A THOMASON MODEL STRUCTURE ON THE CATEGORY OF SMALL n-FOLD CATEGORIES

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ABSTRACT. We construct a cofibrantly generated Thomason model structure on the category of small n-fold categories and prove that it is Quillen equivalent to the standard model structure on the category of simplicial sets. An n-fold functor is a weak equivalence if and only if the diagonal of its n-fold nerve is a weak equivalence of simplicial sets. We introduce an n-fold Grothendieck construction for multisimplicial sets, and prove that it is a homotopy inverse to the n-fold nerve. As a consequence, the unit and counit of the adjunction between simplicial sets and n-fold categories are natural weak equivalences.

CONTENTS

1. INTRODUCTION

An n-fold category is a higher and wider categorical structure obtained by n applications of the internal category construction. In this paper we study the homotopy theory of n-fold categories. Our main result is Theorem 9.26. Namely, we have constructed a cofibrantly generated model structure on the category of n-fold categories in which an n-fold functor is a weak equivalence if and only if its nerve is a diagonal weak equivalence. This model structure is Quillen equivalent to the usual model structure on the category of simplicial sets, and hence also topological spaces. Our main tools are model category theory, the n-fold nerve, and an n-fold Groethendieck construction for multisimplicial sets. Notions of nerve and versions of the Grothendieck construction are very prominent in homotopy theory and higher category theory, as we now explain. The Thomason model structure on **Cat** is also often present, at least implicitly.

The Grothendieck nerve of a category and the Grothendieck construction for functors are fundamental tools in homotopy theory. Theorems A and B of Quillen [77], and Thomason's theorem [85] on Grothendieck constructions as models for certain homotopy colimits, are still regularly applied decades after their creation. Functors with nerves that are weak equivalences of simplicial sets feature prominently in these theorems. Such functors form the weak equivalences of Thomason's model structure on **Cat** [86], which is Quillen equivalent to **SSet**. Earlier, Illusie [48] proved that the nerve and the Grothendieck construction are homotopy inverses. Although the nerve and the Grothendieck construction are not adjoints, the equivalence of homotopy categories can be realized by adjoint functors [28], [29], [86]. Related results on homotopy inverses are found in [63], [64], and [88]. More recently, Cisinski [14] has proved two conjectures of Grothendieck concerning this circle of ideas (see also [49]).

On the other hand, notions of nerve play an important role in various definitions of n-category [65], namely the definitions of Simpson [82], Street [83], and Tamsamani [84], as well as in the theory of quasi-categories developed by Joyal [52], [53], [54], and also Lurie [69], [70]. For notions of nerve for bicategories, see for example work of Duskin and Lack-Paoli [18], [17], [62], and for left adjoints to singular functors in general also [30] and [58]. Fully faithful cellular nerves have been developed for higher categories in [3], together with characterizations of their essential images. Nerve theorems can be established in a very general context, as proved by Leinster and Weber in [66] and [89], and discussed in [67]. As an example, Kock proves in [59] a nerve theorem for polynomial endofunctors in terms of trees.

Model category techniques are only becoming more important in the theory of *higher* categories. They have been used to prove that, in a precise sense, simplicial categories, Segal categories, complete Segal spaces, and quasi-categories are all equivalent models for $(\infty,1)$-categories [4], [6], [5], [51], [79], and [87]. In other directions, although the cellular nerve of [3] does not transfer a model structure from cellular sets to ω-categories, it is proved in [3] that the homotopy category of cellular sets is equivalent to the homotopy category of ω-categories. For this, a Quillen equivalence between *cellular spaces* and *simplicial ω-categories* is constructed. There is also the work of [76] and [82], developing model structures on n-categories for the purpose of n-stacks.

In low dimensions several model structures have already been investigated. On **Cat**, there is the categorical structure of Joyal-Tierney [55], [78], as well as the topological structure of Thomason [86], [13]. A model structure on pro-objects in **Cat** appeared in [33], [34], [35]. The articles [42], [43], [44] and are closely related to the Thomason structure and the Thomason homotopy colimit theorem. More recently, the Thomason structure on **Cat** was proved in Theorem 5.2.12 of [14] in the context of Grothendieck test categories and fundamental localizers. The homotopy categories of spaces and categories are proved equivalent in [47] without using model categories.

On **2-Cat** there is the categorical structure of [60] and [61], as well as the Thomason structure of [90]. Model structures on **2FoldCat** have been studied in [27] in great detail. The homotopy theory of 2-fold categories is very rich, since there are numerous ways to view 2-fold categories: as internal categories in **Cat**, as certain simplicial objects in **Cat**, or as algebras over a 2-monad. In [27], a model structure is associated to each point of view, and these model structures are compared.

However, there is another way to view 2-fold categories not treated in [27], namely as certain bisimplicial sets. There is a natural notion of fully faithful double nerve, which associates to a 2-fold category a bisimplicial set. An obvious question is: does there exist a Thomason-like model structure on **2FoldCat** that is Quillen equivalent to some model structure on bisimplicial sets via the double nerve? Unfortunately, the left adjoint to double nerve is homotopically poorly behaved as it extends the left adjoint c to ordinary nerve, which is itself poorly behaved. So any attempt at a model structure must address this issue.

Fritsch, Latch, and Thomason [28], [29], [86] noticed that the composite of c with second barycentric subdivision Sd^2 is much better behaved than c alone. In fact, Thomason used the adjunction $c\mathrm{Sd}^2 \dashv \mathrm{Ex}^2 N$ to construct his model structure on **Cat**. This adjunction is a Quillen equivalence, as the right adjoint preserves weak equivalences and fibrations by definition, and the unit and counit are natural weak equivalences.

Following this lead, we move to simplicial sets via δ^* (restriction to the diagonal) in order to correct the homotopy type of double categorification using Sd^2. Moreover, our method of proof works for n-fold categories as well, so we shift our focus from 2-fold categories to general n-fold categories. In this paper, we construct a cofibrantly generated model structure on **nFoldCat** using the fully faithful n-fold nerve, via the adjunction below,

$$
(1) \qquad \begin{tikzcd}
\mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}^2"] \arrow[r, bend left, "\delta_!"] \arrow[r, phantom, "\perp"] & \mathbf{SSet^n} \arrow[l, bend left, "\delta^*"] \arrow[r, bend left, "c^n"] \arrow[r, phantom, "\perp"] & \mathbf{nFoldCat} \arrow[l, bend left, "N^n"]
\end{tikzcd}
$$

and prove that the unit and counit are weak equivalences. Our method is to apply the Lemma from Kan on transfer of structure. First we prove Thomason's classical theorem in Theorem 6.2, and then use this proof as a basis for the general n-fold case in Theorem 8.2. We also introduce an n-fold Grothendieck construction in Definition 9.1, prove that it is homotopy inverse to the n-fold nerve in Theorems 9.21 and 9.22, and conclude in Proposition 9.25 that the unit and counit of the adjunction (1) are natural weak equivalences. The articles [28] and [29] proved in a different way that the unit and counit of the classical Thomason adjunction $\mathbf{SSet} \dashv \mathbf{Cat}$ are natural weak equivalences.

Recent interest in n-fold categories has focused on the $n = 2$ case. In many cases, this interest stems from the fact that 2-fold categories provide a good context for incorporating two types of morphisms, and this is useful for applications. For example, between rings there are ring homomorphisms and bimodules, between topological spaces there are continuous maps and parametrized spectra as in [72], between manifolds there are smooth maps and cobordisms, and so on. In this direction, see for example [38], [24], [25], [74], [80], [81]. Classical work on 2-fold categories, originally introduced by Ehresmann as *double categories*, includes [2], [19], [20], [21], [23], [22]. The theory of double categories is now flourishing, with many contributions by Brown-Mosa, Grandis-Paré, Dawson-Paré-Pronk, Dawson-Paré, Fiore-Paoli-Pronk, Shulman, and many others. To mention only a few examples, we have [12], [38], [39], [41], [40], [15], [16], [27], [80], and [81].

There has also been interest in general n-fold categories from various points of view. Connected homotopy $(n+1)$-types are modelled by n-fold categories internal to the category of groups in [68], as summarized in the survey paper [75]. Edge symmetric n-fold categories have been studied by Brown, Higgins, and others for many years now, for example [8], [9], [10], and [11]. There are also the more recent *symmetric weak cubical categories* of [36] and [37]. The homotopy theory of cubical sets has been studied in [50].

The present article is the first to consider a Thomason structure on the category of n-fold categories. Our paper is organized as follows. Section 2 recalls n-fold categories, introduces the n-fold nerve N^n and its left adjoint n-fold categorification c^n, and describes how c^n interacts with $\delta_!$, the left adjoint to precomposition with the diagonal. In Section 3 we recall barycentric subdivision, including explicit descriptions of $\mathrm{Sd}^2\Lambda^k[m]$, $\mathrm{Sd}^2\partial\Delta[m]$, and $\mathrm{Sd}^2\Delta[m]$. More importantly, we present a decomposition of the poset $\mathbf{P}\mathrm{Sd}\Delta[m]$ into the union of three posets **Comp**, **Center**, and **Outer** in Proposition 3.10, as picture in Figure 1 for $m = 2$ and $k = 1$. Though Section 3 may appear technical, the statements become clear after a brief look at the example in Figure 1. This section is the basis for the verification of the pushout axiom (iv) of Corollary 6.1, completed in the proofs of Theorems 6.2 and 8.2.

Sections 4 and 5 make further preparations for the verification of the pushout axiom. Proposition 4.3 gives a deformation retraction of $|N(\mathbf{Comp} \cup \mathbf{Center})|$ to part of its boundary, see Figure 1. This deformation retraction finds application in equation (15). The highlights of Section 5 are Proposition 5.4 on the expression of certain posets as a limit of two ordinals, and Propositions 5.1 and 5.8 on the commutation of nerve with certain colimits. These also find application in equation (15). Section 6 pulls these results together and quickly proves the classical Thomason theorem.

Section 7 proves the n-fold versions of the results in Sections 3, 4, and 5. The n-fold version of Proposition 5.3 on colimit decompositions is Proposition 7.4. The n-fold version of the deformation retraction in Proposition 4.3 is Corollary 7.11. The n-fold version of Proposition 5.1 on commutation of nerve with certain pushouts is Proposition 7.15. Proposition 7.12 displays a calculation of a pushout of double categories, and the diagonal of its nerve is characterized in Proposition 7.13.

Section 8 pulls together the results of Section 7 to prove the Thomason structure on **nFoldCat** in Theorem 8.2. In the last section of the paper, Section 9, we introduce a Grothendieck construction for multi-simplicial sets and prove that it is a homotopy inverse for n-fold nerve in Theorems 9.21 and 9.22. As a consequence, we have in Proposition 9.25 that the unit and counit are weak equivalences.

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2. n-Fold Categories

In this section we quickly recall the inductive definition of n-fold categories, introduce the n-fold nerve N^n, prove the existence of its left adjoint c^n, and recall the adjunction $\delta_! \dashv \delta^*$.

Definition 2.1. A *small n-fold category* $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1)$ is a category object in the category of small $(n-1)$-fold categories. In detail, $\mathbb{D}_0$ and $\mathbb{D}_1$ are $(n-1)$-fold categories equipped with $(n-1)$-fold functors

$$\begin{tikzcd} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "m"] & \mathbb{D}_1 \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathbb{D}_0 \arrow[l, "u" description] \end{tikzcd}$$

that satisfy the usual axioms of a category. We denote the category of n-fold categories by **nFoldCat**.

Since we will always deal with small n-fold categories, we leave off the adjective "small". Also, all of our n-fold categories are strict, unless specified as "pseudo".

A 2-fold category, that is, a category object in **Cat**, is precisely a *double category* in the sense of Ehresmann. A double category consists of a set of *objects*, a set of *horizontal morphisms*, a set of *vertical morphisms*, and a set of *squares* equipped with various sources, targets, and associative and unital compositions. The homotopy theory of double categories was considered in [27].

Example 2.2. There are various standard examples of double categories. To any category, one can associate the double category of commutative squares. Any 2-category can be viewed as a double category with trivial vertical morphisms or as a double category with trivial horizontal morphisms. To any 2-category, one can also associate the double category of *quintets*: a square is a square of morphisms inscribed with a 2-cell in a given direction.

Example 2.3. In nature, one often finds *pseudo double categories.* These are like double categories, except one direction is a bicategory rather than a 2-category (see [38] for a more precise definition). For example, one may consider 1-manifolds, 2-cobordisms, smooth maps, and appropriate squares. Another example is rings, bimodules, ring maps, and twisted equivariant maps. For these examples and more, see [38], [25], and other articles on double categories listed in the introduction.

Example 2.4. Any n-category is an n-fold category in numerous ways, just like a 2-category can be considered as a double category in several ways.

An important method of constructing n-fold categories from n ordinary categories is the external product, which is compatible with the external product of simplicial sets. This was called the *square product* on page 251 of [2].

Definition 2.5. If $\mathbf{C}_1, \ldots, \mathbf{C}_n$ are small categories, then the *external product* $\mathbf{C}_1 \boxtimes \cdots \boxtimes \mathbf{C}_n$ is an n-fold category with object set $Obj\, \mathbf{C}_1 \times \cdots \times Obj\, \mathbf{C}_n$. Morphisms in the i-th direction are n-tuples $(f_1, \ldots, f_n)$ of morphisms in $\mathbf{C}_1 \times \cdots \times \mathbf{C}_n$ where all but the i-th entry are trivial. Squares in the ij-plane are n-tuples where all entries are trivial except the i-th and j-th entries, and so on. An n-cube is an n-tuple of morphisms, possibly all nontrivial.

Proposition 2.6. *The category* **nFoldCat** *is locally finitely presentable.*

Proof: We prove this by induction. The category **Cat** of small categories is known to be locally finitely presentable (see for example [31]). Assume $(\mathbf{n-1})$**FoldCat** is locally finitely presentable. The category **nFoldCat** is the category of models in $(\mathbf{n-1})$**FoldCat** for a sketch with finite diagrams. Since $(\mathbf{n-1})$**FoldCat** is locally finitely presentable, we conclude from Proposition 1.53 of [1] that **nFoldCat** is also locally finitely presentable. □

Proposition 2.7. *The category* **nFoldCat** *is complete and cocomplete.*

Proof: Completeness follows quickly, because **nFoldCat** is a category of algebras. Cocompleteness follows because **nFoldCat** is locally finitely presentable. □

Definition 2.8. The *n-fold nerve* of an n-fold category $\mathbb{D}$ is the multisimplicial set $N^n\mathbb{D}$ with $\overline{p}$-simplices

$$(N^n\mathbb{D})_{\overline{p}} := Hom_{\mathbf{nFoldCat}}([p_1] \boxtimes \cdots \boxtimes [p_n], \mathbb{D}).$$

A $\overline{p}$-simplex is a $\overline{p}$-array of composable n-cubes.

Remark 2.9. The n-fold nerve is the same as iterating the nerve construction for internal categories n times.

Example 2.10. The n-fold nerve is compatible with external products: $N^n(\mathbf{C}_1 \boxtimes \cdots \boxtimes \mathbf{C}_n) = N\mathbf{C}_1 \boxtimes \cdots \boxtimes N\mathbf{C}_n$. In particular,

$$N^n([m_1] \boxtimes \cdots \boxtimes [m_n]) = \Delta[m_1] \boxtimes \cdots \boxtimes \Delta[m_n] = \Delta[m_1, \ldots, m_n].$$

Proposition 2.11. *The functor $N^n\colon$ **nFoldCat**$\longrightarrow$**SSet$^{\mathbf{n}}$** is fully faithful.*

Proof: This follows from the nerve Theorem 4.10 of [89]. For a direct proof in the case $n = 2$, see [26]. □

Proposition 2.12. *The n-fold nerve functor N^n admits a left adjoint c^n called fundamental n-fold category or n-fold categorification.*

Proof: The functor N^n is defined as the singular functor associated to an inclusion. Since **nFoldCat** is cocomplete, a left adjoint to N^n is obtained by left Kan extending along the Yoneda embedding. This is the Lemma from Kan. □

Example 2.13. If $X_1, \ldots, X_n$ are simplicial sets, then

$$c^n(X_1 \boxtimes \cdots \boxtimes X_n) = cX_1 \boxtimes \cdots \boxtimes cX_n$$

where c is ordinary categorification. The symbol $\boxtimes$ on the left means external product of simplicial sets, and the symbol $\boxtimes$ on the right means external product of categories as in Definition 2.5. For a proof in the case $n = 2$, see [26].

Lastly, we consider the behavior of c^n on the image of the left adjoint $\delta_!$. The diagonal functor

$$\delta\colon \Delta \longrightarrow \Delta^n$$

$$[m] \mapsto ([m], \ldots, [m])$$

induces $\delta^*\colon$ **SSet$^{\mathbf{n}}$**$\longrightarrow$**SSet** by precomposition. The functor δ^* admits both a left and right adjoint by Kan extension. The left adjoint $\delta_!$ is uniquely characterized by two properties:

(i) $\delta_!(\Delta[m]) = \Delta[m, \ldots, m]$.
(ii) $\delta_!$ preserves colimits.

Thus,

$$\delta_! X = \delta_!(\operatorname*{colim}_{\Delta[m]\to X} \Delta[m]) = \operatorname*{colim}_{\Delta[m]\to X} \delta_!\Delta[m] = \operatorname*{colim}_{\Delta[m]\to X} \Delta[m, \ldots, m]$$

where the colimit is over the simplex category of the simplicial set X. Further, since c^n preserves colimits, we have

$$c^n\delta_! X = \underset{\Delta[m]\to X}{\operatorname{colim}} c^n\Delta[m,\dots,m] = \underset{\Delta[m]\to X}{\operatorname{colim}} [m]\boxtimes\cdots\boxtimes[m].$$

Clearly, $c^n\delta_!\Delta[m] = [m]\boxtimes\cdots\boxtimes[m]$. The calculation of $c^n\delta_!\mathrm{Sd}^2\Delta[m]$ and $c^n\delta_!\mathrm{Sd}^2\Lambda^k[m]$ is not as simple, because external product does not commute with colimits. We will give a general procedure of calculating the n-fold categorification of nerves of certain posets in Section 7.

