AN EXTENSION OF THE MARSDEN-RATIU REDUCTION FOR POISSON MANIFOLDS

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Abstract. We propose a generalization of the reduction of Poisson manifolds by distributions introduced by Marsden and Ratiu. Our proposal overcomes some of the restrictions of the original procedure, and makes the reduced Poisson structure effectively dependent on the distribution. Different applications are discussed, as well as the algebraic interpretation of the procedure and its formulation in terms of Dirac structures.

1. Introduction

Symplectic manifolds model phase spaces of physical systems, and their theory of reduction is a classical subject. A case in which reduction occurs naturally is when a Lie group G acts on a symplectic manifold (M, ω) with equivariant moment map $J\colon M \to \mathfrak{g}^*$: under regularity assumptions the Marsden-Weinstein theorem states that the quotients $J^{-1}(\mu)/G_\mu$ inherit a symplectic form. Another case is given by submanifolds $C \subset M$ such that $TC^\omega \subset TC$ (coisotropic submanifolds), for in that case the quotient C/TC^ω, when smooth, inherits a symplectic form. The theory of reduction extends naturally to Poisson manifolds, which encode phase spaces of physical systems with symmetry. The hamiltonian reduction and coisotropic reduction mentioned above extend in a straightforward way to Poisson manifolds. Further, both are recovered as special cases of a reduction theorem stated in 1986 by Marsden and Ratiu [7].

The starting data of the Marsden-Ratiu theorem is a pair (N, B) where N is a submanifold of the Poisson manifold (M, π) and B a subbundle of $TM|_N$, the restriction of TM to N. The role of B is to prescribe how to extend certain functions on N to functions on the whole of M, and is needed because the Poisson bracket of M is defined only for elements of $C^\infty(M)$.

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The conclusion of the theorem is that, when the assumptions are met, the quotient $N/(B \cap TN)$ inherits a Poisson bracket from the one on M.

The aim of this paper is two-fold. First we argue that the assumptions of the Marsden-Ratiu theorem are too strong, in the sense that the theorem allows to recover only Poisson structures (on quotients of N) which lose most of the information encoded by the subbundle B (see Prop. 2.2).

Then we set weaker assumptions on the pair (N, B) which ensure the existence of a Poisson structure on $N/(B \cap TN)$ encoding the subbundle B. The main difficulty consists in ensuring that the bracket of functions on the quotient satisfies the Jacobi identity. In Prop. 4.1 we set assumptions similar in spirit to those of [7], whereas in Prop. 4.2 the assumptions involve an additional piece of data, namely a foliation on M. We apply these results to the symplectic setting (with and without moment map) as well.

The paper is organized as follows: in the next section we review the original reduction of Marsden and Ratiu. In section 3 we present the most general form of the extension that we propose, while section 4 is devoted to the application of the previous results to some special situations and examples. We collect in the appendix some complementary results, like the algebraic interpretation of our reduction, its description in term of Dirac structures and other auxiliary material necessary for the main body of the paper.

We finish remarking that an extension of the Marsden-Ratiu reduction using supergeometry is being worked out in [2].

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2. Marsden-Ratiu reduction

We start by recalling the Poisson reduction by distributions as it was stated by Marsden and Ratiu in [7], see also [8]. The set-up we consider here and in the rest of the paper is the following:

$$(M, \{\cdot,\cdot\}) \text{ is a Poisson manifold}$$
$$N \text{ is a submanifold with embedding } \iota : N \hookrightarrow M$$
$$B \subset T_N M \text{ is a smooth subbundle of } TM \text{ restricted to } N.$$
$$F := B \cap TN \text{ is an integrable regular distribution on } N.$$

Definition 2.1. [7] The subbundle $B \subset T_N M$ is called **canonical** if for any elements f_1, f_2 of $C^\infty(M)_B \equiv \{f \in C^\infty(M) \mid \mathrm{d}f|_B = 0\}$ we have $\{f_1, f_2\} \in C^\infty(M)_B$.

In other words, B is canonical if the Poisson bracket of B-invariant functions is B-invariant. Note that in the previous definition, $\mathrm{d}f|_B$ stands for the restriction (not pullback) of $\mathrm{d}f$ to N and then to sections of B.

Definition 2.2. [7] $(M, \{\cdot,\cdot\}, N, B)$ is **Poisson reducible** if there is a Poisson bracket $\{\cdot,\cdot\}_{\underline{N}}$ on $\underline{N}$ such that for any $f_1, f_2 \in C^\infty(M)_B$ we have:

$$\{\iota^* f_1, \iota^* f_2\}_{\underline{N}} = \iota^*\{f_1, f_2\}.$$

In the previous definition, we realize $C^\infty(\underline{N})$ as $C^\infty(N)_F \equiv \{f \in C^\infty(N) \mid \mathrm{d}f|_F = 0\}$, the space of F-invariant, smooth functions on N. The latter makes sense even if $\underline{N}$ is not a manifold.

With the previous definitions we can state the Marsden-Ratiu reduction theorem.

Theorem 2.1. *(Marsden-Ratiu* [7]*) Assume that $B \subset T_N M$ is a canonical subbundle. Then $(M, \{\cdot,\cdot\}, N, B)$ is Poisson reducible if and only if*

$$\sharp B^\circ \subset TN + B.$$

In the above theorem $\sharp : T^*M \to TM$ denotes the contraction with the Poisson bivector on M, and $B^\circ = \mathrm{Ann}(B)$ consists of elements of $T^*_N M$ that kill all vectors in B. The proof of the theorem can be found in [7] and [8].

In the rest of this section we shall discuss the implications of the assumptions of the Marsden-Ratiu theorem.

The main observation is that the assumption made in Theorem 2.1 that B is canonical is a rather strong requirement.

