an object in a category is weakly initial if it admits a morphism to every other object. The proof of Theorem 0.9 will therefore be by studying the category $\mathcal{C}$.

For this study it is helpful to define, for each integer $k \geq 1$, a full subcategory $\mathcal{C}_k \subset \mathcal{C}$. Its objects are the pairs (t,x) with $t \in \mathrm{Prod}_k(S)$.

Next we make some easy observations:

Remark 2.2. The category $\mathcal{C}$ has products. If $\{(t_i,x_i),\ i \in I\}$ is a set of objects in $\mathcal{C}$, then we can form in $\mathcal{T}$ the product $t = \prod_{i\in I} t_i$, and in the abelian group

$$H(t) \quad = \quad \prod_{i\in I} H(t_i)$$

we can look at the element $x = \prod_{i\in I} x_i$. Then (t,x) is the product, in the category $\mathcal{C}$, of the objects (t_i,x_i). Note also that

(i) If (t,x) is an object of $\mathcal{C}$, and if t is isomorphic to a product $t \cong \prod_{i\in I} t_i$, then it is automatic that (t,x) is isomorphic to a product

$$(t,x) \quad \cong \quad \prod_{i\in I} (t_i,x_i)\ .$$

(ii) Any product of objects in $\mathcal{C}_k$ lies in $\mathcal{C}_k$.

Discussion 2.1 tells us that to prove Theorem 0.9 it suffices to produce a weakly initial object in $\mathcal{C}$. The hypothesis of Theorem 0.9 is that $\mathcal{T} = \mathrm{Prod}_n(S)$, or equivalently that $\mathcal{C} = \mathcal{C}_n$. Theorem 0.9 therefore follows from the case $k = n$ of the following lemma:

Lemma 2.3. *Let $H : \mathcal{T} \longrightarrow \mathcal{A}b$ be a pre-representable functor. Let $k > 0$ be an integer. There exists a weakly initial object (t_k, x_k) in the category $\mathcal{C}_k$.*

Proof. **Step 1.** The proof is by induction on k; we begin with the case $k = 1$. Every object of $\mathrm{Prod}_1(S)$ is isomorphic to a product of objects in S, and Remark 2.2(i) guarantees that every object in $\mathcal{C}_1$ is isomorphic to a product of objects (s_i,x_i) with $s_i \in S$. There is a set of objects of the form $c_i = (s_i,x_i)$, and the product of them all is clearly weakly initial.

Step 2. Now we come to the inductive step. Suppose we know the assertion for k; we want to deduce it for $(k+1)$. Choose and fix weakly initial objects $(t_\ell, x_\ell) \in \mathcal{C}_\ell$ for all $\ell \leq k$. Let Ψ be the set of all morphisms $\Sigma^{-1}t_k \longrightarrow s$, over all $s \in S$. For every subset $\Phi \subset \Psi$ we obtain a morphism

$$\Sigma^{-1}t_k \longrightarrow \prod_{\{\Sigma^{-1}t_k \longrightarrow s\}\in\Phi} s \quad .$$

Choose a mapping cone; that is, complete to a triangle

$$\Sigma^{-1}t_k \longrightarrow \prod_{\{t_k \longrightarrow s\}\in\Phi} s \longrightarrow y_\Phi \longrightarrow t_k \ .$$

For each Φ we obtain an object $y_\Phi \in \mathrm{Prod}_{k+1}(S)$. Now form

$$(t_{k+1}, x_{k+1}) \quad = \quad (t_1, x_1) \times \prod_{\Phi\subset\Psi,\ x\in H(y_\Phi)} (y_\Phi, x) \quad .$$

We assert that (t_{k+1}, x_{k+1}) is a weakly initial object in $\mathcal{C}_{k+1}$.

Step 3. It remains to prove the assertion. By Remark 2.2(ii) we know that (t_{k+1}, x_{k+1}) is an object of $\mathcal{C}_{k+1}$; we must prove it weakly initial. Suppose therefore that we are given an object $(t', x') \in \mathcal{C}_{k+1}$; we need to produce a morphism $(t_{k+1}, x_{k+1}) \longrightarrow (t', x')$.