3. Barycentric Subdivision and Decomposition of $\mathbf{P}\mathrm{Sd}\Delta[m]$

The adjunction

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}"] \end{tikzcd} \tag{2}$$

between barycentric subdivision Sd and Kan's functor Ex is crucial to Thomason's transfer from **Cat** to **SSet**. We will need a good understanding of subdivision for the Thomason structure on **nFoldCat** as well, so we recall it in this section. Explicit descriptions of certain subsimplices of the double subdivisions $\mathrm{Sd}^2\Lambda^k[m]$, $\mathrm{Sd}^2\partial\Delta[m]$, and $\mathrm{Sd}^2\Delta[m]$ will be especially useful later. In Proposition 3.10, we present a decomposition of the poset $\mathbf{P}\mathrm{Sd}\Delta[m]$, see Figure 1 for the decomposition in the case $m = 2$ and $k = 1$. The nerve of the poset $\mathbf{P}\mathrm{Sd}\Delta[m]$ is of course $\mathrm{Sd}^2\Delta[m]$. This decomposition allows us to describe a deformation retraction of part of $|\mathrm{Sd}^2\Delta[m]|$ in a very controlled way (Proposition 4.3). In particular, each m-subsimplex gets retracted onto one of its faces. This allows us to do a deformation retraction of the n-fold categorifications as well in Corollary 7.11. These preparations are essential for verifying the pushout-axiom in the Lemma from Kan on transfer of model structures.

We begin with a recollection of barycentric subdivision. The simplicial set $\mathrm{Sd}\Delta[m]$ is the nerve of the poset $\mathbf{P}\Delta[m]$ of nondegenerate simplices of $\Delta[m]$. The ordering is the face relation. Recall that the poset $\mathbf{P}\Delta[m]$ is isomorphic to the poset of nonempty subsets of $[m]$ ordered by inclusion. Thus a q-simplex of $\mathrm{Sd}\Delta[m]$ is a tuple $(v_0,\dots,v_q)$ of nonempty subsets of $[m]$ such that v_i is a subset of v_{i+1} for all $0\le i\le q-1$. For example, the tuple

$$(\{0\},\{0,2\},\{0,1,2,3\}) \tag{3}$$

is a 2-simplex of $\mathrm{Sd}\Delta[3]$. A p-simplex u is a face of a q-simplex v in $\mathrm{Sd}\Delta[m]$ if and only if

$$\{u_0, \ldots, u_p\} \subseteq \{v_0, \ldots, v_q\}. \tag{4}$$

For example the 1-simplex

$$(\{0\}, \{0,1,2,3\}) \tag{5}$$

is a face of the 2-simplex in equation (3). A face that is a 0-simplex is called a vertex. The vertices of v are written simply $v_0, \ldots, v_q$. A q-simplex v of $\mathrm{Sd}\Delta[m]$ is nondegenerate if and only if all v_i are distinct. The simplices in equations (3) and (5) are both non-degenerate.

The barycentric subdivision of a general simplicial set K is defined in terms of the barycentric subdivisions $\mathrm{Sd}\Delta[m]$ that we have just recalled.

Definition 3.1. The *barycentric subdivision* of a simplicial set K is

$$\operatorname*{colim}_{\Delta[n] \to K} \mathrm{Sd}\Delta[n]$$

where the colimit is indexed over the category of simplices of K.

The right adjoint to Sd is the Ex functor of Kan, and is defined in level m by

$$(\mathrm{Ex}X)_m = \mathbf{SSet}(\mathrm{Sd}\Delta[m], X).$$

As pointed out on page 311 of [86], there is a particularly simple description of $\mathrm{Sd}K$ whenever K is a classical simplicial complex each of whose simplices has a linearly ordered vertex set compatible with face inclusion. In this case, $\mathrm{Sd}K$ is the nerve of the poset $\mathbf{P}K$ of nondegenerate simplices of K. The cases $K = \mathrm{Sd}\Delta[m], \Lambda^k[m], \mathrm{Sd}\Lambda^k[m], \partial\Delta[m]$, and $\mathrm{Sd}\partial\Delta[m]$ are of particular interest to us.

We first consider the case $K = \mathrm{Sd}\Delta[m]$ in order to describe the simplicial set $\mathrm{Sd}^2\Delta[m]$. This is the nerve of the poset $\mathbf{P}\mathrm{Sd}\Delta[m]$ of nondegenerate simplices of $\mathrm{Sd}\Delta[m]$. A q-simplex of $\mathrm{Sd}^2\Delta[m]$ is a sequence $V = (V_0, \ldots, V_q)$ where each $V_i = (v^i_0, \ldots, v^i_{r_i})$ is a nondegenerate simplex of $\mathrm{Sd}\Delta[m]$ and $V_{i-1} \subseteq V_i$. For example,

$$(\ \ (\{01\}),\ \ (\{0\},\{01\}),\ \ (\{0\},\{01\},\{012\})\ \ ) \tag{6}$$

is a 2-simplex in $\mathrm{Sd}^2\Delta[2]$. A p-simplex U is a face of a q-simplex V in $\mathrm{Sd}^2\Delta[m]$ if and only if

$$\{U_0, \ldots, U_p\} \subseteq \{V_0, \ldots, V_q\}.$$

For example, the 1-simplex

$$(\ \ (\{01\}),\ \ (\{0\},\{01\},\{012\})\ \ ) \tag{7}$$

is a subsimplex of the 2-simplex in equation (6). The vertices of V are $V_0, \ldots, V_q$. A q-simplex V of $\mathrm{Sd}^2\Delta[m]$ is nondegenerate if and only if all V_i are distinct. The simplices in equations (6) and (7) are both non-degenerate. See Figure 1.

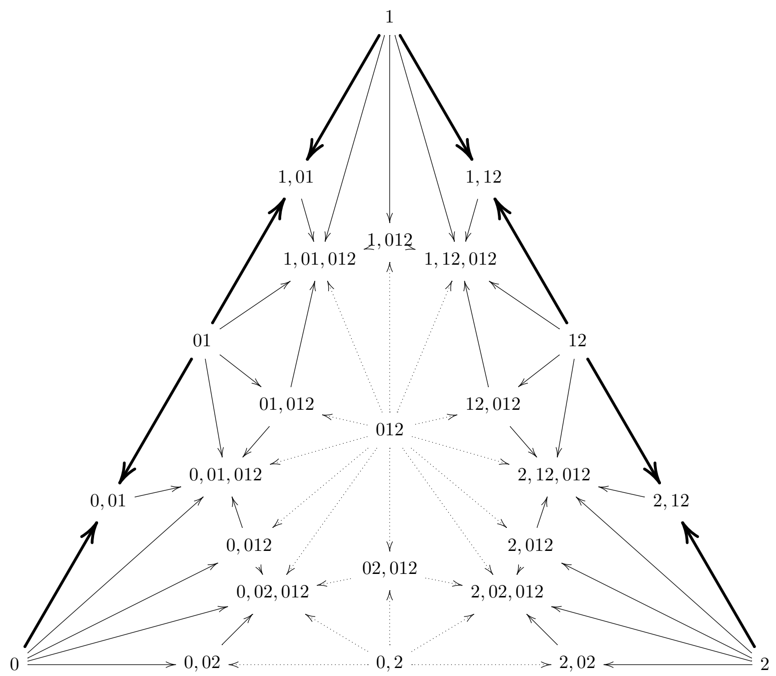

FIGURE 1. Decomposition of the poset $\mathbf{P}\mathrm{Sd}\Delta[2]$. The dark arrows form the poset $\mathbf{P}\Lambda^1[2]$, while its up-closure **Outer** consists of all solid arrows. The poset **Center** consists of all dotted triangles emanating from 012, while **Comp** consists of the four dotted triangles at the bottom. The geometric realization of the dotted part, namely $|N(\mathbf{Comp} \cup \mathbf{Center})|$, is topologically deformation retracted onto the solid part of its boundary.

Next we consider $K = \Lambda^k[m]$ in order to describe $\mathrm{Sd}\Lambda^k[m]$ as the nerve of the poset $\mathbf{P}\Lambda^k[m]$ of nondegenerate simplices of $\Lambda^k[m]$. The simplicial set

$\Lambda^k[m]$ is the smallest simplicial subset of $\Delta[m]$ which contains all nondegenerate simplices of $\Delta[m]$ except the sole m-simplex $1_{[m]}$ and the $(m-1)$-face opposite the vertex $\{k\}$. The n-simplices of $\Lambda^k[m]$ are

$$(\Lambda^k[m])_n = \{\, f : [n] \longrightarrow [m] \,|\, im\; f \neq [m] \text{ and } im\; f \neq [m] \backslash \{k\}\}. \tag{8}$$

A q-simplex $(v_0, \ldots, v_q)$ of $\mathrm{Sd}\Delta[m]$ is in $\mathrm{Sd}\Lambda^k[m]$ if and only if each v_i is a face of $\Lambda^k[m]$. More explicitly, $(v_0, \ldots, v_q)$ is in $\mathrm{Sd}\Lambda^k[m]$ if and only if $|v_q| \leq m$ and in case of equality $k \in v_q$. This follows from equation (8). Similarly, a q-simplex V in $\mathrm{Sd}^2\Delta[m]$ is in $\mathrm{Sd}^2\Lambda^k[m]$ if and only if all v^i_j are faces of $\Lambda^k[m]$. See again Figure 1.

Lastly, we similarly describe $\mathrm{Sd}\partial\Delta[m]$ and $\mathrm{Sd}^2\partial\Delta[m]$. The simplicial set $\partial\Delta[m]$ is the simplicial subset of $\Delta[m]$ obtained by removing the sole m-simplex $1_{[m]}$. A q-simplex $(v_0, \ldots, v_q)$ of $\mathrm{Sd}\Delta[m]$ is in $\mathrm{Sd}\partial\Delta[m]$ if and only if $v_q \neq \{0, 1, \ldots, m\}$. A q-simplex V of $\mathrm{Sd}^2\Delta[m]$ is in $\mathrm{Sd}^2\partial\Delta[m]$ if and only if $v^i_{r_i} \neq \{0, 1, \ldots, m\}$ for all $0 \leq i \leq q$, which is the case if and only if $v^q_{r_q} \neq \{0, 1, \ldots, m\}$. See again Figure 1.

Remark 3.2. Also of interest to us is the way that the nondegenerate m-simplices of $\mathrm{Sd}^2\Delta[m]$ are glued together along their $(m-1)$-subsimplices. In the following, let $V = (V_0, \ldots, V_m)$ be a nondegenerate m-simplex of $\mathrm{Sd}^2\Delta[m]$. See Figure 1 for intuition.

(i) Then $r^i = i$, $|V_i| = i + 1$, and hence also $v^m_m = \{0, 1, \ldots, m\}$.

(ii) If $v^{m-1}_{m-1} \neq \{0, 1, \ldots, m\}$, then the m-th face $(V_0, \ldots, V_{m-1})$ of V is not shared with any other nondegenerate m-simplex V' of $\mathrm{Sd}^2\Delta[m]$.

Proof: If $v^{m-1}_{m-1} \neq \{0, 1, \ldots, m\}$, then the $(m-1)$-simplex $(V_0, \ldots, V_{m-1})$ lies in $\mathrm{Sd}^2\partial\Delta[m]$ by the description of $\mathrm{Sd}^2\partial\Delta[m]$ above, and hence does not lie in any other nondegenerate m-simplex V'.

(iii) If $v^{m-1}_{m-1} = \{0, 1, \ldots, m\}$, then the m-th face $(V_0, \ldots, V_{m-1})$ of V is shared with one other nondegenerate m-simplex V' of $\mathrm{Sd}^2\Delta[m]$.

Proof: If $v^{m-1}_{m-1} = \{0, 1, \ldots, m\}$, then there exists a unique $0 \leq i \leq m-1$ with $v^{m-1}_i \backslash v^{m-1}_{i-1} = \{a, a'\}$ with $a \neq a'$ (since the sequence $v^{m-1}_0, v^{m-1}_1, \ldots, v^{m-1}_{m-1} = \{0, 1, \ldots, m\}$ is strictly ascending). Here we define $v^{m-1}_{i-1} = \emptyset$ whenever $i = 0$. Thus, the $(m-1)$-simplex $(V_0, \ldots, V_{m-1})$ is also a face of the nondegenerate m-simplex V' where

$$V'_\ell = V_\ell \qquad \text{for } 0 \leq \ell \leq m-1$$

$$V'_m = (v^{m-1}_0, \ldots, v^{m-1}_{i-1}, v^{m-1}_{i-1} \cup \{a'\}, v^{m-1}_i, \ldots, v^{m-1}_{m-1}),$$

where we also have

$$V_m = (v_0^{m-1}, \ldots, v_{i-1}^{m-1}, v_{i-1}^{m-1} \cup \{a\}, v_i^{m-1}, \ldots, v_{m-1}^{m-1}).$$

(iv) If $0 \leq j \leq m-1$, then V shares its j-the face $(\ldots, \hat{V}_j, \ldots, V_m)$ with one other nondegenerate m-simplex V' of $\mathrm{Sd}^2\Delta[m]$.

Proof: Since $|V_i| = i+1$, we have $V_{j+1} \backslash V_{j-1} = \{v, v'\}$ with $v \neq v'$ (we define $V_{j-1} = \emptyset$ whenever $j = 0$). Then $(\ldots, \hat{V}_j, \ldots, V_m)$ is shared by the two nondegenerate m-simplices

$$V = (V_0, \ldots, V_{j-1}, V_{j-1} \cup \{v\}, V_{j+1}, \ldots, V_m)$$
$$V' = (V_0, \ldots, V_{j-1}, V_{j-1} \cup \{v'\}, V_{j+1}, \ldots, V_m)$$

and no others.

After this brief discussion of how the nondegenerate m-simplices of $\mathrm{Sd}^2\Delta[m]$ are glued together, we turn to some comments about the relationships between the second subdivisions of $\Lambda^k[m]$, $\partial\Delta[m]$, and $\Delta[m]$. Since the counit $cN \Longrightarrow 1_{\mathbf{Cat}}$ is a natural isomorphism[1], the categories $c\mathrm{Sd}^2\Lambda^k[m]$, $c\mathrm{Sd}^2\partial\Delta[m]$, and $c\mathrm{Sd}^2\Delta[m]$ are respectively the posets $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, $\mathbf{P}\mathrm{Sd}\Delta[m]$, and $\mathbf{P}\mathrm{Sd}\Delta[m]$ of nondegenerate simplices. Moreover, the induced functors

$$c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m] \qquad c\mathrm{Sd}^2\partial\Delta[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]$$

are simply the poset inclusions

$$\mathbf{P}\mathrm{Sd}\Lambda^k[m] \longrightarrow \mathbf{P}\mathrm{Sd}\Delta[m] \qquad \mathbf{P}\mathrm{Sd}\partial\Delta[m] \longrightarrow \mathbf{P}\mathrm{Sd}\Delta[m].$$

The down-closure of $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ in $\mathbf{P}\mathrm{Sd}\Delta[m]$ is easily described.

Proposition 3.3. *The subposet* $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ *of* $\mathbf{P}\mathrm{Sd}\Delta[m]$ *is down-closed.*

Proof: A q-simplex $(v_0, \ldots, v_q)$ of $\mathrm{Sd}\Delta[m]$ is in $\mathrm{Sd}\Lambda^k[m]$ if and only if $|v_q| \leq m$ and in case of equality $k \in v_q$. If $(v_0, \ldots, v_q)$ has this property, then so do all of its subsimplices. □

The rest of this section is dedicated to a decomposition of $\mathbf{P}\mathrm{Sd}\Delta[m]$ into the union of three up-closed subposets: **Comp**, **Center**, and **Outer**. This culminates in Proposition 3.10, and will be used in the construction of the retraction in Section 4 as well as the transfer proofs in Sections 6 and 8. The reader is encouraged to compare with Figure 1 throughout. We begin by describing these posets. The poset **Outer** is the up-closure of $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ in $\mathbf{P}\mathrm{Sd}\Delta[m]$. Although **Outer** depends on k and m, we omit these letters from the notation for readability.

[1]The nerve functor is fully faithful, so the counit is a natural isomorphism by IV.3.1 of [71]

Proposition 3.4. *The smallest up-closed subposet* **Outer** *of* $\mathbf{P}\mathrm{Sd}\Delta[m]$ *which contains* $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ *consists of those* $(v_0, \ldots, v_q) \in \mathbf{P}\mathrm{Sd}\Delta[m]$ *such that there exists a* $(u_0, \ldots, u_p) \in \mathbf{P}\mathrm{Sd}\Lambda^k[m]$ *with*

$$\{u_0, \ldots, u_p\} \subseteq \{v_0, \ldots, v_q\}.$$

Proof: An element of $\mathbf{P}\mathrm{Sd}\Delta[m]$ is in the up-closure of $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ if and only if it lies above some element of $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, and the order is the face relation as in equation (4). $\square$

The following trivial remark will be of use later.

Remark 3.5. Since $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ is down-closed by Proposition 3.3, any morphism of $\mathbf{P}\mathrm{Sd}\Delta[m]$ that ends in $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ must also be contained in $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$. Since **Outer** is the up-closure of the poset $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ in $\mathbf{P}\mathrm{Sd}\Delta[m]$, any morphism that begins in $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ ends in **Outer**.