Lemma 2.1. *Assume that $B \subset T_N M$ is a canonical subbundle. Then either $\sharp B^\circ \subset TN$ or $B = 0$.*

Proof. Assume that there is a point $p \in N$ s. t. $(\sharp B^\circ)_p \not\subset T_p N$. Then there is a B-invariant function $h \in C^\infty(M)_B$ and a constraint $g \in \mathcal{I} \equiv \{f \in C^\infty(M)$ s. t. $f|_N = 0\}$ that satisfies $\{g, h\}(p) \neq 0$. It is clear that g^2 is B-invariant, and the canonicity of B implies that $\mathrm{d}\{g^2, h\}|_B = 0$. In particular one must have $i_v(\mathrm{d}g)_p\{g, h\}(p) = 0$ and we then deduce that $i_v(\mathrm{d}g)_p = 0$ for any $v \in B_p$.

Consider now any other constrain $g' \in \mathcal{I}$, we again have that $g \cdot g'$ is B-invariant and therefore $i_v(\mathrm{d}g')_p\{g, h\}(p) = 0$. From this we deduce that $i_v(\mathrm{d}g')_p = 0$ for any constrain g' and any $v \in B_p$. This is equivalent to saying

$$B_p \subset T_p N.$$

By the assumption of constant rank for $B \cap TN$ we must have $B \subset TN$ everywhere. This implies that $f \cdot g$ is B-invariant for any $f \in C^\infty(M)$ and therefore $i_v(\mathrm{d}f)_p\{g,h\}(p) = 0$ for any $v \in B_p$. But this is possible only if $B_p = 0$ which implies $B = 0$ and the proof is complete. □

Remark 2.1. Consider the familiar situation in which G is a compact Lie group acting freely on a symplectic manifold (M, ω) with equivariant moment map $J : M \to \mathfrak{g}^*$. Fix $\mu \in \mathfrak{g}^*$, let $N = J^{-1}(\mu)$ and B be given by the tangent spaces to the orbits of the G-action at points of N. By Example B of [7] the subbundle B is canonical, and the Marsen-Ratiu theorem recovers the familiar symplectic structure on $J^{-1}(\mu)/G_\mu$.

Now take N as above but $B' \subset TN$ to be given by the tangent spaces to the G_μ-orbits *at points of* N, and assume that μ is not a fixed point of the coadjoint action. Then B' is *not* a canonical subbundle. This fact is consistent with Lemma 2.1, and is of course no contradiction to the fact that the G_μ-invariant functions on M are closed under the Poisson bracket (i.e. that the tangent spaces to the G_μ-orbits at *all points of* M form a canonical distribution).

Remark 2.2. If the subbundle $B \neq 0$ is canonical then it follows from Lemma 2.1 that $B^\circ \to N$ is a Lie subalgebroid of T^*M (with the Lie algebroid structure induced by the Poisson structure on M).

In view of Lemma 2.1 the main statement of Theorem 2.1 becomes:

Proposition 2.1. *Assume that* $B \subset T_N M$ *a canonical subbundle and* $B \neq 0$*.Then* $(M, \{\cdot,\cdot\}, N, B)$ *is Poisson reducible.*

Further the induced Poisson structure on $\underline{N}$ depends only on $F = B \cap TN$ and not on B, as stated by the following proposition.

Proposition 2.2. *If* $B, B' \subset T_N M$ *are non-zero canonical subbundles such that* $B \cap TN = B' \cap TN$*, then the induced Poisson structures on* $\underline{N} = \underline{N'}$ *agree.*

Proof. By Lemma 2.1 we know that $\sharp B^\circ, \sharp B'^\circ \subset TN$. Let $f_1, f_2 \in C^\infty(M)_B$ and $f_1', f_2' \in C^\infty(M)_{B'}$ s. t. $\iota^* f_i = \iota^* f_i'$, $i = 1, 2$. We have

$$\iota^*\{f_1, f_2\} - \iota^*\{f_1', f_2'\} = \iota^*\{f_1 - f_1', f_2\} + \iota^*\{f_1', f_2 - f_2'\},$$

and as for any $p \in N$ we have $d(f_i - f_i')_p \in TN^\circ$ and $\sharp(df_i)_p \in TN$, the right hand side vanishes. Hence

$$\iota^*\{f_1, f_2\} = \iota^*\{f_1', f_2'\}.$$ □

Remark 2.3. To every submanifold N of the Poisson manifold M is canonically associated a Poisson algebra, as follows[1]. Let $\mathcal{I}$ be the ideal of functions on M vanishing on N. Its Poisson normalizer $\mathcal{N} \equiv \{f \in C^\infty(M) \mid \{f, \mathcal{I}\} \subset \mathcal{I}\}$ is a Poisson subalgebra, so the quotient $\mathcal{N}/(\mathcal{N} \cap \mathcal{I})$ is a Poisson algebra (see also [6]). Notice that $\mathcal{N}$ consists of functions whose differentials annihilates all vectors in $\sharp TN^\circ$.

Now let B be a nonzero canonical subbundle. Then $C^\infty(\underline{N})$, with the Poisson bracket induced as in Prop. 2.1, is a *Poisson subalgebra* of $\mathcal{N}/(\mathcal{N} \cap \mathcal{I})$. Indeed by Lemma 2.1 we have $B \supset \sharp TN^\circ$, so $C^\infty(M)_B \subset \mathcal{N}$, hence $C^\infty(\underline{N}) = C^\infty(M)_B/(C^\infty(M)_B \cap \mathcal{I})$ sits inside $\mathcal{N}/(\mathcal{N} \cap \mathcal{I})$ and is a Poisson subalgebra. Notice that $\mathcal{N}/(\mathcal{N} \cap \mathcal{I})$ does not "see" the subbundle B, in agreement with Prop. 2.2 above.

Remark 2.4. We complete Prop. 2.1 and Prop. 2.2 by dealing with the trivial case $B = 0$ (which is clearly canonical). $(M, \{\cdot,\cdot\}, N, B = 0)$ is Poisson reducible iff N is a Poisson submanifold. If B' is some canonical subbundle with $B' \cap TN = 0$ then the Poisson structures induced by B' and $B = 0$ on N agree, as N is a Poisson submanifold.

