CONVOLUTION INEQUALITIES IN LORENTZ SPACES

ERLAN NURSULTANOV AND SERGEY TIKHONOV

Abstract. In this paper we study boundedness of the convolution operator in different Lorentz spaces. In particular, we obtain the limit case of the Young-O'Neil inequality in the classical Lorentz spaces. We also investigate the convolution operator in the weighted Lorentz spaces. Finally, norm inequalities for the potential operator are presented.

1. Introduction

The Young convolution inequality of the form

$$\|f * K\|_p \leqslant \|f\|_p \|K\|_1$$

and in a more general form

$$\|f * K\|_q \leqslant \|f\|_p \|K\|_r, \qquad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$$

plays a very important role both in Harmonic Analysis and PDE.

O'Neil [ON], Yap [Ya], and Blozinski [Bl3] studied the boundedness of the convolution operator

$$Af(y) = \int_D K(y-x) f(x) dx \tag{1.1}$$

in the Lorentz spaces. In particular, the following Young-O'Neil inequality was obtained: for $1 < p, q, r < \infty$, $0 < h_1, h_2, h_3 \leqslant \infty$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, and $\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}$, one has

$$\|Af\|_{L_{qh_1}(\Omega)} \leqslant c \|f\|_{L_{ph_2}(D)} \|K\|_{L_{rh_3}(\Omega - D)} \tag{1.2}$$

where $\Omega - D = \{x - y : x \in \Omega, y \in D\}$.

In this paper we continue investigating the Young-type inequalities in different Lorentz spaces.

The outline of the paper is as follows. In section 2 we study the boundedness of the operator A from $L_{ph_2}(\Omega)$ into $L_{ph_1}(\Omega)$, i.e., the limit case of the

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2000 *Mathematics Subject Classification.* Primary 46E30; Secondary 44A35, 47G10.

Key words and phrases. Young-O'Neil inequality, Lorentz spaces, Convolution, Riesz Potential Operator.

Young-O'Neil inequality ($p = q$ and $r = 1$). It is known (see [Bl2, Theorem 2]) that if $\Omega = \mathbb{R}^n$, $h_1 < h_2 \leqslant \infty$ and $K \geqslant 0$, then

$$A : L_{ph_2}(\mathbb{R}^n) \longrightarrow L_{ph_1}(\mathbb{R}^n)$$

implies $A \equiv 0$, i.e., $K \overset{\text{a.e.}}{=} 0$. We show that in the case when Ω is of finite measure, the same problem has a nontrivial solution: for $1 \leqslant p = q < \infty$ one has

$$\|(K * f)^{**}\|_{p,h_1} \leqslant c\|f^{**}\|_{p,h_2}\|K^{**}\|_{1,h_3}.$$

The case $p = q = \infty$ is studied separately in Section 3. In this case we consider the $L_{\infty q}$-spaces ([BRS], [BMR]) and we obtain

$$\|K * f\|_{L_{\infty,h_1}} \leqslant 2\,\|f\|_{L_{\infty,h_2}}\|K^{**}\|_{L_{1,h_3}}.$$

Further, Section 4 is devoted to the general Young-O'Neil-type inequality in the weighted Lorentz spaces $\Lambda^q(\omega)$, $\Gamma^q(\omega)$, and $S^q(\omega)$, i.e.,

$$\Lambda^q(\omega) = \left\{ \|f\|_{\Lambda^q(\omega)} = \left(\int_0^\infty \big(f^*(t)\big)^q \omega(t)\,dt \right)^{1/q} < \infty \right\},$$

$$\Gamma^q(\omega) = \left\{ \|f\|_{\Gamma^q(\omega)} = \left(\int_0^\infty \big(f^{**}(t)\big)^q \omega(t)\,dt \right)^{1/q} < \infty \right\},$$

$$S^q(\omega) = \left\{ \|f\|_{S^q(\omega)} = \left(\int_0^\infty \big(f^{**}(t) - f^*(t)\big)^q \omega(t)\,dt \right)^{1/q} < \infty \right\}$$

(see, e.g., [CGMP], [Sa]).

In section 5 we obtain the following inequality for the convolution operator

$$\int_0^\infty g^*(s)(Af)^{**}(s)ds \leqslant \int_0^\infty g^*(t) \int_0^\infty f^*(s)K^{**}(|t-s|)dsdt.$$

We use it to prove the norm inequalities for the Riesz operator.

Finally, section 6 contains the Young-O'Neil inequalities for multidimensional Lorentz spaces.

Basic notations. Let μ be n-dimensional Lebesgue measure and $\Omega \subset \mathbb{R}^n$ a measurable subset. For $1 \leqslant p \leqslant \infty$, $L_p(\Omega)$ is the usual Lebesgue space with norm $\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f(x)|^p\,d\mu\right)^{\frac{1}{p}} < \infty$. The distribution of a measurable function f on Ω is defined by

$$m(\sigma, f) = \mu\{x \in \Omega : |f(x)| > \sigma\}.$$

Then $f^*(t) = \inf\{\sigma : m(\sigma, f) \leqslant t\}$ is the decreasing rearrangement of f.

Let $0 < p, q \leqslant \infty$. We will say that a measurable function f on Ω belongs to the Lorentz space $L_{pq}(\Omega, \mu)$ if

$$\|f\|_{L_{pq}} = \left(\int_0^{|\Omega|} (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty$$

for $0 \leqslant q < \infty$, and

$$\|f\|_{L_{p\infty}} = \sup_{t \in (0,|\Omega|)} t^{1/p} f^*(t) < \infty,$$

where $|\Omega|$ is the measure of Ω. We also define

$$f^{**}(x) = \frac{1}{x} \int_0^x f^*(t) dt. \tag{1.3}$$

By C, C_i, c we will denote positive constants that may be different on different occasions. We write $F \asymp G$ if $F \leqslant C_1 G$ and $G \leqslant C_2 F$ for some positive constants C_1 and C_2 independent of appropriate quantities involved in the expressions F and G.

2. Convolution in the Lorentz space of periodic functions

Let $L_{pq}[0,1]$ be the Lorentz space of all 1-periodic functions with the norm given by

$$\|f\|_{pq} = \|f\|_{L_{pq}[0,1]} = \left(\int_0^1 (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

It is known [BS, p. 219] that for $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$ we have

$$\|f\|_{L_{pq}} \leqslant \|f^{**}\|_{p,q} := \left(\int_0^1 (t^{\frac{1}{p}} f^{**}(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \leqslant p' \|f\|_{L_{pq}}, \tag{2.1}$$

where f^{**} is defined by (1.3) and can be written as [BS, p. 53]

$$f^{**}(t) = \sup_{\substack{|e|=t \\ e \subset [0,1]}} \frac{1}{|e|} \int_e |f(x)| dx \tag{2.2}$$

Here $p' = \frac{p}{p-1}$ and $\mu e = |e|$.

Note that for $p = 1$, (2.1) is not true. For a fixed $f \in L_1$ the functional $\|f^{**}\|_{1,q}$ is non-decreasing as a function of $\frac{1}{q} \in [0, +\infty)$; moreover $\|f^{**}\|_{1,\infty} \asymp \|f\|_{L_1}$ and $\|f^{**}\|_{1,1} \asymp \|f\|_{L \log L}$.

We will need the following Hardy-type lemma.

Lemma 2.1. *Let $f(t)$ be a non-increasing non-negative function on $(0,\infty)$, $0 \leqslant \phi(t) \leqslant 1$ and $\int_0^\infty \phi(t)dt = d < \infty$. Then*

$$\int_0^d f(t)dt \geqslant \int_0^\infty f(t)\phi(t)dt. \tag{2.3}$$

Proof. Indeed, $\int_0^x h_1\,dt \geqslant \int_0^x h_2\,dt$ implies $\int_0^\infty h_1 f\,dt \geqslant \int_0^\infty h_2 f\,dt$ for any non-increasing nonnegative function f on $(0,\infty)$ (see, for example, [BS, p. 56]), and (2.3) follows. □

Lemma 2.2. *Let $1 < p < \infty$ and $1 \leqslant h < \infty$. If $f \in L_{ph}(\Omega,\mu)$, then*

$$\lim_{t\to 0} t^{\frac{1}{p}} f^{**}(t) = 0 \qquad \textit{and} \qquad \lim_{t\to\infty} t^{\frac{1}{p}} f^{**}(t) = 0.$$

Proof. It follows from

$$\lim_{s\to 0}\int_{s/2}^{s} (t^{\frac{1}{p}} f^{**}(t))^h \frac{dt}{t} = \lim_{s\to\infty}\int_{s/2}^{s} (t^{\frac{1}{p}} f^{**}(t))^h \frac{dt}{t} = 0$$

and

$$\left(\int_{s/2}^{s} (t^{\frac{1}{p}} f^{**}(t))^h \frac{dt}{t}\right)^{\frac{1}{h}} \geqslant c_{ph}\, s^{\frac{1}{p}} f^{**}(s).$$ □

Theorem 2.1. *Suppose measurable functions f, g, and K are defined on $[0,1]$; then*

$$\int_0^1 g^*(t)(K*f)^{**}(t)dt \leqslant 2\int_0^1 tg^{**}(t)f^{**}(t)K^{**}(t)dt. \tag{2.4}$$

If we also assume that

$$\lim_{t\to+0} t^2 K^{**}(t)f^{**}(t)g^{**}(t) = 0, \tag{2.5}$$

then

$$\begin{aligned}\int_0^1 g^*(t)(K*f)^{**}(t)dt \leqslant{}& \|g\|_{L_1}\|f\|_{L_1}\|K\|_{L_1} \\ &+ \int_0^1 tg^{**}(t)f^{**}(t)\left(K^{**}(t) - K^*(t)\right)dt.\end{aligned} \tag{2.6}$$

Proof. From (2.2) and the Hardy-Littlewood inequality [BS, p.44], we write

$$\begin{aligned}
&\int_0^1 g^*(s)(K*f)^{**}(s)ds \\
&\leqslant \int_0^1 g^*(s) \sup_{\substack{|e|=s\\ e\subset[0,1]}} \int_0^1 |f(x)| \frac{1}{|e|}\int_e |K(y-x)|\,dydxds \\
&\leqslant \int_0^1 g^*(s) \sup_{\substack{|e|=s\\ e\subset[0,1]}} \int_0^1 f^*(t) \left(\frac{1}{|e|}\int_e |K(y-\cdot)|\,dy\right)^{**}(t)\ dtds \\
&= \int_0^1 g^*(s) \sup_{\substack{|e|=s\\ e\subset[0,1]}} \int_0^1 f^*(t) \sup_{\substack{|\omega|=t\\ \omega\subset[0,1]}} \frac{1}{|e|}\frac{1}{|\omega|}\int_e\int_\omega |K(y-x)|\,dxdydtds \\
&\leqslant \int_0^1 g^*(s) \int_0^1 f^*(t) \left(\sup_{\substack{|e|=s\\ e\subset[0,1]}} \sup_{\substack{|\omega|=t\\ \omega\subset[0,1]}} \frac{1}{|e|}\frac{1}{|\omega|}\int_e\int_\omega |K(y-x)|\,dxdy\right) dtds.
\end{aligned}$$

We consider

$$\begin{aligned}
\Phi(s,t) &= \sup_{\substack{|e|=s\\ |\omega|=t}} \frac{1}{|e|}\frac{1}{|\omega|}\int_\omega\int_e |K(y-x)|\,dydx \\
&= \sup_{\substack{|e|=s\\ |\omega|=t}} \frac{1}{|e|}\frac{1}{|w|}\int_0^1 \chi_e(x)\int_0^1 \chi_\omega(y)\,|K(y-x)|\,dydx \\
&= \sup_{\substack{|e|=s\\ |w|=t}} \frac{1}{|e|}\frac{1}{|w|}\int_0^1 |K(x)|\,|e\cap(w+x)|dx \\
&\leqslant \sup_{\substack{|e|=s\\ |w|=t}} \frac{1}{|e|}\frac{1}{|w|}\int_0^1 K^*(\xi)\phi(\xi)d\xi,
\end{aligned}$$

where $\phi(\xi)$ is the non-increasing rearrangement of the function $|e\cap(\omega+x)|$.