We can similarly characterize the up-closure **Center** of $(\{0, 1, \ldots, m\})$ in $\mathbf{P}\mathrm{Sd}\Delta[m]$. We call a nondegenerate m-simplex of $\mathrm{Sd}^2\Delta[m]$ a *central m-simplex* if it has $(\{0, 1, \ldots, m\})$ as its 0-th vertex.

Proposition 3.6. *The smallest up-closed subposet* **Center** *of* $\mathbf{P}\mathrm{Sd}\Delta[m]$ *which contains* $(\{0, 1, \ldots, m\})$ *consists of those* $(v_0, \ldots, v_q) \in \mathbf{P}\mathrm{Sd}\Delta[m]$ *such that* $v_q = \{0, 1, \ldots, m\}$. *The nerve* N**Center** *consists of all central m-simplices of* $\mathrm{Sd}^2\Delta[m]$ *and all their faces. A q-simplex* $(V_0, \ldots, V_q)$ *of* $\mathrm{Sd}^2\Delta[m]$ *is in* N**Center** *if and only if* $v^i_{r_i} = \{0, 1, \ldots, m\}$ *for all* $0 \leq i \leq q$.

For example, the 2-simplex

$$(\ (\{012\}),\ (\{01\}, \{012\}),\ (\{0\}, \{01\}, \{012\})\ ) \tag{9}$$

is a central 2-simplex of $\mathrm{Sd}^2\Delta[2]$ and the 1-simplex

$$(\ (\{01\}, \{012\}),\ (\{0\}, \{01\}, \{012\})\ ) \tag{10}$$

is in N**Center**, as it is a face of the 2-simplex in equation (9). A glance at Figure 1 makes all of this apparent.

Remark 3.7. We need to understand more thoroughly the way the central m-simplices are glued together in N**Center**. Suppose V is a central m-simplex, so that $v^i_i = \{0, 1, \ldots, m\}$ for all $0 \leq i \leq m$ by Proposition 3.6. From the description of V' in Remark 3.2 (iii) and (iv), and also Proposition 3.6 again, we see for $j = 1, \ldots, m$ that the neighboring nondegenerate m-simplex V' containing the $(m-1)$-face $(V_0, \ldots, \hat{V}_j, \ldots)$ of V is also central. The face $(V_1, \ldots, V_m)$ of V opposite $V_0 = (\{0, 1, \ldots, m\})$, is not shared with any other central m-simplex as every central m-simplex has $\{0, \ldots, m\}$ as its 0-th vertex. Thus, each central m-simplex V shares exactly m of its

$(m-1)$-faces with other central m-simplices. A glance at Figure 1 shows that the central simplices fit together to form a 2-ball. More generally, the central m-simplices of $\mathrm{Sd}^2\Delta[m]$ fit together to form an m-ball with center vertex $\{0,\ldots,m\}$.

There is still one last piece of $\mathbf{P}\mathrm{Sd}\Delta[m]$ that we discuss, namely **Comp**.

Proposition 3.8. *Let $0 \leq k \leq m$. The smallest up-closed subposet* **Comp** *of $\mathbf{P}\mathrm{Sd}\Delta[m]$ that contains the object $\{0,1,\ldots,\hat{k},\ldots,m\}$ consists of those $(v_0,\ldots,v_q) \in \mathbf{P}\mathrm{Sd}\Delta[m]$ with*

$$\{0,1,\ldots,\hat{k},\ldots,m\} \in \{v_0,\ldots,v_q\}.$$

We describe how the nondegenerate m-simplices of $N\mathbf{Comp}$ are glued together in terms of collections C^ℓ of nondegenerate m-simplices. A nondegenerate m-simplex $V \in N_m\mathbf{P}\mathrm{Sd}\Delta[m]$ is in $N_m\mathbf{Comp}$ if and only if each $V_0,\ldots,V_m$ is in **Comp**, and this is the case if and only if $V_0 = (\{0,\ldots,\hat{k},\ldots,m\})$ (recall $|V_i| = i+1$ and Proposition 3.8). For $1 \leq \ell \leq m$, we let C^ℓ denote the set of those nondegenerate m-simplices V in $N_m\mathbf{Comp}$ which have their first ℓ vertices $V_0,\ldots,V_{\ell-1}$ on the k-th face of $|\Delta[m]|$. A nondegenerate m-simplex $V \in N_m\mathbf{Comp}$ is in C^ℓ if and only if $v_i^i = \{0,\ldots,\hat{k},\ldots,m\}$ for all $0 \leq i \leq \ell-1$ and $v_i^i = \{0,\ldots,m\}$ for all $\ell \leq i \leq m$.

Proposition 3.9. *Let $V \in C^\ell$. Then the j-th face of V is shared with some other $V' \in C^\ell$ if and only if $j \neq 0, \ell-1, \ell$.*

Proof: By Remark 3.2 we know exactly which other nondegenerate m-simplex V' shares the j-th face of V. So, for each ℓ and j we only need to check whether or not V' is in C^ℓ. Let $V \in C^\ell$.

Cases $1 \leq \ell \leq m$ and $j = 0$.
For all $U \in C^\ell$, we have $U_0 = (\{0,\ldots,\hat{k},\ldots,m\}) = V_0$, so we conclude from the description of V' in Remark 3.2 (iv) that V' is not in C^ℓ.

Case $\ell = m$ and $j = m-1$.
In this case, $v_{m-1}^{m-1} = \{0,\ldots,\hat{k},\ldots,m\}$ and $v_m^m = \{0,1,\ldots,m\}$. By Remark 3.2 (iv), the $m-1$st-face of V is shared with the V' which agrees with V everywhere except in V_{m-1}, where we have $(v')_{m-1}^{m-1} = \{0,\ldots,m\}$ instead of $v_{m-1}^{m-1} = \{0,\ldots,\hat{k},\ldots,m\}$. But this V' is not an element of C^m.

Case $\ell = m$ and $j = m$.
In this case, $v_{m-1}^{m-1} = \{0,\ldots,\hat{k},\ldots,m\} \neq \{0,1,\ldots,m\}$, so we are in the situation of Remark 3.2 (ii). The m-th face $(V_0,\ldots,V_{m-1})$ does not lie in any other nondegenerate m-simplex V', let alone in a V' in C^m.

Case $\ell = m$ and $j \neq 0, m-1, m$.
By Remark 3.2 (iv), the j-th face is shared with the V' that agrees with V in V_0, V_{m-1}, and V_m, so that $V' \in C^m$.

At this point we conclude from the above cases that if $\ell = m$, the j-th face of $V \in C^m$ is shared with another $V' \in C^m$ if and only if $j \neq 0, m-1, m$.

Cases $1 \leq \ell \leq m-1$ and $j = \ell - 1$.
The $\ell-1$st face of V is shared with that V' which agrees with V everywhere except in $V_{\ell-1}$, where we have $(v')^{\ell-1}_{\ell-1} = \{0, \ldots, m\}$ instead of $v^{\ell-1}_{\ell-1} = \{0, \ldots, \hat{k}, \ldots, m\}$. Hence V' is not in C^ℓ.

Cases $1 \leq \ell \leq m-1$ and $j = \ell$.
Similarly, the ℓ-th face of V is shared with that V' which agrees with V everywhere except in V_ℓ, where we have $(v')^\ell_\ell = \{0, \ldots, \hat{k}, \ldots, m\}$ instead of $v^\ell_\ell = \{0, \ldots, m\}$. Hence V' is not in C^ℓ.

Cases $1 \leq \ell \leq m-1$ and $j \neq 0, \ell-1, \ell$.
Then the j-th face is shared with a V' that agrees with V in V_0, $V_{\ell-1}$, and V_ℓ, so that $V' \in C^\ell$.

We conclude that the j-th face of $V \in C^\ell$ is shared with some other $V' \in C^\ell$ if and only if $j \neq 0, \ell-1, \ell$. □

Proposition 3.10. *Let* $0 \leq k \leq m$. *Let* **Comp**, **Center**, *and* **Outer** *denote the up-closure in* $\mathbf{P}\mathrm{Sd}\Delta[m]$ *of* $(\{0, 1, \ldots, \hat{k}, \ldots, m\})$, $(\{0, 1, \ldots, m\})$, *and* $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ *respectively.*

Then the poset $\mathbf{P}\mathrm{Sd}\Delta[m] = c\mathrm{Sd}^2\Delta[m]$ *is the union of these three up-closed subposets:*

$$\mathbf{P}\mathrm{Sd}\Delta[m] = \mathbf{Comp} \cup \mathbf{Center} \cup \mathbf{Outer}.$$

4. Deformation Retraction of $|N(\mathbf{Comp} \cup \mathbf{Center})|$

In this section we construct a retraction of $|N(\mathbf{Comp} \cup \mathbf{Center})|$ to that part of its boundary which lies in **Outer**. As stated in Proposition 4.3, each stage of the retraction is part of a deformation retraction, and is thus a homotopy equivalence. The retraction is done in such a way that we can adapt it later to the n-fold case. We first treat the retraction of $|N\mathbf{Comp}|$ in detail.

Proposition 4.1. *Let* $C^m, C^{m-1}, \ldots, C^1$ *be the collections of nondegenerate* m*-simplices of* $N\mathbf{Comp}$ *defined in Section 3. Then there is an* m *stage retraction of* $|N\mathbf{Comp}|$ *onto* $|N(\mathbf{Comp} \cap (\mathbf{Center} \cup \mathbf{Outer}))|$ *which retracts the individual simplices of* $C^m, C^{m-1}, \ldots, C^1$ *to subcomplexes of their boundaries. Further, each retraction of each simplex is part of a deformation retraction.*

Proof: As an illustration, we first prove the case $m = 1$ and $k = 0$. The poset $\mathbf{P}\mathrm{Sd}\Delta[1]$ is

$$(\{0\}) \longrightarrow (\{0\}, \{01\}) \longleftarrow (\{01\}) \longrightarrow (\{1\}, \{01\}) \xleftarrow{f} (\{1\}) .$$

Of these morphisms, the only one in **Outer** is the one on the far left. The poset **Center** consists of the two middle morphisms, emanating from $(\{01\})$. The only morphism in **Comp** is the one labelled f. The intersection $\mathbf{Comp} \cap (\mathbf{Center} \cup \mathbf{Outer})$ is the vertex $(\{1\}, \{01\})$, which is the target of f.

Clearly, after geometrically realizing, the interval $|f|$ can be deformation retracted to the vertex $(\{1\}, \{01\})$. The case $m = 1$ with $k = 1$ is exactly the same. In fact, k does not matter, since the simplices no longer have a direction after geometric realization.

The case $m = 2$ and $k = 1$ can be similarly observed in Figure 1.

For general $m \in \mathbb{N}$, we construct a *topological* retraction in m steps, starting with Step 0. In Step 0 we retract those nondegenerate m-simplices of $N_m\mathbf{Comp}$ which have an entire $m-1$-face on the k-th face of $\Delta[m]$, *i.e.,* in Step 0 we retract the elements of C^m. Generally, in Step ℓ we retract those nondegenerate m-simplices of $N_m\mathbf{Comp}$ which have exactly ℓ vertices on the k-th face of $\Delta[m]$, *i.e.,* in Step ℓ we retract the elements of $C^{m-\ell}$.

We describe Step $m-\ell$ in detail for $2 \leq \ell \leq m$. We retract each $V \in C^\ell$ to

$$(V_0, \dots, \hat{V}_{\ell-1}, V_\ell, \dots) \cup (V_1, \dots, V_m)$$

in such a way that for each $j \neq 0, \ell-1, \ell$ the j-th face

$$(V_0, \dots, \hat{V}_j, \dots, V_{\ell-1}, V_\ell, \dots)$$

is retracted *within itself* to its subcomplex

$$(V_0, \dots, \hat{V}_j, \dots, \hat{V}_{\ell-1}, V_\ell, \dots) \cup (\hat{V}_0, \dots, \hat{V}_j, \dots, V_{\ell-1}, V_\ell, \dots).$$

We can do this to all $V \in C^\ell$ *simultaneously* because the prescription agrees on the overlaps: V shares the face $(V_0, \dots, \hat{V}_j, \dots, V_{\ell-1}, V_\ell, \dots)$ with only one other nondegenerate m-simplex $V' \in C^\ell$, and V' differs from V only in V'_j by Proposition 3.9.

This procedure is done for Step 0 up to and including Step $m-2$. After Step $m-2$, the only remaining nondegenerate m-simplices in $N_m\mathbf{Comp}$ are those which have only the first vertex (*i.e.,* only V_0) on the k-th face of $\Delta[m]$. This is the set C^1.

Every $V \in C^1$ has

$$V_0 = (\{0, \dots, \hat{k}, \dots, m\})$$

$$V_1 = (\{0, \dots, \hat{k}, \dots, m\}, \{0, \dots, m\}),$$

so all $V \in C^1$ intersect in this edge. In Step $m-1$, we retract each $V \in C^1$ to $(V_1, \dots, V_m)$ in such a way that for $j \neq 0, 1$ we retract the j-th face V to $(V_1, \dots, \hat{V}_j, \dots)$, and further we retract the 1-simplex (V_0, V_1) to the vertex V_1. We can do this simultaneously to all $V \in C^1$, as the procedure agrees in overlaps by Proposition 3.9, and the observation about (V_0, V_1) we made above. For each $V \in C^1$, the 0th face $(V_1, \dots, V_m)$ is also the 0th face of a nondegenerate m-simplex U not in $N_m\mathbf{Comp}$, namely

$$U_0 = (\{0, \dots, m\})$$
$$U_j = V_j \text{ for } j \geq 1$$

by Remark 3.2 (iv). The simplex U is even central. Thus, $(V_1, \dots, V_m)$ is in the intersection $|N(\mathbf{Comp} \cap (\mathbf{Center} \cup \mathbf{Outer}))|$ and we have succeeded in retracting $|N\mathbf{Comp}|$ to $|N(\mathbf{Comp} \cap (\mathbf{Center} \cup \mathbf{Outer}))|$ in such a way that each nondegenerate m-simplex is retracted within itself. Further, each retraction is part of a deformation retraction. $\square$

Proposition 4.2. *There is a multi-stage retraction of* $|N\mathbf{Center}|$ *onto* $|N(\mathbf{Center} \cap \mathbf{Outer})|$ *which retracts each nondegenerate m-simplex to a subcomplex of its boundary. Further, this retraction is part of a deformation retraction.*

Proof: One can similarly decompose **Center** and construct retractions as in the proof of Proposition 4.1. $\square$

Proposition 4.3. *There is a multi-stage retraction of* $|N(\mathbf{Comp} \cup \mathbf{Center})|$ *to* $|N((\mathbf{Comp} \cup \mathbf{Center}) \cap \mathbf{Outer})|$ *which retracts each nondegenerate m-simplex to a subcomplex of its boundary. Further, each retraction of each simplex is part of a deformation retraction. See Figure 1.*

Proof: This follows from Proposition 4.1 and Proposition 4.2. $\square$

5. Nerve, Pushouts, and Colimit Decompositions of Subposets of $\mathbf{P}\mathrm{Sd}\Delta[m]$

In this section we express special posets as a colimit of two finite ordinals in Proposition 5.4, and prove the commutation of the nerve with certain colimits in Propositions 5.1 and 5.8. The question of commutation of nerve with certain pushouts is an old one, and has been studied in Section 5 of [29].

Proposition 5.1. *Suppose* $\mathbf{Q}$, $\mathbf{R}$, *and* $\mathbf{S}$ *are categories, and* $\mathbf{S}$ *is a full subcategory of* $\mathbf{Q}$ *and* $\mathbf{R}$ *such that*

(i) *If* $f : x \longrightarrow y$ *is a morphism in* $\mathbf{Q}$ *and* $x \in \mathbf{S}$, *then* $y \in \mathbf{S}$,
(ii) *If* $f : x \longrightarrow y$ *is a morphism in* $\mathbf{R}$ *and* $x \in \mathbf{S}$, *then* $y \in \mathbf{S}$.

Then the nerve of the pushout is the pushout of the nerves.

$$N(\mathbf{Q} \coprod_{\mathbf{S}} \mathbf{R}) \cong N\mathbf{Q} \coprod_{N\mathbf{S}} N\mathbf{R} \tag{11}$$

Proof: First we claim that there are no free composites in $\mathbf{Q}\coprod_{\mathbf{S}}\mathbf{R}$. Suppose f is a morphism in $\mathbf{Q}$ and g is a morphism in $\mathbf{R}$ and that these are composable in the pushout $\mathbf{Q}\coprod_{\mathbf{S}}\mathbf{R}$.

$$w \xrightarrow{f} x \xrightarrow{g} y$$

Then $x \in \mathit{Obj}\ \mathbf{Q} \cap \mathit{Obj}\ \mathbf{R} = \mathbf{S}$, so $y \in \mathbf{S}$ by hypothesis (ii). Since $\mathbf{S}$ is full, g is a morphism of $\mathbf{S}$. Then $g \circ f$ is a morphism in $\mathbf{Q}$ and is not free. The other case f in $\mathbf{R}$ and g in $\mathbf{Q}$ is exactly the same. Thus the pushout $\mathbf{Q}\coprod_{\mathbf{S}}\mathbf{R}$ has no free composites.