The conclusion of Prop. 2.2 is that, when the Marsden-Ratiu reduction endows $\underline{N}$ with an induced Poisson structure, this structure depends only on F. This result is against the original idea of reduction by distributions, where the role played by B is expected to be more prominent. In order to accomplish this objective we will proceed, in the coming section, to relax the condition of canonicity of the distribution while maintaining the requirement of having a Poisson structure induced on $\underline{N}$.

3. Extension of the Marsden-Ratiu reduction

The set-up of this section consists of the geometric data of the Mardsen-Ratiu theorem; we will set various conditions on these data which guarantee Poisson reducibility. So let (M, Π) be a Poisson manifold, $N \subset M$ a submanifold and $B \subset T_N M$ a subbundle with $F := B \cap TN$ a regular, integrable distribution. We do not need to assume that $\underline{N} := N/F$ is a smooth manifold, even though this is of course the case of interest. In that case $C^\infty(\underline{N}) \cong C^\infty(N)_F$.

We would like to define a bilinear operation $\{\cdot,\cdot\}_{\underline{N}}$ on $C^\infty(N)_F$ by the following rule:

$$\{f, g\}_{\underline{N}} := \{f^B, g^B\}|_N \tag{3.1}$$

[1]This is an algebraic version of Example D in [7]; the latter holds when $\sharp TN^\circ$ and $\sharp TN^\circ \cap TN$ have constant rank.

where f^B, g^B are arbitrary extensions to elements of $C^\infty(M)_B$. The restriction map $\iota^*\colon C^\infty(M)_B \to C^\infty(N)_F$ is surjective[2] so there is at most one bilinear operation $\{\cdot,\cdot\}_{\underline{N}}$. Our task is to determine when $\{\cdot,\cdot\}_{\underline{N}}$ is well-defined and when it is a Poisson bracket.

The r.h.s. of eq. (3.1) is independent of the chosen extensions (for all $f, g \in C^\infty(N)_F$) iff

$$\sharp B^\circ \subset TN + B \tag{3.2}$$

(see the proof of the Marsen-Ratiu theorem or the proof of Thm. 3.1 below). If this is the case, the r.h.s. of (3.1) lies in $C^\infty(N)_F$ iff for one choice of extensions f^B, g^B we have $\{f^B, g^B\}|_N \in C^\infty(N)_F$, or equivalenty if

$$\{C^\infty(M)_B, C^\infty(M)_B\} \subset C^\infty(M)_F. \tag{3.3}$$

In this case clearly $\{\cdot,\cdot\}_{\underline{N}}$ will be a skew-symmetric operation on $C^\infty(N)_F$ which is a biderivation w.r.t. the product; if $\underline{N}$ is smooth, this means that $\{\cdot,\cdot\}_{\underline{N}}$ defines a bivector field on it.

Now we want to determine conditions under which $\{\cdot,\cdot\}_{\underline{N}}$ satisfies the Jacobi identity, for when this is the case $(M, \{\cdot,\cdot\}, N, B)$ is Poisson reducible. Checking the Jacobi identity suggests to require that for any $f, g \in C^\infty(N)_F$ there *exist* extensions f^B, g^B whose bracket annihilate not only F but actually a larger subbundle (not necessarily tangent to N). This leads us to a condition that involves two pieces of data: an additional subbundle D of T_NM and a subspace $\mathcal{B}$ of $C^\infty(M)_B$ which contains the above extensions. In the Appendix we give an algebraic interpretation of these data, and at the end of Subsection 4.2 we give a geometric interpretation.

Theorem 3.1. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold, $N \subset M$ a submanifold and $B \subset T_NM$ a subbundle with $F := B \cap TN$ a regular, integrable distribution. Let D be a subbundle of T_NM satisfying[3] $F \subset D \subset B$ and*

$$\sharp B^\circ \subset D + TN. \tag{3.4}$$

Let $\mathcal{B} \subset C^\infty(M)_B$ be a multiplicative subalgebra such that the restriction map $\iota^*\colon \mathcal{B} \to C^\infty(N)_F$ is surjective. Assume that

$$\{\mathcal{B}, \mathcal{B}\} \subset C^\infty(M)_D. \tag{3.5}$$

Then $(M, \{\cdot,\cdot\}, N, B)$ is Poisson reducible.

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[2]To see this consider the normal bundle to N given by $\pi\colon \tilde{B} \oplus R \to N$, where R is a complement to $TN+B$ in the vector bundle T_NM and $\tilde{B}$ a complement to $F = B \cap TN$ in B. Identifying this normal bundle with a tubular neighborhood of N in M (for instance using a Riemannian metric on M) we see that if $f \in C^\infty(N)_F$ then $\pi^* f$ is an extension lying in $C^\infty(M)_B$.

[3]Equivalently $D \subset B$ and $B \cap TN = D \cap TN$.

Proof. Consider functions $f, g \in C^\infty(N)_F$ and extensions f^B, g^B in $\mathcal{B}$. If we choose a different extension $f^{B'}$ for f, the differential of $f^B - f^{B'}$ annihilates $TN + B$, so because of $\sharp(TN+B)^\circ \subset \sharp(D+TN)^\circ \subset B$ (eq. (3.4)) we have $\iota^*\{f^B - f^{B'}, g^B\} = 0$. Hence the expression for $\{f,g\}_{\underline{N}}$ is independent of the choice of extensions. By eq. (3.5) it actually lies in $C^\infty(N)_F$.

