LOCAL ANALYTIC CLASSIFICATION OF q-DIFFERENCE EQUATIONS WITH $|q| = 1$

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Contents

Introduction

For an algebraic complex semisimple group G and for a fixed $q \in \mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$, $|q| \neq 1$, V. Baranovsky and V. Ginzburg prove the following statement:

Theorem 1 ([BG96, Thm. 1.2]). *There exists a natural bijection between the isomorphism classes of holomorphic principal semistable G-bundles on*

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the elliptic curve $\mathbb{C}/q^{\mathbb{Z}}$ *and the integral twisted conjugacy classes of the points of* G *that are rational over* $\mathbb{C}((x))$.

The *twisted conjugation* is an action of $G(\mathbb{C}((x)))$ on itself defined by

$$(g(x), a(x)) \longmapsto {}^{g(x)}a(x) = g(qx)a(x)g(x)^{-1}\,.$$

An equivalence class is call *integral* when it contains a point of G rational over $\mathbb{C}[[x]]$.

As the authors themselves point out, this result is better understood in terms of q-difference equations. If $G = Gl_\nu$, then the integral twisted conjugacy classes of $G(\mathbb{C}((x)))$ correspond exactly to the isomorphism classes of formal regular singular q-difference systems. In fact, consider a q-difference equation

$$Y(qx) = B(x)Y(x)\,, \qquad \text{with } B(x) \in Gl_\nu(\mathbb{C}((x)))\,.$$

Then this system is regular singular if there exists $G(x) \in Gl_\nu(\mathbb{C}((x)))$ such that $B'(x) = G(qx)B(x)G(x)^{-1} \in Gl_\nu(\mathbb{C}[[x]])$. In this case if $Y(x)$ is a solution of $Y(qx) = B(x)Y(x)$ in some q-difference algebra extending $\mathbb{C}((x))$, then $W(x) = G(x)Y(x)$ is solution of the system $W(qx) = B'(x)W(x)$.

Y. Soibelman and V. Vologodsky in [SV03] use an analogous approach, *via* q-difference equations, to understand vector bundles on non commutative elliptic curves. The classification of analytic q-difference systems, with $|q| = 1$, is a step in Y. Manin's *Alterstraum* [Man04], for understanding real multiplication through non commutative geometry.

In [SV03], the authors identify the category of coherent modules on the elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$, for $q \in \mathbb{C}^*$, not a root of unity, to the category of $\mathcal{O}(\mathbb{C}^*) \rtimes q^{\mathbb{Z}}$-modules of finite presentation over the ring $\mathcal{O}(\mathbb{C}^*)$ of holomorphic functions on $\mathbb{C}^*$ (*cf.* [SV03, §2,§3]), both in the classic (*i.e.* $|q| \neq 1$) and in the non commutative (*i.e.* $|q| = 1$) case. For $|q| = 1$, they study, under convenient diophantine assumptions, its Picard group and make a list of simple objects. In the second part of the paper, they focus on the classification of formal analogous objects defined over $\mathbb{C}((x))$: those are equivalent to q-difference modules over $\mathbb{C}((x))$, *i.e.* $\mathbb{C}((x))$-finite vector spaces M equipped with a semilinear invertible operator Σ_q, such that $\Sigma_q(f(x)m) = f(qx)\Sigma_q(m)$, for any $f(x) \in \mathbb{C}((x))$ and any $m \in M$.

In this paper, we establish, under convenient diophantine assumptions, a complete analytic classification of q-difference modules over the field $\mathbf{K} = \mathbb{C}(\{x\})$ of germs of meromorphic functions at zero, proving some analytic analogs of the results in [SV03] and in [BG96].

* * *

Let $q \in \mathbb{C}$, $|q| = 1$, be not a root of unity. Let us consider the following two categories:

• The category $\mathcal{B}_q^{an,reg}$ of q-difference modules (M, Σ_q) over the field $\mathbf{K}$, such that there exists a basis $\underline{u}$ of M over $\mathbf{K}$ in which the action of the operator Σ_q is described by a constant matrix.

• The category $\mathcal{B}_q^{\delta}$ of q-difference modules (M, Σ_q) free of finite rank over $\mathcal{O}(\mathbb{C}^*)$, such that one can define a regular singular connection on M, commuting to Σ_q. This means that:

1) there exists an action ∇ of $\delta = x\frac{d}{dx}$ over M such that $\nabla(fm) = \delta(f)m + f\nabla(m)$, for any $f \in \mathcal{O}(\mathbb{C}^*)$ and $m \in M$;

2) there exists a basis $\underline{e}$ of M over $\mathcal{O}(\mathbb{C}^*)$ in which $\nabla\underline{e} = \underline{e}A$, with $A \in M_{\nu\times\nu}(\mathcal{O}(\mathbb{C}))$;

3) $\nabla \circ \Sigma_q = \Sigma_q \circ \nabla$.

In both cases, the morphisms are morphisms of q-difference modules. We prove the statement (*cf.* Theorem 3.20 below):

Theorem 2. *The categories $\mathcal{B}_q^{an,reg}$ and $\mathcal{B}_q^{\delta}$ are equivalent.*

Let us make some comments:

1) The objects (M, Σ_q) of $\mathcal{B}_q^{\delta}$ are actually $\mathcal{O}(\mathbb{C}^*) \rtimes q^{\mathbb{Z}}$-modules of finite presentation over $\mathcal{O}(C^*)$, *i.e.* non commutative counterpart of coherent modules over the elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$, equipped with a regular singular connection (*cf.* [SV03, Lemma 2]). To establish Theorem 1, Baranovsky and Ginzburg use the fact that any semistable G-bundle on an elliptic curves is equipped with a regular singular connection (*cf.* [BG96, Prop. 4.1 and Thm. 4.2]). From this point of view, we can consider Theorem 2 as the analytic non commutative analogue of Theorem 1.

2) Consider a q-difference module over $\mathbf{K}$ and fix a basis $\underline{e}$ such that $\Sigma_q\underline{e} = \underline{e}B(x)$, with $B(x) \in Gl_\nu(\mathbf{K})$. If it is a regular singular q-difference module at zero, we can choose a basis $\underline{f}$ of $M \otimes_{\mathbf{K}} \mathbb{C}((x))$ such that $\Sigma_q\underline{f} = \underline{f}B'$ and B' is a constant matrix in $Gl_\nu(\mathbb{C})$ in its Jordan normal form, with non resonant eigenvalues, *i.e.* for any pair α, β of eigenvalues we have either $\alpha = \beta$ or $\alpha\beta^{-1} \notin q^{\mathbb{Z}}$. When $|q| \neq 1$ we do not need to extends the scalars to $\mathbb{C}((x))$ and we can find such a basis $\underline{f}$ over $\mathbf{K}$. When $|q| = 1$ this is not possible in general because of some small divisors appearing in the construction of the basis change.

The proof of Theorem 2 is actually quite easy, and its inspired by [MvS, Prop. 18]. The real object of this paper is the characterization of $\mathcal{B}_q^{an,reg}$ inside the category $\mathcal{B}_q$ of q-difference modules of finite rank over $\mathbf{K}$.

We consider a subcategory $q\text{-}Diff^{aa}_{\mathbf{K}}$ of $\mathcal{B}_q$ of almost admissible q-difference modules over $\mathbf{K}$: they are q-difference modules over $\mathbf{K}$ satisfying a diophantine condition (*cf.* §2.2 and §3.2 below). Those modules admit a decomposition associated to their Newton polygon, namely they are *direct sum* of q-difference modules, whose Newton polygon has one single slope.

The indecomposable objects, *i.e.* those objects that cannot be written as direct sum of submodules, are obtained by iterated non trivial extension of a simple objet by itself. The simple objects are all obtained by scalar restriction to $\mathbf{K}$ from rank 1 $q^{1/n}$-difference objects over $\mathbf{K}(t)$, $x = t^n$, associated to equations of the form $y(q^{1/n}t) = \frac{\lambda}{t^\mu} y(t)$, with $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{Z}$, with $(\mu, n) = 1$. If we call $q - Diff^{a,reg}_{\mathbf{K}}$ the subcategory of $q\text{-}Diff^{aa}_{\mathbf{K}}$ of objects whose Newton polygon has only one slope equal to zero, then (*cf.* §3.6 below):

Theorem 3. *The category $q\text{-}Diff^{aa}_{\mathbf{K}}$ is equivalent to the category of $\mathbb{Q}$-graded objects of $q - Diff^{a,reg}_{\mathbf{K}}$,* i.e. *each object of $q\text{-}Diff^{aa}_{\mathbf{K}}$ is a direct sum indexed on $\mathbb{Q}$ of objects of $q - Diff^{a,reg}_{\mathbf{K}}$ and the morphisms of q-difference modules respect the grading.*

Theorem 4. *The category $q - Diff^{a,reg}_{\mathbf{K}}$ is equivalent to the category of finite dimensional $\mathbb{C}^*/q^{\mathbb{Z}}$-graded complex vector spaces V endowed with nilpotent operators which preserves the grading, that moreover have the following property:*

> *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be a set of representatives of the classes of $\mathbb{C}^*/q^{\mathbb{Z}}$ corresponding to non zero homogeneous components of V. The series $\Phi_{(q;\underline{\Delta})}(x)$ (defined in Definition 2.5) is convergent.*

To prove the classification described above, one only need to study the small divisor problem (*cf.* §1). Once this is done, the techniques used are similar to the techniques used in q-difference equations theory for $|q| \neq 1$ (*cf.* the work of F. Marotte and Ch. Zhang [MZ00], J. Sauloy [Sau04], M. van der Put and M. Reversat [vdPR06], that have their roots in the work of G.D. Birkhoff and P.E. Guether [BG41] and C.R. Adams [Ada29]). The statements we have cited in this introduction are actually consequencesof analytic factorizations properties of q-difference linear operators (*cf.* §2 below). Finally, we point out a work in progress by C. De Concini, D. Hernandez, N. Reshetikhin on related topics.