Let $(f_1,\ldots,f_p)$ be a p-simplex in $N(\mathbf{Q}\coprod_{\mathbf{S}}\mathbf{R})$. Then each f_j is a morphism in $\mathbf{Q}$ or $\mathbf{R}$, as there are no free composites. Further, by repeated application of the argument above, if f_1 is in $\mathbf{Q}$ then every f_j is in $\mathbf{Q}$. Similarly, if f_1 is in $\mathbf{R}$ then every f_j is in $\mathbf{R}$. Thus we have a morphism $N(\mathbf{Q}\coprod_{\mathbf{S}}\mathbf{R}) \longrightarrow N\mathbf{Q}\coprod_{N\mathbf{S}}N\mathbf{R}$. Its inverse is the canonical morphism $N\mathbf{Q}\coprod_{N\mathbf{S}}N\mathbf{R} \longrightarrow N(\mathbf{Q}\coprod_{\mathbf{S}}\mathbf{R})$. □

Proposition 5.2. *The full subcategory* $(\mathbf{Comp}\cup\mathbf{Center})\cap\mathbf{Outer}$ *of the categories* $\mathbf{Comp}\cup\mathbf{Center}$ *and* $\mathbf{Outer}$ *satisfies (i) and (ii) of Proposition 5.1.*

Proof: We use the decomposition

$$(\mathbf{Comp}\cup\mathbf{Center})\cap\mathbf{Outer} = (\mathbf{Comp}\cap\mathbf{Outer})\cup(\mathbf{Center}\cap\mathbf{Outer}).$$

By Propositions 3.4 and 3.8 an element $x \in \mathbf{P}\mathrm{Sd}\Delta[m]$ is in $\mathbf{Comp}\cap\mathbf{Outer}$ if and only if there exists $u \in \mathbf{P}\mathrm{Sd}\Lambda^k[m]$ with $u \subseteq x$ and $\{0,1,\ldots,\hat{k},\ldots,m\} \in x$. If $x \in \mathbf{Comp}\cap\mathbf{Outer}$ and $x \longrightarrow y$ in $\mathbf{P}\mathrm{Sd}\Delta[m]$, then $u \subseteq y$ and $\{0,1,\ldots,\hat{k},\ldots,m\} \in y$, so that $y \in \mathbf{Comp}\cap\mathbf{Outer}$.

By Propositions 3.4 and 3.6 an element $x \in \mathbf{P}\mathrm{Sd}\Delta[m]$ is in $\mathbf{Center}\cap\mathbf{Outer}$ if and only if there exists $u \in \mathbf{P}\mathrm{Sd}\Lambda^k[m]$ with $u \subseteq x$ and $\{0,1,\ldots,m\} \in x$. If $x \in \mathbf{Center}\cap\mathbf{Outer}$ and $x \longrightarrow y$ in $\mathbf{P}\mathrm{Sd}\Delta[m]$, then $u \subseteq y$ and $\{0,1,\ldots,m\} \in y$, so that $y \in \mathbf{Center}\cap\mathbf{Outer}$. □

Proposition 5.3. *Let* $\mathbf{T}$ *be a subposet of* $\mathbf{P}\mathrm{Sd}\Delta[m]$ *such that the following hold.*

(i) *Any linearly ordered subposet* $U = \{U_0 < U_1 < \cdots < U_p\}$ *of* $\mathbf{T}$ *is contained in a linearly ordered subposet* V *of* $\mathbf{T}$ *with* $m+1$ *distinct elements.*

(ii) *If* $x \in \mathbf{T}$ *is contained in two linearly ordered subposets* V *and* V', *each with* $m+1$ *elements, then there exist linearly ordered subposets* $V^0, V^1, \ldots, V^k$ *of* $\mathbf{T}$ *such that*

(a) $V^0 = V$

(b) $V^k = V'$

(c) *For all* $0 \leq j \leq k$*, the linearly ordered poset* V^j *has* $m+1$ *elements*

(d) *For all* $0 \leq j \leq k$*, we have* $x \in V^j$

(e) *For all* $0 \leq j \leq k-1$*, the poset* $V^j \cap V^{j+1}$ *has* m *elements.*

Let $\mathbf{J}$ *denote the poset of linearly ordered subposets* U *of* $\mathbf{T}$ *with* m *or* $m+1$ *elements. Then* $\mathbf{T}$ *is the colimit of the functor*

$$F : \mathbf{J} \longrightarrow \mathbf{Cat}$$

$$U \longmapsto U.$$

The components of the universal cocone $\pi : F \Longrightarrow \Delta_{\mathbf{T}}$ *are the inclusions* $F(U) \longrightarrow \mathbf{T}$*.*

Proof: Suppose $\mathbf{S} \in \mathbf{Cat}$ and $\alpha : F \Longrightarrow \Delta_{\mathbf{S}}$ is a natural transformation. We define a functor $G : \mathbf{T} \longrightarrow \mathbf{S}$ as follows. If $x \in \mathbf{T}$, then $G(x) := \alpha_V(x)$ where V is any linearly ordered subposet of $\mathbf{T}$ with $m+1$ elements. If V' is another such subposet, then we have a sequence $V_0, \ldots, V_k$ as in hypothesis (ii), and the naturality diagrams below.

$$\begin{tikzcd}
V_i \arrow[r, "\alpha_{V_i}"] & \mathbf{S} \arrow[d, equal] \\
V_i \cap V_{i+1} \arrow[u] \arrow[d] \arrow[r] & \mathbf{S} \arrow[d, equal] \\
V_{i+1} \arrow[r, "\alpha_{V_{i+1}}"'] & \mathbf{S}
\end{tikzcd}$$

Thus we have a chain of equalities

$$\alpha_{V_0}(x) = \alpha_{V_1}(x) = \cdots = \alpha_{V_k}(x),$$

and we conclude $\alpha_V(x) = \alpha_{V'}(x)$ so that $G(x)$ is well defined. The same argument works for defining G on morphisms.

Then for all $U \in \mathbf{J}$, we have $\alpha_U = G \circ \pi_U$. Further, G is the unique such functor, as the posets $U \in \mathbf{J}$ cover $\mathbf{T}$ by hypothesis (i). $\square$

Proposition 5.4. *The posets* $\mathbf{P}\mathrm{Sd}\Delta[m]$*,* $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$*,* **Center***,* **Outer***,* **Comp***, and* **Comp** $\cup$ **Center** *satisfy (i) and (ii) of Proposition 5.3. Thus, each of these posets is a colimit of finite ordinals* $[m-1]$ *and* $[m]$*. Similarly,* **Outer**$\cap$(**Comp**$\cup$**Center**) *is a colimit of finite ordinals* $[m-2]$ *and* $[m-1]$*.*

Proof: We prove the proposition for $\mathbf{P}\mathrm{Sd}\Delta[m]$, the other posets are similar.

(i) As before, we write $U_i = (u_0^i, \ldots, u_{r_i}^i)$. We extend U to a linearly ordered subposet V with $m+1$ elements so that U_i occupies the r_i-th place (the lowest element is in the 0-th place). For $j \leq r_0$, let $V_j = (u_0^0, \ldots, u_j^0)$. For $r_i \leq j \leq r_{i+1}$, we define V_{j+1} as V_j with one additional element of $U_{i+1} \backslash U_i$. If $|U_p| = m+1$, then we are now finished. If $|U_p| < m+1$, then choose any maximal strictly increasing sequence of subsets

$$u_{r_p}^p \subsetneq v_1 \subsetneq v_2 \subseteq v_k = \{0, 1, \ldots, m\}$$

and define

$$V_{r_p+j} := V_{r_p} \cup \{v_1, \ldots, v_j\}.$$

(ii) The star of any vertex x in the simplicial complex $N\mathbf{P}\mathrm{Sd}\Delta[m]$ is a triangulated m-ball with x at the center and m-simplices radiating out. These m-simplices are glued together along $(m-1)$-subsimplices (for example, see Remark 3.2 and Figure 1). Since V and V' are part of this triangulation, and the m-ball is connected, we can move from V to V' via pairs of m-simplices that share an $m-1$-subsimplex. $\square$

Remark 5.5. The posets $\mathbf{C}^\ell$ do not satisfy (i) and (ii).

Remark 5.6. The nerve functor commutes with a colimit whenever the underlying graph of the colimit is the colimit of the underlying graphs and all relations in the colimit come from relations in the constituent categories. Recall that the colimit of a functor $F\colon \mathbf{J} \longrightarrow \mathbf{Cat}$ is calculated by constructing the colimit of the underlying graphs, building the free category on this, and then modding out by all relations in the constituent categories. Thus, the nerve commutes with the colimit of F when the following condition holds.

- If $f_1\colon a_1 \longrightarrow b_1$ is in $F(U_1)$ and $g_2\colon b_2 \longrightarrow c_2$ is in $F(U_2)$, and $b_1 \sim b_2$, then there exists some $F(U)$ containing
$$a \xrightarrow{f} b \xrightarrow{g} c$$
with $f \sim f_1$ and $g \sim g_2$.

Remark 5.7. Recall that a colimit of a functor $F\colon \mathbf{J} \longrightarrow \mathbf{Set}$ is given by

$$C = (\coprod_{U \in \mathbf{J}} FU)/\sim$$

where $\sim$ is the smallest equivalence relation containing the basic relation $(a_1 \in FU_1) \sim (a_2 \in FU_2) :\Longleftrightarrow$ there exists $U \in \mathbf{J}$ and there exist $p_1 : U_1 \longrightarrow U$ and $p_2 : U_2 \longrightarrow U$ such that $Fp_1(a_1) = Fp_2(a_2)$. Compare with Proposition 2.13.3 of [7].

Proposition 5.8. *Let* $\mathbf{T}$ *and* F *be as in Proposition 5.3. Then*

$$N(\operatorname*{colim}_{\mathbf{J}} F) = \operatorname*{colim}_{\mathbf{J}} (N \circ F).$$

Proof: The relation in Remark 5.7 is clearly reflexive and symmetric. However, in the situation of Proposition 5.3 it is not transitive. Since $\mathbf{J}$ is finite and F takes values in finite categories, two objects (or arrows) are equivalent if and only if they can be connected by a finite string of basically related objects (or arrows).

In the special situation of Proposition 5.3, we can see that two objects $b_1 \in FU_1$ and $b_2 \in FU_2$ are equivalent if and only if they are the same element b in $\mathbf{T}$, and similarly for arrows. Thus, if $a_1 < b_1$ in FU_1 and $b_2 < c_1$ in FU_2, and $b_1 \sim b_2$, then by hypothesis (i) there exists some $U \in \mathbf{J}$ containing

$$a_1 < b_1 = b = b_2 < c_2.$$

By Remark 5.6 we conclude that the nerve commutes with colimits in the situation of Proposition 5.3. $\square$

Corollary 5.9. *The simplicial sets* $N(\mathbf{P}\mathrm{Sd}\Delta[m])$*,* $N(\mathbf{P}\mathrm{Sd}\Lambda^k[m])$*,* $N(\mathbf{Center})$*,* $N(\mathbf{Outer})$*,* $N(\mathbf{Comp})$*, and* $N(\mathbf{Comp} \cup \mathbf{Center})$ *are each a colimit of simplicial sets of the form* $\Delta[m]$ *and* $\Delta[m-1]$*. Similarly, the simplicial set* $N(\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center}))$ *is a colimit of simplicial sets of the form* $\Delta[m-2]$ *and* $\Delta[m-1]$*.*

Proof: This follows from Propositions 5.3, 5.4, and 5.8. $\square$

6. Thomason Structure on Cat

The Thomason structure on $\mathbf{Cat}$ is transferred from the standard model structure on $\mathbf{SSet}$ by transferring across the adjunction

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}^2"] \arrow[r, bend left, "c"] \arrow[r, phantom, "\perp"] & \mathbf{Cat} \arrow[l, bend left, "N"] \end{tikzcd} \tag{12}$$

as in [86]. In other words, a functor F in **Cat** is a weak equivalence or fibration if and only if $\mathrm{Ex}^2 NF$ is. We present a quick proof that this defines a model structure using a corollary to Kan's Lemma on Transfer. Although Thomason did not do it exactly this way, it is practically the same, in spirit. Our proof relies on the results in the previous sections: the decomposition of $\mathrm{Sd}^2\Delta[m]$, the commutation of nerve with certain colimits, and the deformation retraction.

This proof of the Thomason structure on **Cat** will be the basis for our proof of the Thomason structure on **nFoldCat**. The key corollary to Kan's Lemma on Transfer is the following.

Corollary 6.1 (Proposition 3.4.1 in [90])**.** *Let* $\mathbf{C}$ *be a cofibrantly generated model category with generating cofibrations* I *and generating acyclic cofibrations* J*. Suppose* $\mathbf{D}$ *is complete and cocomplete, and that* $F \dashv G$ *is an adjunction as in* (13).

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd} \tag{13}$$

Assume the following.

(i) *For every* $i \in I$ *and* $j \in J$*, the objects dom* Fi *and dom* Fj *are small with respect to the entire category* $\mathbf{D}$.

(ii) *For any ordinal* λ *and any colimit preserving functor* $X\colon \lambda \longrightarrow \mathbf{C}$ *such that* $X_\beta \longrightarrow X_{\beta+1}$ *is a weak equivalence, the transfinite composition*
$$X_0 \longrightarrow \operatorname*{colim}_{\lambda} X$$
is a weak equivalence.

(iii) *For any ordinal* λ *and any colimit preserving functor* $Y\colon \lambda \longrightarrow \mathbf{D}$*, the functor* G *preserves the colimit of* Y.

(iv) *If* j' *is a pushout of* $F(j)$ *in* $\mathbf{D}$ *for* $j \in J$*, then* $G(j')$ *is a weak equivalence in* $\mathbf{C}$.

Then there exists a cofibrantly generated model structure on $\mathbf{D}$ *with generating cofibrations* FI *and generating acyclic cofibrations* FJ*. Further,* f *is a weak equivalence in* $\mathbf{D}$ *if and only* $G(f)$ *is a weak equivalence in* $\mathbf{C}$*, and* f *is a fibration in* $\mathbf{D}$ *if and only* $G(f)$ *is a fibration in* $\mathbf{C}$.

Proof: For a proof, see [27]. It is a straightforward application of the Lemma from Kan. □

We may now prove Thomason's Theorem.

Theorem 6.2. *A model structure on* **Cat** *is given if we declare a functor F to be a weak equivalence or fibration if and only if* $\mathrm{Ex}^2 NF$ *is a weak equivalence or fibration in* **SSet**. *This model structure is cofibrantly generated with generating cofibrations*

$$\{\, c\mathrm{Sd}^2\partial\Delta[m] \longrightarrow c\mathrm{Sd}^2\Delta[m] \mid m \geq 0\}$$

and generating acyclic cofibrations

$$\{\, c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m] \mid 0 \leq k \leq m \text{ and } m \geq 1\}.$$

These functors were explicitly described in Section 3.

Proof:

(i) The categories $c\mathrm{Sd}^2\partial\Delta[m]$ and $c\mathrm{Sd}^2\Lambda^k[m]$ each have a finite number of morphisms, hence they are finite, and are small with respect to **Cat**. For a proof, see Proposition 7.6 of [27].

(ii) The model category **SSet** is cofibrantly generated, and the domains and codomains of the generating cofibrations and generating acyclic cofibrations are finite. By Corollary 7.4.2 in [46], this implies that transfinite compositions of weak equivalences in **SSet** are weak equivalences.

(iii) The nerve functor preserves filtered colimits. Every ordinal is filtered, so the nerve functor preserves λ-sequences.

The Ex functor preserves colimits of λ-sequences as well. We use the idea in the proof of Theorem 4.5.1 of [90]. First recall that for each m, the simplicial set $\mathrm{Sd}\Delta[m]$ is finite, so that $\mathbf{SSet}(\mathrm{Sd}\Delta[m], -)$ preserve colimits of all λ-sequences. If $Y\colon \lambda \longrightarrow \mathbf{SSet}$ is a λ-sequence, then

$$\begin{aligned}(\mathrm{Ex}\ \underset{\lambda}{\mathrm{colim}}\ Y)_m &= \mathbf{SSet}(\mathrm{Sd}\Delta[m], \underset{\lambda}{\mathrm{colim}}\ Y)\\ &\cong \underset{\lambda}{\mathrm{colim}}\ \mathbf{SSet}(\mathrm{Sd}\Delta[m], Y)\\ &\cong (\underset{\lambda}{\mathrm{colim}}\ \mathrm{Ex}Y)_m.\end{aligned}$$

Colimits in **SSet** are formed pointwise, we see that Ex preserves λ-sequences.

Thus $\mathrm{Ex}^2 N$ preserves λ-sequences.

(iv) Let $j\colon \Lambda^k[m] \longrightarrow \Delta[m]$ be a generating acyclic cofibration for **SSet**. Let the functor j' be the pushout along L as in the following diagram.

$$\begin{tikzcd} c\mathrm{Sd}^2\Lambda^k[m] \arrow[r, "L"] \arrow[d, "c\mathrm{Sd}^2 j"'] & \mathbf{B} \arrow[d, "j'"] \\ c\mathrm{Sd}^2\Delta[m] \arrow[r] & \mathbf{P} \end{tikzcd}$$

We factor j' into two inclusions

$$\mathbf{B} \xrightarrow{i} \mathbf{Q} \longrightarrow \mathbf{P}$$

and show that the nerve of each is a weak equivalence.

By Remark 3.5 the only free composites that occur in the pushout $\mathbf{P}$ are of the form (f_1, f_2)

$$\xrightarrow{f_1}\ \xrightarrow{f_2}$$

where f_1 is a morphism in $\mathbf{B}$ and f_2 is a morphism of **Outer** with source in $c\mathrm{Sd}^2\Lambda^k[m]$ and target outside of $c\mathrm{Sd}^2\Lambda^k[m]$ (see for example the drawing of $c\mathrm{Sd}^2\Delta[m]$). Hence, $\mathbf{P}$ is the union

$$\mathbf{P} = \overbrace{(\mathbf{B} \coprod_{c\mathrm{Sd}^2\Lambda^k[m]} \mathbf{Outer})}^{\mathbf{Q}} \cup \overbrace{(\mathbf{Comp} \cup \mathbf{Center})}^{\mathbf{R}} \tag{14}$$

by Proposition 3.10, all free composites are in $\mathbf{Q}$, and they have the form (f_1, f_2).