Now $\{f^B, g^B\}$ and $(\{f,g\}_{\underline{N}})^B$ by definition agree on N, and are elements respectively of $C^\infty(M)_D$ (by eq. (3.5)) and $\mathcal{B}$. So their difference annihilates $D + TN$ and by eq. (3.4) the Poisson bracket of their difference with any element of $\mathcal{B}$ vanishes on N. This explains the second equality in the identity

$$\{\{f,g\}_{\underline{N}}, h\}_{\underline{N}} = \iota^*\{(\{f,g\}_{\underline{N}})^B, h^B\} = \iota^*\{\{f^B, g^B\}, h^B\}.$$

From this is clear that the Jacobi identity for $\{\cdot,\cdot\}_{\underline{N}}$ holds as a consequence of that for $\{\cdot,\cdot\}$. □

Remark 3.1. Enlarging D makes the constraint (3.5) more severe, so in applications one should choose D satisfying (3.4) to have dimension as small as possible. In general there is no unique minimal choice of D.

4. Applications and examples

In this section we consider special cases of Thm. 3.1. As usual (M, Π) is a Poisson manifold, $N \subset M$ a submanifold and $B \subset T_N M$ a subbundle with $F := B \cap TN$ a regular, integrable distribution.

4.1. **A straightforward application.** Setting $D = F$ and $\mathcal{B} = C^\infty(M)_B$ in Thm. 3.1. we obtain a minor improvement of the Marsden-Ratiu theorem, where the condition on the canonicity of B is weakened:

Proposition 4.1. *If*

$$\{C^\infty(M)_B, C^\infty(M)_B\} \subset C^\infty(M)_F \tag{4.1}$$

and $\sharp B^\circ \subset TN$ *then* $(M, \{\cdot,\cdot\}, N, B)$ *is Poisson reducible.*

Remark 4.1. In the above proposition condition (4.1) is equivalent to the following, which is more suited for computations: locally there exists a frame of sections X_i of F and extensions thereof to vector fields on M such that

$$(\mathcal{L}_{X_i}\Pi)|_N \subset B \wedge T_N M. \tag{4.2}$$

This can be shown using formula (4.5) below and $\sharp B^\circ \subset TN$.

We present an example where the assumptions of Prop. 4.1 are satisfied but B is not canonical.

Example 4.1. Let (M,Π) be $(\mathbb{R}^3, z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y})$ and N the plane given by $z=0$. Let $B=\mathbb{R}\frac{\partial}{\partial z}$. The conditions of Prop. 4.1 are satisfied because Π vanishes at points of N and because $F=B\cap TN=\{0\}$. However $C^\infty(M)_B$ is not closed w.r.t. the Poisson bracket: for instance x,y lie in $C^\infty(M)_B$ but $\{x,y\}=z$ does not.

If $\sharp B^\circ\subset TN$ then necessarily $\sharp TN^\circ\cap TN\subset F$. When this last inclusion is an equality eq. (4.1) holds automatically, so the interesting case is when the inclusion is strict, as in the following example, in which we use Remark 4.1 to check condition (4.1).

Example 4.2. Let $(M,\Pi)=(\mathbb{R}^6,\sum_i\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_i})$ and N be the (coisotropic) hyperplane $\{y_3=0\}$. Let $B=\text{span}\{\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_1}+\alpha\frac{\partial}{\partial y_2}\}\subset TN$ where $\alpha\in C^\infty(N)$. $F=B$ is integrable iff α is independent of x_3. We have $\sharp B^\circ\subset TN$ since B contains the characteristic distribution of N.

We check condition (4.2), which is easier than checking directly condition (4.1). We have $\mathcal{L}_{\frac{\partial}{\partial x_3}}\Pi=0$, and $(\mathcal{L}_{\frac{\partial}{\partial x_1}+\alpha\frac{\partial}{\partial y_2}}\Pi)|_N=-\sharp d\alpha\wedge\frac{\partial}{\partial y_2}$ surely lies in $B\wedge T_NM$ if α depends only on the coordinates y_1 and x_2. In this case by Prop. 4.1 the quotient $\underline{N}\cong\mathbb{R}^3$ has an induced Poisson structure, which in suitable coordinates is given by $\{y_1,y_2\}=\alpha$, $\{x_2,y_2\}=1$ and $\{y_1,x_2\}=0$.

4.2. **An application involving distributions.** If M is endowed with a suitable distribution we can weaken the condition $\sharp B^\circ\subset TN$ (which, as seen in Lemma 2.1, is an assumption of the Marsden-Ratiu theorem for $B\neq 0$).

Definition 4.1. Let θ_D be an integrable distribution on M such that $F\subset\theta_D|_N\subset B$. We say that θ_D **and** B **are compatible** if $\iota^*: C^\infty(M)_B\cap C^\infty(M)_{\theta_D}\to C^\infty(N)_F$ is surjective.

The above compatibility is satisfied for instance when $\theta_D|_N=B$ or $F:=B\cap TN=\{0\}$. In the appendix (Prop. A.3) we shall give an equivalent characterization of Def. 4.1.

Proposition 4.2. *Suppose that on M there is an integrable distribution θ_D such that $F\subset D:=\theta_D|_N\subset B$ and so that θ_D is compatible with B. Assume that*

$$\sharp B^\circ\subset D+TN \tag{4.3}$$

and that, for any section X of θ_D,

$$(\mathcal{L}_X\Pi)|_N\subset B\wedge T_NM. \tag{4.4}$$

Then $(M,\{\cdot,\cdot\},N,B)$ is Poisson reducible.

Proof. Set $\mathcal{B} = C^\infty(M)_B \cap C^\infty(M)_{\theta_D}$ in Thm. 3.1. By assumption $\iota^* \colon \mathcal{B} \to C^\infty(N)_F$ is surjective. Condition (3.5) reads

$$\{C^\infty(M)_B \cap C^\infty(M)_{\theta_D}, C^\infty(M)_B \cap C^\infty(M)_{\theta_D}\} \subset C^\infty(M)_D.$$

This is equivalent to (4.4), as one can see evaluating at points of N the following equation: for $X \in \Gamma(\theta_D)$ and $f, g \in C^\infty(M)_B \cap C^\infty(M)_{\theta_D}$,

$$X\{f,g\} = (\mathcal{L}_X \Pi)(df, dg) + \Pi(d(Xf), dg) + \Pi(df, d(Xg)). \qquad (4.5)$$

□

Remark 4.2. It is sufficient to apply Prop. 4.2 locally. More precisely: let $\{U^\alpha\}$ be an open cover of a tubular neighborhood of N, and suppose that on each $\{U^\alpha\}$ there exists an integrable distribution θ_D^α as in the proposition. Then in particular eq. (3.2) is satisfied, so eq. (3.1) determines a well-defined map $C^\infty(N)_F \times C^\infty(N)_F \to C^\infty(N)$. Applying Prop. 4.2 on each open set U^α ensures that this map defines a Poisson bracket on $C^\infty(N)_F$.