Then $\phi(\xi)$ satisfies

$$\phi(s) \leqslant \min(|e|,|w|)$$

and

$$\int_0^1 \phi(\xi)d\xi = |e||w|.$$

We assume that $s = |e| < |w| = t$, then the function $\phi_0(\xi) = \phi(\xi)/|e|$ satisfies conditions of Lemma 2.1 with $d = |w|$. Then for $s < t$ we have

$$\begin{aligned}\Phi(s,t) &\leqslant \sup_{\substack{|e|=s, e\subset[0,1]\\ |w|=t, w\subset[0,1]}} \frac{1}{|w|}\int_0^1 K^*(\xi)\phi_0(\xi)d\xi \\ &\leqslant \sup_{|w|=t}\frac{1}{|w|}\int_0^{|w|} K^*(\xi)d\xi = K^{**}(t)\end{aligned}$$

As above, for $s \geqslant t$ we write

$$\Phi(s,t) \leqslant \sup_{|e|=s}\frac{1}{|e|}\int_0^{|e|} K^*(\xi)d\xi = K^{**}(s)$$

Therefore we have

$$\int_0^1 g^*(s)(K*f)^{**}(s)ds \leqslant \int_0^1 g^*(s)\int_0^1 f^*(t)K^{**}(\max\{s,t\})dt\,ds. \tag{2.7}$$

Using this inequality, we get

$$\begin{aligned}\int_0^1 g^*(s)(K*f)^{**}(s)ds &\leqslant \int_0^1 g^*(s)\int_0^1 f^*(t)K^{**}(\max(t,s))dtds \\ &= \int_0^1 g^*(s)K^{**}(s)\int_0^s f^*(t)dtds \\ &\quad + \int_0^1 g^*(s)\int_s^1 f^*(t)K^{**}(t)dtds \\ &\leqslant 2\int_0^1 tK^{**}(t)g^{**}(t)f^{**}(t)dt.\end{aligned}$$

Thus, inequality (2.4) is verified. To prove (2.6), we use

$$g^*(s)\int_0^s f^*(t)dt + f^*(s)\int_0^s g^*(s)dt = \left(\int_0^s g^*(t)dt\int_0^s f^*(t)dt\right)'.$$

Therefore, (2.7) implies

$$\begin{aligned}\int_0^1 g^*(s)(K*f)^{**}(s)ds &\leqslant \int_0^1 g^*(s)\int_0^1 f^*(t)K^{**}(\max(t,s))dtds \\ &= \int_0^1 K^{**}(s)\left(g^*(s)\int_0^s f^*(t)dt + f^*(s)\int_0^s g^*(t)dt\right)ds \\ &= \int_0^1 K^{**}(s)\left(\int_0^s g^*(t)dt\int_0^s f^*(t)dt\right)' ds.\end{aligned}$$

Integrating by parts, we get

$$\int_0^1 g^*(s)(K*f)^{**}(s)ds \leqslant K^{**}(s)\int_0^s f^*(t)dt\int_0^s g^*(t)dt\Big|_0^1 - \int_0^1\int_0^s f^*(t)dt\int_0^s g^*(t)dt\,(K^{**}(s))'\,ds.$$

Hence, using $(K^{**}(s))' = -\left(\frac{1}{s^2}\int_0^s K^*(t)dt - \frac{1}{s}K^*(s)\right) = -\frac{1}{s}\left(K^{**}(s) - K^*(s)\right)$ and (2.5), we obtain

$$\int_0^1 g^*(t)(K*f)^{**}(t)dt \leqslant \|g\|_{L_1}\|f\|_{L_1}\|K\|_{L_1} + \int_0^1 tg^{**}(t)f^{**}(t)\left(K^{**}(t)-K^*(t)\right)dt.$$

The proof is complete. □

Theorem 2.2. *Let* $1 \leqslant p, q, r \leqslant \infty$, $1 \leqslant h_1, h_2, h_3 \leqslant \infty$, *and*

$$\frac{1}{q}+1 = \frac{1}{p}+\frac{1}{r}, \qquad \frac{1}{h_1} = \frac{1}{h_2}+\frac{1}{h_3}. \tag{2.8}$$

Suppose $f \in L_{p,h_2}[0,1]$ *and* $K \in L_{r,h_3}[0,1]$ *are* 1*-periodic functions.*

a) *(Young-O'Neil) If* $1 < p, q, r < \infty$, *then* $K*f \in L_{q,h_1}$ *and*

$$\|K*f\|_{L_{q,h_1}} \leqslant c\|f\|_{L_{p,h_2}}\|K\|_{L_{r,h_3}}.$$

b) *If* $1 < p = q < \infty$ *or* $p = q = h_1 = h_2 = h_3 = \infty$, *then* $K*f \in L_{p,h_1}$ *and*

$$\|K*f\|_{L_{p,h_1}} \leqslant c\|f\|_{L_{p,h_2}}\|K^{**}\|_{1,h_3}. \tag{2.9}$$

c) *If* $p = q = 1$ *and* $1 < h_1, h_2, h_3 < \infty$, *then* $K*f \in L_{1h_1}$ *and*

$$\|(K*f)^{**}\|_{1,h_1} \leqslant c\|f^{**}\|_{1,h_2}\|K^{**}\|_{1,h_3}.$$

Proof. Let $f^{**} \in L_{p,h_2}, K^{**} \in L_{r,h_3}$, and $g^{**} \in L_{q',h_1'}$, where p, r, q, h_1, h_2, h_3 satisfy (2.8). If $p_1 = q', p_2 = p, p_3 = r$, then $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} = 2$ and $\frac{1}{h_1'}+\frac{1}{h_2}+\frac{1}{h_3} = 1$. This implies that at least one of the parameters $\frac{1}{h_1'}, \frac{1}{h_2}, \frac{1}{h_3}$ differs from zero. Let us assume that $\frac{1}{h_1'} > 0$. Then Lemma 2.2 implies

$\lim_{t\to 0} t^{1/p_1} g^{**}(t) = 0$. Applying the Hölder inequality twice, we get

$$\begin{aligned} t^2 K^{**}(t) f^{**}(t) g^{**}(t) &= (t^{\frac{1}{p_3}} K^{**}(t))(t^{\frac{1}{p_2}} f^{**}(t))(t^{\frac{1}{p_1}} g^{**}(t)) \\ &\leqslant \frac{1}{t^{1-1/p_3}} \left(\int_0^t s^{\frac{h_3'}{p_3'}} \frac{ds}{s} \right)^{1/h_3'} \frac{1}{t^{1-1/p_2}} \left(\int_0^t s^{\frac{h_2'}{p_2'}} \frac{ds}{s} \right)^{1/h_2'} \\ &\qquad \|K^{**}\|_{L_{p_3,h_3}} \|f^{**}\|_{L_{p_2,h_2}} \left(t^{\frac{1}{p_1}} g^{**}(t) \right) \\ &= C \|K^{**}\|_{L_{p_3,h_3}} \|f^{**}\|_{L_{p_2,h_2}} \left(t^{\frac{1}{p_1}} g^{**}(t) \right). \end{aligned}$$

Therefore,

$$\lim_{t\to 0} t^2 K^{**}(t) f^{**}(t) g^{**}(t) = 0. \tag{2.10}$$

By Theorem 2.1 and the Hölder inequality, we have

$$\begin{aligned} \|f * K\|_{L_{q,h_1}} &\asymp \sup_{\|g\|_{L_{q',h_1'}}=1} \int_0^1 g(y)(K*f)(y)dy \\ &\leqslant \sup_{\|g\|_{L_{q',h_1'}}=1} \int_0^1 g^*(y)(K*f)^{**}(y)dy \\ &\leqslant \sup_{\|g\|_{L_{q',h_1'}}=1} \Big[\|g\|_{L_1} \|f\|_{L_1} \|K\|_{L_1} \\ &\qquad + \int_0^1 t \left(K^{**}(t) - K^*(t) \right) f^{**}(t) g^{**}(t) dt \Big] \\ &\leqslant \sup_{\|g\|_{L_{q',h_1'}}=1} \Big[\|g\|_{L_1} \|f\|_{L_1} \|K\|_{L_1} \\ &\qquad + \int_0^1 \left(t^{\frac{1}{p_3}-\frac{1}{h_3}} K^{**}(t) \right) (t^{\frac{1}{p_2}-\frac{1}{h_2}} f^{**}(t))(t^{\frac{1}{p_1}-\frac{1}{h_1'}} g^{**}(t)) dt \Big] \\ &\leqslant c \sup_{\|g\|_{L_{q',h_1'}}=1} \|K^{**}\|_{r,h_3} \|f^{**}\|_{p,h_2} \|g^{**}\|_{q',h_1'}. \end{aligned}$$

Thus we derive

$$\|f * K\|_{L_{q,h_1}} \leqslant c \sup_{\|g\|_{L_{q',h_1'}}=1} \|K^{**}\|_{r,h_3} \|f^{**}\|_{p,h_2} \|g^{**}\|_{q',h_1'}. \tag{2.10}$$

In the case a), i.e., $1 < p, r, q, < \infty$, we have $\|f^{**}\|_{p,h_2} \asymp \|f\|_{L_{p,h_2}}$, $\|g^{**}\|_{q',h_1'} \asymp \|g\|_{L_{q',h_1'}}$, $\|K^{**}\|_{r,h_3} \asymp \|K\|_{L_{r,h_3}}$ and inequality (2.10) implies

$$\|f*K\|_{L_{q,h_1}} \leqslant c \sup_{\|g\|_{L_{q',h_1'}}=1} \|K\|_{L_{r,h_3}} \|f\|_{L_{p,h_2}} \|g\|_{L_{q',h_1'}} = c\|K\|_{L_{r,h_3}} \|f\|_{L_{p,h_2}}.$$

b) Let now $r = 1$ and $1 < p = q < \infty, f \in L_{ph_2}, K \in L_{1,h_3}, g \in L_{p',h_1'}$. Then since at least one of the parameters $\frac{1}{h_1'}, \frac{1}{h_2}, \frac{1}{h_3}$ differs from zero, by Lemma 2.2, we write (2.10) and, therefore, (2.10). Then because of $\|f^{**}\|_{p,h_2} \asymp \|f\|_{L_{p,h_2}}$ and $\|g^{**}\|_{p',h_1'} \asymp \|g\|_{L_{p',h_1'}}$, we finally have

$$\begin{aligned}\|f * K\|_{L_{p,h_1}} &\leqslant c_1 \sup_{\|g\|_{L_{p',h_1'}}=1} \|K^{**}\|_{1,h_3} \|f\|_{L_{p,h_2}} \|g\|_{L_{p',h_1'}} \\ &= c_1 \|K^{**}\|_{1,h_3} \|f\|_{L_{p,h_2}}.\end{aligned}$$