A last remark: the greatest part of the statements proved in this paper are true also in the ultrametric case, therefore we will mainly work over an algebraically closed normed field $\mathbf{C}, |\ |$.

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1. A small divisor problem

Let:

- $q=\exp(2i\pi\omega)$, with $\omega\in(0,1)\smallsetminus\mathbb{Q}$;
- $\lambda=\exp(2i\pi\alpha)$, with $\alpha\in(0,1]$ and $\lambda\notin q^{\mathbb{Z}_{\leq 0}}$.

We want to study the convergence of the q-hypergeometric series

$$\phi_{(q;\lambda)}(x)=\sum_{n\geq 0}\frac{x^n}{(\lambda;q)_n}\in\mathbb{C}[[x]]\,, \tag{1.0.1}$$

where the *q-Pochhammer symbols* appearing at the denominator of the coefficients of $\phi_{(q;\lambda)}(x)$ are defined by:

$$\begin{cases}(\lambda;q)_0=1\,,\\ (\lambda;q)_n=(1-\lambda)(1-q\lambda)\cdots(1-q^{n-1}\lambda)\,, & \text{for } n\geq 1\,.\end{cases}$$

The main result of this section is the following:

Proposition 1.1. *Suppose that $\lambda\notin q^{\mathbb{Z}}$. The series $\phi_{(q;\lambda)}(x)$ converges if and only if both the series $\sum_{n\geq 0}\frac{x^n}{(q;q)_n}$ and the series $\sum_{n\geq 0}\frac{x^n}{1-q^n\lambda}$ converge. Under these assumptions the radius of convergence of $\phi_{(q;\underline{\lambda})}(x)$ is at least:*

$$R(\omega)\inf\left(1,r(\alpha)\right)\,,$$

where $R(\omega)$ (resp. $r(\alpha)$) is the radius of convergence of $\sum_{n\geq 0}\frac{x^n}{(q;q)_n}$ (resp. $\sum_{n\geq 0}\frac{x^n}{1-q^n\lambda}$).

Remark 1.2. If $\lambda\in q^{\mathbb{Z}_{>0}}$, the series $\phi_{(q;\lambda)}(x)$ is defined and its radius of convergence is equal to $R(\omega)$. Estimates and lower bounds for $R(\omega)$ and $r(\alpha)$ are discussed in the following subsection.

The proof of the Proposition 1.1 obviously follows from the lemma below, which is a q-analogue of a special case of the Kummer transformation formula:

$$\sum_{n\geq 0}\frac{x^s}{(1-\alpha)(2-\alpha)\cdots(n-\alpha)}=\alpha\exp(x)\sum_{n\geq 0}\frac{(-x)^n}{n!}\frac{1}{\alpha-n}\,,$$

which is used in some estimates for p-adic Liouville numbers [DGS94, Ch. VI, Lemma 1.1].

Lemma 1.3 ([DV04, Lemma 20.1])**.** *We have the following formal identity:*

$$\begin{aligned}\phi_{(q;q\lambda)}(x) &= \sum_{n\geq 0}\frac{x^n}{(1-q\lambda)\cdots(1-q^n\lambda)}\\ &= (1-\lambda)\left(\sum_{n\geq 0}\frac{x^n}{(q;q)_n}\right)\left(\sum_{n\geq 0}q^{\frac{n(n+1)}{2}}\frac{(-x)^n}{(q;q)_n}\frac{1}{1-q^n\lambda}\right).\end{aligned}$$

Proof. We set $x=(1-q)t$, $[n]_q=1+q+\cdots+q^{n-1}$ and $[0]_q=1$, $[n]_q^!=[n]_q[n-1]_q^!$. Then we have to show the identity:

$$\phi_{(q;q\lambda)}((1-q)t) = (1-\lambda)\left(\sum_{n\geq 0}\frac{t^n}{[n]_q^!}\right)\left(\sum_{n\geq 0}q^{\frac{n(n+1)}{2}}\frac{(-t)^n}{[n]_q^!}\frac{1}{1-q^n\lambda}\right).$$

Consider the q-difference operator $\sigma_q : t\longmapsto qt$. One verifies directly that the series $\Phi(t):=\phi_{(q;q\lambda)}((1-q)t)$ is solution of the q-difference operator:

$$\begin{aligned}\mathcal{L} &= \big[\sigma_q-1\big]\circ\big[\lambda\sigma_q-((q-1)t+1)\big]\\ &= \lambda\sigma_q^2-((q-1)qt+1+\lambda)\,\sigma_q+(q-1)qt+1\,,\end{aligned}$$

in fact:

$$\begin{aligned}\mathcal{L}\Phi(t) &= \big[\sigma_q-1\big]\circ\big[\lambda\sigma_q-((q-1)t+1)\big]\Phi(t)\\ &= \big[\sigma_q-1\big](\lambda-1)=0\,.\end{aligned}$$

Since the roots of the characteristic equation[1] $\lambda T^2-(\lambda+1)T+1=0$ of $\mathcal{L}$ are exactly $\lambda^{-1}\notin q^{\mathbb{Z}}$ and 1, any solution of $\mathcal{L}y(t)=0$ of the form $1+\sum_{n\geq 1}a_nt^n\in\mathbb{C}[[t]]$ must coincide with $\Phi(t)$. Therefore, to finish the proof of the lemma, it is enough to verify that

$$\Psi(t) = (1-\lambda)\left(\sum_{n\geq 0}\frac{t^n}{[n]_q^!}\right)\left(\sum_{n\geq 0}q^{\frac{n(n+1)}{2}}\frac{(-t)^n}{[n]_q^!}\frac{1}{1-q^n\lambda}\right)$$

is a solution of $\mathcal{L}y(t)=0$ and that $\Psi(0)=1$.

Let $e_q(t)=\sum_{n\geq 0}\frac{t^n}{[n]_q^!}$. Then $e_q(t)$ satisfies the q-difference equation

$$e_q(qt)=((q-1)t+1)\,e_q(t)\,,$$

[1] *i.e.* the equation whose coefficients are the constant terms of the coefficients of the q-difference operator. For a complete description of its construction and properties *cf.* §2.

hence

$$\begin{aligned}\mathcal{L}\circ e_q(t) &= \left[\sigma_q-1\right]\circ e_q(qt)\circ\left[\lambda\sigma_q-1\right]\\ &= e_q(t)\left((q-1)t+1\right)\left[\left((q-1)qt+1\right)\sigma_q-1\right]\circ\left[\lambda\sigma_q-1\right]\\ &= (*)\left[\left((q-1)qt+1\right)\sigma_q-1\right]\circ\left[\lambda\sigma_q-1\right],\end{aligned}$$

where we have denoted with $(*)$ a coefficient in $\mathbb{C}(t)$, not depending on σ_q.

Consider the series $E_q(t)=\sum_{n\geq 0} q^{\frac{n(n+1)}{2}}\frac{t^n}{[n]_q^!}$, which satisfies

$$(1-(q-1)t)\,E_q(qt)=E_q(t)\,,$$

and the series

$$g_\lambda(t)=\sum_{n\geq 0} q^{\frac{n(n+1)}{2}}\frac{(-t)^n}{[n]_q^!}\frac{1}{1-q^n\lambda}\,.$$

Then

$$\begin{aligned}\mathcal{L}\circ e_q(t)g_\lambda(t) &= (*)\left[\left((q-1)qt+1\right)\sigma_q-1\right]\circ\left[\lambda\sigma_q-1\right]g_\lambda(t)\\ &= (*)\left[\left((q-1)qt+1\right)\sigma_q-1\right]E_q(-qt)\\ &= (*)\left[\left((q-1)qt+1\right)E_q(-q^2t)-E_q(-qt)\right]\\ &= 0\,.\end{aligned}$$

It is enough to observe that $e_q(0)g_\lambda(0)=\frac{1}{1-\lambda}$ to conclude that the series $\Psi(t)=(1-\lambda)e_q(t)g_\lambda(t)$ coincides with $\Phi(t)$. □

Remark 1.4. Let $C,|\ |$ be a field equipped with an ultrametric norm and let $q\in C$, with $|q|=1$ and q not a root of unity. Then the formal equivalence in Lemma 1.7 is still true. The series $\sum_{n\geq 0}\frac{x^n}{(q;q)_n}$ is convergent for any $q\in C$ such that $|q|=1$ (*cf.* [ADV04, §2]). On the other side the series $\sum_{n\geq 0}\frac{x^n}{q^n-\lambda}$ is not always convergent. If $\left|\frac{\lambda-1}{q-1}\right|<1$ then its radius of convergence coincides with the radius of convergence of the series $\sum_{n\geq 0}\frac{x^n}{n-\alpha}$, where $\alpha=\frac{\log\lambda}{\log q}$ (*cf.* [DV04, §19], [DGS94, Ch. VI]), otherwise it converges for $|x|<1$.