Further it is sufficient to check condition (4.4) locally on a frame $\{X_i\}$ of sections of θ_D.

To further illustrate the properties of the reduction discussed in Prop. 4.2 we provide some concrete examples that highlight different aspects of the reduction. The first examples are particularly simple, since there $B \oplus TN = T_N M$, so that formula (3.1) defines a bivector field on $\underline{N} = N$.

Example 4.3. Consider the symplectic manifold $(\mathbb{R}^4, \sum_i dx_i \wedge dy_i)$, let N be given by the constraints $x_1 = x_2 = 0$ and let $B = \theta_D|_N$ where θ_D is the distribution on $\mathbb{R}^4$ given by $\theta_D := span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} - \lambda \frac{\partial}{\partial y_1}\}$ (with $\lambda \in \mathbb{R}$). $C^\infty(M)_{\theta_D}$ is closed under the bracket and $\sharp(TN + B)^\circ = 0$, so the assumptions of Prop. 4.2 are met. The quotient $\underline{N}$ is $\mathbb{R}^2$ with natural coordinates y_1, y_2 and Poisson bivector $\lambda \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}$.

Notice that in this example the condition $\sharp B^\circ \subset TN$ is violated. The final Poisson structure depends on B (while $B \cap TN = \{0\}$ is independent of λ). As shown in Prop. 2.2 this can not happen in the Marsden-Ratiu reduction (Thm. 2.1).

The next example illustrates the fact that, even if we have a well-defined smooth bivector on $\underline{N}$, we need extra conditions to fulfill the Jacobi identity.

Example 4.4. Let (M, Π) be the Poisson manifold $(\mathbb{R}^4, \sum_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i})$, consider the hyperplane $N = \{y_2 = 0\}$ and the subbundle B of $T_N M$ spanned by $\frac{\partial}{\partial y_2} + \alpha \frac{\partial}{\partial x_1}$, where $\alpha \in C^\infty(N)$. The bivector field induced by eq. (3.1) on N is $\frac{\partial}{\partial x_1} \wedge (\frac{\partial}{\partial y_1} + \alpha \frac{\partial}{\partial x_2})$, hence it is Poisson iff α is independent of x_1.

All the Poisson structures obtained above can be obtained using Prop. 4.2. Indeed, if we extend B to the distribution $\theta_D := \mathbb{R}(\frac{\partial}{\partial y_2} + \alpha \frac{\partial}{\partial x_1})$, eq. (4.4) is satisfied iff $\frac{\partial}{\partial x_1} \alpha = 0$.

In the previous example we have seen an obstruction to obtaining a Poisson structure after the reduction, namely eq. (4.4). In the following example the distribution F on N is non-trivial, and we shall also exhibit an obstruction to have a well defined bivector field on $\underline{N}$ in the first place.

Example 4.5. Let (M, Π) be the Poisson manifold $(\mathbb{R}^6, \sum_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i})$, consider the hyperplane $N = \{y_2 = 0\}$ and the subbundle of $T_N M$ given by $B = \text{span}\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} + \alpha \frac{\partial}{\partial x_1}\}$, where $\alpha \in C^\infty(N)$. Clearly $\sharp B^\circ \subset TN + B$, and $F := B \cap TN = \mathbb{R}\frac{\partial}{\partial y_1}$.

Now the bracket of the B-invariant extensions of the coordinate functions x_1, x_2 is: $\{x_1^B, x_2^B\}|_N = \alpha$, which is well defined on $\underline{N}$ iff α does not depend on y_1. This condition ensures that we have a bivector field on $\underline{N}$ but still is not enough to guarantee reducibility.

Prop. 4.2 can be applied to determine when the bracket $\{\cdot,\cdot\}_{\underline{N}}$ is a well-defined Poisson bracket. We extend B constantly in the y_2 direction to obtain the distribution $\theta_D = span\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} + \alpha \frac{\partial}{\partial x_1}\}$ on M. The distribution θ_D is integrable iff α does not depend on y_1. Now $\mathcal{L}_{\frac{\partial}{\partial y_1}} \Pi = 0$, and $\mathcal{L}_{\frac{\partial}{\partial y_2} + \alpha \frac{\partial}{\partial x_1}} \Pi = -X_\alpha \wedge \frac{\partial}{\partial x_1} \subset B \wedge T_N M$ iff α does not depend on the coordinates x_3 and y_3. Hence Prop. 4.2 allows us to conclude that, when α depends only on the coordinates x_1 and x_2, we obtain a Poisson bivector on $\underline{N} \cong \mathbb{R}^4$. In the natural coordinates, the induced Poisson bivector is $\alpha \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y_3}$.

The following is a simple example in which M is a linear Poisson manifold.

Example 4.6. Let $\mathfrak{g}$ be a Lie algebra, $V \subset \mathfrak{g}$ a subspace and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra satisfying $[\mathfrak{h}, V \cap \mathfrak{h}] \subset V$. We set $M := \mathfrak{g}^*$, $N := V^\circ$, and $B_x := \mathfrak{h}^\circ \subset T_x M$ at all $x \in N$. Using Lemma 5.4 of [3] and the assumptions, we see $\sharp B_x^\circ = \{ad_h^*(x) : h \in \mathfrak{h}\} \subset T_x N + B_x$ at all $x \in N$. Extending $B = \mathfrak{h}^\circ$ by translation to a distribution θ_D on M and noticing that the projection $\mathfrak{g}^* \to \mathfrak{g}^*/\mathfrak{h}^\circ \cong \mathfrak{h}^*$ is a Poisson map we see that eq. (4.4) is satisfied. By Prop. 4.2 we conclude that there is an induced (linear) Poisson structure on $\underline{N} = \frac{V^\circ}{V^\circ \cap \mathfrak{h}^\circ} \cong (\frac{V+\mathfrak{h}}{V})^*$. It corresponds to the Lie algebra structure on $\frac{\mathfrak{h}}{\mathfrak{h} \cap V}$, which as a vector space is canonically isomorphic to $\frac{V+\mathfrak{h}}{V}$.