For the case $p = q = h_1 = h_2 = h_3 = \infty$, we simply write

$$\|f * K\|_{L_\infty} \leqslant c \|f\|_{L_\infty} \|K\|_1.$$

c) Let $q = p = r = 1$ and $1 < h_1 < \infty$. Then using [CGMP, Th 4.1] and periodicity of functions, we get[1]

$$\begin{aligned}\|(K * f)^{**}\|_{1,h_1} &\asymp \sup_{\|g\|_{L_{\infty h_1'}}=1} \int_0^1 g(y)(K * f)(y)dy \\ &= \sup_{\|g\|_{L_{\infty h_1'}}=1} \int_0^1 f(x)(g * K)(x)dx \\ &\leqslant \sup_{\|g\|_{L_{\infty, h_1'}}=1} \int_0^1 f^*(s)(g * K)^{**}(s)ds.\end{aligned}$$

We again apply Theorem 2.1 and the Hölder inequality:

$$\begin{aligned}\|(K * f)^{**}\|_{1,h_1} &\leqslant c \sup_{\|g\|_{L_{\infty h_1'}}=1} \Bigg(\|f\|_{L_1} \|f\|_{L_1} \|K\|_{L_1} \\ &\qquad + \int_0^1 t f^{**}(t) K^{**}(t) \left(g^{**}(t) - g^*(t)\right) dt \Bigg) \\ &\leqslant c \sup_{\|g\|_{L_{\infty,h_1'}}=1} \|f^{**}\|_{1,h_2} \|K\|^*_{1,h_3} \\ &\qquad \left(\|g\|_1 + \left(\int_0^1 \frac{(g^{**} - g^*)^{h_1'}}{t} dt \right)^{\frac{1}{h_1'}} \right) \\ &= c \|f^{**}\|_{1,h_2} \|K^{**}\|_{1,h_3}.\end{aligned}$$

The proof is complete. □

[1]See the definition of the space $L_{\infty h_1'}$ in the next section

Let us give two examples showing sharpness of the results of Theorem 2.2. First, we prove that in inequality (2.9), the factor $\|K^{**}\|_{L_{1,h_3}}$ could not be changed to $\|K\|_{L_{1,h_3}}$. That is, in general, for $1 \leqslant p = q < \infty$ and $\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}$ the inequality

$$\|K * f\|_{L_{p,h_1}} \leqslant C\|f\|_{L_{p,h_2}}\|K\|_{L_{1,h_3}} \tag{2.11}$$

does not hold.

Example 2.1. Let $N \in \mathbb{N}$. We define $f(t) = (\min(N, 1/t))^{1/p}$ and $K(t) = \min(N, 1/t)$. Then

$$\|f\|_{L_{p,h_2}} \asymp (\ln N)^{1/h_2}$$

and

$$\|K\|_{L_{1,h_3}} \asymp (\ln N)^{1/h_3} = (\ln N)^{1/h_1 - 1/h_2}.$$

Define

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in [0, a) \\ (K * f)(x) & \text{for } x \in [a, 1) \end{cases}, \qquad a = \frac{1}{N}\big(e^p + 1\big).$$

Therefore if $x \in [a, 1)$ we write

$$\varphi(x) \geqslant \int_{\frac{1}{N}}^{x-\frac{1}{N}} \frac{ds}{s^{\frac{1}{p}}(x-s)} \geqslant \frac{1}{(x-\frac{1}{N})^{\frac{1}{p}}} \int_{\frac{1}{N}}^{x-\frac{1}{N}} \frac{ds}{x-s} = \frac{N^{1/p}\ln(Nx-1)}{(Nx-1)^{1/p}}.$$

Noting $\left(\frac{\ln \xi}{\xi^{1/p}}\right)' = \frac{p - \ln \xi}{p\xi^{1/p+1}} < 0$ for $\xi \in [e^p, 1]$, we obtain that the function $\frac{\ln \xi}{\xi^{1/p}}$ is decreasing on $[e^p, 1]$. Hence,

$$\varphi^*(t) \geqslant \frac{\ln\big(N(t+a)-1\big)}{\big(t+a-\frac{1}{N}\big)^{1/p}} \qquad \text{for} \qquad t \in (0, 1-a).$$

Using this, we get

$$\begin{aligned}
\|f * K\|_{L_{p,h_1}}^{h_1} &\geqslant \int_0^1 (t^{1/p}\varphi^*(t))^{h_1}\frac{dt}{t} \\
&\geqslant \int_0^{1-a} \left(\frac{t^{1/p}\ln(N(t+a)-1)}{(t+a-\frac{1}{N})^{1/p}}\right)^{h_1}\frac{dt}{t} \\
&\geqslant \int_a^{1-a} \left(\frac{t^{1/p}\ln(N(t+a)-1)}{(t+a-\frac{1}{N})^{1/p}}\right)^{h_1}\frac{dt}{t} \\
&\geqslant 2^{-\frac{h_1}{p}}\int_a^{1-a} \frac{(\ln(N(t+a)-1))^{h_1}dt}{t} \\
&\asymp \ln^{h_1+1}(N(t+a)-1)\ \Big|_a^{1-a} \asymp (\ln N)^{h_1+1}.
\end{aligned}$$

Thus, (2.11) implies

$$\ln N \leqslant C.$$

This contradiction concludes the proof.

We now provide an example showing sharpness of (2.9) in the following sense: none of the parameters could be changed to make the inequality stronger.

Corollary 2.2.1. *Let* $1 < p < \infty, 1 \leqslant h_1, h_2 \leqslant \infty$ *and* $1/h = (1/h_1 - 1/h_2)_+$. *Then for the Cesàro operator of* 1*-periodic function* f*, given by*

$$\begin{aligned}\sigma_N(f;x) &= \frac{1}{N+1}\sum_{k=0}^{N} S_k(f;x)\\ &= \int_0^1 f(y)F_N(x-y)dy, \quad F_N(x)\\ &= \frac{1}{N}\frac{\sin^2\pi(2N-1)x}{\sin^2\pi x},\end{aligned}$$

we write

$$\|\sigma_N\|_{L_{p,h_2}\to L_{p,h_1}} \asymp \|F_N^{**}\|_{1,h} \asymp (\ln N)^{1/h}.$$

Proof. By Theorem 2.2, we have

$$\|\sigma_N\|_{L_{p,h_2}\to L_{p,h_1}} \leqslant C\|F_N^{**}\|_{1,h}.$$

Further, we estimate

$$\begin{aligned}\|F_N^{**}\|_{L_{1,h}} \leqslant c\Bigg(&\int_0^{1/N}\left(\int_0^t \left(F_N(s)\chi_{[0,1/2]}(s)\right)^* ds\right)^h \frac{dt}{t}\\ &+\int_{1/N}^1 \left(\left(\int_0^{1/N}+\int_{1/N}^t\right)\left(F_N(s)\chi_{[0,1/2]}(s)\right)^* ds\right)^h \frac{dt}{t}\Bigg)^{1/h}.\end{aligned}$$

Using the known inequalities $F_N(s) \leqslant cN$ for $x > 0$ and $F_N(s) \leqslant c/(Nx^2)$ for $x \in (1/N, 1)$, we write

$$\|F_N\|^*_{L_{1,h}} \leqslant c\left(N^h\int_0^{1/N} t^{h-1}dt + \int_{1/N}^1 \frac{dt}{t}\right)^{1/h} \leqslant c\,(\ln N)^{1/h}.$$

Finally, we get

$$\|\sigma_N\|_{L_{p,h_2}\to L_{p,h_1}} \leqslant c\,(\ln N)^{1/h}.$$

On the other hand, defining

$$f_0(x) = \sum_{k=1}^{N} \frac{1}{k^{1/p}} e^{2\pi i k x}$$

and using Hardy-Littlewood's theorem on Fourier coefficients for the Lorentz space [Se], we obtain

$$\|f_0\|_{L_{p,h_2}} \asymp \left(\sum_{k=1}^{N}\left(k^{1/p}\frac{1}{k^{1/p}}\right)^{h_2}\frac{1}{k}\right)^{1/h_2} \asymp (\ln N)^{1/h_2}$$

$$\|\sigma_N(f_0)\|_{L_{p,h_1}} \asymp \left(\sum_{k=1}^{N}\left(\frac{N-k}{N}\right)^{h_1}\frac{1}{k}\right)^{1/h_1} \asymp (1+\ln N)^{1/h_1}.$$

Therefore, we derive

$$\|\sigma_N\|_{L_{p,h_2}\to L_{p,h_1}} \geqslant \frac{\|\sigma_N(f_0)\|_{L_{p,h_1}}}{\|f_0\|_{L_{p,h_2}}} \asymp (\ln N)^{1/h_1-1/h_2} = (\ln N)^{1/h},$$

completing the proof. □

3. Convolution in the Lorentz space of periodic functions: the case of $p = q = \infty$

Theorem 2.2 does not encompass the limit case $p = q = \infty$ ($h_i < \infty$). It is clear that in this case the classical Lorentz space is trivial. We consider another scale of Lorentz spaces.

Following Bennett et al. [BRS] (see also [BMR], [CGMP]), we define $L_{\infty h}[0,1]$ as follows

$$L_{\infty q}[0,1] = \Bigg\{ f \in L_1[0,1] : \|f\|_{L_{\infty q}[0,1]} := \|f\|_{L_1[0,1]} + \left(\int_0^1 \frac{(f^{**}-f^*)^q}{t}dt\right)^{\frac{1}{q}} < \infty \Bigg\}.$$

The following embedding hold: for $1 \leqslant p < \infty$ and $1 \leqslant q < q_1 \leqslant \infty$

$$L_\infty[0,1] = L_{\infty,1}[0,1] \hookrightarrow L_{\infty q}[0,1] \hookrightarrow L_{\infty,q_1}[0,1] \hookrightarrow L_p[0,1].$$

Moreover,

$$\begin{aligned}\|f\|_{L_{\infty,1}} &= \|f\|_1 + \int_0^1 (f^{**}-f^*)\frac{dt}{t} \\ &= \|f\|_1 - \int_0^1 (f^{**}(t))' dt \\ &= \|f\|_1 + f^{**}(0) - f^{**}(1) = \|f\|_{L_\infty},\end{aligned}$$

i.e., $L_{\infty,1} = L_\infty$.