We are given that t' lies in $\mathrm{Prod}_{k+1}(S)$; Definition 0.5 tells us that there exists a distinguished triangle

$$a \xrightarrow{\ f\ } t' \xrightarrow{\ g\ } b \xrightarrow{\ h\ } \Sigma a \ ,$$

with $a \in \mathrm{Prod}_1(S)$ and $b \in \mathrm{Prod}_k(S)$. The object $(t', x') \in \mathcal{C}_{k+1}$ provides us with an $x' \in H(t')$, and the map $H(g) : H(t') \longrightarrow H(b)$ takes x' to an element $\overline{x} \in H(b)$. We have constructed an object $(b, \overline{x}) \in \mathcal{C}$. The fact that $b \in \mathrm{Prod}_k(S)$ means that $(b, \overline{x})$ belongs to the subcategory $\mathcal{C}_k \subset \mathcal{C}$, which has a weakly initial object (t_k, x_k). There is a morphism $(t_k, x_k) \longrightarrow (b, \overline{x})$; we have morphisms in $\mathcal{T}$

$$\begin{tikzcd} & t_k \arrow[d, "j"] \\ t' \arrow[r, "g"'] & b \end{tikzcd} \quad ,$$

and the induced maps of abelian groups take $x' \in H(t')$ and $x_k \in H(t_k)$ to the same image $\overline{x} \in H(b)$. Form in $\mathcal{T}$ a homotopy cartesian square

$$\begin{tikzcd} t'' \arrow[r, "\alpha"] \arrow[d, "\beta"'] & t_k \arrow[d, "j"] \\ t' \arrow[r, "g"'] & b \end{tikzcd} \quad ;$$

see [13, Definition 1.4.1]. The fact that H is homological permits us to find an element $x'' \in H(t'')$ which maps, under $H(\beta)$ and $H(\alpha)$ respectively, to $x' \in H(t')$ and $x_k \in H(t_k)$. We have produced a morphism $(t'', x'') \longrightarrow$

(t', x') in $\mathcal{C}$. Next we have to study the object t'', to determine which subcategory $\mathcal{C}_\ell \subset \mathcal{C}$ might contain the object (t'', x'').

We appeal to [13, Lemma 1.4.4]; it permits us to extend the homotopy cartesian square above to a morphism of triangles

$$\begin{tikzcd} a \arrow[r] \arrow[d, "1"'] & t'' \arrow[r] \arrow[d, "\beta"'] & t_k \arrow[r] \arrow[d, "j"] & \Sigma a \arrow[d, "1"] \\ a \arrow[r, "f"'] & t' \arrow[r, "g"'] & b \arrow[r] & \Sigma a \end{tikzcd} .$$

In the top triangle we have that $a \in \mathrm{Prod}_1(S)$ and $t_k \in \mathrm{Prod}_k(S)$; we immediately conclude that $t'' \in \mathrm{Prod}_{k+1}(S)$, meaning that the object (t'', x'') lies in $\mathcal{C}_{k+1}$. But if we look at the triangle a little more carefully we note that t'' is the cone on a morphism $\Sigma^{-1}t_k \longrightarrow a$, with $a \in \mathrm{Prod}_1(S)$. That is, t'' is the mapping cone on a morphism

$$\Sigma^{-1}t_k \longrightarrow \prod_{i \in I} s_i \ .$$

For each $i \in I$ we have a map $\Sigma^{-1}t_k \longrightarrow s_i$ for some $s_i \in S$. The maps that occur, as i ranges over I, give us a subset Φ of the set Ψ of all maps $\Sigma^{-1}t_k \longrightarrow s$. This means that the map $\Sigma^{-1}t_k \longrightarrow \prod_{i \in I} s_i$ factors as

$$\Sigma^{-1}t_k \longrightarrow \prod_{s \in \Phi} s \xrightarrow{\ \Delta\ } \prod_{i \in I} s_i \ ,$$

where Δ is some diagonal inclusion. Diagonal inclusions are split monomorphisms; up to isomorphism the composite must identify with

$$\Sigma^{-1}t_k \xrightarrow{\begin{pmatrix} 0 \\ \mu \end{pmatrix}} \left(\prod_{i \in I-J} s_i\right) \oplus \left(\prod_{s \in \Phi} s\right) \qquad .$$