Our last example shows that conditions of Prop. 4.2 are not necessary in order to obtain a Poisson structure after the reduction.

Example 4.7. Let (M, Π) be $(\mathbb{R}^3, z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$, N the plane given by $z - x = 0$ and $B = \mathbb{R}\frac{\partial}{\partial z}$. Formula (3.1) defines the Poisson structure $\{x, y\} = x$ on N, however Prop. 4.2 can not be applied because a distribution θ_D as in the proposition does not exist. Indeed θ_D has to be one-dimensional because of eq. (4.3). For any vector field X which restricts to $\frac{\partial}{\partial z}$ on N, we have $(\mathcal{L}_X \Pi)|_p = X(z)|_p \cdot \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ at any point $p \in N$ of the form $(0, y, 0)$, so eq. (4.4) is not satisfied.

We conclude the subsection giving a geometric interpretation of Prop. 4.2. Assume that the quotient $\underline{M} := M/\theta_D$ is smooth and that $C^\infty(M)_{\theta_D}$ is closed under the Poisson bracket, so that $\underline{M}$ has a Poisson structure for which the projection $M \to \underline{M}$ is a Poisson map. Assume $\underline{N} \subset \underline{M}$ is a Poisson-Dirac submanifold [5], so that it has an induced Poisson structure. Then the Poisson bracket of functions on $\underline{N}$ is computed by lifting to functions in $C^\infty(M)_B$ where B is a subbundle as in Prop. 4.2. In this interpretation the case $D = B$ corresponds to the case where $\underline{N}$ is actually a *Poisson* submanifold of $\underline{M}$.

4.3. **An application to hamiltonian actions.** Here is an instance where the assumptions of Prop. 4.2 are naturally met. Given an action of a Lie group on a manifold M we denote by $\mathfrak{g}_M(p)$ the span at $p \in M$ of the vector fields generating the action (i.e. the tangent space of the G-orbit through p).

Proposition 4.3. *Let the Lie group G act on the symplectic manifold (M, ω) so that $\mathfrak{g}_M$ has constant rank and with equivariant moment map $J \colon M \to \mathfrak{g}^*$. Let $m \in J^{-1}(0)$ and N be a slice transverse to $J^{-1}(0)$ at m, i.e.*

$$T_m N \oplus T_m J^{-1}(0) = T_m M. \tag{4.6}$$

Then N, after shrinking it to a smaller neighborhood of m if necessary, has an induced Poisson structure, obtained extending functions from N to M so that they annihilate $[\mathfrak{g}_M + (TN + \mathfrak{g}_M)^\omega]|_N$

Proof. Consider $B := [\mathfrak{g}_M + (TN + \mathfrak{g}_M)^\omega]|_N \subset T_N M$ and the distribution $\theta_D := \mathfrak{g}_M$. We now check that the assumptions of Prop. 4.2 are automatically satisfied; we will make use repeatedly of $\mathfrak{g}_M(m) \subset T_m J^{-1}(0) = \mathfrak{g}_M(m)^\omega$, which holds by the equivariance of J.

First of all B has constant rank, at least near m. Indeed the sum of TN and $\mathfrak{g}_M$ has constant rank because their intersection at m is trivial. Further $TN + \mathfrak{g}_M$ is a symplectic subbundle of $T_N M$. To this aim we check that at the point m we have

$$\begin{aligned} T_m N^\omega \cap [\mathfrak{g}_M(m)^\omega \cap (T_m N + \mathfrak{g}_M(m))] &= T_m N^\omega \cap \mathfrak{g}_M(m) = \\ &= (T_m N + \mathfrak{g}_M(m)^\omega)^\omega = \{0\}. \end{aligned} \tag{4.7}$$

We conclude that $B = \mathfrak{g}_M \oplus (TN \oplus \mathfrak{g}_M)^\omega$ has constant rank near m. Further we have $F = B \cap TN = \{0\}$ since $B_m \subset T_m J^{-1}(0)$.

Compatibility of θ_D and B holds because $F = \{0\}$. Condition (4.3) as well as $D := \theta_D|_N \subset B$ are trivially satisfied. Condition (4.4) is satisfied since the G-action preserves ω. □

Remark 4.3. 1) The geometric interpretation of Prop. 4.2 applied to the special case of Prop. 4.3 is the following: if the G action is free and proper it is known that M/G is a Poisson manifold, whose symplectic leaves are given by $J^{-1}(\mathcal{O})/G$ as $\mathcal{O} \subset \mathfrak{g}^*$ ranges over all coadjoint orbits. Therefore $\underline{N} \cong N$ is a submanifold of M/G which intersects transversely the symplectic leaf $J^{-1}(0)/G$, and has such it has a Poisson structure induced from M/G. This Poisson structure agree with the one that Prop. 4.3 induces on N.

2) In the case that the G-action in Prop. 4.3 is free and proper one has a dual pair $M/G \leftarrow M \rightarrow \mathfrak{g}^*$, and Thm. 8.1 of [9] says that the Poisson structure on $\underline{N}$ (as in part 1) above) is isomorphic up to sign to the one on the open subset $J(N)$ of $\mathfrak{g}^*$. However the identification $\underline{N} \cong N \cong J(N)$ given by the dual pair does not preserve the Poisson structures in general. (A sufficient condition is that N is isotropic.)

The following is an example for Prop. 4.3.