The last embedding follows from Hölder's inequality (one can assume that $p < q_1$)

$$\|f\|_{L_p} = \|f\|_{L_{p,p}} \asymp \|f\|_{L_1} + \left(\int_0^1 (f^{**} - f^*)^p \, dt\right)^{1/p}$$
$$\leqslant \|f\|_{L_1} + \left(\int_0^1 t^{\frac{p}{q_1-p}} dt\right)^{1/p-1/q_1} \left(\int_0^1 \frac{(f^{**} - f^*)^{q_1}}{t} dt\right)^{1/q_1}$$
$$\leqslant c\|f\|_{L_{\infty,q_1}}.$$

Lemma 3.1. *Suppose $f \in L_1[0,1]$ and $K \in L_\infty[0,1]$; then*

$$(f * K)^{**}(t) \leqslant f^*(1)\|K\|_1 + K^{**}(t)\int_0^t (-f^*(s))' s ds + \int_t^1 (-f^*(s))' s K^{**}(s) ds \tag{3.1}$$

Proof. By (2.2),

$$\Big(f * K\Big)^{**}(t) \leqslant \sup_{|e|=t} \int_0^1 |f(x)| \frac{1}{|e|} \int_e |K(y-x)| \, dy \, dx$$
$$\equiv \sup_{|e|=t} \int_0^1 |f(x)| \widetilde{K}(x,e), dx$$

and by Hardy's inequality,

$$\leqslant \sup_{|e|=t} \int_0^1 f^*(s) \widetilde{K}^*(s,e) ds =$$
$$= \sup_{|e|=t} \left[f^*(s) \int_0^s \widetilde{K}^*(\xi,e) d\xi \Big|_0^1 + \int_0^1 (-f^*(s))' s \widetilde{K}^{**}(s,e) ds \right]$$
$$\leqslant \sup_{|e|=t} \left| f^*(s) \int_0^s \widetilde{K}^*(\xi,e) d\xi \Big|_0^1 \right| + \sup_{|e|=t} \int_0^1 (-f^*(s))' s \widetilde{K}^{**}(s) ds.$$

Then the estimate

$$\sup_{|e|=t} \widetilde{K}^{**}(s,e) \leqslant K^{**}(\max(s,t))$$

gives

$$\begin{aligned}\sup_{|e|=t}\left|f^*(s)\int_0^s \widetilde{K}^*(\xi,e)d\xi\Big|_0^1\right| &= \sup_{|e|=t}\left|sf^*(s)\widetilde{K}^{**}(s,e)\Big|_0^1\right| \\ &\leqslant f^*(1)K^{**}(\max(1,t)) \\ &\quad + \lim_{s\to 0} sf^*(s)K^{**}(\max(s,t)) \\ &= f^*(1)\|K\|_1 + K^{**}(t)\lim_{s\to 0} sf^*(s) \\ &= f^*(1)\|K\|_1\end{aligned}$$

(note that $\lim_{t\to 0} tf^*(t) = 0$ because of $f \in L_1$) and

$$\begin{aligned}\sup_{|e|=t}\int_0^1(-f^*(s))'s\widetilde{K}^{**}(s)ds &\leqslant \int_0^1(-f^*(s))'sK^{**}(\max(s,t))ds \\ &= \int_0^t(-f^*(s))'sK^{**}(t)ds \\ &\quad + \int_t^1(-f^*(s))'sK^{**}(s)ds.\end{aligned}$$

Hence we derive

$$\begin{aligned}(f*K)^{**}(t) \leqslant f^*(1)\|K\|_1 + K^{**}(t)\int_0^t(-f^*(s))'sds \\ + \int_t^1(-f^*(s))'sK^{**}(s)ds\end{aligned}$$

and the proof is now complete. □

Theorem 3.1. *Let $1 \leqslant h_1, h_2, h_3 < \infty$ and $\frac{1}{h_1} = \frac{1}{h_2}+\frac{1}{h_3}$. For any 1-periodic functions $f \in L_{\infty,h_2}[0,1]$ and $K \in L_{1,h_3}[0,1]$ we have*

$$\|K*f\|_{L_{\infty,h_1}} \leqslant 2\,\|f\|_{L_{\infty,h_2}}\|K^{**}\|_{L_{1,h_3}}. \tag{3.2}$$

Proof. Let us suppose first that $h_1 > 1$. Without loss of generality, we can assume that K and f are from C^∞.

Put $h \equiv K*f$. If

$$\|h\|_1 \geqslant \left(\int_0^1 (h^{**}(t)-h^*(t))^{h_1}\,\frac{dt}{t}\right)^{1/h_1}$$

then

$$\|h\|_{L_{\infty,h_1}} \leqslant 2\|h\|_1 \leqslant 2\|f\|_{L_{\infty,h_2}}\|K^{**}\|_{L_{1,h_3}}$$

and (3.2) is proved. Now suppose the reverse:

$$\|h\|_1 \leqslant \left(\int_0^1 \left(h^{**}(t) - h^*(t)\right)^{h_1} \frac{dt}{t}\right)^{1/h_1}. \tag{3.3}$$

Then

$$\begin{aligned}\|h\|_{L_{\infty,h_1}} &\leqslant 2\left(\int_0^1 \left(h^{**}(t) - h^*(t)\right)^{h_1} \frac{dt}{t}\right)^{1/h_1} \\ &= 2\int_0^1 g(t)\left(h^{**}(t) - h^*(t)\right) dt\end{aligned}$$

where

$$g(t) = \frac{\left(h^{**}(t) - h^*(t)\right)^{h_1 - 1}}{t\left(\int_0^1 \left(h^{**}(s) - h^*(s)\right)^{h_1} \frac{ds}{s}\right)^{1/h_1'}}$$

The function g satisfies the following conditions:

1) $g(t) \geqslant 0$,
2) $g(1) \leqslant 1$ (see (3.3)),
3) $\left(\int_0^1 (tg(t))^{h_1'} \frac{dt}{t}\right)^{1/h_1'} = 1$,
4) $g \in C^\infty$.

By A denote a collection of all functions satisfying 1) - 4). From (3.3), we get

$$\|K * f\|_{L_{\infty,h_1}} \leqslant 2\sup_{g\in A} \int_0^1 g(t)\Big((K*f)^{**}(t) - (K*f)^*(t)\Big)\, dt.$$

For $g \in A$,

$$\begin{aligned}\int_0^1 g(t)\Big(h^{**}(t) - h^*(t)\Big)\, dt &= -\int_0^1 tg(t)\Big(h^{**}(t)\Big)'\, dt \\ &= -tg(t)h^{**}(t)\Big|_0^1 + \int_0^1 \Big(tg(t)\Big)'\Big(h^{**}(t)\Big) dt \\ &= -g(1)h^{**}(1) + \int_0^1 \Big(tg(t)\Big)'\Big(h^{**}(t)\Big) dt \\ &\leqslant \int_0^1 \Big(tg(t)\Big)'\Big(h^{**}(t)\Big) dt.\end{aligned}$$

Hence,

$$\begin{aligned}\|K*f\|_{L_{\infty,h_1}} &\leqslant \sup_{g\in A}\int_0^1 \Big(tg(t)\Big)' h^{**}(t)\,dt \\ &= \sup_{g\in A,\ (tg(t))'\geqslant 0}\int_0^1 \Big(tg(t)\Big)' h^{**}(t)\,dt.\end{aligned} \tag{3.4}$$

We now use Lemma 3.1:

$$\begin{aligned}&\int_0^1 \Big(tg(t)\Big)'(K*f)^{**}(t)dt \\ &\leqslant \int_0^1 (tg(t))'\Big[f^*(1)\|K\|_1 + K^{**}(t)\int_0^t (-f^*(s))' s\,ds \\ &\qquad + \int_t^1 (-f^*(s))' sK^{**}(s)ds\Big]dt \\ &= f^*(1)g(1)\|K\|_1 + \int_0^1 (tg(t))' K^{**}(t)\int_0^t (-f^*(s))' s\,dsdt \\ &\qquad + \int_0^1 (-f^*(t))' sK^{**}(t)\int_0^t (sg(s))' dsdt \\ &= f^*(1)g(1)\|K\|_1 + \int_0^1 K^{**}(t)\left(\int_0^t (-f^*(s))' sds\int_0^t (sg(s))' ds\right)' dt \\ &= f^*(1)g(1)\|K\|_1 + K^{**}(t)\Big(\int_0^t (-f^*(s))' sds\Big)\Big(\int_0^t (sg(s))' ds\Big)\Big|_0^1 \\ &+ \int_0^1 t^{-1}\left(K^{**}(t)-K^*(t)\right)\Big(\int_0^t (-f^*(s))' sds\Big)\Big(\int_0^t (sg(s))' ds\Big)dt.\end{aligned}$$

Integrating by parts, we write

$$\int_0^t (-f^*(s))' sds = -f^*(t)t + \int_0^t f^*(s)ds = t\left(f^{**}(t)-f^*(t)\right)$$

and therefore

$$\begin{aligned}\int_0^1 \Big(tg(t)\Big)'(K*f)^{**}(t)dt &\leqslant g(1)\|K\|_1\|f\|_1 \\ &\quad + \int_0^1 t\left(K^{**}(t)-K^*(t)\right)\left(f^{**}(t)-f^*(t)\right)g(t)dt.\end{aligned}$$

By the Hölder inequality and the definition of A, we obtain

$$\begin{aligned}\|K*f\|_{L_{\infty,h_1}} &\leqslant 2 \sup_{g\in A}\Bigg\{g(1)\|K\|_1\|f\|_1+\left(\int_0^1 (f^{**}(t)-f^*(t))^{h_2}\,\frac{dt}{t}\right)^{1/h_2} \\ &\times\left(\int_0^1 [t(K^{**}(t)-K^*(t))]^{h_3}\,\frac{dt}{t}\right)^{1/h_3}\left(\int_0^1 (tg(t))^{h_1'}\,\frac{dt}{t}\right)^{1/h_1'}\Bigg\} \\ &\leqslant 2\,\|f\|_{L_{\infty,h_2}}\|K^{**}\|_{L_{1,h_3}}.\end{aligned}$$

Let now $h_1=1$. Then

$$\begin{aligned}\|f*K\|_{L_{\infty 1}} &= \|f*K\|_{L_\infty}=\sup_{x\in[0,1]}\left|\int_0^1 K(x-y)f(y)dy\right| \\ &\leqslant \int_0^1 K^*(t)f^*(t)dt = f^*(t)\int_0^t K^*(s)ds\Big|_0^1 \\ &\quad+\int_0^1 (-f^*(t))'tK^{**}(t)dt \\ &= f^*(1)\|K\|_1+\int_0^1\left(\int_0^t(-f^*(s))'sds\right)'K^{**}(t)dt \\ &= f^*(1)\|K\|_1+\left(\int_0^t(-f^*(s))'sds\right)K^{**}(t)\Big|_0^1 \\ &\quad+\int_0^1\left(\int_0^t(-f^*(s))'sds\right)\Big(K^{**}(t)-K^*(t)\Big)\frac{dt}{t} \\ &= f^*(1)\|K\|_1+\left(\int_0^1(-f^*(s))'sds\right)\|K\|_{L_1} \\ &\quad+\int_0^1\left(\int_0^t(-f^*(s))'sds\right)\Big(K^{**}(t)-K^*(t)\Big)\frac{dt}{t} \\ &= \|f\|_1\|K\|_1+\int_0^1(f^{**}-f^*)(t)(K^{**}(t)-K^*(t))dt \\ &\leqslant \|f\|_{L_{\infty h_2}}\|K^{**}\|_{L_{1h_2}}\end{aligned}$$

The proof is complete. □

4. Convolution in the weighted Lorentz spaces

Let μ be the Lebesgue measure and Ω, D be measurable subsets of $\mathbb{R}^n$. For functions f and K defined on Ω and $D-\Omega$ we consider the convolution

$$Af(y)=\int_\Omega K(y-x)f(x)dx.$$

Theorem 4.1. *Let f, g, and K be measurable functions on Ω, D, and $D-\Omega$ of $\mathbb{R}^n$. One has*

$$\int_0^\infty g^*(s)(Af)^{**}(s)ds \leqslant t^2 f^{**}(t)g^{**}(t)K^{**}(t)\Big|_0^\infty + \int_0^\infty tf^{**}(t)g^{**}(t)(K^{**}(t)-K^*(t))dt;$$

$$\int_0^\infty g^*(s)(Af)^{**}(s)ds \leqslant 2\int_0^\infty tf^{**}(t)g^{**}(t)K^{**}(t)dt; \tag{4.1}$$

and

$$\int_0^\infty g^*(s)(Af)^{**}(s)ds \leqslant \int_0^\infty g^*(t)\int_0^\infty f^*(s)K^{**}(|t-s|)dsdt. \tag{4.2}$$

Proof. It is similar to the proof of Theorem 2.1. Indeed, the same technique that we used implies the following inequality:

$$\int_0^\infty g^*(s)(Af)^{**}(s)ds \leqslant \int_0^\infty g^*(s)\int_0^\infty f^*(t)K^{**}(\max(s,t))dt\,ds. \tag{4.3}$$