1.1. **Some remarks on Proposition 1.1.** Let us make some comments on the convergence of the series $\sum_{n\geq 0}\frac{x^n}{(q;q)_n}$ and $\sum_{n\geq 0}\frac{x^n}{1-q^n\lambda}$. A first contribution to the study the convergence of the series $\sum_{n\geq 0}\frac{x^n}{(q;q)_n}$ can be found in [HW88]. The subject has been studied in detail in [Lub98].

Definition 1.5. (*cf.* for instance [Mar00, §4.4]) Let $\left\{\frac{p_n}{q_n}\right\}_{n\geq 0}$ be the convergents of ω, occurring in its continued fraction expansion. Then the *Brjuno* *function* $\mathcal{B}$ of ω is defined by

$$\mathcal{B}(\omega) = \sum_{n\geq 0} \frac{\log q_{n+1}}{q_n}$$

and ω is a *Brjuno number* if $\mathcal{B}(\omega) < \infty$.

Now we are ready to recall the well-known theorem:

Theorem 1.6 (Yoccoz lower bound, *cf.* [Yoc95], [CM00, Thm. 2.1], [Mar00, Thm. 5.1])**.** *If ω is a Brjuno number then the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ converges.*

*Moreover its radius of convergence is bounded from below by $e^{-B(\omega)-C_0}$, where $C_0 > 0$ is an universal constant (*i.e.* independent of ω).*

Sketch of the proof. Suppose that ω is a Brjuno number, then our statement is much easier than the ones cited above and its actually an immediate consequence of the Davie's lemma (*cf.* [Mar00, Lemma 5.6 (c)] or [CM00, Lemma B.4,3)]). □

We set $\|x\|_{\mathbb{Z}} = \inf_{k\in\mathbb{Z}} |x+k|$. Then, as far as the series $\sum_{n\geq 0} \frac{x^n}{1-q^n\lambda}$ is concerned, we have:

Lemma 1.7. (1) *The series $\sum_{n\geq 0} \frac{x^n}{1-q^n\lambda}$ is convergent.*

(2) $\limsup_{n\to\infty} \dfrac{\log|1-\lambda q^n|^{-1}}{n} < +\infty.$

(3) $\liminf_{n\to\infty} \|n\omega+\alpha\|_{\mathbb{Z}}^{1/n} > 0.$

Proof. The equivalence between 1. and 2. is straightforward. Let us prove the equivalence "$1 \Leftrightarrow 3$".

Notice that for any $x \in [0,1/4]$ we have $f(x) := sin(\pi x) - x \geq 0$, in fact $f(0) = 0$ and $f'(x) = \pi cos(\pi x) - 1 \geq 0$. Therefore we conclude that the following inequality holds for any $x \in [0,1/2]$:

$$\sin(\pi x) > \min\big(x, 1/4\big)\,.$$

This implies that:

$$\begin{aligned}|q^n\lambda - 1| &= |\exp\left(2i\pi(n\omega+\alpha)\right) - 1| \\ &= 2\sin\left(\pi\|n\omega+\alpha\|_{\mathbb{Z}}\right) \in \Big[\min\left(2\|n\omega+\alpha\|_{\mathbb{Z}}, 1/2\right), 2\pi\|n\omega+\alpha\|_{\mathbb{Z}}\Big[\end{aligned}$$

and ends the proof. □

1.2. **A corollary.** Let:

- $q = \exp(2i\pi\omega)$, with $\omega \in (0,1) \smallsetminus \mathbb{Q}$;
- $m \in \mathbb{Z}_{>0}$ and $\lambda_i = \exp(2i\pi\alpha_i)$, for $i = 1, \dots, m$, with $\alpha_i \in (0,1]$ and $\lambda_i \notin q^{\mathbb{Z}}$.

From the proposition above we immediately obtain:

Corollary 1.8. *The series*

$$\phi_{(q;\underline{\lambda})}(x) = \sum_{n\geq 0} \frac{x^n}{(\lambda_1;q)_n \cdots (\lambda_m;q)_n} \in \mathbb{C}[[x]]$$

converges if and only if both the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ *and the series* $\sum_{n\geq 0} \frac{x^n}{1-q^n\lambda_i}$, *for* $i = 1, \dots, m$, *converge. Under these assumptions the radius of convergence of* $\phi_{(q;\underline{\lambda})}(x)$ *is at least:*

$$R(\omega)^m \cdot \prod_{i=1}^{m} \inf\left(1, r(\alpha_i)\right) .$$

2. Analytic factorization of q-difference operators

Notation 2.1. Let $(\mathbf{C}, |\ |)$ be either the field of complex numbers with the usual norm or an algebraically closed field with an ultrametric norm. We fix $q \in \mathbf{C}$, such that $|q| = 1$ and q is not a root of unity, and a set of elements $q^{1/n} \in \mathbf{C}$ such that $(q^{1/n})^n = q$.

If $\mathbf{C} = \mathbb{C}$ then let $\omega \in (0,1] \smallsetminus \mathbb{Q}$ be such that $q = \exp(2i\pi\omega)$. *We suppose that the series* $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ *is convergent*, which happens for instance if ω is a Brjuno number.

The contents of this section is largely inspired by [Sau04], where the author proves an analytic classification result for q-difference equations with $|q| \neq 1$ and integral slopes: the major difference is the small divisor problem that the assumption $|q| = 1$ introduces. Of course, once the small divisor problem is solved, the techniques are the same. For this reason some proofs will be only sketched.

2.1. **The Newton polygon.** We consider a q-difference operator $\mathcal{L} = \sum_{i=0}^{\nu} a_i(x)\sigma_q^i \in \mathbf{C}\{x\}[\sigma_q]$, *i.e.* an element of the skew ring $\mathbf{C}\{x\}[\sigma_q]$, where $\mathbf{C}\{x\}$ is the $\mathbf{C}$-algebra of germs of analytic function at zero and $\sigma_q f(x) = f(qx)\sigma_q$. The associated q-difference equations is

$$\mathcal{L}y(x) = a_\nu(x)y(q^\nu x) + a_{\nu-1}(x)y(q^{\nu-1}x) + \cdots + a_0(x)y(x) = 0 \,. \tag{2.1.1}$$

We suppose that $\alpha_\nu(x) \neq 0$, and we call ν is the *order* of $\mathcal{L}$ (or of $\mathcal{L}y = 0$).

Definition 2.2. The *Newton polygon* $NP(\mathcal{L})$ of the equation $\mathcal{L}y = 0$ (or of the operator $\mathcal{L}$) is the convex envelop in $\mathbb{R}^2$ of the following set:

$$\{(i,k) \in \mathbb{Z} \times \mathbb{R} : i = 0, \dots, \nu,\ a_i(x) \neq 0,\ k \geq \mathrm{ord}_x a_i(x)\}\ ,$$

where $\mathrm{ord}_x a_i(x) \geq 0$ denotes the order of zero of $a_i(x)$ at $x = 0$.

Notice that the polygon $NP(\mathcal{L})$ has a finite number of finite slopes, which are all rational and can be negative, and two infinite vertical sides. We will denote $\mu_1, \dots, \mu_k$ the finite slopes of $NP(\mathcal{L})$ (or, briefly of $\mathcal{L}$), ordered so that $\mu_1 < \mu_2 < \cdots < \mu_k$ (*i.e.* from left to right), and $r_1, \dots, r_k$ the length of their respective projections on the x-axis. Notice that $\mu_i r_i \in \mathbb{Z}$ for any $i = 1, \dots, k$.

We can always assume, and we will actually assume, that the boundary of the Newton polygon of $\mathcal{L}$ and the x-axis intersect only in one point or in a segment, by clearing some common powers of x in the coefficients of $\mathcal{L}$. Once this convention fixed, the Newton polygon is completely determined by the set $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\} \in \mathbb{Q} \times \mathbb{Z}_{>0}$, therefore we will identify the two data.

Definition 2.3. A q-difference operator, whose Newton polygon has only one slope (equal to μ) is called *pure (of slope μ)*.

Remark 2.4. All the properties of Newton polygons of q-difference equations listed in [Sau04, §1.1] are formal and therefore independent of the field $\mathbf{C}$ and of the norm of q: they can be rewritten, with exactly the same proof, in our case. We recall, in particular, two properties of the Newton polygon that we will use in the sequel (*cf.* [Sau04, §1.1.5]):

• Let θ be a solution, in some formal extension of $\mathbf{C}(\{x\}) = Frac(\mathbf{C}\{x\})$, of the q-difference equation $y(qx) = xy(x)$. The twisted conjugate operator $x^C \theta^\mu \mathcal{L} \theta^{-\mu} \in \mathbf{C}\{x\}[\sigma_q]$, where C is a convenient non negative integer, is associated to the q-difference equation[2]

(2.4.1)
$$a_\nu(x) q^{-\mu\frac{\nu(\nu+1)}{2}} x^{C-\mu\nu} y(q^\nu x) + a_{\nu-1}(x) q^{-\mu\frac{\nu(\nu-1)}{2}} x^{C-\mu(\nu-1)} y(q^{\nu-1}x) + \cdots + x^C a_0(x) y(x) = 0\,,$$

and has Newton polygon $\{(\mu_1 - \mu, r_1), \dots, (\mu_k - \mu, r_k)\}$.