The mapping cone t'' is isomorphic to $\widehat{a} \oplus y_\Phi$, with $\widehat{a} \in \mathrm{Prod}_1(S)$ and y_Φ as in Step 2. By Remark 2.2(i) the object (t'', x'') decomposes as

$$(t'', x'') \quad \cong \quad (\widehat{a}, \widehat{x}) \times (y_\Phi, x) \quad ,$$

and $(t_{k+1}, x_{k+1}) = (t_1, x_1) \times \prod (y_\Phi, x)$ clearly maps to it. $\square$

3. How many steps it takes to generate

It is now time to prove Theorem 0.17. We remind the reader:

Theorem 0.17. *Let $\mathcal{T}$ be a triangulated category possessing a Rosický functor H. Then there exist two sets of objects $S, S' \subset \mathcal{T}$ so that*

$$\mathcal{T} = \mathrm{Coprod}_4(S) \ , \qquad\qquad \mathcal{T} = \mathrm{Prod}_{16}(S') \ .$$

We begin with a little lemma.

Lemma 3.1. *Let $H : \mathcal{T} \longrightarrow \mathcal{A}$ be a pre-Rosický functor, with the notation as in Definition 0.11. Suppose S is some set of objects in $\mathcal{T}$, closed under suspension. Assume further that there is a pair of maps in $\mathcal{T}$, with a vanishing composite*

$$X \longrightarrow Y \longrightarrow Z ,$$

so that:

(i) *Y and Z both belong to* $\mathrm{Prod}_1(S)$.
(ii) *In the abelian category $\mathcal{A}$, the sequence*

$$0 \longrightarrow H(X) \longrightarrow H(Y) \longrightarrow H(Z)$$

is exact.

Then X must belong to $\mathrm{Prod}_4(S)$.

Proof. Since S is closed under suspension so is $\mathrm{Prod}_1(S)$ (see Remark 0.7 (iii)), and $\Sigma^{-1}Z$ must be in $\mathrm{Prod}_1(S)$. Complete the morphism $Y \longrightarrow Z$ to a distinguished triangle

$$\Sigma^{-1}Z \longrightarrow \overline{X} \longrightarrow Y \longrightarrow Z ;$$

from this triangle we learn that $\overline{X} \in \mathrm{Prod}_2(S)$. Now we have, in $\mathcal{T}$, a vanishing composite $X \longrightarrow Y \longrightarrow Z$, and hence the map $X \longrightarrow Y$ must factor as $X \xrightarrow{\alpha} \overline{X} \longrightarrow Y$. Choose such an $\alpha : X \longrightarrow \overline{X}$.

On the other hand we have a diagram with exact rows

$$\begin{array}{ccccccc}
 & & H(\overline{X}) & \longrightarrow & H(Y) & \longrightarrow & H(Z) \\
 & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
0 & \longrightarrow & H(X) & \longrightarrow & H(Y) & \longrightarrow & H(Z)
\end{array} ,$$

which tells us that there is a unique factorization of the map $H(\overline{X}) \longrightarrow H(Y)$ through $H(\overline{X}) \longrightarrow H(X) \longrightarrow H(Y)$. By Definition 0.11(i) the map $H(\overline{X}) \longrightarrow H(X)$ has a lifting to $\mathcal{T}$; choose a map $\beta : \overline{X} \longrightarrow X$ inducing it.

Now consider the composite $X \xrightarrow{\alpha} \overline{X} \xrightarrow{\beta} X$. It is easy to show that $H(X) \longrightarrow H(\overline{X}) \longrightarrow H(X)$ is the identity, and from Definition 0.11(ii) we learn that the composite $X \longrightarrow \overline{X} \longrightarrow X$ must be an isomorphism in $\mathcal{T}$. Hence $\overline{X}$ splits as $\overline{X} \cong X \oplus X'$.