We claim that the nerve of the inclusion $i\colon \mathbf{B} \longrightarrow \mathbf{Q}$ is a weak equivalence. Let $r\colon \mathbf{Q} \longrightarrow \mathbf{B}$ be the identity on $\mathbf{B}$, and for any $(v_0, \ldots, v_q) \in \mathbf{Outer}$ we define $r(v_0, \ldots, v_q) = (u_0, \ldots, u_p)$ where $(u_0, \ldots, u_p)$ is the maximal subset

$$\{u_0, \ldots, u_p\} \subseteq \{v_0, \ldots, v_q\}$$

that is in $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ (recall Proposition 3.4). On free composites in $\mathbf{Q}$ we then have $r(f_1, f_2) = (f_1, r(f_2))$. Then $ri = 1_{\mathbf{B}}$, and there is a unique natural transformation $ir \Longrightarrow 1_{\mathbf{Q}}$ which is the identity on the objects of $\mathbf{B}$. Thus $|Ni|\colon |N\mathbf{B}| \longrightarrow |N\mathbf{Q}|$ includes $|N\mathbf{B}|$ as a deformation retract of $|N\mathbf{Q}|$.

Next we show that the nerve of the inclusion $\mathbf{Q} \longrightarrow \mathbf{P}$ is also a weak equivalence. The intersection of $\mathbf{Q}$ and $\mathbf{R}$ in (14) is equal to

$$\mathbf{S} = \mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center}).$$

Proposition 5.2 then implies that $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ satisfy the hypotheses of Proposition 5.1. Then

$$\begin{aligned}
|N\mathbf{Q}| &\cong |N\mathbf{Q}| \coprod_{|N\mathbf{S}|} |N\mathbf{S}| \text{ (pushout along identity)} \\
&\simeq |N\mathbf{Q}| \coprod_{|N\mathbf{S}|} |N\mathbf{R}| \text{ (Prop. 4.3 and Gluing Lemma)} \\
&\cong |N\mathbf{Q} \coprod_{N\mathbf{S}} N\mathbf{R}| \text{ (realization is a left adjoint)} \\
&\cong |N(\mathbf{Q} \coprod_{\mathbf{S}} \mathbf{R})| \text{ (Prop. 5.1 and Prop. 5.2)} \\
&= |N\mathbf{P}|.
\end{aligned} \tag{15}$$

In the second line, for the application of the Gluing Lemma (Lemma 8.12 in [32] or Proposition 13.5.4 in [45]), we use two identities and the inclusion $|N\mathbf{S}| \longrightarrow |N\mathbf{R}|$. It is a homotopy equivalence whose inverse is the retraction in Proposition 4.3. We conclude that the inclusion $|N\mathbf{Q}| \longrightarrow |N\mathbf{P}|$ is a weak homotopy equivalence, as it is the composite of the morphisms in equation (15).

We conclude that $|Nj'|$ is the composite of two weak equivalences

$$|N\mathbf{B}| \xrightarrow{|Ni|} |N\mathbf{Q}| \longrightarrow |N\mathbf{P}|$$

and is therefore a weak equivalence. As Ex preserves weak equivalences, $\mathrm{Ex}^2 Nj'$ is also a weak equivalence of simplicial sets. Part (iv) of Corollary 6.1 then holds, and we have the Thomason model structure on $\mathbf{Cat}$. □

7. Pushouts and Colimit Decompositions of $c^n \delta_! \mathrm{Sd}^2 \Delta[m]$

Next we enhance the proof of the $\mathbf{Cat}$-case to obtain the $\mathbf{nFoldCat}$-case. The preparations of Section 3, 4, and 5 are adapted in this section to n-fold categorification.

Proposition 7.1. *Let* $d^i : [m-1] \longrightarrow [m]$ *be the injective order preserving map which skips* i. *Then the pushout in* **nFoldCat**

$$\begin{tikzcd}[m-1] \boxtimes \cdots \boxtimes [m-1] \arrow[r, "d^i \boxtimes \cdots \boxtimes d^i"] \arrow[d, "d^i \boxtimes \cdots \boxtimes d^i"'] & {[m] \boxtimes \cdots \boxtimes [m]} \arrow[d] \\ {[m] \boxtimes \cdots \boxtimes [m]} \arrow[r] & \mathbb{P}\end{tikzcd} \tag{16}$$

does not have any free composites, and is an n*-fold poset.*

Proof: We do the proof for $n = 2$.

We consider horizontal morphisms, the proof for vertical morphisms and more generally squares is similar. We denote the two copies of $[m] \boxtimes [m]$ by $\mathbb{N}_1$ and $\mathbb{N}_2$ for convenience. A free composite occurs whenever there are

$$f_1 \colon A_1 \longrightarrow B_1$$

$$g_2 \colon B_2 \longrightarrow C_2$$

in $\mathbb{N}_1$ and $\mathbb{N}_2$ respectively such that B_1 and B_2 are identified in the pushout, and further, the images of $[m-1] \boxtimes [m-1]$ contain neither f_1 nor g_2. Inspection of $d^i \boxtimes d^i$ shows that this does not occur. $\square$

Remark 7.2. The gluings of Proposition 7.1 are the only kinds of gluings that occur in $c^n \delta_! \mathrm{Sd}^2 \Delta[m]$ and $c^n \delta_! \mathrm{Sd} \Lambda^k[m]$ because of the description of glued simplices in Remark 3.2 and the fact that $c^n \delta_!$ is a left adjoint.

Corollary 7.3. *Consider the pushout* $\mathbb{P}$ *in Proposition 7.1. The application of* $\delta^* N^n$ *to Diagram (16) is a pushout and is drawn in Diagram (17).*

$$\begin{tikzcd}\Delta[m-1] \times \cdots \times \Delta[m-1] \arrow[r, "\delta^* N^n(d^i \boxtimes \cdots \boxtimes d^i)"] \arrow[d, "\delta^* N^n(d^i \boxtimes \cdots \boxtimes d^i)"'] & \Delta[m] \times \cdots \times \Delta[m] \arrow[d] \\ \Delta[m] \times \cdots \times \Delta[m] \arrow[r] & \delta^* N^n \mathbb{P}\end{tikzcd} \tag{17}$$

Proof: The functor N^n preserves a pushout whenever there are no free composites in that pushout, which is the case here by Proposition 7.1. Also, $\delta_!$ is a left adjoint, so it preserves any pushout. $\square$

The n-fold version of Proposition 5.3 is as follows.

Proposition 7.4. *Let* $\mathbf{T}$ *and* F *be as in Proposition 5.3. In particular,* $\mathbf{T}$ *could be* $\mathbf{P}\mathrm{Sd}\Delta[m], \mathbf{P}\mathrm{Sd}\Lambda^k[m], \mathbf{Center}, \mathbf{Outer}, \mathbf{Comp}$ *or* $\mathbf{Comp} \cup \mathbf{Center}$ *by Proposition 5.4. Then* $c^n\delta_! N\mathbf{T}$ *is the union*

$$c^n\delta_! N\mathbf{T} = \bigcup_{\substack{U \subseteq \mathbf{T} \text{ lin. ord.} \\ |U|=m+1}} U \boxtimes U \boxtimes \cdots \boxtimes U. \tag{18}$$

Similarly, if $\mathbf{S} = \mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center})$*, then*

$$c^n\delta_! N\mathbf{S} = \bigcup_{\substack{U \subseteq \mathbf{S} \text{ lin. ord.} \\ |U|=m}} U \boxtimes U \boxtimes \cdots \boxtimes U,$$

also by Proposition 5.4.

Proof: For any linearly ordered subposet U of $\mathbf{T}$ we have

$$\begin{aligned} c^n\delta_! NU &= c^n(NU \boxtimes NU \boxtimes \cdots \boxtimes NU) \\ &= cNU \boxtimes cNU \boxtimes \cdots \boxtimes cNU \\ &= U \boxtimes U \boxtimes \cdots \boxtimes U. \end{aligned}$$

Thus we have

$$\begin{aligned} c^n\delta_! N\mathbf{T} &= c^n\delta_! N(\operatorname*{colim}_{\mathbf{J}} F) \text{ by Proposition 5.3} \\ &= c^n\delta_!(\operatorname*{colim}_{\mathbf{J}} NF) \text{ by Proposition 5.8} \\ &= \operatorname*{colim}_{\mathbf{J}} c^n\delta_! NF \text{ because } c^n\delta_! \text{ is a left adjoint} \\ &= \operatorname*{colim}_{U \in \mathbf{J}} U \boxtimes U \boxtimes \cdots \boxtimes U \\ &= \bigcup_{\substack{U \subseteq \mathbf{T} \text{ lin. ord.} \\ |U|=m+1}} U \boxtimes U \boxtimes \cdots \boxtimes U. \end{aligned}$$

This last equality follows for the same reason that $\mathbf{T}$ (=colimit of F) is the union of the linearly ordered subposets U of $\mathbf{T}$ with exactly $m+1$ elements. See also Proposition 7.1. $\square$

Remark 7.5. Note that

$$\mathbf{T} \boxtimes \mathbf{T} \boxtimes \cdots \boxtimes \mathbf{T} \supsetneq \bigcup_{\substack{U \subseteq \mathbf{T} \text{ lin. ord.} \\ |U|=m+1}} U \boxtimes U \boxtimes \cdots \boxtimes U.$$

Remark 7.6. Note that $\mathbf{T} \boxtimes \mathbf{T} \boxtimes \cdots \boxtimes \mathbf{T}$ is an *n-fold poset*, in other words, given any two objects A and B, there is at most one square with A in the $(0, 0, \ldots, 0)$-corner and B in the $(1, 1, \ldots, 1)$-corner. In particular, each of the underlying 1-categories is a poset. Since $\mathbf{T} \boxtimes \mathbf{T} \boxtimes \cdots \boxtimes \mathbf{T}$ contains $c^n\delta_! N\mathbf{T}$, the n-fold category $c^n\delta_! N\mathbf{T}$ is also an n-fold poset.

Proposition 7.7. *The n-fold poset $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ is down-closed in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Delta[m]$.*

Proof: If $(a_1, a_2, \ldots, a_n)$ is below an object $(b_1, b_2, \ldots, b_n)$ of the n-fold poset $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, then each a_i is below an object b_i of $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, and is therefore in $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ by Proposition 3.3. □

Proposition 7.8. *The up-closure of $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Delta[m]$ is $c^n\delta_! N\mathbf{Outer}$.*

Proof: An explicit description of all three n-fold posets is given in Proposition 7.4. An object $(x_1, x_2, \ldots, x_n)$ of $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Delta[m]$ is in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ (respectively $c^n\delta_! N\mathbf{Outer}$) if and only if each x_i is. An object $(b_1, b_2, \ldots, b_n)$ of $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Delta[m]$ is above $(a_1, a_2, \ldots, a_n)$ if and only if $a_i \leq b_i$ for all $1 \leq i \leq n$. Thus, an object $(b_1, b_2, \ldots, b_n)$ is above an object in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ if and only if each b_i is above an object of $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, or equivalently $(b_1, b_2, \ldots, b_n)$ is in $c^n\delta_! N\mathbf{Outer}$. □

Remark 7.9.
- (i) If α is an n-cube in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Delta[m]$ whose i-th target is in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, then α is in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$.
- (ii) If α is an n-cube in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Delta[m]$ whose i-th source is in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, then α is in $c^n\delta_! N\mathbf{Outer}$.

Proof:

- (i) If α is an n-cube in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Delta[m]$ whose i-th target is in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, then its $(1, 1, \ldots, 1)$-corner is in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, as this corner lies on the i-th target. By Proposition 7.7, we then have α is in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$.
- (ii) If α is an n-cube in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Delta[m]$ whose i-th source is in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, then the $(0, 0, \ldots, 0)$-corner is in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$, as this corner lies on the i-th source. By Proposition 7.8, we then have α is in $c^n\delta_! N\mathbf{Outer}$. □

Next we describe the diagonal of the nerve of certain n-fold categories as an n-fold product of standard simplices.

Proposition 7.10. *Let $\mathbf{T}$ and F be as in Proposition 5.3. In particular, $\mathbf{T}$ could be $\mathbf{P}\mathrm{Sd}\Delta[m], \mathbf{P}\mathrm{Sd}\Lambda^k[m], \mathbf{Center}, \mathbf{Outer}, \mathbf{Comp}$ or $\mathbf{Comp} \cup \mathbf{Center}$ by Proposition 5.4. Then*

$$\begin{aligned}\delta^* N^n c^n \delta_! N\mathbf{T} &= \delta^* N^n c^n \delta_! (\operatorname*{colim}_{\mathbf{J}} NF) \\ &= \operatorname*{colim}_{\mathbf{J}} (NF \times \cdots \times NF)\end{aligned}$$

where $NF(U)$ *is isomorphic to* $\Delta[m]$ *or* $\Delta[m-1]$ *for all* $U \in \mathbf{J}$*. Similarly,* $\delta^* N^n c^n \delta_! N(\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center}))$ *is a colimit of simplicial sets of the form* $\Delta[m-2] \times \cdots \times \Delta[m-2]$ *and* $\Delta[m-1] \times \cdots \times \Delta[m-1]$*.*

Proof: Let $G = c^n \delta_! NF \colon \mathbf{J} \longrightarrow \mathbf{nFoldCat}$. Note that $G(U) = U \boxtimes U \boxtimes \cdots \boxtimes U$. First we claim that the nerve N^n commutes with the colimit of n-fold categories

$$\operatorname*{colim}_{\mathbf{J}} c^n \delta_! NF = \operatorname*{colim}_{\mathbf{J}} G.$$

Let α be the n-cube in $G(U_1)$ determined by the two vertices $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. Let β be the n-cube in $G(U_2)$ determined by the two vertices $(b_1, a_2, \ldots, a_n)$ and $(c_1, b_2, \ldots, b_n)$. Then the subsets

$$\{a_1, \ldots, a_n, b_1, \ldots, b_n\} \subseteq U_1$$

$$\{b_1, a_2, \ldots, a_n, c_1, b_2, \ldots, b_n\} \subseteq U_2$$

are linearly ordered posets as they are subsets of linearly ordered posets. Also, $a_1 \leq b_1 \leq c_1$, so c_1 is also comparable to a_1. Thus the subset

$$\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c_1\} \subseteq \mathbf{T} \tag{19}$$

is linearly ordered. By hypothesis (i), there exists a linearly ordered subset $V \subseteq \mathbf{T}$ containing (19) with $|V| = m + 1$. Hence, the squares α and β are composable in $G(V)$, and N^n commutes with the colimit of G by Remark 5.6.

We have

$$\begin{aligned}
\delta^* N^n c^n \delta_! N\mathbf{T} &= \delta^* N^n c^n \delta_! N(\operatorname*{colim}_{\mathbf{J}} F) \text{ by Proposition 5.3} \\
&= \delta^* N^n c^n \delta_! (\operatorname*{colim}_{\mathbf{J}} NF) \text{ by Proposition 5.8} \\
&= \delta^* N^n (\operatorname*{colim}_{\mathbf{J}} c^n \delta_! NF) \text{ because } c^n \delta_! \text{ is a left adjoint} \\
&= \delta^* (\operatorname*{colim}_{\mathbf{J}} N^n c^n \delta_! NF) \text{ by the above} \\
&= \operatorname*{colim}_{\mathbf{J}} \delta^* N^n c^n \delta_! NF \text{ because } \delta^* \text{ is a left adjoint} \\
&= \operatorname*{colim}_{\mathbf{J}} (NF \times \cdots \times NF).
\end{aligned}$$

The last equality follows because every $U \in \mathit{Obj}\, \mathbf{J}$ is a finite linearly ordered poset so that

$$\begin{aligned}
\delta^* N^n c^n \delta_! NU &= \delta^* N^n c^n (NU \boxtimes NU \boxtimes \cdots \boxtimes NU) \\
&= \delta^* N^n (cNU \boxtimes cNU \boxtimes \cdots \boxtimes cNU) \\
&= \delta^* N^n (U \boxtimes U \boxtimes \cdots \boxtimes U) \\
&= \delta^* (NU \boxtimes NU \boxtimes \cdots \boxtimes NU) \\
&= NU \times NU \times \cdots \times NU.
\end{aligned}$$

□

The n-fold version of Proposition 4.3 is the following.

Corollary 7.11. *The space* $|\delta^* N^n c^n \delta_!(\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center}))|$ *includes into the space* $|\delta^* N^n c^n \delta_!(\mathbf{Comp} \cup \mathbf{Center})|$ *as a deformation retract.*

Proof: Recall that realization $|\cdot|$ commutes with colimits, since it is a left adjoint, and that $|\cdot|$ also commutes with products. We do the multi-stage deformation retraction of Proposition 4.3 to each factor $|\Delta[m]|$ of $|\Delta[m]| \times \cdots \times |\Delta[m]|$ in the colimit of Proposition 7.10. This is the desired deformation retraction of $|\delta^* N^n c^n \delta_!(\mathbf{Comp} \cup \mathbf{Center})|$ to $|\delta^* N^n c^n \delta_!(\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center}))|$. □

Proposition 7.12. *Consider* $n = 2$. *Let* $j \colon \Lambda^k[m] \longrightarrow \Delta[m]$ *be a generating acyclic cofibration for* **SSet**, $\mathbb{B}$ *a double category, and* L *a double functor as below. Then the pushout* $\mathbb{P}$ *in the diagram*

$$\begin{tikzcd} c^n \delta_! \mathrm{Sd}^2 \Lambda^k[m] \arrow[r, "L"] \arrow[d, "c^n \delta_! \mathrm{Sd}^2 j"'] & \mathbb{B} \arrow[d] \\ c^n \delta_! \mathbf{Outer} \arrow[r] & \mathbb{P} \end{tikzcd} \tag{20}$$

has the following form.