• If $e_{q,c}(x)$ is a solution of $y(qx) = cy(x)$, with $\in \mathbf{C}^*$. Then the twisted operator $e_{q,c}(x)^{-1} \mathcal{L} e_{q,c}(x)$ has the same Newton polygon as $\mathcal{L}$, but all its exponents are multiplied by c.

[2]Notice that there is no need of determine the function θ.

2.2. **Admissible q-difference operators.** Suppose that 0 is a slope of $NP(\mathcal{L})$. We call *characteristic polynomial of the zero slope* the polynomial

$$a_\nu(0)T^\nu + a_{\nu-1}(0)T^{\nu-1} + \cdots + a_0(0) = 0\,.$$

The characteristic polynomial of a slope $\mu \in \mathbb{Z}$ is characteristic polynomial of the zero slope of the q-difference operator $x^C\theta^\mu\mathcal{L}\theta^{-\mu}$ (*cf.* Equation 2.4.1). In the general case, when $\mu \in \mathbb{Q} \smallsetminus \mathbb{Z}$, we reduce to the previous assumption by performing a ramification. Namely, for $n \in \mathbb{Z}_{>0}$, we set $t = x^{1/n}$. With this variable change, the operator $\mathcal{L}$ becomes $\sum a_i(t^n)\sigma^i_{q^{1/n}}$. A convenient choice of n allows to obtain a $q^{1/n}$-difference operator with integral slopes and to give the general definition of characteristic polynomial.

Finally, we call the non zero roots of the characteristic polynomial of the slope μ the *exponents of the slope* μ. Notice that the cardinality of the set of the exponents of the slope μ, counted with multiplicities, is equal to the length of the projection of μ on the x-axis.

Definition 2.5. Let $(\lambda_1, \dots, \lambda_r)$ be the exponents of the slope μ of $\mathcal{L}$ and let

$$\underline{\Lambda} = \{\lambda_i\lambda_j^{-1} \,:\, i,j = 1,\dots,r\,;\ \lambda_i\lambda_j^{-1} \notin q^{\mathbb{Z}_{\leq 0}}\}\,.$$

We say that a slope $\mu \in \mathbb{Z}$ of $\mathcal{L}$ is *admissible* if the series $\phi_{(q;\underline{\Lambda})}(x)$ is convergent and that a slope $\mu \in \mathbb{Q}$ is *almost admissible* if it becomes admissible in $\mathbf{C}\{x^{1/n}\}[\sigma_{q^{1/n}}]$, for a convenient $n \in \mathbb{Z}_{>0}$.

A q-difference operator is said to be *admissible* (resp. *almost admissible*) if all its slopes are admissible (resp. *almost admissible*).

2.3. **Analytic factorization of admissible q-difference operators.** The main result of this subsection is the analytic factorization of admissible q-difference operators. The analogous result in the case $|q| \neq 1$, with integral slopes, is well known (*cf.* [MZ00], [Sau04, §1.2], or, for a more detailed exposition, [Sau02b, §1.2]). The germs of those works are already in [BG41], where the authors establish a canonical form for solution of analytic q-difference systems. The result is stated in [vdPR06, §1.3], for q-difference operators with $|q| \neq 1$ and rational slopes.

Theorem 2.6. *Suppose that the q-difference operator $\mathcal{L}$ is admissible, with Newton polygon $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$. Then for any permutation σ of the set $\{1, \dots, k\}$ there exists a factorization of $\mathcal{L}$:*

$$\mathcal{L} = \mathcal{L}_{\sigma(1)} \circ \mathcal{L}_{\sigma(2)} \circ \cdots \circ \mathcal{L}_{\sigma(k)}\,,$$

such that $\mathcal{L}_{\sigma(i)} \in \mathbf{C}\{x\}[\sigma_q]$ is admissible and pure of slope μ_i and order r_i.

Remark 2.7.

- Given the permutation σ, the q-difference operator $\mathcal{L}_{\sigma(i)}$ is uniquely determined, modulo a factor in $\mathbf{C}\{x\}$.

• Exactly the same statement holds for almost admissible q-difference operator (*cf.* Theorem 3.12 below).

Theorem 2.6 follows from the recursive application of the statement:

Proposition 2.8. *Let $\mu \in \mathbb{Z}$ be an admissible slope of the Newton polygon of $\mathcal{L}$ and let r be the length of its projection on the x-axis. Then the q-difference operator $\mathcal{L}$ admits a factorization $\mathcal{L} = \widetilde{\mathcal{L}} \circ \mathcal{L}_\mu$, such that:*

1. *the operator $\widetilde{\mathcal{L}}$ is in $\mathbf{C}\{x\}[\sigma_q]$ and $NP(\widetilde{\mathcal{L}}) = NP(\mathcal{L}) \smallsetminus \{(\mu, r)\}$;*
2. *the operator $\mathcal{L}_\mu$ has the form:*

$$\mathcal{L}_\mu = (x^\mu \sigma_q - \lambda_r) h_r(x) \circ (x^\mu \sigma_q - \lambda_{r-1}) h_{r-1}(x) \circ \cdots \circ (x^\mu \sigma_q - \lambda_1) h_1(x)\,,$$

where:
• *$\lambda_1, \ldots, \lambda_r \in \mathbf{C}$ are the roots of the characteristic polynomial of the slope μ, ordered so that if $\frac{\lambda_i}{\lambda_j} \in q^{\mathbb{Z}_{>0}}$ then $i < j$;*
• *$h_1(x), \ldots, h_r(x) \in 1 + x\mathbf{C}\{x\}$.*

Moreover if $\mathcal{L}$ is admissible (resp. almost admissible), the operator $\widetilde{\mathcal{L}}$ is also admissible (resp. almost admissible).

Proposition 2.8 itself follows from an iterated application of the lemma below:

Lemma 2.9. *Let $(\mu, r) \in NP(\mathcal{L}) = \{(\mu_1, r_1), \ldots, (\mu_k, r_k)\}$ be an integral slope of $\mathcal{L}$ with exponents $(\lambda_1, \ldots, \lambda_r)$. Fix an exponent λ of μ such that:*

1. *$q^n \lambda$ is not an exponent of the same slope for any $n > 0$;*
2. *the series $\phi_{\left(q;\left(\frac{\lambda_1}{\lambda}, \ldots, \frac{\lambda_r}{\lambda}\right)\right)}(x)$ is convergent.*

Then there exists a unique $h(x) \in 1 + x\mathbf{C}\{x\}$ such that $\mathcal{L} = \widetilde{\mathcal{L}} \circ (x^\mu \sigma_q - \lambda) h(x)$, for some $\widetilde{\mathcal{L}} \in \mathbf{C}\{x\}[\sigma_q]$. Moreover let $\iota = 1, \ldots, k$ such that $\mu_\iota = \mu$:
• *if $r_\iota = 1$ then $NP(\widetilde{\mathcal{L}}) = \{(\mu_1, r_1), \ldots, (\mu_{\iota-1}, r_{\iota-1}), (\mu_{\iota+1}, r_{\iota+1}), \ldots, (\mu_k, r_k)\}$;*
• *if $r_\iota > 1$ then $NP(\widetilde{\mathcal{L}}) = \{(\mu_1, r_1), \ldots, (\mu_\iota, r_\iota - 1), \ldots, (\mu_k, r_k)\}$. The set of exponents of the slope μ of $\widetilde{\mathcal{L}}$ coincides with the set of exponents of the slope μ of $\mathcal{L}$, minus the exponent λ.*

Proof. It is enough to prove the lemma for $\mu = 0$ and $\lambda = 1$ (*cf.* Remark 2.4). Write $y(x) = \sum_{n \geq 0} y_n x^n$, with $y_0 = 1$, and $a_i(x) = \sum_{n \geq 0} a_{i,n} x^n$. Then we obtain by direct computation that $\mathcal{L} y(x) = 0$ if and only if for any $n \geq 1$ we have:

$$F_0(q^n) y_n = -\sum_{l=0}^{n-1} F_{n-l}(q^l) y_k\,,$$

where $F_l(T)=\sum_{i=0}^{\nu} a_{i,l}T^i$. Remark that assumption 1. is equivalent to the property: $F_0(q^n)\neq 0$ for any $n\in\mathbb{Z}_{>0}$.