Then

$$\begin{aligned}\int_0^\infty g^*(s)(Af)^{**}(s)ds &\leqslant \int_0^\infty g^*(s)\int_0^s f^*(t)K^{**}(s)dtds \\ &\quad+\int_0^\infty g^*(s)\int_s^\infty f^*(t)K^{**}(t)dtds \\ &= \int_0^\infty K^{**}(s)\left(\int_0^s f^*(t)dt\int_0^s g^*(t)dt\right)' ds \\ &= t^2 f^{**}(t)g^{**}(t)K^{**}(t)\Big|_0^\infty \\ &\quad-\int_0^\infty (K^{**}(s))'\left(\int_0^s f^*(t)dt\int_0^s g^*(t)dt\right)ds \\ &= t^2 f^{**}(t)g^{**}(t)K^{**}(t)\Big|_0^\infty \\ &\quad+\int_0^\infty sf^{**}(s)g^{**}(s)\left(K^{**}(s)-K^*(s)\right)ds,\end{aligned}$$

The second inequality follows from

$$\begin{aligned}\int_0^\infty g^*(s)(Af)^{**}(s)ds &\leqslant \int_0^\infty g^*(s)K^{**}(s)\int_0^s f^*(t)dtds \\ &\quad + \int_0^\infty g^*(s)\int_s^\infty f^*(t)K^{**}(t)dtds \\ &= \int_0^\infty g^*(s)K^{**}(s)\int_0^s f^*(t)dtds \\ &\quad + \int_0^\infty f^*(t)K^{**}(t)\int_0^t g^*(s)dsdt \\ &\leqslant 2\int_0^\infty tf^{**}(t)g^{**}(t)K^{**}(t)dt.\end{aligned}$$

Using $K^{**}(\max(s,t)) \leqslant K^{**}(|t-s|)$, (4.3) implies inequality (4.2). The proof is now complete. □

Now we are going to study the Young-O'Neil inequality in weighted Lorentz spaces $\Lambda^q(\omega)$ and $\Gamma^q(\omega)$. We remark (see [Sa]) that

$$\Lambda^q(\omega) = \Gamma^q(\omega)$$

if and only if the weight ω satisfies the B_p-condition, that is,

$$\int_x^\infty \frac{\omega(t)}{t^q}dt \leqslant \frac{C}{x^q}\int_0^x \omega(t)dt.$$

We also recall the definition of the associated space

$$E' = \left\{ \|f\|_{E'} = \sup_{\substack{g\in E,\\ \|g\|_E=1}} \left|\int_X fg\,d\mu(x)\right| < \infty \right\}.$$

It is known ([CRS, 2.4], [KM]) that

$$\Gamma^{q'}(\tilde{\omega}) = (\Lambda^q(\omega))' \qquad q>1, \tag{4.4}$$

where

$$\widetilde{\omega}(t) = t^{q'}W(t)^{-q'}\omega(t), \tag{4.5}$$

and

$$W(x) = \int_0^x \omega(t)dt.$$

In the case of the classical example $\omega(t) = t^{q/p-1}$, $1<p,q<\infty$, we have $\|f\|_{\Lambda^q(\omega)} = \|f\|_{\Gamma^q(\omega)} = \|f\|_{L^{pq}}$, $\widetilde{\omega}(t) = t^{q'/p'-1}$, i.e., $(L^{pq})' = L^{p'q'}$.

Theorem 4.2. *Let* $\omega \in B_q$. *We have*

$$\|K * f\|_{\Gamma^q(\omega)} \asymp \|K * f\|_{\Lambda^q(\omega)} \leqslant C\|f\|_{\Gamma^{s_1}(\omega_1)} \|K\|_{\Gamma^{s_2}(\omega_2)} \tag{4.6}$$

for $1/q = 1/s_1 + 1/s_2$, $1 < q < \infty$, $1 \leqslant s_1, s_2 \leqslant \infty$, *and*

$$\int_0^t \omega(x)\,dx \leqslant c\,\omega(t)\frac{\omega_1(t)^{1/s_1}\omega_2(t)^{1/s_2}}{\omega(t)^{1/s_1+1/s_2}}. \tag{4.7}$$

Proof. We first remark that (4.7) can be rewritten as follows

$$t \leqslant c\,\widetilde{\omega}(t)^{1/q'}\omega_1(t)^{1/s_1}\omega_2(t)^{1/s_2}. \tag{4.8}$$

Then by Theorem 4.1 (see (4.1)) and Hölder's inequality, we write

$$\begin{aligned}
&\int_{\mathbb{R}} g(y)\left(\int_{\mathbb{R}} f(x)K(x-y)dx\right)dy \\
&\quad\leqslant 2c\int_0^\infty \left[g^{**}(t)\widetilde{\omega}(t)^{1/q'}\right]\left[f^{**}(t)\omega_1(t)^{1/s_1}\right]\left[K^{**}(t)\omega_2(t)^{1/s_2}\right]dt \\
&\quad\leqslant 2c\left(\int_0^\infty \left(g^{**}(t)\right)^{q'}\widetilde{\omega}(t)\,dt\right)^{\frac{1}{q'}}\left(\int_0^\infty \left(f^{**}(t)\right)^{s_1}\omega_1(t)\,dt\right)^{\frac{1}{s_1}} \\
&\qquad\qquad\left(\int_0^\infty \left(K^{**}(t)\right)^{s_2}\omega_2(t)\,dt\right)^{\frac{1}{s_2}} \\
&\quad\leqslant 2c\|g\|_{\Gamma^{q'}(\widetilde{\omega})}\|f\|_{\Gamma^{s_1}(\omega_1)}\|K\|_{\Gamma^{s_2}(\omega_2)}.
\end{aligned}$$

Considering the supremum over g such that $\|g\|_{\Gamma^{q'}(\widetilde{\omega})} = 1$, we obtain

$$\|K * f\|_{(\Gamma^{q'}(\widetilde{\omega}))'} \leqslant 2c\,\|f\|_{\Gamma^{s_1}(\omega_1)}\|K\|_{\Gamma^{s_2}(\omega_2)}.$$

Finally, since $\omega \in B_p$, we have $\left(\Lambda^q(\omega)\right)'' = \Lambda^q(\omega)$ and then (4.6) follows from (4.4). ☐

Corollary 4.2.1. *Under the hypotheses of Theorem 4.2, if additionally,* $\omega_1 \in B_{s_1}$ *and* $\omega_2 \in B_{s_2}$, *then*

$$\|K * f\|_{\Lambda^q(\omega)} \leqslant C\|f\|_{\Lambda^{s_1}(\omega_1)}\|K\|_{\Lambda^{s_2}(\omega_2)}.$$

Examples. **1.** For $\omega(t) = t^{q/h-1}$, $\omega_1(t) = t^{s_1/p-1}$, $\omega_2(t) = t^{s_2/r-1}$ and $1/q = 1/s_1 + 1/s_2$, inequality (4.7) is equivalent to $\frac{h}{q}t^{1-1/h} \leqslant c\,t^{1/p+1/r}$. Hence, for $1 + 1/h = 1/p + 1/r$ and for $1 < p, r, h < \infty$, we have

$$\|K * f\|_{h,q} \leqslant C\|K^{**}\|_{p,s_1}\|f^{**}\|_{r,s_2} \leqslant C\|K\|_{p,s_1}\|f\|_{r,s_2},$$

(see (2.1)).

2. Let $\omega(t) = t^{q/h-1}\xi_1^{Aq}(t)$, $\omega_1(t)=t^{s_1/p-1}\xi_2^{Bs_1}(t)$, $\omega_2(t)=t^{s_2/r-1}\xi_3^{Ds_2}(t)$, where $1/q = 1/s_1 + 1/s_2$, $1 + 1/h = 1/p + 1/r$, and ξ_i are slowly oscillating functions. Then inequality (4.7) is equivalent to

$$\xi_1^A(x) \leqslant C\xi_2^B(x)\xi_3^D(x),$$

i.e., in this case we obtain the Young-O'Neil-type inequality for the Lorentz-Zygmund spaces [BS, p. 253].

Further we study the convolution inequality for periodic functions. Suppose $\omega, \omega_1, \omega_2$ are weights on $[0,1]$ and $0 < q < \infty$; then by definition,

$$S^q(\omega) = \left\{ \|f\|_{S^q(\omega)} = \|f\|_1 + \left(\int_0^1 (f^{**}(s) - f^*(s))^q \omega(s) ds \right)^{\frac{1}{q}} < \infty \right\}.$$

Theorem 4.3. *Let $\omega_2 \in L_1$. We have*

$$\|K * f\|_{S^q(\omega)} \leqslant C\|f\|_{S^{s_1}(\omega_1)} \|K\|_{S^{s_2}(\omega_2)}$$

for $1/q = 1/s_1 + 1/s_2$, $1 < q < \infty$, $1 \leqslant s_1, s_2 \leqslant \infty$, and

$$t\omega^{1/q}(t) \leqslant C{\omega_1}^{1/s_1}(t){\omega_2}^{1/s_2}(t). \tag{4.9}$$

Proof. Since $\omega_2 \in L_1$ we have $\|K\|_1 \leqslant C\|K\|_{\Gamma^{s_2}(\omega_2)}$.

In the case when $(h = K * f)$

$$\|f\|_1 \leqslant \left(\int_0^1 (f^{**}(s) - f^*(s))^q \omega(s) ds \right)^{\frac{1}{q}},$$

we write

$$\begin{aligned} \|h\|_{S_q(\omega)} &\leqslant 2 \left(\int_0^1 \left(h^{**}(t) - h^*(t)\right)^q \omega(t) dt \right)^{1/q} \\ &= 2 \int_0^1 g(t) \left(h^{**}(t) - h^*(t)\right) dt \end{aligned}$$

where

$$g(t) = \frac{\omega(t)\Big(h^{**}(t) - h^*(t)\Big)^{q-1}}{\left(\int_0^1 \left(h^{**}(s) - h^*(s)\right)^q \omega(s) ds\right)^{1/q'}}.$$

We remark that the function g satisfies the following conditions:

1) $g(t) \geqslant 0$,
2) $g(1) \leqslant \omega(1)$,
3) $\left(\int_0^1 \left(\frac{g(t)}{\omega(t)}\right)^{q'} \omega(t) dt \right)^{1/q'} = 1$,
4) $g \in C^\infty$.

Similar to the proof of Theorem 3.1, one has

$$\|h\|_{S_q(\omega)} \leqslant 2 \sup_{g\in A}\Big\{ g(1)\|K\|_{L_1}\|f\|_{L_1} + \int_0^1 t\left(K^{**}(t)-K^*(t)\right)\left(f^{**}(t)-f^*(t)\right)g(t)dt\Big\}$$

$$\leqslant C\,\|K\|_{L_1}\|f\|_{L_1} + C\sup_{g\in A}\left(\int_0^1 \left(K^{**}(t)-K^*(t)\right)^{s_2}(t)\omega_2(t)dt\right)^{1/s_2}$$

$$\times\left(\int_0^1 \left(f^{**}(t)-f^*(t)\right)^{s_1}\omega_1(t)dt\right)^{1/s_1}\left(\int_0^1 [\omega^{1-q'}(t)g^{q'}(t)]dt\right)^{1/q'}$$

$$\leqslant C\,\|f\|_{S_{s_1}(\omega_1)}\,\|K\|_{S_{s_2}(\omega_2)},$$

and we obtain the required inequality. □

We finally remark that for Example **2** above, condition (4.9) is again equivalent to

$$\xi_1^A(x) \leqslant C\xi_2^B(x)\xi_3^D(x).$$

5. Riesz potential in weighted Lebesgue spaces

Let $\nu(x)$ be a weight, i.e., a non-negative, measurable and locally integrable in $\mathbb{R}^n$. We define the norm in the weighted Lebesgue space $L_p(\mathbb{R}^n,\nu)$ as

$$\|f\|_{L_p(\nu)} = \left(\int_{\mathbb{R}^n} |f(x)\nu(x)|^p\,dx\right)^{1/p}.$$

We consider the Riesz kernel $K(x,y) = |x-y|^{\gamma-n}$, $0<\gamma<n$, and the corresponding potential operator

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}dy, \qquad x\in\mathbb{R}^n. \tag{5.1}$$

Continuity properties of the potential operator in the Lebesgue spaces have been extensively studied (see [BS], [So], [SW]). In particular, it is well known that I_γ is of strong type (p,q), where $1<p<n/\gamma$ and $1/q = 1/p-\gamma/n$. Analogous questions have been also investigated in the weighted Lebesgue spaces. In this case, the solution of the classical problem: to describe necessary and sufficient conditions on a weight for $I_\gamma f(x)$ to be bounded on $L_p(\mathbb{R}^n,\mu)$ into $L_q(\mathbb{R}^n,\nu)$, can be found in [MW], [EKM], [KK] and others.