Now use [13, Proposition 1.6.8], or, more accurately, use the dual of the proof. The direct summand X of $\overline{X}$ can be described as the homotopy limit of some sequence

$$\cdots \xrightarrow{\ e\ } \overline{X} \xrightarrow{\ e\ } \overline{X} \xrightarrow{\ e\ } \overline{X} \xrightarrow{\ e\ } \overline{X} ;$$

what is relevant for us it that there exists a distinguished triangle

$$\Sigma^{-1}\prod_{i=1}^{\infty}\overline{X} \longrightarrow X \longrightarrow \prod_{i=1}^{\infty}\overline{X} \xrightarrow{1-shift} \prod_{i=1}^{\infty}\overline{X} \quad .$$

Since $\overline{X}$ lies in $\mathrm{Prod}_2(S)$ so do $\prod_{i=1}^{\infty}\overline{X}$ and $\Sigma^{-1}\prod_{i=1}^{\infty}\overline{X}$; see Remark 0.7(i) and (iii). From Remark 0.7(ii) we now conclude that $X \in \mathrm{Prod}_4(S)$. $\square$

Now we are ready to prove the first half of Theorem 0.17.

Lemma 3.2. *Let $\mathcal{T}$ be a triangulated category possessing a Rosický functor H. Then $\mathcal{T} = \mathrm{Coprod}_4(\mathcal{P})$, with $\mathcal{P}$ the set of objects of Definition 0.12.*

Proof. **Step 1.** Let X be any object in $\mathcal{T}$, and suppose we are given a map in $\mathcal{A}$

$$\coprod_{j\in J} H(p_j) \longrightarrow H(X) \ .$$

with $p_j \in \mathcal{P}$. We assert first that there is a unique lifting to a morphism $\coprod_{j\in J} p_j \longrightarrow X$.

Each of the maps $H(p_j) \longrightarrow H(X)$ must lift, by Definition 0.12(ii), to a morphism in $\mathcal{T}$. We produce therefore a map $\coprod_{j\in J} p_j \longrightarrow X$ in the category $\mathcal{T}$. Applying H to this map, and using the fact that H commutes with coproducts (Definition 0.11(iii)), we conclude that H takes

$$\coprod_{j\in J} p_j \longrightarrow X \qquad \text{to} \qquad \coprod_{j\in J} H(p_j) \longrightarrow H(X) \quad .$$

The uniqueness is because the lifting of each map $H(p_j) \longrightarrow H(X)$, to a morphism $p_j \longrightarrow X$, is unique; see Definition 0.12(ii).

Step 2. Now we proceed to the proof of the lemma. Let Z be an object of $\mathcal{T}$; we wish to show that it belongs to $\mathrm{Coprod}_4(\mathcal{P})$. By Lemma 3.1, applied to $\mathcal{T}^{\mathrm{op}}$, it suffices to produce a vanishing composite $X \longrightarrow Y \longrightarrow Z$ in $\mathcal{T}$, so that

(i) X and Y belong to $\mathrm{Coprod}_1(\mathcal{P})$, and
(ii) the sequence

$$H(X) \longrightarrow H(Y) \longrightarrow H(Z) \longrightarrow 0$$

is exact in $\mathcal{A}$.

By Definition 0.12(i) the objects in $\mathcal{P}$ are projective generators and therefore, in the abelian category $\mathcal{A}$, we have an exact sequence

$$\coprod_{i\in I} H(p_i) \longrightarrow \coprod_{j\in J} H(p_j) \longrightarrow H(Z) \longrightarrow 0 \quad ,$$

where p_i, p_j are objects in $\mathcal{P}$. Step 1 permits us first to lift the map $\coprod_{j\in J} H(p_j) \longrightarrow H(Z)$ to a morphism $\coprod_{j\in J} p_j \longrightarrow Z$, and then to lift

$$\coprod_{i\in I} H(p_i) \longrightarrow H\left(\coprod_{j\in J} p_j\right)$$

to a morphism $\coprod_{i\in I} p_i \longrightarrow \coprod_{j\in J} p_j$. The vanishing of the composite

$$\coprod_{i\in I} p_i \longrightarrow \coprod_{j\in J} p_j \longrightarrow Z$$

is because it is the unique lifting of the vanishing

$$\coprod_{i\in I} H(p_i) \longrightarrow H(Z) \ .$$

$\square$

Now it is time to conclude by proving the second half of Theorem 0.17. We will present it as a lemma:

Lemma 3.3. *Let $\mathcal{T}$ be a triangulated category possessing a Rosický functor H. Let S' be a set of objects, containing one representative in each isomorphism class of objects*

$$\prod_{i\in I} p_i \ ,$$

where each p_i is in $\mathcal{P}$ and the cardinality of I is $\leq \alpha$. Here, α is the α of Definition 0.12(iii); each object $p \in \mathcal{P}$ is α–small.

Then $\mathcal{T} = \mathrm{Coprod}_{16}(S')$.

Proof. Lemma 3.2 tells us that $\mathcal{T} = \mathrm{Coprod}_4(\mathcal{P})$, and by Remark 0.7(ii) it suffices to show that $\mathrm{Coprod}_1(\mathcal{P}) \subset \mathrm{Prod}_4(S')$. This is what we will prove. Suppose therefore that X is in $\mathrm{Coprod}_1(\mathcal{P})$. We want to show that it belongs to $\mathrm{Prod}_4(S')$, and we will do it by applying Lemma 3.1. If suffices therefore to produce a vanishing composite $X \longrightarrow Y \longrightarrow Z$ in $\mathcal{T}$, with Y and Z in $\mathrm{Prod}_1(S')$, and so that the sequence

$$0 \longrightarrow H(X) \longrightarrow H(Y) \longrightarrow H(Z)$$

is exact in $\mathcal{A}$. This is what we are about to do.

Since X belongs to $\mathrm{Coprod}_1(\mathcal{P})$ we may express it as

$$X \quad \cong \quad \coprod_{\lambda\in\Lambda} p_\lambda \ .$$

The vanishing composite we wish to consider is

$$X \xrightarrow{\ \beta\ } \prod_{\substack{I\subset\Lambda\\ \#I\leq\alpha}} \left(\coprod_{\lambda\in I} p_\lambda\right) \xrightarrow{\ \gamma\ } \prod_{\substack{I,\,J\subset\Lambda\\ \#I,\,\#J\leq\alpha}} \left(\coprod_{\lambda\in I\cap J} p_\lambda\right) \quad .$$

Perhaps we should explain the notation. We have a set Λ. Given a subset $I \subset \Lambda$ there is a restriction map, the projection to the direct summand

$$\coprod_{\lambda\in\Lambda} p_\lambda \longrightarrow \coprod_{\lambda\in I} p_\lambda \quad .$$

The map β is the product of all these restrictions, over all subsets I of cardinality $\leq \alpha$. If I and J are two subsets of Λ, we have a commutative square

$$\begin{array}{ccc}
\coprod_{\lambda\in\Lambda} p_\lambda & \longrightarrow & \coprod_{\lambda\in I} p_\lambda \\
\Big\downarrow & & \Big\downarrow f \\
\coprod_{\lambda\in J} p_\lambda & \xrightarrow[g]{} & \coprod_{\lambda\in I\cap J} p_\lambda
\end{array} \quad ,$$

and γ is the map whose components are $f - g$. The commutativity of the square guarantees that $\gamma\beta = 0$. It remains to prove that, after applying the functor H, we obtain an exact sequence.

We are given a sequence

$$0 \longrightarrow H(X) \longrightarrow H(Y) \longrightarrow H(Z)$$

and want to check its exactness. Since $H(p)$, $p \in \mathcal{P}$ are projective and generate, it suffices to check that each $\mathcal{A}\big(H(p), -\big)$ takes the above to an exact sequence. By Definition 0.12(ii),

$$\mathcal{T}(p,-) \quad \cong \quad \mathcal{A}\big(H(p), H(-)\big) \quad ;$$

it therefore suffices to check that the sequence

$$0 \longrightarrow \mathcal{T}(p,X) \xrightarrow{\ \beta\ } \mathcal{T}(p,Y) \xrightarrow{\ \gamma\ } \mathcal{T}(p,Z)$$

is exact, for every $p \in \mathcal{P}$. We will now do this.