(i) *The object set of* $\mathbb{P}$ *is the pushout of the object sets.*

(ii) *The set of horizontal morphisms of* $\mathbb{P}$ *consists of the set of horizontal morphisms of* $\mathbb{B}$, *the set of horizontal morphisms of* $c^n \delta_! \mathbf{Outer}$, *and the set of formal composites of the form*

$$\xrightarrow{f_1} \xrightarrow{(1, f_2)}$$

where f_1 *is a horizontal morphism in* $\mathbb{B}$, f_2 *is a morphism in* **Outer**, *and the target of* f_1 *is the source of* $(1, f_2)$ *in* $Obj\, \mathbb{P}$.

(iii) *The set of vertical morphisms of* $\mathbb{P}$ *consists of the set of vertical morphisms of* $\mathbb{B}$, *the set of vertical morphisms of* $c^n \delta_! \mathbf{Outer}$, *and* *the set of formal composites of the form*

$$\begin{tikzcd} {} \arrow[d, "g_1"] \\ {} \\ {} \arrow[d, "(g_2{,}1)"] \\ {} \end{tikzcd}$$

where g_1 *is a vertical morphism in* $\mathbb{B}$, g_2 *is a morphism in* **Outer**, *and the target of* g_1 *is the source of* $(g_2, 1)$ *in* $Obj\,\mathbb{P}$.

(iv) *The set of squares of* $\mathbb{P}$ *consists of the set of squares of* $\mathbb{B}$, *the set of squares of* $c^n\delta_!$**Outer**, *and the set of formal composites of the following three forms.*

(a)

$$\begin{tikzcd} {} \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\alpha_1"] & (W,A') \arrow[r, "(1_W{,}f_2)"] \arrow[d, "(g{,}1_{A'})"] & (W,B') \arrow[d, "(g{,}1_{B'})"] \\ {} \arrow[r, "p_1"'] & (A,A') \arrow[r, "(1_A{,}f_2)"'] & (A,B') \end{tikzcd}$$

(b)

$$\begin{tikzcd} {} \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\beta_1"] & {} \arrow[d, "q_1"] \\ (A,W') \arrow[r, "(1_A{,}f)"] \arrow[d, "(g_2{,}1_{W'})"'] & (A,A') \arrow[d, "(g_2{,}1_{A'})"] \\ (B,W') \arrow[r, "(1_B{,}f)"'] & (B,A') \end{tikzcd}$$

(c)

$$
\begin{tikzcd}
{} \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\gamma_1"] & (W,A') \arrow[r, "{(1_W,f_2)}"] \arrow[d, "{(g,1_{A'})}"] & (W,B') \arrow[d, "{(g,1_{B'})}"] \\
(A,W') \arrow[r, "{(1_A,f)}"] \arrow[d, "{(g_2,1_{W'})}"'] & (A,A') \arrow[r, "{(1_A,f_2)}"] \arrow[d, "{(g_2,1_{A'})}"] & (A,B') \arrow[d, "{(g_2,1_{B'})}"] \\
(B,W') \arrow[r, "{(1_B,f)}"'] & (B,A') \arrow[r, "{(1_B,f_2)}"'] & (B,B')
\end{tikzcd}
$$

where $\alpha_1, \beta_1, \gamma_1$ are squares in $\mathbb{B}$, the horizontal morphisms f_1, p_1 are in $\mathbb{B}$, the vertical morphisms g_1, q_1 are in $\mathbb{B}$, and the morphisms f, f_2, g, g_2 are in **Outer**. *Of course, the sources and targets must match appropriately.*

Proof: All of this follows from the colimit formula in **DblCat**, which is Theorem 4.6 of [27]. The horizontal and vertical 1-categories of $\mathbb{P}$ are the pushouts of the horizontal and vertical 1-categories, so (1) follows, and then (2) and (3) follow from Remark 3.5. To see (4), one observes that the only free composite pairs of squares that can occur are of the first two forms, again from Remark 3.5. Certain of these can be composed with a square in **Outer** to obtain the third form. No further free composites can be obtained from these ones because of Remark 3.5 and the special form of $c^n\delta_!$**Outer**. □

Proposition 7.13. *Consider $n = 2$ and the pushout $\mathbb{P}$ in diagram (20). Then any q-simplex in $\delta^* N^n \mathbb{P}$ is a $q \times q$-matrix of composable squares of $\mathbb{P}$ which has the form in Figure 2. The submatrix labelled $\mathbb{B}$ is a matrix of squares in $\mathbb{B}$. The submatrix labelled a is a single column of squares of the form (a) in Proposition 7.12 (iv) (the α_1's may be trivial). The submatrix labelled b is a single row of squares of the form (b) in Proposition 7.12 (iv) (the β_1's may be trivial). The submatrix labelled c is a single square of the form (c) in Proposition 7.12 (iv) (part of the square may be trivial). The remaining squares in the q-simplex are squares of $c^n\delta_!$***Outer**.

Proof: These are the only composable $q \times q$-matrices of squares because of the special form of the horizontal and vertical 1-categories. □

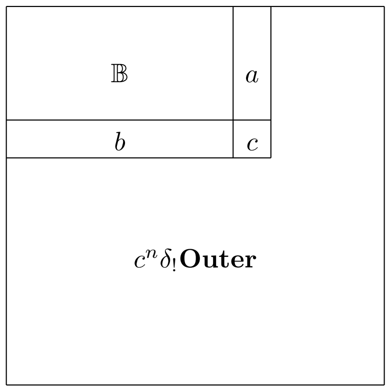

FIGURE 2. A q-simplex in $\delta^* N^n \mathbb{P}$.

Remark 7.14. The analogues of Propositions 7.12 and 7.13 clearly hold in higher dimensions as well, only the notation gets more complicated.

The n-fold version of 5.1 is the following.

Proposition 7.15. *Suppose $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{S}$ are n-fold categories, and $\mathbb{S}$ is an n-foldly full n-fold subcategory of $\mathbb{Q}$ and $\mathbb{R}$ such that*

(i) *If $f : x \longrightarrow y$ is a morphism in $\mathbb{Q}$ (in any direction) and $x \in \mathbb{S}$, then $y \in \mathbb{S}$,*
(ii) *If $f : x \longrightarrow y$ is a morphism in $\mathbb{R}$ (in any direction) and $x \in \mathbb{S}$, then $y \in \mathbb{S}$.*

Then the nerve of the pushout of n-fold categories is the pushout of the nerves.

$$N^n(\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R}) \cong N^n\mathbb{Q} \coprod_{N^n\mathbb{S}} N^n\mathbb{R} \tag{21}$$

Proof: We claim that there are no free composite n-cubes in the pushout $\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R}$. Suppose that α is an n-cube in $\mathbb{Q}$ and β is an n-cube in $\mathbb{R}$ and that these are composable in the i-th direction. In other words, the i-th target of α is the i-th source of β, which we will denote by γ. Then γ must be an $(n-1)$-cube in $\mathbb{S}$, as it lies in both $\mathbb{Q}$ and $\mathbb{R}$. Since the corners of γ are in $\mathbb{S}$, we can use hypothesis (ii) to conclude that all corners of β are in $\mathbb{S}$ by travelling along edges that emanate from γ. By the fullness of $\mathbb{S}$, the cube β is in $\mathbb{S}$, and also $\mathbb{Q}$. Then $\beta \circ_i \alpha$ is in $\mathbb{Q}$ and is not free.

If α is in $\mathbb{R}$ and β is in $\mathbb{Q}$, we can similarly conclude that β is in $\mathbb{S}$, $\beta \circ_i \alpha$ is in $\mathbb{R}$, and $\beta \circ_i \alpha$ is not a free composite.

Thus, the pushout $\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R}$ has no free composite n-cubes, and hence no free composites of any cells at all.

Let $(\alpha_{\bar{j}})_{\bar{j}}$ be a p-simplex in $N^n(\mathbb{Q}\coprod_{\mathbb{S}}\mathbb{R})$. Then each $\alpha_{\bar{j}}$ is an n-cube in $\mathbb{Q}$ or $\mathbb{R}$, since there are no free composites. By repeated application of the argument above, if $\alpha_{(0,\ldots,0)}$ is in $\mathbb{Q}$ then every $\alpha_{\bar{j}}$ is in $\mathbb{Q}$. Similarly, if $\alpha_{(0,\ldots,0)}$ is in $\mathbb{R}$ then every $\alpha_{\bar{j}}$ is in $\mathbb{R}$. Thus we have a morphism $N^n(\mathbb{Q}\coprod_{\mathbb{S}}\mathbb{R}) \longrightarrow N^n\mathbb{Q}\coprod_{N^n\mathbb{S}}N^n\mathbb{R}$. Its inverse is the canonical morphism $N^n\mathbb{Q}\coprod_{N^n\mathbb{S}}N^n\mathbb{R} \longrightarrow N^n(\mathbb{Q}\coprod_{\mathbb{S}}\mathbb{R})$. □

8. Thomason Structure on **nFoldCat**

We apply Corollary 6.1 to transfer across the adjunction below.

$$(22)\qquad \begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}^2"] \arrow[r, bend left, "\delta_!"] \arrow[r, phantom, "\perp"] & \mathbf{SSet^n} \arrow[l, bend left, "\delta^*"] \arrow[r, bend left, "c^n"] \arrow[r, phantom, "\perp"] & \mathbf{nFoldCat} \arrow[l, bend left, "N^n"] \end{tikzcd}$$

Proposition 8.1. *Let F be an n-fold functor. Then the morphism of simplicial sets δ^*N^nF is a weak equivalence if and only if $\mathrm{Ex}^2\delta^*N^nF$ is a weak equivalence.*

Proof: There is a natural weak equivalence $1_{\mathbf{SSet}} \Longrightarrow \mathrm{Ex}$ by Lemma 3.7 of [57], or more recently Theorem 6.2.4 of [56], or Theorem 4.6 of [32]. Then the Proposition follows from the naturality diagram below.

$$\begin{tikzcd} \delta^*N^n\mathbb{D} \arrow[r, "\delta^*N^nF"] \arrow[d] & \delta^*N^n\mathbb{E} \arrow[d] \\ \mathrm{Ex}^2\delta^*N^n\mathbb{D} \arrow[r, "\mathrm{Ex}^2\delta^*N^nF"'] & \mathrm{Ex}^2\delta^*N^n\mathbb{E} \end{tikzcd}$$
□

Theorem 8.2. *Call an n-fold functor F a weak equivalence or fibration if and only if $\mathrm{Ex}^2\delta^*N^nF$ is. Then this defines a cofibrantly generated model structure on* **nFoldCat** *with generating cofibrations*

$$\{\ c^n\delta_!\mathrm{Sd}^2\partial\Delta[m] \longrightarrow c^n\delta_!\mathrm{Sd}^2\Delta[m] \mid m \geq 0\}$$

and generating acyclic cofibrations

$$\{\ c^n\delta_!\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c^n\delta_!\mathrm{Sd}^2\Delta[m] \mid 0 \leq k \leq m \text{ and } m \geq 1\}.$$

Proof: We apply Corollary 6.1.

(i) The n-fold categories $c^n\delta_!\mathrm{Sd}^2\partial\Delta[m]$ and $c^n\delta_!\mathrm{Sd}^2\Lambda^k[m]$ each have a finite number of n-cubes, hence they are finite, and are small with respect to **nFoldCat**. For a proof, see Proposition 7.7 of [27] and the remark immediately afterwards.

(ii) This holds as in the proof of (ii) in Theorem 6.2.
(iii) The n-fold nerve functor N^n preserves filtered colimits. Every ordinal is filtered, so N^n preserves λ-sequences. The functor δ^* preserves all colimits, as it is a left adjoint. The functor Ex preserves λ-sequences as in the proof of (iii) in Theorem 6.2.
(iv) Let $j\colon \Lambda^k[m] \longrightarrow \Delta[m]$ be a generating acyclic cofibration for **SSet**. Let the functor j' be the pushout along L as in the following diagram.

$$\begin{tikzcd} c^n\delta_!\mathrm{Sd}^2\Lambda^k[m] \arrow[r, "L"] \arrow[d, "c^n\delta_!\mathrm{Sd}^2 j"'] & \mathbb{B} \arrow[d, "j'"] \\ c^n\delta_!\mathrm{Sd}^2\Delta[m] \arrow[r] & \mathbb{P} \end{tikzcd} \tag{23}$$

We factor j' into two inclusions

$$\mathbb{B} \xrightarrow{i} \mathbb{Q} \longrightarrow \mathbb{P} \tag{24}$$

and show that δ^*N^n applied to each yields a weak equivalence. For the first inclusion i, we will see in Lemma 8.3 that $\delta^*N^n i$ is a weak equivalence of simplicial sets.

By Remark 7.9, the only free composites of an n-cube in $c^n\delta_!\mathrm{Sd}^2\Delta[m]$ with an n-cube in $\mathbb{B}$ that can occur in $\mathbb{P}$ are of the form $\beta \circ_i \alpha$ where α is an n-cube in $\mathbb{B}$ and β is an n-cube in $c^n\delta_! N\mathbf{Outer}$ with i-th source in $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ and i-th target outside of $c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m]$. Of course, there are other free composites in $\mathbb{P}$, most generally of a form analogous to Proposition 7.12 (c), but these are obtained by composing the free composites of the form $\beta \circ_i \alpha$ above. Hence $\mathbb{P}$ is the union

$$\mathbb{P} = \overbrace{(\mathbb{B} \coprod_{c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[n]} c^n\delta_! N\mathbf{Outer})}^{\mathbb{Q}} \cup \overbrace{(c^n\delta_! N(\mathbf{Comp} \cup \mathbf{Center}))}^{\mathbb{R}}. \tag{25}$$

We show that δ^*N^n applied to the second inclusion $\mathbb{Q} \longrightarrow \mathbb{P}$ in equation (24) is a weak equivalence. The intersection of $\mathbb{Q}$ and $\mathbb{R}$ in (25) is equal to

$$\begin{aligned} \mathbb{S} &= c^n\delta_! N(\mathbf{Outer}) \cap c^n\delta_! N(\mathbf{Comp} \cup \mathbf{Center}) \\ &= c^n\delta_! N(\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center})). \end{aligned}$$

Propositions 5.2 and 7.4 then imply that $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{S}$ satisfy the hypotheses of Proposition 7.15. Then

$$\begin{aligned}
|\delta^* N^n \mathbb{Q}| &\cong |\delta^* N^n \mathbb{Q}| \coprod_{|\delta^* N^n \mathbb{S}|} |\delta^* N^n \mathbb{S}| \text{ (pushout along identity)} \\
&\simeq |\delta^* N^n \mathbb{Q}| \coprod_{|\delta^* N^n \mathbb{S}|} |\delta^* N^n \mathbb{R}| \text{ (Cor. 7.11 and Gluing Lemma)} \\
&\cong |\delta^* \left(N^n \mathbb{Q} \coprod_{N^n \mathbb{S}} N^n \mathbb{R} \right)| \text{ (the functors } |\cdot| \text{ and } \delta^* \text{ are left adjoints)} \\
&\cong |\delta^* N^n (\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R})| \text{ (Prop. 7.15)} \\
&= |\delta^* N^n \mathbb{P}|.
\end{aligned}$$

In the second line, for the application of the Gluing Lemma, we use two identities and the inclusion $|\delta^* N^n \mathbb{S}| \longrightarrow |\delta^* N^n \mathbb{R}|$. It is a homotopy equivalence whose inverse is the retraction in Corollary 7.11. We conclude that the inclusion $|\delta^* N^n \mathbb{Q}| \longrightarrow |\delta^* N^n \mathbb{P}|$ is a weak homotopy equivalence, as it is the composite of the morphisms above.

We conclude that $|\delta^* N^n j'|$ is the composite of two weak equivalences

$$|\delta^* N^n \mathbb{B}| \xrightarrow{|\delta^* N^n i|} |\delta^* N^n \mathbb{Q}| \longrightarrow |\delta^* N^n \mathbb{P}|$$

and is therefore a weak equivalence. Thus $\delta^* N^n j'$ is a weak equivalence of simplicial sets. As Ex preserves weak equivalences, $\mathrm{Ex}^2 \delta^* N^n j'$ is also a weak equivalence of simplicial sets. Part (iv) of Corollary 6.1 then holds, and we have the Thomason model structure on **nFoldCat**. $\square$

Lemma 8.3. *The inclusion $\delta^* N^n i \colon \delta^* N^n \mathbb{B} \longrightarrow \delta^* N^n \mathbb{Q}$ embeds the simplicial set $\delta^* N^n \mathbb{B}$ into $\delta^* N^n \mathbb{Q}$ as a simplicial deformation retract.*

Proof: Recall $i \colon \mathbb{B} \longrightarrow \mathbb{Q}$ is the inclusion in equation (24) and $\mathbb{Q}$ is defined as in equation (25). We define an n-fold functor $\overline{r} \colon \mathbb{Q} \longrightarrow \mathbb{B}$ using the universal property of the pushout $\mathbb{Q}$ and the functor $r \colon \mathbf{Outer} \longrightarrow \mathbf{P}\mathrm{Sd}\Lambda^k[m]$. If $(v_0, \ldots, v_q) \in \mathbf{Outer}$ then $r(v_0, \ldots, v_q) := (u_0, \ldots, u_p)$ where $(u_0, \ldots, u_p)$ is the maximal subset

$$\{u_0, \ldots, u_p\} \subseteq \{v_0, \ldots, v_q\}$$

that is in $\mathbf{P}\mathrm{Sd}\Lambda^k[m]$ (recall Proposition 3.4). We have

$$\begin{aligned} c^n\delta_! N\mathbf{P}\mathrm{Sd}\Lambda^k[m] &= \bigcup_{\substack{U\subseteq \mathbf{P}\mathrm{Sd}\Lambda^k[m] \text{ lin. ord.} \\ |U|=m+1}} U\boxtimes U\boxtimes\cdots\boxtimes U \\ &\subseteq \bigcup_{\substack{U\subseteq \mathbf{Outer} \text{ lin. ord.} \\ |U|=m+1}} U\boxtimes U\boxtimes\cdots\boxtimes U \\ &= c^n\delta_! N\mathbf{Outer}. \end{aligned}$$

Recall L is the n-fold functor in diagram (23). We define $\overline{r}$ on $c^n\delta_! N\mathbf{Outer}$ to be

$$L\circ(r\boxtimes r\boxtimes\cdots\boxtimes r)\colon\ c^n\delta_! N\mathbf{Outer} \longrightarrow \mathbb{B}$$

and we define $\overline{r}$ to be the identity on $\mathbb{B}$. This induces the desired n-fold functor $\overline{r}\colon\ \mathbb{Q}\longrightarrow\mathbb{B}$ by the universal property of the pushout $\mathbb{Q}$.