The convergence of the coefficients $a_i(x)$ of $\mathcal{L}$ implies the existence of two constants $A,B>0$ such that $|F_{n-l}(q^l)|\leq AB^{n-l}$, for any $n\geq 0$ and any $l=0,\dots,n-1$. We set

$$s_n=F_0(1)F_0(q)\dots F_0(q^n)y_n\,.$$

Then

$$|s_n|\leq\left|\sum_{l=0}^{n-1}s_lF_0(q^{l+1})\cdots F_0(q^{n-1})F_{n-l}(q^l)\right|\leq A^nB^n\sum_{l=0}^{n-1}\frac{|s_l|}{(AB)^l}\,,$$

and therefore:

$$|t_n|\leq\sum_{l=0}^{n-1}|t_l|\,,\ \text{with } t_l=\frac{s_l}{(AB)^l}\,.$$

If $|t_l|<CD^l$, for any $l=0,\dots,n-1$, with $D>1$, then $|t_n|\leq C\sum_{l=0}^{n-1}D^l\leq CD^n(D-1)^{-1}\leq CD^n$. Therefore $|t_n|\leq CD^n$ for any $n\geq 1$, and hence $|s_n|\leq C(ABD)^n$. Hypothesis 2 assures that the series $\sum_{n\geq 1}\frac{x^n}{F_0(1)\cdots F_0(q^n)}$ is convergent and therefore that $y(x)$ is convergent. We conclude setting $h(x)=y(x)^{-1}$.

For the assertion on the Newton polygon (*cf.* [Sau04]). □

For further reference we point out that we have actually proved the following corollaries:

Corollary 2.10. *In the hypothesis of Lemma 2.9, suppose that $\mathcal{L}$ has a right factor of the form $(\sigma_q^\mu-\lambda)\circ h(x)$, with $\mu\in\mathbb{Q}$, $\lambda\in\mathbf{C}^*$ and $h(x)\in\mathbf{C}[[x]]$. Then $h(x)$ is convergent.*

Remark 2.11. Corollary 2.10 above generalizes [Béz92, Thm. 6.1], where the author proves that a formal solution of an analytic q-difference operator satisfying some diophantine assumptions is always convergent.

Corollary 2.12. *Any almost admissible q-difference operator $\mathcal{L}$ admits an analytic factorization in $\mathbf{C}\{x^{1/n}\}[\sigma_q]$, with $\sigma_q x^{1/n}=q^{1/n}x^{1/n}$, for a convenient $n\in\mathbb{Z}_{>0}$.*

The irreducible factors of $\mathcal{L}$ in $\mathbf{C}\{x^{1/n}\}[\sigma_q]$ are of the form $(x^{\mu/n}\sigma_q-\lambda)h(x^{1/n})$, with $\mu\in\mathbb{Z}$, $\lambda\in\mathbf{C}^*$ and $h(x^{1/n})\in 1+x^{1/n}\mathbf{C}\{x^{1/n}\}$.

The following example shows the importance of considering admissible operators.

Example 2.13. The series $\Phi(x)=\Phi_{(q;q\lambda)}((1-q)x)$, studied in Proposition 1.1, is solution of the q-difference operator $\mathcal{L}=(\sigma_q-1)\circ[\lambda\sigma_q-((q-1)x+1)]$. This operator is already factored.

Suppose that $\lambda \notin q^{\mathbb{Z}_{<0}}$. If the series $\Phi(x)$ is convergent, *i.e.* if $\mathcal{L}$ is admissible, the operator $(\sigma_q - 1)\circ\Phi(x)^{-1}$ is a right factor of $\mathcal{L}$, as we could have deduced from Lemma 2.9. We conclude that if $\Phi(x)$ is not convergent the operator $\mathcal{L}$ cannot be factored "starting with the exponents 1".

3. Analytic classification of admissible q-difference modules

In this section, we are going to apply to q-difference modules the results of the previous section.

Let $\mathbf{K} = \mathbf{C}(\{x\})$ be the field of germs of meromorphic function at 0, *i.e.* the field of fractions of $\mathbf{C}\{x\}$. In the following we will denote by $\widehat{\mathbf{K}} = \mathbf{C}((x))$ the field of Laurent series, and by $\mathbf{K}_n = \mathbf{K}(x^{1/n})$ (resp. $\widehat{\mathbf{K}}_n = \widehat{\mathbf{K}}(x^{1/n})$) the finite extension of $\mathbf{K}$ (resp. $\widehat{\mathbf{K}}$) or degree n, with its natural $q^{1/n}$-difference structure.

3.1. Generalities on q-difference modules.

We recall some generalities on q-difference modules (for a more detailed exposition *cf.* for instance [DV02, Part I], [Sau04] and [DVRSZ03]).

Let F be a q-difference field over $\mathbf{C}$, *i.e.* a field $F/\mathbf{C}$ of functions with an action of σ_q.

Definition 3.1. A *q-difference modules $\mathcal{M} = (M, \Sigma_q)$ over F (of rank ν)* is a finite F-vector space M, of dimension ν, equipped with a σ_q-linear bijective endomorphism Σ_q, *i.e.* with a $\mathbf{C}$-linear isomorphism such that $\Sigma_q(fm) = \sigma_q(f)\Sigma_q(m)$, for any $f \in F$ and any $m \in M$.

A *morphism of q-difference modules* $\varphi\colon (M, \Sigma_q^M) \to (N, \Sigma_q^N)$ is a $\mathbf{C}$-linear morphism $M \to N$, commuting to the action of Σ_q^M and Σ_q^N, *i.e.* $\Sigma_q^N \circ \varphi = \varphi \circ \Sigma_q^M$.

If G is a q-difference field extending F (*i.e.* G/F and the action of σ_q on G extends the one on F), the module $\mathcal{M}_G = (M \otimes_F G, \Sigma_q \otimes \sigma_q)$ is naturally a q-difference module over G.

If F_n, $n \in \mathbb{Z}_{>1}$, is a $q^{1/n}$-difference field containing F and such that ${\sigma_{q^{1/n}}}_{|F} = \sigma_q$ (for instance, think of $\mathbf{K}$ and $\mathbf{K}_n$), for any q-difference modules $\mathcal{M} = (M, \Sigma_q)$ over F we can consider the $q^{1/n}$-difference module $\mathcal{M}_{F_n} = (M \otimes_F F_n, \Sigma_q \otimes \sigma_{q^{1/n}})$ over F_n.

For other algebraic constructions (tensor product, internal *Hom*,...) we refer to [DV02] and [Sau04].

Remark 3.2 (The cyclic vector lemma).
The cyclic vector lemma says that a q-difference module $\mathcal{M}$ over F, of rank ν, contains a cyclic element $m \in M$, *i.e.* an element such that

$m, \Sigma_q m, \dots, \Sigma_q^{\nu-1} m$ is an F-basis of M. This is equivalent to say that there exists a q-difference operator $\mathcal{L} \in F[\sigma_q]$ of order ν such that we have an isomorphism of q-difference modules

$$M \cong \frac{F[\sigma_q]}{F[\sigma_q]\mathcal{L}}\,.$$

We will call $\mathcal{L}$ a q-difference operator associated to $\mathcal{M}$, and $\mathcal{M}$ the q-difference module associated to $\mathcal{L}$.

Example 3.3. [Rank 1 q-difference modules[3]] Let $\mu \in \mathbb{Z}$, $\lambda \in \mathbf{C}^*$ and $h(x) \in \mathbf{K}$ (resp. $h(x) \in \widehat{\mathbf{K}}$). Let us consider the rank 1 q-difference module $\mathcal{M}_{\mu,\lambda} = (M_{\mu,\lambda}, \Sigma_q)$ over $\mathbf{K}$ (resp. $\widehat{\mathbf{K}}$) associated to the operator $(x^\mu \sigma_q - \lambda) \circ h(x) = h(qx)x^\mu \sigma_q - h(x)\lambda$. This means that there exists a basis f of $M_{\mu,\lambda}$ such that $\Sigma_q f = \frac{h(x)}{h(qx)} \frac{\lambda}{x^\mu} f$. If one consider the basis $e = h(x)f$, then $\Sigma_q e = \frac{\lambda}{x^\mu} e$.

A straightforward calculation shows that all rank 1 q-difference modules over $\mathbf{K}$ (resp. $\widehat{\mathbf{K}}$) are obtained in this way and that $\mathcal{M}_{\mu,\lambda}$ is isomorphic, as a q-difference module, to $\mathcal{M}_{\mu',\lambda'}$ if and only if $\mu = \mu'$ and $\frac{\lambda}{\lambda'} \in q^{\mathbb{Z}}$.

The remark and the example above, together with the results of the previous section, imply that we can attach to a q-difference modules a *Newton polygon* by choosing a cyclic vector and that the Newton polygon of a q-difference modules is well-defined (*cf.* [Sau04]).

3.2. **Simple objects in the category of admissible q-difference modules.** Once again the considerations above imply that (almost) admissible q-difference modules are well defined:

Definition 3.4. We say that a q-difference module $\mathcal{M}$ over $\mathbf{K}$ is *admissible (resp. almost admissible; resp. pure (of slope μ))* if there exists an operator $\mathcal{L} \in \mathbf{C}\{x\}[\sigma_q]$ such that $M \cong \mathbf{K}[\sigma_q]/(\mathcal{L})$ and that $\mathcal{L}$ is admissible (resp. almost admissible; resp. pure (of slope μ)).

Let $q\text{-}Diff^a_{\mathbf{K}}$ (resp. $q\text{-}Diff^{aa}_{\mathbf{K}}$) be the category of admissible (resp. almost admissible) q-difference modules over $\mathbf{K}$, whose objects are the admissible (resp. almost admissible) q-difference modules over $\mathbf{K}$ and whose morphisms are the morphisms of q-difference modules over $\mathbf{K}$.