We continue this line of work with the goal to estimate the norm of the potential operator from above and from below.

Theorem 5.1. *Suppose* $1 < p < q < \infty,\ \ 0 < p\gamma < n,\ \ 1/q \leqslant 1/p - \gamma/n;$ *then for the potential operator (5.1) the following inequalities hold*

$$
\begin{aligned}
&C \sup_{1/2 \leqslant |w|/|e| \leqslant 2} \frac{1}{|w|^{\frac{1}{q'}+\frac{1}{p}}} \int_e \nu(y) \int_w \mu^{-1}(x)|x-y|^{\gamma-n} dxdy \\
&\leqslant \|I_\gamma\|_{L_p(\mathbb{R}^n,\mu)\to L_q(\mathbb{R}^n,\nu)} \leqslant \\
&C \sup_{1/2 \leqslant |w|/|e| \leqslant 2} \frac{1}{|w|^{\frac{1}{q'}+\frac{1}{p}}} \int_0^{|e|} \nu^*(t) \int_0^{|w|} (\mu^{-1})^*(s)|t-s|^{\gamma/n-1} dsdt.
\end{aligned}
\tag{5.2}
$$

Proof. Let us prove the left-hand side inequality. If I_γ is bounded on $L_p(\mathbb{R}^n,\mu)$ into $L_q(\mathbb{R}^n,\nu)$, then

$$
Af(y) = \int_{\mathbb{R}^n} \frac{\nu(y)\mu^{-1}(x)f(x)}{|x-y|^{n-\gamma}} dx
$$

is such that $A : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$. Therefore, it is (p,q) weak-type operator, i.e., $A\colon L_{p\,1} \to L_{q\infty}$, and by Corollary 3.1.1 from [NT] (see also [KN]), we have

$$
\begin{aligned}
\|I_\gamma\|_{L_p(\mu)\to L_q(\nu)} &= \|A\|_{L_p\to L_q} \geqslant \|A\|_{L_{p1}\to L_{q\infty}} \\
&\asymp \sup_{|e|>0,|w|>0} \frac{1}{|e|^{1/q'}} \frac{1}{|w|^{1/p}} \int_e \nu(y) \int_w \mu^{-1}(x)|x-y|^{\gamma-n} dxdy \\
&\geqslant \sup_{\substack{|e|=|w|>0 \\ 1/2\leqslant |w|/|e| \leqslant 2}} \frac{1}{|w|^{\frac{1}{q'}+\frac{1}{p}}} \int_e \nu(y) \int_w \mu^{-1}(x)|x-y|^{\gamma-n} dxdy.
\end{aligned}
$$

Let us now show the accuracy of the right-hand side inequality. We denote $M^* := \{e \in \mathbb{R}^n : 0 < |e| < \infty\}$ and take $e, w \in M^*$. Suppose also that $|e| > |w|$. Then there exists an integer $M > 1$ such that $(M-1)|w| < |e| \leqslant M|w|$, where $p,\ q$ are any numbers satisfying $1 < p < n/\gamma,\ \ 1/q < 1/p$.

Let $K(x) = |x|^{\gamma-n}$. Noting $K^{**}(|t-s|) \asymp |t-s|^{\frac{\gamma}{n}-1}$ and using inequality (4.2) from theorem 4.1, we estimate

$$
\begin{aligned}
&\frac{1}{|e|^{1/q'}} \frac{1}{|w|^{1/p}} \int_e \nu(y) \int_w \mu^{-1}(x)|x-y|^{\gamma-n} dxdy \\
&\quad\leqslant \frac{C}{((M-1)|w|)^{1/q'}|w|^{1/p}} \int_0^{|e|} \nu^*(t) \int_0^{|w|} (\mu^{-1})^*(s)|t-s|^{\gamma/n-1} dsdt \\
&\quad\leqslant \frac{C}{(M\,|w|)^{1/q'}|w|^{1/p}} \int_0^{|w|} (\mu^{-1})^*(s) \sum_{k=1}^{M} \int_{(k-1)|w|}^{k|w|} \nu^*(t)|t-s|^{\gamma/n-1} dtds.
\end{aligned}
$$

We divide the last expression into two terms:

$$\frac{C}{(M\,|w|)^{1/q'}|w|^{1/p}}\int_0^{|w|}(\mu^{-1})^*(s)\int_0^{2|w|}\nu^*(t)|t-s|^{\gamma/n-1}dtds$$
$$+\frac{C}{(M\,|w|)^{1/q'}|w|^{1/p}}\int_0^{|w|}(\mu^{-1})^*(s)\sum_{k=3}^{M}\int_{(k-1)|w|}^{k|w|}\nu^*(t)|t-s|^{\gamma/n-1}dtds$$
$$=: I_1+I_2.$$

By changing the variables $t\to t+(k-1)|w|$,

$$\int_{(k-1)|w|}^{k|w|}\nu^*(t)|t-s|^{\gamma/n-1}dt=\int_0^{|w|}\nu^*(t+(k-1)|w|)\Big|t+(k-1)|w|-s\Big|^{\gamma/n-1}dt.$$

Further, we estimate

$$\Big|(t-s)+(k-1)|w|\Big|^{\gamma/n-1}\leqslant\Big((k-1)|w|-|t-s|\Big)^{\gamma/n-1}\leqslant\Big((k-2)|w|\Big)^{\gamma/n-1}$$

and then

$$\int_0^{|w|}\nu^*(t+(k-1)|w|)\Big|t+(k-1)|w|-s\Big|^{\gamma/n-1}dt\leqslant\int_0^{|w|}\frac{\nu^*(t)}{\Big((k-2)|w|\Big)^{1-\gamma/n}}dt.$$

This yields

$$I_2\leqslant\frac{C}{(M\,|w|)^{1/q'}|w|^{1/p}}\int_0^{|w|}(\mu^{-1})^*(s)\int_0^{|w|}\frac{\nu^*(t)}{|w|^{1-\gamma/n}}\sum_{k=3}^{M}\frac{1}{(k-2)^{1-\gamma/n}}dt\,ds.$$

Noting that $1/q'\geqslant 1-1/p+\gamma/n$, we get

$$M^{-1/q'}\sum_{k=3}^{M}(k-2)^{\gamma/n-1}\leqslant CM^{\gamma/n-1/q'}\leqslant C.$$

Summing up the estimates for I_1 and I_2, we finally have

$$\frac{1}{|e|^{1/q'}}\frac{1}{|w|^{1/p}}\int_e\nu(y)\int_w\mu^{-1}(x)|x-y|^{\gamma-n}dxdy$$
$$\leqslant C\Bigg(\frac{1}{(M\,|w|)^{1/q'}|w|^{1/p}}\int_0^{|w|}(\mu^{-1})^*(s)\int_0^{2|w|}\nu^*(t)|t-s|^{\gamma/n-1}dtds$$
$$+\frac{1}{|w|^{1/p+1/q'}}\int_0^{|w|}(\mu^{-1})^*(s)\int_0^{|w|}\nu^*(t)|t-s|^{\gamma/n-1}dtds\Bigg)$$
$$\leqslant C\sup_{1/2\leqslant|w|/|e|\leqslant2}\frac{1}{|w|^{\frac{1}{q'}+\frac{1}{p}}}\int_0^{|e|}\nu^*(t)\int_0^{|w|}(\mu^{-1})^*(s)|t-s|^{\gamma/n-1}dsdt.$$

If $|\omega|\geqslant|e|$, then we use similar estimates (we also use $\gamma/n<1/p$).

Thus we obtain the following estimates

$$\|A\|_{L_{p,1}(\mathbb{R}^n)\to L_{q,\infty}(\mathbb{R}^n)} \leqslant CJ,$$

where

$$J = \sup_{1/2\leqslant|w|/|e|\leqslant 2} \frac{1}{|w|^{\frac{1}{q'}+\frac{1}{p}}} \int_0^{|e|} \nu^*(t) \int_0^{|w|} (\mu^{-1})^*(s)|t-s|^{\gamma/n-1} ds dt.$$

We note that the expression on the right-hand side is independent on all pairs (p,q) such that $(1/p-1/q)$ is a constant. Therefore, if $J<\infty$, then the operator A is a (p,q) weak-type operator for $1/p-1/q=C$ and $1<p<\gamma/n$.

Further, there exist two pairs (p_0,q_0) and (p_1,q_1) such that $1<p_0<p<p_1<\infty$, $1<q_0<q<q_1<\infty$, and $1/p_0-1/q_0=1/p_1-1/q_1=1/p-1/q$. Therefore, if the right-hand side of (5.2) is bounded, then

$$A\colon L_{p_0,1}\to L_{q_0,\infty} \quad \text{and} \quad A\colon L_{p_1,1}\to L_{q_1,\infty},$$

where the norms are controlled by CJ. Then by the interpolation theorem, we write

$$A\colon L_{p(\theta)}\to L_{q(\theta)}$$

and

$$\|A\| \leqslant C\,J$$

for

$$1/p(\theta)=(1-\theta)/p_0+\theta/p_1, \quad 1/q(\theta)=(1-\theta)/q_0+\theta/q_1.$$

We have $p(\theta)=p$ for some $0<\theta<1$. Since $1/p_0-1/q_0=1/p_1-1/q_1=1/p(\theta)-1/q(\theta)$, in this case $q(\theta)$ coincides with q. The proof is now complete. □

Note that in the case when weights satisfy some regular conditions, the left-hand side and the right-hand side integrals are equivalent. Then Theorem 5.1 implies the following relation for the norm of the potential operator.

Corollary 5.1.1. *Let $1<p\leqslant q\leqslant\infty$, $1/q\leqslant 1/p-\gamma$. Suppose weights μ and ν satisfy*

$$\mu^*(t) \leqslant \frac{c}{t}\int_{\frac{t}{2}\leqslant|x|\leqslant t} \mu(x)dx, \tag{5.3}$$
$$\nu^*(s) \leqslant \frac{c}{s}\int_{\frac{s}{2}\leqslant|x|\leqslant s} \nu(x)dx,$$

then

$$\|I_\gamma\|_{L_p(\mathbb{R},\mu^{-1})\to L_q(\mathbb{R},\nu)} \asymp \sup_{\frac{1}{2}\leqslant\frac{|e|}{|w|}\leqslant 2} \frac{1}{|e|^{\frac{1}{q'}}}\frac{1}{|w|^{\frac{1}{p}}}\int_0^{|e|}\int_0^{|w|}\frac{\mu^*(t)\nu^*(s)}{|t-s|^{1-\gamma}}dsdt.$$

Remark. Any monotone or quasi-monotone (i.e., there exists $\tau \geqslant 0$ such that $\mu(t)t^{-\tau}$ is monotone) μ and ν satisfy (5.3).