Be definition we have $\overline{r}i = 1_{\mathbb{B}}$. We next define a simplicial homotopy σ from $\delta^*N^n(i\overline{r})$ to $1_{\delta^*N^n\mathbb{Q}}$. It is similar to a simplicial homotopy induced by a natural transformation.

For each q, we need to define $q+1$ maps $\sigma_\ell : (\delta^*X)_q \longrightarrow (\delta^*X)_{q+1}$ compatible with the face and degeneracy maps, $\delta^*N^n(i\overline{r})$, and $1_{\delta^*N^n\mathbb{Q}}$. We define σ_ℓ on a q-simplex α of the form in Proposition 7.13. Suppose that the unique square of type (c) is in entry (u,v) and $u\leq v$.

If $\ell < u$, then $\sigma_\ell(\alpha)$ is obtained from α by inserting a row of vertical identities between rows ℓ and $\ell+1$ of α, as well as a column of horizontal identity squares between columns ℓ and $\ell+1$ of α. Thus $\sigma_\ell(\alpha)$ is vertically trivial in row $\ell+1$ and horizontally trivial in column $\ell+1$ of α.

If $\ell = u$ and $u<v$, then to obtain $\sigma_\ell(\alpha)$ from α, we replace row u by the two rows that make row u into a row of formal vertical composites, and we insert a column of horizontal identity squares between column u and column $u+1$ of α.

If $\ell = u$ and $u = v$, then to obtain $\sigma_\ell(\alpha)$ from α, we replace row u by the two rows that make row u into a row of formal vertical composites, and we replace column u by the two columns that make column u into a column of formal horizontal composites.

If $u<\ell<v$, then to obtain $\sigma_\ell(\alpha)$ from α, we replace row u by the row of squares β_1 in $\mathbb{B}$ that make up the first part of the formal vertical composite row u as in (b) of Proposition 7.13 (iv), then rows $u+1, u+2, \ldots, \ell$ are identity rows, row $\ell+1$ is the composite of the bottom half of row u of α with rows $u+1,u+2,\ldots,\ell$ of α, and the remaining rows of $\sigma_\ell(\alpha)$ are the remaining rows of α (shifted down by 1). We also insert a column of horizontal identity squares between column ℓ and column $\ell+1$ of α.

If $u < \ell = v$, then to obtain $\sigma_\ell(\alpha)$ from α, we do the row construction as in the case $u < \ell < v$, and we also replace column v by the two columns that make column v into a column of formal horizontal composites.

If $u \leq v < \ell$, then to obtain $\sigma_\ell(\alpha)$ from α, we do the row construction as in the case $u < \ell < v$, and we also do the analogous column construction.

The maps σ_ℓ for $0 \leq \ell \leq q$ are compatible with the boundary operators, $\delta^* N^n(i\overline{r})$, and $1_{\delta^* N^n \mathbb{Q}}$ for the same reason that the analogous maps associated to a natural transformation of functors are compatible with the face and degeneracy maps and the functors. Indeed, the σ_ℓ's are defined precisely as those for a natural transformation, we merely take into account the horizontal and vertical aspects.

In conclusion, we have morphisms of simplicial sets

$$\delta^* N^n(i)\colon\ \delta^* N^n \mathbb{B} \hookrightarrow \delta^* N^n \mathbb{Q}$$

$$\delta^* N^n(\overline{r})\colon\ \delta^* N^n \mathbb{Q} \longrightarrow \delta^* N^n \mathbb{B}$$

such that $(\delta^* N^n(\overline{r})) \circ (\delta^* N^n(i)) = 1_{\delta^* N^n \mathbb{B}}$ and $(\delta^* N^n(i)) \circ (\delta^* N^n(\overline{r}))$ is simplicially homotopic to $1_{\delta^* N^n \mathbb{Q}}$ via the simplicial homotopy σ. □

9. Unit and Counit are Weak Equivalences

In this section we prove that the unit and counit of the adjunction in (22) are weak equivalences. Our main tool is the n-fold Grothendieck construction and the theorem that, in certain situations, a natural weak equivalence between functors induces a weak equivalence between the colimits of the functors. We prove that N^n and the n-fold Grothendieck construction are "homotopy inverses". From this, we conclude that our Quillen adjunction (22) is actually a Quillen equivalence. The left and right adjoints of (22) preserve weak equivalences, so the unit and counit are weak equivalences.

Definition 9.1. Let $Y\colon\ \Delta^{\times n} \longrightarrow \mathbf{Set}$ be a multi-simplicial set. We define the *n-fold Grothendieck construction* $\Delta^{\boxtimes n}/Y \in$ **nFoldCat** as follows. The *objects* of the n-fold category $\Delta^{\boxtimes n}/Y$ are

$$\mathit{Obj}\, \Delta^{\boxtimes n}/Y = \{(y, \overline{k}) | \overline{k} = ([k_1], \ldots, [k_n]) \in \Delta^{\times n}, y \in Y_{\overline{k}}\}.$$

An *n-cube* in $\Delta^{\boxtimes n}/Y$ with $(0, 0, \ldots, 0)$-vertex $(y, \overline{k})$ and $(1, 1, \ldots, 1)$-vertex $(z, \overline{\ell})$ is a morphism $\overline{f} = (f_1, \ldots, f_n)\colon\ \overline{k} \longrightarrow \overline{\ell}$ in $\Delta^{\times n}$ such that

$$\overline{f}^*(z) = y. \tag{26}$$

For $\epsilon_\ell \in \{0, 1\}$, the $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$-vertex of such an n-cube is

$$(f_1^{1-\epsilon_1}, f_2^{1-\epsilon_2}, \ldots, f_n^{1-\epsilon_n})^*(z).$$

For $1 \leq i \leq n$, a *morphism in direction* i is an n-cube that has f_j the identity except at $j = i$. A *square in direction* ii' is an n-cube such that f_j is the identity except at $j = i$ and $j = i'$, etc. In this way, the edges, subsquares, subcubes, etc. of an n-cube $\overline{f}$ are determined.

Example 9.2. If $n = 1$, then the Grothendieck construction of Definition 9.1 is the usual Grothendieck construction of a simplicial set.

Example 9.3. The Grothendieck construction $\Delta/\Delta[m]$ of the simplicial set $\Delta[m]$ is the comma category $\Delta/[m]$.

Example 9.4. The Grothendieck construction commutes with external products, that is, for simplicial sets $X_1, X_2, \ldots, X_n$ we have

$$\Delta^{\times n}/(X_1 \boxtimes X_2 \boxtimes \cdots \boxtimes X_n) = (\Delta/X_1) \boxtimes (\Delta/X_2) \boxtimes \cdots \boxtimes (\Delta/X_n).$$

Remark 9.5. We describe the n-fold nerve of the n-fold Grothendieck construction. Let $Y \colon \Delta^{\times n} \longrightarrow \mathbf{Set}$ be a multi-simplicial set and $\overline{p} = ([p_1], \ldots, [p_n]) \in \Delta^{\times n}$. Then a $\overline{p}$-multisimplex of $N^n(\Delta^{\boxtimes n}/Y)$ consists of n composable paths of morphisms in Δ of lengths $p_1, p_2, \ldots, p_n$

$$\langle f_1^1, \ldots, f_{p_1}^1 \rangle \colon [k_0^1] \xrightarrow{f_1^1} [k_1^1] \xrightarrow{f_2^1} \cdots \xrightarrow{f_{p_1}^1} [k_{p_1}^1]$$

$$\langle f_1^2, \ldots, f_{p_2}^2 \rangle \colon [k_0^2] \xrightarrow{f_1^2} [k_1^2] \xrightarrow{f_2^2} \cdots \xrightarrow{f_{p_2}^2} [k_{p_2}^2]$$

$$\cdots$$

$$\langle f_1^n, \ldots, f_{p_n}^n \rangle \colon [k_0^n] \xrightarrow{f_1^n} [k_1^n] \xrightarrow{f_2^n} \cdots \xrightarrow{f_{p_n}^n} [k_{p_n}^n]$$

and a multi-simplex z of Y in degree

$$\overline{k_{\overline{p}}} := (k_{p_1}^1, k_{p_2}^2, \ldots, k_{p_n}^n).$$

The last vertex in this $\overline{p}$-array of n-cubes in $\Delta^{\boxtimes n}/Y$ is

$$(z, ([k_{p_1}^1], [k_{p_2}^2], \ldots, [k_{p_n}^n])).$$

The other vertices of this array are determined from z by applying the f's and their composites as in equation (26). Thus, the set of $\overline{p}$-multisimplices of $N^n(\Delta^{\boxtimes n}/Y)$ is

$$\coprod_{\substack{\langle f_1^1, \ldots, f_{p_1}^1 \rangle \\ \langle f_1^2, \ldots, f_{p_2}^2 \rangle \\ \cdots \\ \langle f_1^n, \ldots, f_{p_n}^n \rangle}} Y_{\overline{k_{\overline{p}}}}. \tag{27}$$

Proposition 9.6. *The functor $Y \mapsto N^n(\Delta^{\boxtimes n}/Y)$ preserves colimits.*

Proof: The set of $\overline{p}$-multisimplices of $N^n(\Delta^{\boxtimes n}/Y)$ is (27). The assignment of Y to the expression in (27) preserves colimits. $\square$

Remark 9.7. We can also describe the p-simplices of $\delta^* N^n(\Delta^{\boxtimes n}/Y)$. A p-simplex of $\delta^* N^n(\Delta^{\boxtimes n}/Y)$ is a composable path of p n-cubes

$$\overline{f^i} \colon (y^{i-1}, \overline{k^{i-1}}) \longrightarrow (y^i, \overline{k^i})$$

$(i = 1, \ldots, k)$. Each y^i is determined from y^p by the $\overline{f^i}$'s, as in equation (26). The last target, namely $(y^p, \overline{k^p})$, is the same as a morphism of multi-simplicial sets $\Delta[\overline{k^i}] \longrightarrow Y$. So by Yoneda, a p-simplex is the same as a composable path of morphisms of multi-simplicial sets

$$\Delta[\overline{k^0}] \longrightarrow \Delta[\overline{k^1}] \longrightarrow \cdots \longrightarrow \Delta[\overline{k^p}] \longrightarrow Y\ .$$

The set of p-simplices of $\delta^* N^n(\Delta^{\boxtimes n}/Y)$ is

$$\coprod_{\Delta[\overline{k^0}] \to \Delta[\overline{k^1}] \to \cdots \to \Delta[\overline{k^p}]} Y_{\overline{k^p}}. \tag{28}$$

Let us recall the natural morphism of simplicial sets $N(\Delta/X) \longrightarrow X$ in 6.1 of [56], which we shall call ρ_X as in Appendix A of [73]. First note that any path of morphisms in Δ

$$[k_0] \longrightarrow [k_1] \longrightarrow \cdots \longrightarrow [k_p] \tag{29}$$

determines a morphism

$$\begin{aligned} [p] &\longrightarrow [k_p] \\ i &\mapsto im\, k_i \end{aligned} \tag{30}$$

where $im\, k_i$ refers to the image of k_i under the composite of the last $p - i$ morphisms in (29). Note also that paths of the form (29) are in bijective correspondence with paths of the form

$$\Delta[k_0] \longrightarrow \Delta[k_1] \longrightarrow \cdots \longrightarrow \Delta[k_p] \tag{31}$$

by the Yoneda Lemma. The morphism $\rho_X \colon N(\Delta/X) \longrightarrow X$ sends a p-simplex

$$\Delta[k^0] \longrightarrow \Delta[k^1] \longrightarrow \cdots \longrightarrow \Delta[k^p] \longrightarrow X$$

to the composite

$$\Delta[p] \longrightarrow \Delta[k^p] \longrightarrow X$$

where the first morphism is the image of (30) under the Yoneda embedding. As is well known, the morphism $N(\Delta/X) \longrightarrow X$ is a natural weak equivalence (see Theorem 6.2.2 of [56], page 21 of [48], page 359 of [88]).

We analogously define a morphism of multisimplicial sets

$$\rho_Y\colon\ N^n(\Delta^{\boxtimes n}/Y) \longrightarrow Y$$

natural in Y. Consider a $\overline{p}$-multisimplex of $N^n(\Delta^{\boxtimes n}/Y)$ as in Remark 9.5. For each $1 \leq j \leq n$, the path $\langle f_1^j, \ldots, f_{p_j}^j \rangle$ gives rise to a morphism in Δ

$$[p_j] \longrightarrow [k_{p_j}^j]$$

as in (29) and (30). Together these form a morphism in $\Delta^{\times n}$, which induces a morphism of multisimplicial sets

$$\Delta^{\times n}[\overline{p}] \longrightarrow \Delta^{\times n}[\overline{k_{\overline{p}}}]\ .$$

The morphism ρ_Y assigns to the $\overline{p}$-multisimplex we are considering the $\overline{p}$-multisimplex

$$\Delta^{\times n}[\overline{p}] \longrightarrow \Delta^{\times n}[\overline{k_{\overline{p}}}] \xrightarrow{z} Y\ .$$

This completes the definition of the natural transformation ρ.

Remark 9.8. The natural transformation ρ is compatible with external products. If $X_1, X_2, \ldots, X_n$ are simplicial sets and $Y = X_1 \boxtimes X_2 \boxtimes \cdots \boxtimes X_n$, then

$$\rho_Y\colon\ N^n(\Delta^{\times n}/Y) \longrightarrow Y$$

is equal to

$$\rho_{X_1} \boxtimes \rho_{X_2} \boxtimes \cdots \boxtimes \rho_{X_n}\colon$$

$$N(\Delta/X_1) \boxtimes N(\Delta/X_2) \boxtimes \cdots \boxtimes N(\Delta/X_n) \longrightarrow X_1 \boxtimes X_2 \boxtimes \cdots \boxtimes X_n.$$

Thus $\delta^*\rho_Y = \rho_{X_1} \times \rho_{X_2} \times \cdots \times \rho_{X_n}$ is a weak equivalence, since in **SSet** any finite product of weak equivalences is a weak equivalence. We conclude that ρ_Y is a weak equivalence of multisimplicial sets whenever Y is an external product. As we shall soon see, ρ_Y is a weak equivalence for all Y.

We quickly recall what we will need regarding Reedy model structures. The following definition and proposition are part of Definitions 5.1.2, 5.2.2, and Theorem 5.2.5 of [46], or Definitions 15.2.3, 15.2.5, and Theorem 15.3.4 of [45]

Definition 9.9. Let $(\mathcal{B}, \mathcal{B}_+, \mathcal{B}_-)$ be a Reedy category and $\mathcal{C}$ a category with all small colimits and limits. For $i \in \mathcal{B}$, the *latching category* $\mathcal{B}_i$ is the full subcategory of $\mathcal{B}_+/i$ on the *non-identity* morphisms $b \longrightarrow i$. For $F \in \mathcal{C}^{\mathcal{B}}$ the *latching object of* F *at* i is the colimit L_iF of the composite functor

$$\mathcal{B}_i \longrightarrow \mathcal{B} \xrightarrow{F} \mathcal{C}\ . \tag{32}$$

For $i \in \mathcal{B}$, the *matching category* $\mathcal{B}^i$ is the full subcategory of $i/\mathcal{B}_-$ on the *non-identity* morphisms $i \longrightarrow b$. For $F \in \mathcal{C}^{\mathcal{B}}$ the *matching object of F at i* is the limit $M_i F$ of the composite functor

$$\mathcal{B}^i \longrightarrow \mathcal{B} \xrightarrow{F} \mathcal{C}\ . \tag{33}$$

Theorem 9.10 (Kan)**.** *Let $(\mathcal{B}, \mathcal{B}_+, \mathcal{B}_-)$ be a Reedy category and $\mathcal{C}$ a model category. Then the level-wise weak equivalences, Reedy fibrations, and Reedy cofibrations form a model structure on the category $\mathcal{C}^{\mathcal{B}}$ of functors $\mathcal{B} \longrightarrow \mathcal{C}$.*

Remark 9.11. A consequence of the definitions is that a functor $\mathcal{B} \longrightarrow \mathcal{C}$ is *Reedy cofibrant* if and only if the induced map $L_i F \longrightarrow Fi$ is a cofibration in $\mathcal{C}$.