In differential and difference equation theory, simple objects are called *irreducible*. They are those objects $\mathcal{M} = (M, \Sigma_q)$ over $\mathbf{K}$ such that any $m \in M$ is a cyclic vector: this is equivalent to the property of not having a proper q-difference sub-module, or to the fact that any q-difference operator associated to $\mathcal{M}$ cannot be factorized in $\mathbf{K}[\sigma_q]$.

[3]For more details on the rank one case *cf.* [SV03, §3.1], where the Picard group of q-difference modules satisfying a convenient diophantine assumption is studied.

Corollary 3.5. *The only irreducible objects in the category $q\text{-}Diff^a_{\mathbf{K}}$ are the rank one modules described in Example 3.3.*

Proof. It is a consequence of Proposition 2.8. □

Before describing the irreducible object of the category $q\text{-}Diff^{aa}_{\mathbf{K}}$, we need to introduce a functor of restriction of scalars going from $q\text{-}Diff^a_{\mathbf{K}_n}$ to $q\text{-}Diff^{aa}_{\mathbf{K}}$. In fact, the set $\{1, x^{1/n}, \dots, x^{n-1/n}\}$ is a basis of $\mathbf{K}_n/\mathbf{K}$ such that $\sigma_q x^{i/n} = q^{i/n}x^{i/n}$. Therefore $\mathbf{K}_n$ can be identified to the admissible q-difference module $M_{0,1} \oplus M_{0,q^{1/n}} \oplus \cdots \oplus M_{0,q^{n-1/n}}$ (in the notation of Example 3.3).

In the same way, we can associate to any (almost) admissible $q^{1/n}$-difference module $\mathcal{M}$ of rank ν over $\mathbf{K}_n$ an almost admissible difference module $Res_n(\mathcal{M})$ of rank $n\nu$ over $\mathbf{K}$ by restriction of scalars. The functor Res_n "stretches" the Newton polygon horizontally, meaning that if the Newton polygon of $\mathcal{M}$ over $\mathbf{K}_n$ is $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$, then the Newton polygon of $Res_n(\mathcal{M})$ over $\mathbf{K}$ is $\{(\mu_1/n, nr_1), \dots, (\mu_k/n, nr_k)\}$

Example 3.6. Consider the $q^{1/2}$-module over $\mathbf{K}_2$ associated to the equation $x^{1/2}y(qx) = \lambda y(x)$, for some $\lambda \in \mathbf{C}^*$. This means that we consider a rank 1 module $\mathbf{K}_2 e$ over $\mathbf{K}_2$, such that $\Sigma_q e = \frac{\lambda}{x^{1/2}} e$. Notice that its Newton polygon over $\mathbf{K}_2$ has only one single slope equal to 1.

Since $\mathbf{K}_2 e = \mathbf{K}e + \mathbf{K}x^{1/2}e$, the module $\mathbf{K}_2 e$ is a q-difference module of rank 2 over $\mathbf{K}$, whose q-difference structure is defined by:

$$\Sigma_q(e, x^{1/2}e) = (e, x^{1/2}e)\begin{pmatrix} 0 & q^{1/2}\lambda \\ \lambda/x & 0 \end{pmatrix}.$$

Consider the vector $m = e + x^{1/2}e$. We have: $\Sigma_q(m) = q^{1/2}\lambda e + \frac{\lambda}{x}(x^{1/2}e)$ and $\Sigma_q^2(m) = \frac{q^{1/2}\lambda^2}{qx} e + \frac{q^{1/2}\lambda^2}{x}(x^{1/2}e)$. Since m and $\Sigma_q(m)$ are linearly independent, m is a cyclic vector for $\mathbf{K}_2 e$ over $\mathbf{K}$. Moreover, for

$$\begin{cases} P(x) = -\lambda^2(q^{3/2}x - 1) \\ Q(x) = \lambda(q-1)x \\ R(x) = -q^{1/2}x(q^{1/2}x - 1) \end{cases}$$

we have $P(x)m + Q(x)\Sigma_q(m) = R(x)\Sigma_q^2(m)$. In other words, the Newton polygon of the rank 2 q-difference module $\mathbf{K}_2 e$ over $\mathbf{K}$ has only one slope equal to $1/2$.

Let $n \in \mathbb{Z}_{>0}$, μ be an integer prime to n and $\mathcal{M}_{\mu,\lambda,n}$ be the rank one module over $\mathbf{K}_n$ associated to the equation $x^{\mu/n}y(qx) = \lambda y(x)$. In [SV03, Lemma 4], Soibelman and Vologodsky show that $\mathcal{N}_{\mu/n,\lambda} = Res_n(\mathcal{M}_{\mu,\lambda,n})$ is a simple object over $\mathcal{O}(\mathbb{C}^*)$. We show that all the simple objects of the category $q\text{-}Diff_{\mathbf{K}}^{aa}$ are of this form (for the case $|q| \neq 1$, *cf.* [vdPR06]). Remark that $\mathcal{M}_{\mu,\lambda} = \mathcal{M}_{\mu,\lambda,1} = \mathcal{N}_{\mu,\lambda}$ as q-difference modules over $\mathbf{K}$.

Let us start by proving the lemma:

Lemma 3.7. *Let $\mathcal{M}$ be a q-difference module associated to a q-difference operator $\mathcal{L} \in \mathbf{C}\{x\}[\sigma_q]$. Suppose that the operator $\mathcal{L}$ has a right factor in $\mathbf{C}\{x^{1/n}\}[\sigma_q]$ of the form $(x^{\mu/n}\sigma_q - \lambda) \circ h(x^{1/n})$, with $n \in \mathbb{Z}_{>1}$, $\mu \in \mathbb{Z}$, $(n, \mu) = 1$, $\lambda \in \mathbf{C}^*$ and $h(x) \in \mathbf{C}\{x^{1/n}\}$.*

Then $\mathcal{M}$ has a submodule isomorphic to $\mathcal{N}_{\mu/n,\lambda}$.

Proof. First of all remark that any operator $\mathcal{L} \in \mathbf{C}\{x\}[\sigma_q]$ divisible by $(x^{\mu/n}\sigma_q - \lambda) \circ h(x^{1/n})$ has order $\geq n$.

Let $\mathcal{L}_{\mu/n,\lambda} \in \mathbf{C}\{x\}[\sigma_q]$ be a q-difference operator (of order n) associated to $\mathcal{N}_{\mu/n,\lambda}$. Since the ring $\mathbf{C}\{x\}[\sigma_q]$ is euclidean the exist $\mathcal{Q}, \mathcal{R} \in \mathbf{C}\{x\}[\sigma_q]$, such that

$$\mathcal{L} = \mathcal{Q} \circ \mathcal{L}_{\mu/n,\lambda} + \mathcal{R}\,,$$

with $\mathcal{R} = 0$ or $\mathcal{R}$ of order strictly smaller than n and divisible on the right by $(x^{\mu/n}\sigma_q - \lambda) \circ h(x^{1/n})$. Of course, if $\mathcal{R} \neq 0$, we obtain a contradiction. Therefore $\mathcal{L}_{\mu/n,\lambda}$ divides $\mathcal{L}$ and the lemma follows. □

Finally we have a complete description of the isomorphism classes of almost admissible irreducible q-difference modules over $\mathbf{K}$:

Proposition 3.8. *A system of representatives of the isomorphism classes of the irreducible objects of $q\text{-}Diff_{\mathbf{K}}^{aa}$ is given by the reunion of the following sets:*
- admissible rank 1 q-difference modules $\mathcal{M}_{\mu,\lambda}$, with $\mu \in \mathbb{Z}$ and $c \in \mathbf{C}^*/q^{\mathbb{Z}}$, i.e. *the irreducible objects of $q\text{-}Diff_{\mathbf{K}}^{a}$ up to isomorphism (*cf. *Example 3.3);*
- the q-difference modules $\mathcal{N}_{\mu/n,\lambda} = Res_n(\mathcal{M}_{\mu,\lambda,n})$, where $n \in \mathbb{Z}_{>0}$, $\mu \in \mathbb{Z}$, $(n, \mu) = 1$, and $\lambda \in \mathbf{C}^*/(q^{1/n})^{\mathbb{Z}}$.

Proof. Rank 1 irreducible objects of $q\text{-}Diff_{\mathbf{K}}^{aa}$ are necessarily admissible, therefore they are of the form $\mathcal{M}_{\mu,\lambda}$, for some $\mu \in \mathbb{Z}$ and some $\lambda \in \mathbf{C}^*/q^{\mathbb{Z}}$.