Proof. By Theorem 5.1, it is sufficient to prove the inequality

$$\sup_{\frac{1}{2}\leqslant\frac{|e|}{|w|}\leqslant 2} \frac{1}{|e|^{\frac{1}{q'}}}\frac{1}{|w|^{\frac{1}{p}}}\int_0^{|e|}\int_0^{|w|}\frac{\mu^*(t)\nu^*(s)}{|t-s|^{1-\gamma}}dsdt \leqslant C \sup_{\frac{1}{2}\leqslant\frac{|e|}{|w|}\leqslant 2} \frac{1}{|e|^{\frac{1}{q'}}}\frac{1}{|w|^{\frac{1}{p}}}\int_e\int_w\frac{\mu(x)\nu(y)}{|x-y|^{1-\gamma}}dxdy.$$

Using (5.3), we have

$$\int_0^{|e|}\int_0^{|w|}\frac{\mu^*(t)\nu^*(s)}{|t-s|^{1-\gamma}}dsdt \leqslant c^2\int_0^{|e|}\int_0^{|w|}\frac{\int_{\frac{t}{2}\leqslant|\eta|\leqslant t}\int_{\frac{s}{2}\leqslant|\xi|\leqslant s}\mu(\eta)\nu(\xi)d\eta d\xi}{ts|t-s|^{1-\gamma}}dsdt$$
$$= C\int_{-|e|}^{|e|}\int_{-|w|}^{|w|}\mu(\eta)\nu(\xi)\int_{|\eta|}^{2|\eta|}\int_{|\xi|}^{2|\xi|}\frac{dsdt}{ts|t-s|^{1-\gamma}}d\eta d\xi.$$

Now let $0<\xi<\eta$. If $\xi>\frac{\eta}{4}$, then

$$\int_\xi^{2\xi}\int_\eta^{2\eta}\frac{dtds}{ts|t-s|^{1-\gamma}} \leqslant \int_\eta^\infty\int_\xi^\infty\frac{dtds}{|t-s|^{1-\gamma}ts} = \frac{1}{\eta^{1-\gamma}}\int_1^\infty\int_{\frac{\xi}{\eta}}^\infty\frac{dtds}{|t-s|^{1-\gamma}ts}$$
$$\leqslant \frac{1}{\eta^{1-\gamma}}\int_1^\infty\int_{\frac{1}{4}}^\infty\frac{dtds}{|t-s|^{1-\gamma}ts} = \frac{c_\gamma}{\eta^{1-\gamma}} \leqslant \frac{c_\gamma}{|\eta-\xi|^{1-\gamma}}.$$

If $\xi<\frac{\eta}{4}$, then

$$\int_\xi^{2\xi}\int_\eta^{2\eta}\frac{dtds}{|t-s|^{1-\gamma}ts} \leqslant \frac{1}{|\eta-2\xi|^{1-\gamma}}\int_\xi^{2\xi}\int_\eta^{2\eta}\frac{dtds}{ts}$$
$$= \frac{(\ln 2)^2}{|\eta-\frac{\eta}{2}|^{1-\gamma}} = \frac{c}{\eta^{1-\gamma}} \leqslant \frac{c}{|\eta-\xi|^{1-\gamma}}.$$

Similarly, in the case of $0<\eta\leqslant\xi$, one has

$$\int_\xi^{2\xi}\int_\eta^{2\eta}\frac{dtds}{|t-s|^{1-\gamma}ts} \leqslant \frac{c_\gamma}{|\eta-\xi|^{1-\gamma}}.$$

Thus,

$$\int_{|\eta|}^{2|\eta|}\int_{|\xi|}^{2|\xi|}\frac{dsdt}{|t-s|^{1-\gamma}ts} \leqslant \frac{c_\gamma}{|\eta-\xi|^{1-\gamma}}$$

and

$$\int_0^{|e|}\int_0^{|w|}\frac{\mu^*(t)\nu^*(s)}{|t-s|^{1-\gamma}}dtds \leqslant c_\gamma\int_{-|e|}^{|e|}\int_{-|w|}^{|w|}\frac{\mu(\xi)\nu(\eta)}{|\xi-\eta|^{1-\gamma}}d\xi d\eta,$$

which concludes the proof. □

6. Convolution in multidimensional Lorentz spaces

Let vectors $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{q} = (q_1, \ldots, q_n)$ be such that $0 < p_j, q_j \leqslant \infty$, $j = 1, \ldots, n$ and if $p_j = \infty$, then $q_j = \infty$. Then for two vectors $\mathbf{p}$ and $\mathbf{q}$ and for a given rearrangement $* = (j_1, j_2, \cdots, j_n)$ of the sequence $(1, 2, \cdots, n)$ we define the following functional

$$\Phi_{\mathbf{pq}*}(\varphi) = \left(\int_0^\infty \cdots \left(\int_0^\infty \left| t_1^{\frac{1}{p_1}} \ldots t_n^{\frac{1}{p_n}} \varphi(t_1, \ldots, t_n) \right|^{q_{j_1}} \frac{dt_{j_1}}{t_{j_1}} \right)^{\frac{q_{j_2}}{q_{j_1}}} \cdots \frac{dt_{j_n}}{t_{j_n}} \right)^{\frac{1}{q_{j_n}}},$$

where the expression $\left(\int_0^\infty (G(s))^q \frac{ds}{s}\right)^{1/q}$ is understood as $\sup_{s>0} G(s)$ for $q = \infty$.

Let $\mathbf{m} = (m_1, \cdots, m_n) \in \mathbb{N}^n$ and let $f(x_1, \cdots, x_n)$ be a measurable function on $\mathbb{R}^{\mathbf{m}} = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$. Applying decreasing rearrangement according to $x_1 \in \mathbb{R}^{m_1}, \cdots, x_n \in \mathbb{R}^{m_n}$ with other variables being fixed we will construct a function $f^{*_1, \cdots, *_n}(t_1, \cdots, t_n)$. This function is called a decreasing rearrangement of f in $\mathbb{R}^{\mathbf{m}}$.

The Lorentz space $L_{\mathbf{pq}*}(\mathbb{R}^{\mathbf{m}})$ is defined as the collection of measurable functions on $\mathbb{R}^{\mathbf{m}}$ such that

$$\|f\|_{L_{\mathbf{pq}*}(\mathbb{R}^{\mathbf{m}})} = \Phi_{\mathbf{pq}*}(f^{*_1 \cdots *_n}) < \infty.$$

In the case $\mathbf{m} = (1, \cdots, 1)$ and $* = (1, \cdots, n)$, this space was introduced by Blozinski [Bl1]. He also obtained corresponding convolution inequalities. In a general case, this definition can be found in [Nu]. Below we prove the Young-O'Neil-type inequality for this case.

Theorem 6.1. *Let* $\mathbf{m} = (m_1, \cdots m_n) \in \mathbb{N}^n$ *and let* $\mathbf{1} < \mathbf{q}, \mathbf{p}, \mathbf{r} < \infty$, $\mathbf{1/q + 1 = 1/p + 1/r}$, $\mathbf{1} \leqslant \mathbf{h}, \mathbf{h_1}, \mathbf{h_2} \leqslant \infty$, *and* $\mathbf{1/h = 1/h_1 + 1/h_2}$. *Let also* $* = (j_1, \cdots, j_n)$ *be a rearrangement of* $(1, 2, \cdots, n)$. *If* $K \in L_{\mathbf{r}, \mathbf{h_1^*}}(\mathbb{R}^{\mathbf{m}})$ *and* $g \in L_{\mathbf{p}, \mathbf{h_2^*}}(\mathbb{R}^{\mathbf{m}})$, *then for the convolution*

$$(K * g)(y) = \int_{\mathbb{R}^{\mathbf{m}}} K(x - y) g(x) dx$$

one has

$$\|K * g\|_{L_{\mathbf{q}, \mathbf{h}^*}(\mathbb{R}^{\mathbf{m}})} \leqslant C \|K\|_{L_{\mathbf{r}, \mathbf{h_1^*}}(\mathbb{R}^{\mathbf{m}})} \|g\|_{L_{\mathbf{p}, \mathbf{h_2^*}}(\mathbb{R}^{\mathbf{m}})}. \tag{6.1}$$

Proof. First, we show that for all measurable functions f, g and K on $\mathbb{R}^{\mathbf{m}}$ one has

(6.2)
$$\int_{\mathbb{R}^{\mathbf{m}}} f(y)\int_{\mathbb{R}^{\mathbf{m}}} g(x)K(x-y)\,dx\,dy \leqslant 2^n\int_0^\infty\cdots\int_0^\infty !t_1\cdots t_n f^{**}(t_1,\cdots,t_n) \\ g^{**}(t_1,\cdots,t_n)K^{**}(t_1,\cdots,t_n)dt_1\cdots dt_n,$$

where

$$f^{**}(t_1,\cdots,t_n)=\frac{1}{\prod_{i=1}^n t_i}\int_0^{t_1}\cdots\int_0^{t_n} f^{*_1\cdots *_n}(s_1,\cdots,s_n)ds_1\cdots ds_n.$$

Using (4.1),

$$\begin{aligned}
&\int_{\mathbb{R}^{\mathbf{m}}} f(y)\int_{\mathbb{R}^{\mathbf{m}}} g(x)K(x-y)\,dx\,dy \\
&\quad=\int_{\mathbb{R}^{m_n}}\cdots\int_{\mathbb{R}^{m_2}}\int_{\mathbb{R}^{m_n}}\cdots\int_{\mathbb{R}^{m_2}} \\
&\qquad\qquad\left(\int_{\mathbb{R}^{m_1}} f(y_1,\mathbf{y})\int_{\mathbb{R}^{m_1}} K(x_1-y_1,\mathbf{x}-\mathbf{y})g(x_1,\mathbf{x})dx_1dy_1\right)\mathbf{dx}\,\mathbf{dy} \\
&\quad\leqslant 2\int_{\mathbb{R}^{m_n}}\cdots\int_{\mathbb{R}^{m_2}}\int_{\mathbb{R}^{m_n}}\cdots\int_{\mathbb{R}^{m_2}} \\
&\qquad\qquad\left(\int_0^\infty t_1K^{**}(t_1,\mathbf{x}-\mathbf{y})g^{**}(t_1,\mathbf{x})f^{**}(t_1,\mathbf{y})\,dt_1\right)\mathbf{dx}\,\mathbf{dy} \\
&\quad=2\int_0^\infty t_1\Bigg(\int_{\mathbb{R}^{m_n}}\cdots\int_{\mathbb{R}^{m_2}} f^{(**)_1}(t_1,\mathbf{y})\int_{\mathbb{R}^{m_n}}\cdots \\
&\qquad\qquad\int_{\mathbb{R}^{m_2}} K^{(**)_1}(t_1,\mathbf{x}-\mathbf{y})g^{(**)_1}(t_1,\mathbf{x})\,dt_1\Bigg)\mathbf{dx}\,\mathbf{dy},
\end{aligned}$$

where $(\mathbf{x})=(x_2,\cdots,x_n)$ and $\mathbf{dx}=dx_2\cdots dx_n$. Continuing this process, we will obtain the required inequality.