Definition 0.12(iii) tells us that p is α–small. Any map $p \longrightarrow X$, where $X = \coprod_{\lambda\in\Lambda} p_\lambda$, must factor as

$$p \longrightarrow \coprod_{\lambda\in I} p_\lambda \longrightarrow \coprod_{\lambda\in\Lambda} p_\lambda \quad ,$$

with $I \subset \Lambda$ a subset of cardinality $< \alpha$. If the map $p \longrightarrow X$ is non-zero, then certainly the composite

$$p \longrightarrow \coprod_{\lambda \in I} p_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} p_\lambda \longrightarrow \coprod_{\lambda \in I} p_\lambda$$

cannot vanish; we conclude that β is injective. It remains only to show that the kernel of γ equals the image of β.

Suppose therefore that we are given a morphism $p \longrightarrow Y$ in the kernel of γ. That is for every subset $I \subset \Lambda$, of cardinality $\leq \alpha$, we have a map

$$p \xrightarrow{f_I} \coprod_{\lambda \in I} p_\lambda \quad ,$$

and these maps are compatible on intersections. We need to show that they assemble to a single map $p \longrightarrow X$.

It clearly suffices to show that there is a set $I \subset \Lambda$, of cardinality $< \alpha$, so that $f_J = 0$ whenever $I \cap J = \emptyset$. If such an I exists we would define $f : p \longrightarrow X$ as the composite

$$p \xrightarrow{f_I} \coprod_{\lambda \in I} p_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} p_\lambda \quad ,$$

and it is easy to check that this f would work. Next we prove the existence of the set I. For the rest of the proof we will assume there is no such I, and produce a contradiction.

What we will prove next, by transfinite induction, is that it is possible to choose α many disjoint subsets $I_i \subset \Lambda$, each with $\#I_i < \alpha$ and with $f_{I_i} \neq 0$. Then we will deduce our contradiction. To start the induction note that $I = \emptyset$ cannot work; we can therefore choose a subset $I_1 \subset \Lambda$, with $\#I_1 < \alpha$, so that $f_{I_1} \neq 0$. Now we proceed to choose such sets, disjoint from each other, for every ordinal $i < \alpha$.

Suppose we have chosen the sets I_i for all ordinals $i < j$, with j an ordinal $< \alpha$. The set $I = \cup_{i<j} I_i$ is a union of $< \alpha$ sets, each of cardinality $< \alpha$. Since α is regular, the cardinality of I must be $< \alpha$. Our assumption is that there exists a set I_j, of cardinality $< \alpha$ and disjoint from I, so that $f_{I_j} \neq 0$. Choose one. This completes the induction.

Now consider the set $J = \cup_{i<\alpha} I_i$. The cardinality of J is $\leq \alpha$, and hence we can look at $f_J : p \longrightarrow \coprod_{\lambda \in I} p_\lambda$. Because p is α–small this map factors through the coproduct over a subset $K \subset J$ of cardinality $< \alpha$. Choose and fix such a K. Now the data of the maps $\{f_J;\, f_{I_i},\, i < \alpha\}$ comes from an element in the kernel of γ; the maps must be compatible. For every ordinal $i < \alpha$ we have $I_i \subset J$, and hence the composite $p \xrightarrow{f_J} \coprod_{\lambda \in J} p_\lambda \xrightarrow{\pi} \coprod_{\lambda \in I_i} p_\lambda$

must agree with f_{I_i}. Therefore f_{I_i} can be written as a composite

$$p \longrightarrow \coprod_{\lambda\in K} p_\lambda \longrightarrow \coprod_{\lambda\in J} p_\lambda \xrightarrow{\pi} \coprod_{\lambda\in I_i} p_\lambda \quad .$$

Since all the f_{I_i} are non-zero, each set I_i must contain an element of K. Choose one for each i. The I_i were constructed disjoint, there are α of them, and we have produced α distinct elements in the set K of cardinality $< \alpha$. Hence our contradiction. $\square$

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