Consider an irreducible object $\mathcal{M}$ in $q\text{-}Diff_{\mathbf{K}}^{aa}$ of higher rank. Because of the previous lemma and of Corollary 2.12, it must contain an object of the form $N_{\mu,\lambda,n}$, for convenient μ, λ, n. The irreducibility implies that $\mathcal{M} \cong \mathcal{N}_{\mu,\lambda,n}$. □

Remark 3.9. Consider the rank 1 modules $\mathcal{N}_{\mu,\lambda,n}$ over $\mathbf{K}_n$ and $\mathcal{N}_{r\mu,\lambda,rn}$ over $\mathbf{K}_{rn}$, for some $\mu, r, n \in \mathbb{Z}$, $r > 1$, $n > 0$, $(\mu, n) = 1$, and $\lambda \in \mathbb{C}^*$. Then $Res_n(\mathcal{N}_{\mu,\lambda,n})$ is a rank n q-difference module over $\mathbf{K}$, while $Res_{rn}(\mathcal{N}_{r\mu,\lambda,rn})$ has rank rn, although $\mathcal{N}_{\mu,\lambda,n}$ and $\mathcal{N}_{r\mu,\lambda,rn}$ are associated to the same rank one operator.

Writing explicitly the basis of $\mathbf{K}_{rn}$ over $\mathbf{K}_n$ and over $\mathbf{K}$, one can show that $Res_{rn}(\mathcal{N}_{r\mu,\lambda,rn})$ is a direct sum of r copies of $Res_n(\mathcal{N}_{\mu,\lambda,n})$.

3.3. **Main results.** Now we are ready to state a structure theorem for almost admissible q-difference modules:

Theorem 3.10. *Suppose that the q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $\mathbf{K}$ is almost admissible, with Newton polygon $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$. Then*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_k\,,$$

where the q-difference modules $\mathcal{M}_i = (M_i, \Sigma_{q|M_i})$ are defined over $\mathbf{K}$, almost admissible and pure of slope μ_i and rank r_i.

Each $\mathcal{M}_i$ is a direct sum of almost admissible indecomposable q-difference modules, i.e. *iterated non trivial extension of a simple almost admissible q-difference module by itself.*

Remark 3.11. More precisely, consider the rank ν unipotent q-difference module $\mathcal{U}_\nu = (U_\nu, \Sigma_q)$, defined by the property of having a basis $\underline{e}$ such that the action of Σ_q on $\underline{e}$ is described by a matrix composed by a single Jordan block with eigenvalue 1. Then the indecomposable modules $\mathcal{N}$ in the previous theorem are isomorphic to $\mathcal{N} \otimes_{\mathbf{K}} \mathcal{U}_\nu$, for some irreducible module $\mathcal{N}$ of $q\text{-}Diff^{aa}_{\mathbf{K}}$ and some ν.

The theorem above is equivalent to a stronger version of Theorem 2.6 for almost admissible q-difference operators:

Theorem 3.12. *Suppose that the q-difference operator $\mathcal{L}$ is almost admissible, with Newton polygon $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$. Then for any permutation σ on the set $\{1, \dots, k\}$ there exists a factorization of $\mathcal{L}$:*

$$\mathcal{L} = \mathcal{L}_{\sigma(1)} \circ \mathcal{L}_{\sigma(2)} \circ \cdots \circ \mathcal{L}_{\sigma(k)}\,,$$

such that $\mathcal{L}_{\sigma(i)} \in \mathbf{C}\{x\}[\sigma_q]$ is almost admissible and pure of slope μ_i and order r_i.

Moreover, for any $i = 1, \dots, k$, write $\mu_i = d_i/s_i$, with $d_i, s_i \in \mathbb{Z}$, $s_i > 0$ and $(d_i, s_i) = 1$. We have:

$$\mathcal{L}_{\sigma(i)} = \mathcal{L}_{d_i,\lambda_l^{(i)},s_i} \circ \cdots \circ \mathcal{L}_{d_i,\lambda_1^{(i)},s_i}\,,$$

where:

- $\lambda_1^{(i)}, \dots, \lambda_l^{(i)}$ *are exponents of the slope μ_i, ordered so that $\lambda_j^{(i)}\left(\lambda_{j'}^{(i)}\right)^{-1} \in q^{\mathbb{Z}_{>0}}$ then $j < j'$;*
- *the operator $\mathcal{L}_{d_i,\lambda_j,s_i}$ is associated to the module $\mathcal{N}_{d_i,\lambda_j,s_i}$.*

Proof. Suppose that the operator has at least one non integral slope. *A priori* the operators $\mathcal{L}_{\sigma(i)}$ are defined over $\mathbf{C}\{x^{1/n}\}$, for some $n > 1$. But it follows from Lemma 3.7 that they are product of operators associated to q-difference modules defined over $\mathbf{K}$, of the form $\mathcal{N}_{\mu,\lambda,n}$, for same $\mu, n \in \mathbb{Z}$, $n > 0$, and $\lambda \in \mathbf{C}^*$. □

3.4. **Analytic vs formal classification.** The formal classification of q-difference modules with $|q| = 1$ is studied in [SV03], by different techniques. It can also be deduced by the results of the previous section, dropping the diophantine assumptions, and establishing a formal factorization theorem for q-difference operators. Irreducible objects are q-difference modules over $\widehat{\mathbf{K}}$ obtained by rank one modules associated to q-difference equations of the form $x^\mu y(qx) = \lambda y(x)$, with $\mu \in \mathbb{Q}$ and $\lambda \in \mathbf{C}^*$, by restriction of scalars.

The formal result, actually already proved in [SV03, Thm. 4], is completely analogous:

Theorem 3.13. *Consider a q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $\widehat{\mathbf{K}}$, with Newton polygon $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$. Then*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_k \,,$$

where the q-difference modules $\mathcal{M}_i = (M_i, \Sigma_{q|M_i})$ are defined over $\widehat{\mathbf{K}}$ and are pure of slope μ_i and rank r_i.

Each $\mathcal{M}_i$ is a direct sum of almost admissible indecomposable q-difference modules, i.e. *iterated non trivial extension of a simple almost admissible q-difference module by itself.*

Moreover we have:

Theorem 3.14. *Let $\mathcal{M} = (M, \Sigma_q^M)$ and $\mathcal{N} = (N, \Sigma_q^N)$ be two almost admissible q-difference modules over $\mathbf{K}$. Then $\mathcal{M}$ is isomorphic to $\mathcal{N}$ over $\mathbf{K}$ if and only if $\mathcal{M}_{\widehat{\mathbf{K}}}$ is isomorphic to $\mathcal{N}_{\widehat{\mathbf{K}}}$ over $\widehat{\mathbf{K}}$.*

Remark 3.15. Notice that the situation is completely different from the case of q-difference modules, with $|q| \neq 1$. Of course, also for $|q| \neq 1$ one has a natural map from the moduli space of analytic q-difference modules to the moduli space of formal q-difference modules. Take a formal equivalence class of q-difference modules with Newton polygon $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$. Then its fiber in the moduli space of analytic modules is a complex affine variety of dimension $\sum_{1 \le i < j \le k} r_i r_j (\mu_j - \mu_i)$ (*cf.* [RSZ04], [Sau02a], [vdPR06]).

Proof. It follows from the analytic (resp. formal) factorizations of q-difference modules over $\mathbf{K}$ (resp. $\widehat{\mathbf{K}}$) that:

$$\mathcal{M} \cong \mathcal{N} \Leftrightarrow \mathcal{M}_{\mathbf{K}_n} \cong \mathcal{N}_{\mathbf{K}_n} \quad \text{and} \quad \mathcal{M}_{\widehat{\mathbf{K}}} \cong \mathcal{N}_{\widehat{\mathbf{K}}} \Leftrightarrow \mathcal{M}_{\widehat{\mathbf{K}}_n} \cong \mathcal{N}_{\widehat{\mathbf{K}}_n} \,,$$

for an integer $n \geq 1$ such that the the slopes of the two modules become integral over $\mathbf{K}_n$. So we can suppose that the two modules are actually admissible.

If $\mathcal{M}$ and $\mathcal{N}$ are isomorphic over $\mathbf{K}$ than they are necessarily isomorphic over $\widehat{\mathbf{K}}$. On the other side suppose that $\mathcal{M}_{\widehat{\mathbf{K}}} \cong \mathcal{N}_{\widehat{\mathbf{K}}}$. Then the results follows from the fact that any formal factorization must actually be analytic (*cf.* Corollary 2.10). □

3.5. **Algebraization.** Until now we have considered q-difference modules over fields of functions: the definition can obviously be extended to $\mathbf{C}$-algebras with an action of σ_q (*cf.* for instance [DV02, Part I]).