Example 4.8. Consider the action of $G = U(2)$ on $M = GL(2,\mathbb{C})$ by left multiplication, and endow M with the symplectic form induced by the natural embedding in $\mathbb{C}^4$. This action is Hamiltonian with moment map $J\colon GL(2,\mathbb{C}) \rightarrow \mathfrak{u}^*(2) \cong \mathfrak{u}(2)$ given by $J(A) = \frac{1}{2i}(AA^* - I)$ [1]. A slice transverse to $J^{-1}(0)$ at the identity is given by

$$N := \left\{ \begin{pmatrix} x_1 & x_2+ix_3 \\ 0 & x_4 \end{pmatrix} \right\}$$

where x_1, x_4 are real numbers close to 1 and x_2, x_3 are close to 0. A straightforward computation shows that extending the coordinates x_i on N so that they annihilate $[\mathfrak{g}_M + (TN + \mathfrak{g}_M)^\omega]|_N = [\mathfrak{g}_M]|_N$ delivers the following bracket on N:

$$\{x_1, x_2\} = \frac{x_3}{x_1}, \quad \{x_1, x_3\} = -\tfrac{x_2}{x_1}, \quad \{x_1, x_4\} = 0$$

$$\{x_2, x_3\} = 1 - \frac{x_4^2}{x_1^2}, \quad \{x_2, x_4\} = \tfrac{x_3 x_4}{x_1^2}, \quad \{x_3, x_4\} = -\frac{x_2 x_4}{x_1^2}.$$

Prop. 4.3 states that this is a Poisson bracket.

In the new coordinates $\xi_1 = \frac{1}{2}x_1x_2, \xi_2 = \frac{1}{2}x_1x_3, \xi_3 = \frac{1}{4}(x_1^2 - x_2^2 - x_3^2 - x_4^2), \eta = \frac{1}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$ the Poisson bracket is linear and coincides with that of $\mathfrak{u}^*(2)$, in agreement with Remark 4.3.

4.4. **The symplectic case.** We end this section asking when eq. (3.1) defines a *symplectic* structure on the quotient $\underline{N}$. We focus on the case where M has a symplectic (not just Poisson) structure ω.

Lemma 4.1. *Assume that eq.* (3.1) *endows* $\underline{N}$ *with a well-defined bivector field* $\underline{\Pi}$. $\underline{\Pi}$ *corresponds to a non-degenerate 2-form iff*

$$TN + B = B^{\omega} + B. \tag{4.8}$$

In this case the 2-form on $\underline{N}$ *is obtained pushing down* $\omega^B \in \Omega^2(N)$ *given by* $\omega^B(X_1, X_2) = \omega(X_1 + b_1, X_2 + b_2)$, *where* $b_i \in B$ *are such that* $X_i + b_i \in B^{\omega}$.

Proof. From Lemma A.2 in the Appendix it follows that $\underline{\Pi}$ is invertible iff the almost Dirac structure $\iota^*(L^B_\Pi)$ on N is the graph of a 2-form with kernel F. Writing out explicitly $\iota^*(L^B_\Pi)$ one sees that it is the graph of a 2-form iff $TN \subset B^{\omega} + B$, which in turn is equivalent to eq. (4.8) since eq. (3.2) holds. In this case the kernel of the 2-form is automatically F. This shows the equivalence claimed in the lemma.

A computation shows that $\iota^*(L^B_\Pi)$ is the graph of the 2-form ω^B defined above. □

A simple instance of Lemma 4.1 is the case when N is a symplectic submanifold of (M, ω) and B is small perturbation of TN^{ω}. Then N is endowed with a non-degenerate 2-form ω^B, which is intertwined with $\iota^*\omega$ by the bundle isomorphism $TN \cong B^{\omega}$ (given by projection along B).

Suppose that B can be extended locally to an integrable distribution θ on M so that the θ-invariant functions are closed w.r.t. the Poisson bracket. Then ω^B is a closed form, for it is just the pullback to N of the symplectic form on the quotient M/θ (this is an instance of Prop. 4.2). In general, writing B as the graph of a bundle map $A\colon TN^{\circ} \cong TN^{\omega} \to TN$, it would be interesting to spell out in terms on A when ω^B is a *symplectic* structure.

Appendix A

A.1. **Algebraic interpretations.** We provide an algebraic interpretation of Thm. 3.1.

Proposition A.1. *Let* $\mathcal{M}$ *be a Poisson algebra,* $\mathcal{B} \subset \mathcal{D}$ *multiplicative subalgebras of* $\mathcal{M}$ *and* $\mathcal{I}$ *a multiplicative ideal of* $\mathcal{M}$. *Assume that the images of* $\mathcal{B}$ *and* $\mathcal{D}$ *under the projection* $\mathcal{M} \to \mathcal{M}/\mathcal{I}$ *are equal and that*

$$\{\mathcal{B}, \mathcal{I} \cap \mathcal{D}\} \subset \mathcal{I} \tag{A.1}$$

and

$$\{\mathcal{B}, \mathcal{B}\} \subset \mathcal{D}. \tag{A.2}$$

Then there is an induced Poisson algebra structure on $\frac{\mathcal{B}}{\mathcal{B}\cap\mathcal{I}}$*, whose bracket is determined by the commutative diagram*

$$\begin{array}{ccc} \mathcal{B}\times\mathcal{B} & \xrightarrow{\{\cdot,\cdot\}} & \mathcal{D} \\ \downarrow & & \downarrow \\ \frac{\mathcal{B}}{\mathcal{B}\cap\mathcal{I}}\times\frac{\mathcal{B}}{\mathcal{B}\cap\mathcal{I}} & \longrightarrow & \frac{\mathcal{D}}{\mathcal{D}\cap\mathcal{I}}=\frac{\mathcal{B}}{\mathcal{B}\cap\mathcal{I}} \end{array}$$

Thm. 3.1 is recovered setting $\mathcal{M} = C^\infty(M)$, $\mathcal{D} = C^\infty(M)_D$ and $\mathcal{I} = \{f \in C^\infty(M) : f|_N = 0\}$. Conditions (A.1) and (A.2) become conditions (3.4) and (3.5) respectively.