Secondly, inequality (6.2) implies

$$\begin{aligned}
\|K*g\|_{L_{\mathbf{q},\mathbf{h}^*}(\mathbb{R}^{\mathbf{m}})} &= \sup_{\|f\|_{L_{\mathbf{q}'\mathbf{h}'^*}(\mathbb{R}^{\mathbf{m}})}=1}\int_{\mathbb{R}^{\mathbf{m}}} f(y)\int_{\mathbb{R}^{\mathbf{m}}} K(x-y)g(x)dx\,dy \\
&\leqslant 2^n \sup_{\|f\|_{L_{\mathbf{q}'\mathbf{h}'^*}(\mathbb{R}^{\mathbf{m}})}=1}\int_0^\infty\!\!\cdots\!\int_0^\infty tf^{**}(t)g^{**}(t)K^{**}(t)dt_1\cdots dt_n \\
&= 2^n \sup_{\|f\|_{L_{\mathbf{q}'\mathbf{h}'^*}(\mathbb{R}^{\mathbf{m}})}=1}\int_0^\infty\!\!\cdots\!\int_0^\infty tf^{**}(t)g^{**}(t)K^{**}(t)dt_{j_1}\cdots dt_{j_n}.
\end{aligned}$$

Then the Hölder inequality implies

$$\|K * g\|_{L_{\mathbf{q},\mathbf{h}^*}(\mathbb{R}^{\mathbf{m}})} \leqslant 2^n \sup_{\|f\|_{L_{\mathbf{q}'\mathbf{h}'^*}(\mathbb{R}^{\mathbf{m}})}=1} \|f^{**}\|_{L_{q'h'^*}(\mathbb{R}^{\mathbf{m}})}\|K^{**}\|_{L_{\mathbf{r},\mathbf{h}_1^*}(\mathbb{R}^{\mathbf{m}})}\|g^{**}\|_{L_{\mathbf{p},\mathbf{h}_2^*}(\mathbb{R}^{\mathbf{m}})}$$

Using Hardy's inequality, one can see for $1 < p < \infty$ that

$$\|\psi\|_{L_{\mathbf{p},\mathbf{h}^*}(\mathbb{R}^{\mathbf{m}})} \asymp \|\psi^{**}\|_{L_{\mathbf{p},\mathbf{h}^*}(\mathbb{R}^{\mathbf{m}})} := \Phi_{\mathbf{p},\mathbf{h}^*}(\psi^{**}).$$

Thus, we derive

$$\|K * g\|_{L_{\mathbf{q},\mathbf{h}^*}(\mathbb{R}^{\mathbf{m}})} \quad \leqslant \quad c\|K\|_{L_{\mathbf{r},\mathbf{h}_1^*}(\mathbb{R}^{\mathbf{m}})}\|g\|_{L_{\mathbf{p},\mathbf{h}_2^*}(\mathbb{R}^{\mathbf{m}})}.$$

The proof is complete. □

We note that in the case of $\mathbf{m} = (1, \cdots, 1)$ and $* = (1, \cdots, n)$ the O'Neil-type inequality (6.1) was proved by Blozinski [Bl1].

Remarks. 1. The classical Young-O'Neil inequality for the convolution $Af = K * f$ in the case $1 < p = (p, \cdots, p) < q = (q, \cdots, q) < \infty$ and $1/r = 1 + 1/q - 1/p$ is given by

$$\|A\|_{L_p(\mathbb{R}^n)\to L_q(\mathbb{R}^n)} \leqslant c\|K\|_{L_{r\infty}(\mathbb{R}^n)}.$$

Also, Stepanov [St] proved the following inequality

$$\|A\|_{L_p(\mathbb{R}^n)\to L_q(\mathbb{R}^n)} \leqslant c\|\cdots\|K\|_{L_{r\infty}(\mathbb{R}_{x_1})}\cdots\|_{L_{r\infty}(\mathbb{R}_{x_n})}. \tag{6.3}$$

On the other hand, inequality (6.1) yields

$$\|A\|_{L_p(\mathbb{R}^n)\to L_q(\mathbb{R}^n)} \leqslant c\|K\|_{L_{(r,\cdots,r),(\infty,\cdots,\infty)}(\mathbb{R}^n)}, \tag{6.4}$$

which is an improvement. Indeed, suppose $n = 2$, then

$$\begin{aligned}
\|K\|_{L_{(r,r),(\infty,\infty)}(\mathbb{R}^2)} &= \sup_{t_1>0,t_2>0} t_1^{1/r}t_2^{1/r}K^{*_1*_2}(t_1,t_2) \\
&\leqslant \sup_{t_1>0,t_2>0} t_1^{1/r}t_2^{1/r-1}\int_0^{t_2} K^{*_1*_2}(t_1,s_2)ds_2 \\
&= \sup_{t_1>0,t_2>0} t_1^{1/r}t_2^{1/r-1}\sup_{|e|=t_2}\int_e K^{*_1}(t_1,x_2)dx_2 \\
&= \sup_{t_1>0,t_2>0} t_1^{1/r-1}t_2^{1/r-1}\sup_{|e|=t_2}\int_e \sup_{|w|=t_1}\int_w |K(x_1,x_2)|dx_1dx_2.
\end{aligned}$$

From the definition of supremum, for a.e. $x_2 \in e$ there exists a measurable set $w(x_2)$ (in general, depending on x_2) such that $|w(x_2)| = t_1$ and

$$\sup_{|w|=t_1}\int_w |K(x_1,x_2)|dx_1 \leqslant 2\int_{w(x_2)} |K(x_1,x_2)|dx_1.$$

Therefore,

$$\|K\|_{L_{(r,r),(\infty,\infty)}(\mathbb{R}^2)} \leqslant 2 \sup_{\mu\Omega>0} \frac{1}{(\mu\Omega)^{1-1/r}} \int_{\Omega} |K(x)| d\mu \asymp \|K\|_{L_{r,\infty}(\mathbb{R}^2)}$$

where μ is the Lebesgue measure, $x = (x_1, x_2)$.

Let us also verify that (6.4) implies (6.3). Indeed,

$$\begin{aligned}
\|K\|_{L_{(r,r),(\infty,\infty)}(\mathbb{R}^2)} &\leqslant \sup_{t_1>0,t_2>0} t_1^{1/r} t_2^{1/r-1} \sup_{|e|=t_2} \int_e K^{*_1}(t_1, x_2) dx_2 \\
&\leqslant \sup_{t_2>0} t_2^{1/r-1} \sup_{|e|=t_2} \int_e \sup_{t_1>0} t_1^{1/r} K^{*_1}(t_1, x_2) dx_2 \\
&= \sup_{t_2>0} t_2^{1/r-1} \int_0^{t_2} \left(\|K(\cdot, x_2)\|_{L_{r\infty}(\mathbb{R}_{x_1})} \right)^{*_2} (s_2) ds_2 \\
&\leqslant \left\| \|K\|_{L_{r\infty}(\mathbb{R}_{x_1})} \right\|_{L_{r\infty}(\mathbb{R}_{x_2})} \sup_{t_2>0} t_2^{1/r-1} \int_0^{t_2} s_2^{1/r} ds_2 \\
&= c \left\| \|K\|_{L_{r\infty}(\mathbb{R}_{x_1})} \right\|_{L_{r\infty}(\mathbb{R}_{x_2})}
\end{aligned}$$

Now we show that for the function

$$K(x_1, x_2) = (|x_1||x_2|)^{-1/r}, \qquad 1 < r < \infty$$

one has $\|K\|_{L_{(r,r),(\infty,\infty)}(\mathbb{R}^2)} < \infty$ and $\|K\|_{L_{r,\infty}(\mathbb{R}^2)} = \infty$.

Obviously, $\sup_{t_1>0,t_2>0} t_1^{1/r} t_2^{1/r} K^{*_1 *_2}(t_1, t_2) < \infty$. Let us prove that $\sup_{t>0} t^{1/r} K^*(t) = \infty$. Indeed, for $\varepsilon > 0$ we define a set $e_\varepsilon = \{(x_1, x_2) : \varepsilon \leqslant x_1 \leqslant 1,\ 0 \leqslant x_2 \leqslant \varepsilon/x_1\}$. Then

$$\frac{1}{|e_\varepsilon|^{1-1/r}} \int_{e_\varepsilon} K(x_1, x_2) dx_1 dx_2 \asymp |\ln \varepsilon|^{1/r} \to \infty \qquad \text{as} \qquad \varepsilon \to 0,$$

and therefore

$$\|K\|_{L_{r\infty}} \asymp \sup_{|e|>0} \frac{1}{|e|^{1-1/r}} \left| \int_e K(x) dx \right| = \infty.$$

2. The statement of this theorem in the limit case is not true, that is, if for some i_0, we have $p_{i_0} = 1$, then (6.1) does not generally hold. The counter-example can be constructed as in [Bl2]. We present an analogue of Young-O'Neil inequality for the limit case.

Let vectors $\mathbf{p}=(p_1, \ldots, p_n)$, $\mathbf{q}=(q_1, \ldots, q_n)$ be such that $0 < p_j, q_j \leqslant \infty$, $j = 1, \ldots, n$ and if $p_j = \infty$, then $q_j = \infty$. The Lorentz space $L_{\mathbf{pq}*}([0,1]^{\mathbf{m}})$

is defined as the collection of measurable 1-periodic (on each variable) functions f such that

$$\|f\|_{L_{\mathbf{pq}*}([0,1]^{\mathbf{m}})}$$
$$=\left(\int_0^1\cdots\left(\int_0^1\left|t_1^{\frac{1}{p_1}}\cdots t_n^{\frac{1}{p_n}}f^{*_1,\cdots,*_n}(t_1,\ldots,t_n)\right|^{q_{j_1}}\frac{dt_{j_1}}{t_{j_1}}\right)^{\frac{q_{j_2}}{q_{j_1}}}\cdots\frac{dt_{j_n}}{t_{j_n}}\right)^{\frac{1}{q_{j_n}}}<\infty.$$

Theorem 6.2. *Let* $\mathbf{m}=(m_1,\cdots m_n)\in\mathbb{N}^n$ *and let* $\mathbf{1}<\mathbf{p}\leqslant\mathbf{q}<\infty$*,* $\mathbf{1/q}+\mathbf{1}=\mathbf{1/p}+\mathbf{1/r}$*,* $\mathbf{1}\leqslant\mathbf{h},\mathbf{h_1},\mathbf{h_2}\leqslant\infty$*, and* $\mathbf{1/h}=\mathbf{1/h_1}+\mathbf{1/h_2}$*.*

Let also $*=(j_1,\cdots,j_n)$ *be a rearrangement of* $(1,2,\cdots,n)$*. Then for the convolution*

$$(K*g)(y)=\int_{[0,1]^{\mathbf{m}}}K(x-y)g(x)dx$$

one has

$$\|K*g\|_{L_{\mathbf{q},\mathbf{h}^*}([0,1]^{\mathbf{m}})}\leqslant C\|K^{**}\|_{L_{\mathbf{r},\mathbf{h}_1^*}([0,1]^{\mathbf{m}})}\|g^{**}\|_{L_{\mathbf{p},\mathbf{h}_2^*}([0,1]^{\mathbf{m}})}.$$

The proof is similar to the proof of Theorem 6.1 (using inequality (4.2)).

Acknowledgements

The research was supported by INTAS (05-1000008-815), the RFFI (Grants N 06-01-00268, N 08-01-00302), Russian Foundation for the Humanities (Grants N 08-06-00144a), the Leading Scientific Schools (Grant NSH-2787.2008.1), and Scuola Normale Superiore. The paper was started when the authors were staying at the Centre de Recerca Matemática (Barcelona) in Spring 2006.

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E. Nursultanov
Kazakh Branch of Moscow State University
Munatpasova, 7
010010 Astana Kazakhstan
E-mail address: er-nurs@yandex.ru

S. Tikhonov
Scuola Normale Superiore
Piazza dei Cavalieri, 7
56126 Pisa Italy
E-mail address: s.tikhonov@sns.it