Proposition 3.16. *Let $\mathcal{M} = (M, \Sigma_q)$ be q-difference module over $\widehat{\mathbf{K}}$ (resp. an object of q-$Diff^{aa}_{\mathbf{K}}$). Then there exists a q-difference module $\mathcal{M}_{alg} = (M_{alg}, \Sigma_q)$ over $\mathbf{C}\left[x, \frac{1}{x}\right]$ such that $\mathcal{M} \cong \mathcal{M}_{alg} \otimes_{\mathbf{C}\left[x,\frac{1}{x}\right]} \widehat{\mathbf{K}}$ (resp. $\mathcal{M} \cong \mathcal{M}_{alg} \otimes_{\mathbf{C}\left[x,\frac{1}{x}\right]} \mathbf{K}$).*

Proof. Because of Theorem 3.13 (resp. Theorem 3.10), it is enough to show the statement for indecomposable q-difference modules. Indecomposable q-difference modules are tensor products of an irreducible module by an unipotent one (*cf.* Remak 3.11). The proposition is clearly true for unipotent module. Irreducible q-difference modules can be obtained from rank 1 q-difference modules defined over $\mathbf{C}\left[x^{1/n}, \frac{1}{x^{1/n}}\right]$, for some $n \geq 1$, by scalar restriction to $\mathbf{C}\left[x, \frac{1}{x}\right]$, and then by tensor product by $\widehat{\mathbf{K}}$ (resp. $\mathbf{K}$). This ends the proof. □

Corollary 3.17. *Let $\mathcal{M} = (M, \Sigma_q)$ be a pure q-difference module over $\widehat{\mathbf{K}}$ (resp. a pure almost admissible q-difference module over $\mathbf{K}$), of slope μ and rank ν. Then for any $n \in \mathbb{Z}_{\geq 1}$ such that $n\mu \in \mathbb{Z}$, there exists a $\mathbf{C}$-vector space V contained in $M_{\widehat{\mathbf{K}}_n}$ (resp. $M_{\mathbf{K}_n}$), of dimension ν, such that $x^\mu \Sigma_q(V) \subset V$ and $M_{\widehat{\mathbf{K}}_n} \cong V \otimes_{\mathbf{C}} \widehat{\mathbf{K}}_n$ (resp. $M_{\mathbf{K}_n} \cong V \otimes_{\mathbf{C}} \mathbf{K}_n$).*

3.6. **Comparison with the results in** [BG96]**,** [SV03] **and** [MvS]**.** Let $q - Diff^{a,reg}_{\mathbf{K}}$ be the full subcategory of q-$Diff^{a}_{\mathbf{K}}$ of pure q-difference modules of slope zero, usually called regular singular. We have an analytic version of [SV03, Thm. 4]:

Theorem 3.18. *The category q-$Diff^{aa}_{\mathbf{K}}$ is equivalent to the category of $\mathbb{Q}$-graded objects of $q - Diff^{a,reg}_{\mathbf{K}}$,* i.e. *each object of q-$Diff^{aa}_{\mathbf{K}}$ is a direct sum indexed on $\mathbb{Q}$ of objects of $q - Diff^{a,reg}_{\mathbf{K}}$ and the morphisms of q-difference modules respect the grading.*

Proof. For any $\mu \in \mathbb{Q}$, the component of degree μ of an object of q-$Diff^{aa}_{\mathbf{K}}$ is its maximal pure submodule of slope μ. The theorem follows from the remark that there are no non trivial morphisms between two pure modules of different slope. $\square$

As far the structure of the category $q - Diff_{\mathbf{K}}^{a,reg}$is concerned we have an analytic analog of [SV03, Thm. 3] and [BG96, Thm. 1.6']:

Theorem 3.19. *The category $q - Diff_{\mathbf{K}}^{a,reg}$ is equivalent to the category of finite dimensional $\mathbb{C}^*/q^{\mathbb{Z}}$-graded complex vector spaces V endowed with nilpotent operators which preserves the grading, that moreover have the following property:*

> *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be a set of representatives of the classes of $\mathbb{C}^*/q^{\mathbb{Z}}$ corresponding to non zero homogeneous components of V. The series $\Phi_{(q;\underline{\Lambda})}(x)$, where $\underline{\Lambda} = \{\lambda_i\lambda_j^{-1} : i, j = 1, \ldots, r;\ \lambda_i\lambda_j^{-1} \notin q^{\mathbb{Z}_{\leq 0}}\}$, is convergent.*

Proof. We have seen that a module $\mathcal{M} = (M, \Sigma_q)$ in $q - Diff_{\mathbf{K}}^{a,reg}$ contains a $\mathbb{C}$-vector space V, invariant under Σ_q, such that $M \cong V \otimes \mathbf{K}$. Hence there exists a basis $\underline{e}$, such that $\Sigma_q \underline{e} = \underline{e}B$, with $B \in Gl_\nu(\mathbb{C})$ in the Jordan normal form. This means that $B = D + N$, where D is a diagonal constant matrix and N a nilpotent one. The operator $\Sigma_q - D$ is nilpotent on V.

Since any eigenvalue λ of D is uniquely determined modulo $q^{\mathbb{Z}}$, we obtain the $\mathbb{C}^*/q^{\mathbb{Z}}$-grading, by considering the kernel of the operators $(\Sigma_q - \lambda)^n$, for $n \in \mathbb{Z}$ large enough. $\square$

We have shown that the category $q - Diff_{\mathbf{K}}^{a,reg}$ is equivalent to its subcategory of admissible q-difference modules admitting a basis over $\mathbf{K}$ in which the action of Σ_q is constant. For $\mathbf{C} = \mathbb{C}$, let us consider the following two categories (*cf.* [MvS, §2.1]):

- The category $\mathcal{B}_q^\delta$ of q-difference module (M, Σ_q) free and of finite rank over the ring $\mathcal{O}(\mathbb{C}^*)$ of holomorphic function on $\mathbb{C}^*$, such that one can define a regular singular connection $\nabla : M \to M$, commuting to Σ. This means that for $\delta = x\frac{d}{dx}$, $f \in \mathcal{O}(\mathbb{C}^*)$, and $m \in M$, we have $\nabla(fm) = \delta(f)m + f\nabla(m)$; that there exists a basis $\underline{e}$ of M over $\mathcal{O}(\mathbb{C}^*)$ such that $\nabla(\underline{e}) = \underline{e}A$ with $A \in M_{\nu\times\nu}(\mathcal{O}(\mathbb{C}))$; and that $\nabla \circ \Sigma_q = \Sigma_q \circ \nabla$. The morphisms of $\mathcal{B}_q^\delta$ are the morphisms of q-difference modules over $\mathcal{O}(\mathbb{C}^*)$.[4]
- The category $\mathcal{B}_q^{an,reg}$ of q-difference modules (M, Σ_q) over $\mathbf{K}$, such that there exists a basis $\underline{e}$ of M over $\mathbf{K}$ in which the action of Σ_q is described by a constant matrix. Again, the morphisms are the morphisms of q-difference modules over $\mathbf{K}$.

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[4] Remark that a free $\mathcal{O}(\mathbb{C}^*)$-module is equivalent to a module of finite presentation over the sheaf of holomorphic function over $\mathbb{C}^*$ (*cf.* [SV03, Lemma 2]).

Remark that $q - Diff_{\mathbf{K}}^{a,reg}$ is a subcategory of $\mathcal{B}_q^{an,reg}$. The characterization [BG96, Prop. 4.1 and Thm. 4.2] of semistable holomorphic principal G-bundles over an elliptic curves as the ones admitting a regular singular connections, may allow to consider the following statement as an analogue of [BG96, Thm. 1.2], already cited in the introduction:

Theorem 3.20. *The category $\mathcal{B}_q^{\delta}$ and $\mathcal{B}_q^{an,reg}$ are equivalent.*

Proof. Take an object (M, Σ_q) in $\mathcal{B}_q^{an,reg}$ and fix a basis $\underline{e}$ of M over $\mathbf{K}$ such that $\Sigma_q(\underline{e}) = \underline{e}B$, with $B \in Gl_\nu(C)$. Then given a constant matrix A in the centralizer of B, we can define a connection on M setting $\nabla(\underline{e}) = \underline{e}A$. The $\mathbb{C}$-vector space $V \subset M$ generated by $\underline{e}$ is stable under Σ_q and ∇, therefore $(V \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{C}^*), \nabla \otimes \delta, \Sigma_q \otimes \sigma_q)$ is an objects of $\mathcal{B}_q^{\delta}$.

Let (M, Σ_q) be an object of $\mathcal{B}_q^{\delta}$, equipped with a regular singular connection ∇. Then the general theory of regular singular connections says that there exists a basis $\underline{e}$ of M over $\mathcal{O}(\mathbb{C}^*)$ such that $\nabla(\underline{e}) = \underline{e}A$, where A is a constant square matrix, in the Jordan normal form, such that the difference of two eigenvalues is not a non zero integer (*cf.* [MvS, Prop. 1.8 and its proof]). Let $B = \sum_{n \in \mathbb{Z}} B_n x^n$ be the matrix of Σ_q with respect to $\underline{e}$. The commutativity relation between ∇ and Σ_q forces B to satisfy the equations: $(A - n)B_n = B_n A$, for any $n \geq 1$. Since the eigenvalues of the map:

$$\begin{array}{cccc} \Phi_{A,n}: & M_{\nu\times\nu}(\mathbb{C}) & \longrightarrow & M_{\nu\times\nu}(\mathbb{C}) \\ & X & \longmapsto & (A-n)X - XA \end{array}$$

are exactly of the form $\alpha - \beta - n$, where α, β are eigenvalues of A, we conclude that $\Phi_{A,n}$ is invertible and that $B = B_0 \in Gl_\nu(C)$. If $V \subset M$ is the $\mathbb{C}$-vector space generated by $\underline{e}$, the q-difference module $(V \otimes_{\mathbb{C}} \mathbf{K}, \Sigma_q)$ is an object of $\mathcal{B}_q^{an,reg}$.

The fact that the two functors send isomorphic objects into isomorphic objects follows from the fact that the equivalence classes on both sides are determined by the Jordan structure of the constant matrix B and images of its eigenvalues in $\mathbb{C}^*/q^{\mathbb{Z}}$. □

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