Proposition 9.12 (Compare Example 15.1.19 of [45])**.** *If $Y\colon \Delta^{\times n} \longrightarrow \mathbf{Set}$ is a multisimplicial set, its category of multisimplices*

$$\Delta^{\times n} Y := \Delta^{\times n}/Y$$

is a Reedy category. The degree of a $\overline{p}$-multisimplex is $p_1 + p_2 \cdots + p_n$. The direct subcategory consists of those morphisms $(f_1, \ldots, f_n)$ that are iterated coface maps in each coordinate. The inverse subcategory consists of those morphisms $(f_1, \ldots, f_n)$ that are iterated codegeneracy maps in each coordinate.

Proposition 9.13 (Compare Proposition 15.10.4(1) of [45])**.** *If $\mathcal{B}$ is the category of multisimplices of a multisimplicial set, then for every $i \in \mathcal{B}$, the matching category $\mathcal{B}^i$ is either connected or empty.*

Proof: This follows from the multisimplicial version of the Eilenberg-Zilber Lemma, which is proved in Proposition 5.1.1 of [56] for example. □

Theorem 9.14. *Suppose $\mathcal{C}$ is a model category and $\mathcal{B}$ is a Reedy category such that for all $i \in \mathcal{B}$, the matching category $\mathcal{B}^i$ is either connected or empty. Then the colimit functor*

$$\operatorname{colim}\colon \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}$$

takes levelwise weak equivalences between Reedy cofibrant functors to weak equivalences between cofibrant objects of $\mathcal{C}$.

Proof: This is merely a summary of Definition 15.10.1(2), Proposition 15.10.2(2), and Theorem 15.10.9(2) of [45]. □

Notation 9.15. Let $Y\colon\ \Delta^{\times n} \longrightarrow \mathbf{Set}$ be a multisimplicial set, $\mathcal{B}=\Delta^{\times n}Y$, $\mathcal{C} = \mathbf{SSet}$, and $i\colon\ \Delta^{\times n}[\overline{m}] \longrightarrow Y$ an object of $\mathcal{B}$. Then the set of nonidentity morphisms in $\mathcal{B}_+$ with target i is the set of morphisms $(f_1, \ldots, f_n)$ in $\Delta^{\times n}$ with target $[\overline{m}]$ such that each f_j is injective and not all f_j's are the identity.

Notation 9.16. Let F and G be the following two functors.

$$F\colon\ \Delta^{\times n}Y \longrightarrow \mathbf{SSet^n}$$

$$\left[\Delta^{\times n}[\overline{m}] \to Y\right] \mapsto N^n(\Delta^{\boxtimes n}/\Delta^{\times n}[\overline{m}])$$

$$G\colon\ \Delta^{\times n}Y \longrightarrow \mathbf{SSet^n}$$

$$\left[\Delta^{\times n}[\overline{m}] \to Y\right] \mapsto \Delta^{\times n}[\overline{m}]$$

Note that $\delta^* \circ F$ and $\delta^* \circ G$ are in $\mathcal{C}^{\mathcal{B}}$. The natural transformation ρ induces a natural transformation we denote by

$$\rho^Y\colon\ F \Longrightarrow G\ .$$

Remark 9.17. The natural transformation ρ^Y is levelwise a weak equivalence by Remark 9.8.

Lemma 9.18. *The morphism in* $\mathbf{SSet^n}$

$$\operatorname*{colim}_{\Delta^{\times n}Y} \rho^Y\colon\ \operatorname*{colim}_{\Delta^{\times n}Y} F \longrightarrow \operatorname*{colim}_{\Delta^{\times n}Y} G$$

is equal to

$$\rho_Y\colon\ N^n(\Delta^{\boxtimes n}/Y) \longrightarrow Y.$$

Proof: By Proposition 9.6, we have

$$\begin{aligned}
\operatorname*{colim}_{\Delta^{\times n}Y} F &= \operatorname*{colim}_{\Delta^{\times n}[\overline{m}]\to Y} N^n(\Delta^{\boxtimes n}/\Delta^{\times n}[\overline{m}]) \\
&= N^n(\Delta^{\boxtimes n}/(\operatorname*{colim}_{\Delta^{\times n}[\overline{m}]\to Y} \Delta^{\times n}[\overline{m}])) \\
&= N^n(\Delta^{\boxtimes n}/Y).
\end{aligned}$$

□

Lemma 9.19. *The functor*

$$\delta^* \circ F\colon\ \Delta^{\times n}Y \longrightarrow \mathbf{SSet}$$

$$\left[\Delta^{\times n}[\overline{m}] \to Y\right] \mapsto N(\Delta/\Delta[m_1]) \times N(\Delta/\Delta[m_2]) \times \cdots \times N(\Delta/\Delta[m_n])$$

is Reedy cofibrant.

Proof: We use Notations 9.15 and 9.16. The colimit of equation (32) is

$$L_i(\delta^* \circ F) = \bigcup_{1 \le j \le n} N(\Delta/\Delta[m_1]) \times \cdots \times N(\Delta/\partial\Delta[m_j]) \times \cdots \times N(\Delta/\Delta[m_n])$$

and $\delta^* \circ F(i) = N(\Delta/\Delta[m_2]) \times \cdots \times N(\Delta/\Delta[m_n])$. The map

$$L_i(\delta^* \circ F) \longrightarrow \delta^* \circ F(i)$$

is injective, or equivalently, a cofibration. Remark 9.11 now implies that $\delta^* \circ F$ is Reedy cofibrant. □

Lemma 9.20. *The functor*

$$\delta^* \circ G \colon \Delta^{\times n} Y \longrightarrow \mathbf{SSet}$$

$$\left[\Delta^{\times n}[\overline{m}] \to Y\right] \mapsto \Delta[m_1] \times \Delta[m_2] \times \cdots \times \Delta[m_n]$$

is Reedy cofibrant.

Proof: We use Notations 9.15 and 9.16. The colimit of equation (32) is

$$L_i(\delta^* \circ G) = \bigcup_{1 \le j \le n} \Delta[m_1] \times \cdots \times \partial\Delta[m_j] \times \cdots \times \Delta[m_n]$$

and $\delta^* \circ G(i) = \Delta[m_1] \times \Delta[m_2] \times \cdots \times \Delta[m_n]$. The map

$$L_i(\delta^* \circ G) \longrightarrow \delta^* \circ G(i)$$

is injective, or equivalently, a cofibration. Remark 9.11 now implies that $\delta^* \circ G$ is Reedy cofibrant. □

Theorem 9.21. *For every multisimplicial set* $Y \colon \Delta^{\times n} \longrightarrow \mathbf{Set}$*, the morphism*

$$\rho_Y \colon N^n(\Delta^{\times n}/Y) \longrightarrow Y$$

is a weak equivalence of multisimplicial sets.

Proof: Fix a multisimplicial set Y, and let F, G, and ρ^Y be as in Notation 9.16. The natural transformation $\delta^*\rho^Y \colon \delta^*F \Longrightarrow \delta^*G$ is levelwise a weak equivalence of simplicial sets by Remark 9.17, and is a natural transformation between Reedy cofibrant functors by Lemmas 9.19 and 9.20. By Proposition 9.13, each matching category of the Reedy category $\Delta^{\times n}Y$ is connected or empty. Theorem 9.14 then guarantees that the morphism

$$\operatorname*{colim}_{\Delta^{\times n}Y} \delta^*\rho^Y \colon \operatorname*{colim}_{\Delta^{\times n}Y} \delta^* \circ F \longrightarrow \operatorname*{colim}_{\Delta^{\times n}Y} \delta^* \circ G$$

is a weak equivalence of simplicial sets. Since δ^* is a left adjoint, it commutes with colimits, and we have

$$\operatorname*{colim}_{\Delta^{\times n} Y} \delta^* \rho^Y = \delta^* \operatorname*{colim}_{\Delta^{\times n} Y} \rho^Y = \delta^* \rho_Y$$

by Lemma 9.18. We conclude $\delta^* \rho_Y$ is a weak equivalence, and that ρ_Y is a weak equivalence of multisimplicial sets. $\square$

We also define an n-fold functor

$$\lambda_{\mathbb{D}} \colon \Delta^{\boxtimes n}/N^n(\mathbb{D}) \longrightarrow \mathbb{D}$$

natural in $\mathbb{D}$, by analogy to Appendix A of [73], and many others. If $(y, \overline{k})$ is an object of $\Delta^{\boxtimes n}/N^n(\mathbb{D})$, then $\lambda_{\mathbb{D}}$ is the double category in the last vertex of the array of n-cubes y, namely

$$\lambda_{\mathbb{D}}(y, \overline{k}) = y_{\overline{k}}.$$

Theorem 9.22. *For any double category $\mathbb{D}$, we have $N^n(\lambda_{\mathbb{D}}) = \rho_{N^n(\lambda_{\mathbb{D}})}$. In particular, $\lambda_{\mathbb{D}}$ is a weak equivalence of double categories.*

Corollary 9.23. *The functor N^n:* **nFoldCat**$\longrightarrow$**SSet$^{\mathbf{n}}$** *induces an equivalence of categories*

$$Ho\, \mathbf{DblCat} \simeq Ho\, \mathbf{SSet^n}.$$

Proof: An "inverse" to N^n is the n-fold Grothendieck construction, since ρ and λ induce natural isomorphisms after passing to homotopy categories by Theorems 9.21 and 9.22. $\square$

The following simple proposition, pointed out to us by Denis-Charles Cisinski, will be of use.

Proposition 9.24. *Let (L, R) be a Quillen equivalence. If both L and R preserve weak equivalences, then*

(i) *Both L and R detect weak equivalences,*
(ii) *The unit and counit of the adjunction (L, R) are weak equivalences.*

Proposition 9.25. *The unit and counit of (22) are weak equivalences.*

Proof: Let (L, R) denote the adjunction in (22). This is a Quillen adjunction by Theorem 8.2. It is even a Quillen equivalence: $\mathrm{Ex}^2\delta^*$ is known to induce an equivalence of homotopy categories, and N^n induces an equivalence of homotopy categories by Corollary 9.23. By the Lemma from Ken Brown, the left adjoint L preserves weak equivalences (every simplicial set is cofibrant). The right adjoint R preserves weak equivalences by definition. Thus, by Cisinski's Proposition, the unit and counit are weak equivalences. $\square$

We now summarize our main results of Theorem 8.2, Corollary 9.23, Proposition 9.25.

Theorem 9.26. (i) *There is a cofibrantly generated model structure on* **nFoldCat** *such that an n-fold functor F is a weak equivalence (respectively fibration) if and only if* $\mathrm{Ex}^2\delta^* N^n(F)$ *is a weak equivalence (respectively fibration). In particular, an n-fold functor is a weak equivalence if and only if the diagonal of its nerve is a weak equivalence of simplicial sets.*

(ii) *The adjunction*

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}^2"] \arrow[r, bend left, "\delta_!"] \arrow[r, phantom, "\perp"] & \mathbf{SSet^n} \arrow[l, bend left, "\delta^*"] \arrow[r, bend left, "c^n"] \arrow[r, phantom, "\perp"] & \mathbf{nFoldCat} \arrow[l, bend left, "N^n"] \end{tikzcd}$$

is a Quillen equivalence.

(iii) *The unit and counit of this Quillen equivalence are weak equivalences.*

Corollary 9.27. *The homotopy category of n-fold categories is equivalent to the homotopy category of topological spaces.*

Another approach to proving that N^n and the n-fold Grothendieck construction are homotopy inverse would be to apply a multisimplicial version of the following Weak Equivalence Extension Theorem. We apply the present Weak Equivalence Extension Theorem to prove that there is a natural isomorphism

$$\delta^* N^n(\Delta^{\boxtimes n}/\delta_!\text{-}) \Longrightarrow 1_{\mathrm{Ho}\,\mathbf{SSet}}\,.$$

Theorem 9.28 (Theorem 6.2.1 of [56])**.** *Let $\phi\colon F \Longrightarrow G$ be a natural transformation between functors $F, G\colon \Delta \longrightarrow \mathbf{SSet}$. We denote by $\phi^+\colon F^+ \Longrightarrow G^+$ the left Kan extension along the Yoneda embedding $Y\colon \Delta \longrightarrow \mathbf{SSet}$.*

$$\begin{tikzcd} \mathbf{SSet} \arrow[dr, "F^+{,}G^+"] & \\ \Delta \arrow[u, "Y"] \arrow[r, "F{,}G"'] & \mathbf{SSet} \end{tikzcd}$$

Suppose that G satisfies the following condition.

- *$im\, G\epsilon^0 \cap im\, G\epsilon^1 = \emptyset$, where $\epsilon^i\colon [0] \longrightarrow [1]$ is the injection which misses i.*

If $\phi[m]\colon F[m] \longrightarrow G[m]$ is a weak equivalence for all $m \geq 0$, then

$$\phi^+ X\colon F^+X \longrightarrow G^+X$$

is a weak equivalence for every simplicial set X.

Lemma 9.29. *The functor*

$$\mathbf{SSet^n} \longrightarrow \mathbf{SSet}$$

$$Y \mapsto \delta^* N^n(\Delta^{\boxtimes n}/Y)$$

preserves colimits.

Proof: The functor which assigns to Y the expression in (28) is colimit preserving. $\square$

Proposition 9.30. *For every simplicial set* X*, the canonical morphism*

$$\delta^* N^n(\Delta^{\boxtimes n}/\delta_! X) \longrightarrow \delta^*\delta_! X$$

is a weak equivalence.

Proof: We apply the Weak Equivalence Extension Theorem 9.28. Let $F, G\colon \Delta \longrightarrow \mathbf{SSet}$ be defined by

$$F[m] = \delta^* N^n(\Delta^{\boxtimes n}/\delta_!\Delta[m])$$

$$G[m] = \delta^*\delta_!\Delta[m].$$

The functor

$$\delta^* N^n(\Delta^{\boxtimes n}/\delta_!\text{-})\colon \mathbf{SSet} \longrightarrow \mathbf{SSet}$$

preserves colimits by Lemma 9.29 and the fact that $\delta_!$ is a left adjoint. The functor

$$\delta^*\delta_!\colon \mathbf{SSet} \longrightarrow \mathbf{SSet}$$

preserves colimits since both δ^* and $\delta_!$ are both left adjoints. Thus the canonical comparison morphisms

$$F^+X \longrightarrow \delta^* N^n(\Delta^{\boxtimes n}/\delta_! X)$$

$$G^+X \longrightarrow \delta^*\delta_! X$$

are isomorphisms.

The condition on G listed in Theorem 9.28 is easy to verify, since

$$G\epsilon^0 = \epsilon^0 \times \epsilon^0\colon \Delta[0]\times\Delta[0] \longrightarrow \Delta[1]\times\Delta[1]$$

$$G\epsilon^1 = \epsilon^1 \times \epsilon^1\colon \Delta[0]\times\Delta[0] \longrightarrow \Delta[1]\times\Delta[1]\ .$$

All that remains is to define natural morphisms

$$\phi[m]\colon \delta^* N^n(\Delta^{\boxtimes n}/\Delta[m,\ldots,m]) \longrightarrow \Delta[m]\times\cdots\times\Delta[m]$$

and to show that each is a weak equivalence of simplicial sets. By the description in Definition 9.1, an object of $\Delta^{\boxtimes n}/\Delta[m,\ldots,m]$ is a morphism

$$y = (y_1,\ldots,y_n)\colon\ \overline{k} \longrightarrow ([m],\ldots,[m])$$

in $\Delta^{\times n}$. An n-cube $\overline{f}$ is a morphism in $\Delta^{\times n}$ making the diagram

$$\begin{tikzcd} \overline{k} \arrow[rr, "\overline{f}"] \arrow[dr, "y"'] & & \overline{k'} \arrow[dl, "y'"] \\ & ([m],\ldots,[m]) & \end{tikzcd}$$

commute. A p-simplex in $\delta^* N^n(\Delta^{\boxtimes n}/\Delta[m,\ldots,m])$ is a path $\overline{f^1},\ldots,\overline{f^p}$ of composable morphisms in $\Delta^{\times n}$ making the appropriate triangles commute. We see that

$$\delta^* N^n(\Delta^{\boxtimes n}/\Delta[m,\ldots,m]) \cong N(\Delta/\Delta[m]) \times \cdots N(\Delta/\Delta[m]).$$

We define $\phi[m]$ to be the product of n-copies of the weak equivalence

$$\rho_{\Delta[m]}\colon\ N(\Delta/\Delta[m]) \longrightarrow \Delta[m]$$

defined on page 41. Since $\phi[m]$ is a weak equivalence for all m, we conclude from Theorem 9.28 that the canonical morphism

$$\phi^+ X\colon\ \delta^* N^n(\Delta^{\boxtimes n}/\delta_! X) \longrightarrow \delta^*\delta_! X$$

is a weak equivalence for every simplicial set X. □

Lemma 9.31. *There is a natural weak equivalence* $\delta^*\delta_! X \longleftarrow X$ *.*

Proof: In Theorem 9.28, let F be the Yoneda embedding and G once again $\delta^*\delta_!$. The diagonal morphism

$$\Delta[m] \longrightarrow \Delta[m] \times \cdots \times \Delta[m]$$

is a weak equivalence, as both the source and target are contractible. □

Proposition 9.32. *There is a zig-zag of natural weak equivalences between* $\delta^* N^n(\Delta^{\boxtimes n}/\delta_!\text{-})$ *and the identity functor on* **SSet**. *Consequently, there is a natural isomorphism*

$$\delta^* N^n(\Delta^{\boxtimes n}/\delta_!\text{-}) \Longrightarrow 1_{\mathrm{Ho}\ \mathbf{SSet}}\,.$$

Proof: This follows from Proposition 9.30 and Lemma 9.31. □

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