The proof of Prop. A.1 is similar to that of Prop. 3.1 and will not be given here. We just mention that condition (A.1) can be interpreted as "$\mathcal{I}$ behaves like an ideal in $\mathcal{B}$", and condition (A.2) as "$\mathcal{B}$ behaves like a Poisson subalgebra", showing that one has a well-defined almost Poisson bracket on $\frac{\mathcal{B}}{\mathcal{B}\cap\mathcal{I}}$. To show that it satisfies the Jacobi identity one needs to use once more both conditions.

A.2. **Descriptions in terms of Dirac structures.** In the next proposition we interpret in terms of Dirac structures the operation $\{\cdot,\cdot\}_{\underline{N}}$ given by eq. (3.1). Let (M,Π) be a Poisson manifold, $N \subset M$ a submanifold and $B \subset T_N M$ a subbundle with $F := B\cap TN$ a regular, integrable distribution.

Proposition A.2. *Assume that* $\underline{N} := N/F$ *is smooth and that the prescription* (3.1) *gives a well-defined bivector field on* $\underline{N}$*, and denote by* $L_{\underline{N}}$ *its graph. Then the pullback of the almost Dirac structure* $L_{\underline{N}}$ *under* $p\colon N \to \underline{N}$ *is* $\iota^*(L_\Pi^B)$.

Here L_Π^B *is the stretching* [4] *of* $L_\Pi = graph(\Pi)$ *in direction of* B*, defined as* $[L_\Pi|_N \cap (T_N M \oplus B^\circ)] + (B \oplus 0)$.

Proof. We will show that the Poisson algebras of admissible functions for $\iota^*(L_\Pi^B)$ and $p^*(L_{\underline{N}})$ match, hence the subbundles have to agree too. Short computations using $\sharp(TN + B)^\circ \subset B$ (which holds since we assume that eq. (3.1) gives a well-defined expression) show that $\iota^*(L_\Pi^B)$ is a smooth almost Dirac structure on N and that its kernel is exactly F. Hence its set of admissible functions is $C^\infty(N)_F$. If $f, g \in C^\infty(N)_F$ their $\iota^*(L_\Pi^B)$-bracket is $\langle X_{f^B} + b, dg^B\rangle$ (where $f^B, g^B \in C^\infty(M)_B$ are extensions and $b \in \Gamma(B)$ is such that $X_{f^B} + b \in TN$), which is equal to $\{f^B, g^B\}$.

The kernel of $p^*(L_{\underline{N}})$ is clearly also F, and if $f, g \in C^\infty(N)_F$ their $p^*(L_{\underline{N}})$-bracket is $\{f, g\}_{\underline{N}}$. Using eq. (3.1) this concludes the proof. □

Remark A.1. The following statements complement Proposition A.2 and are proved similarly. Assume that $\sharp(TN + B)^\circ \subset B$. Then eq. (3.1) defines a bivector field on $\underline{N}$ iff $\iota^*(L_\Pi^B)$ pushes forward under $p\colon N \to \underline{N}$ (to the graph of eq. (3.1)). If $\iota^*(L_\Pi^B)$ is integrable (i.e. if it is a Dirac structure on N) then it automatically pushes forward, and therefore eq. (3.1) defines a Poisson structure on $\underline{N}$.

Hence, assuming $\sharp(TN+B)^\circ \subset B$, eq. (3.1) defines a Poisson structure on $\underline{N}$ iff $\iota^*(L_\Pi^B)$ is integrable. Unfortunately we were not able to express the integrability of the latter in simple terms.

A.3. **Compatibility with foliations.** We now address the question of compatibility stated in Def. 4.1 and give an equivalent characterization. By Remark 4.2 we can work locally, so in the following we will assume that $\underline{N}$ and $\underline{M} := M/\theta_D$ are smooth.

Proposition A.3. *θ_D and B as in Def. 4.1 are compatible if and only if*

$$\text{There exists a subbundle } \hat{B} \text{ with } B \subset \hat{B} \subset T_N M \text{ and } \hat{B} \cap TN = F \text{ such that } pr\colon M \to \underline{M} \text{ maps } \hat{B} \text{ to a well-defined subbundle of } T_{\underline{N}}\underline{M}. \tag{A.3}$$

Proof. To show the "if" part notice that $pr_*\hat{B}$ intersects trivially $T\underline{N}$ (since $F \subset D$), hence any function on $\underline{N}$ can be extended to an element of $C^\infty(\underline{M})_{pr_*\hat{B}}$, and the pullback under pr is then an element of $C^\infty(M)_B \cap C^\infty(M)_{\theta_D}$. Conversely, if θ_D and B are compatible, we can extend a set of coordinates on $\underline{N}$ to functions x_i on $\underline{M}$ so that $pr^*x_i \in C^\infty(M)_B$, and $\hat{B} := pr_*^{-1}(\cap \mathrm{ker} dx_i) \subset T_N M$ will satisfy the condition above. □

In general it is not trivial to check whether the conditions of the previous proposition are satisfied. One can however compute easily a sufficient condition for the compatibility in the case one can take $\hat{B} = B$.

To state the result we introduce $\tilde{\Gamma}(B) := \{X \in \Gamma(TM) : X|_N \subset B\}$ and $\Gamma'(\theta_D) := \Gamma(\theta_D) \cap \tilde{\Gamma}(F)$. Then one can prove that (A.3) holds with $\hat{B} = B$ if and only if

$$[\Gamma'(\theta_D), \tilde{\Gamma}(B)] \subset \tilde{\Gamma}(B), \tag{A.4}$$

which implies the compatibility of θ_D and B.

We conclude remarking that, given a subbundle D with $F \subset D \subset B$, locally one can always find an extension of D to an involutive distribution θ_D compatible with B.

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