DUALIZING COMPLEXES—THE MODERN WAY

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Abstract. We give a survey of some recent results on Grothendieck duality. We begin with a brief reminder of the classical theory, and then launch into an overview of some of the striking developments since 2005.

Contents

1. Grothendieck duality done classically

Let $\mathcal{V}$ be a finite dimensional complex vector space. The dual space, denoted $\mathcal{V}^*$, is the vector space $\mathcal{V}^* = \mathrm{Hom}(\mathcal{V}, \mathbb{C})$. Very early on in our mathematical education we learn that the natural map $\mathcal{V} \longrightarrow \mathcal{V}^{**}$ is an isomorphism.

A slightly fancier version of the same phenomenon comes from the following.

Definition 1.1. *Let X be a complex manifold, and let $\mathcal{L}$ be a line bundle on X. Given a vector bundle $\mathcal{V}$ we define the dual bundle to be $\mathcal{V}^* = \mathcal{H}\mathrm{om}(\mathcal{V}, \mathcal{L})$; it is the vector bundle of bundle maps $\mathcal{V} \longrightarrow \mathcal{L}$.*

Remark 1.2. Note that the definition depends on choosing, and fixing, a line bundle $\mathcal{L}$ on X. I suppose the notation ought to make this explicit, but $\mathcal{V}^{(*_\mathcal{L})}$ looks ridiculous.

Remark 1.3. With this definition, the natural map $\mathcal{V} \longrightarrow \mathcal{V}^{**}$ is still an isomorphism.

Remark 1.4. If X is the 1–point space then a vector bundle is simply a finite dimensional vector space, and a line bundle is a 1-dimensional vector space. Thus

Key words and phrases. Grothendieck duality, dualizing complex.
The research was partly supported by the Australian Research Council.

the fancier construction of Definition 1.1, in the special case where X is the 1–point space, is nothing more than the ordinary dual vector space, of elementary linear algebra, that we encountered in the first paragraph of this section.

The reverse is also true; the second construction may be viewed as a very minor variant of the first. A vector bundle of rank n gives us, at every point $p \in X$, an n–dimensional vector space $V = V(p)$. A line bundle determines, at every point p, a 1–dimensional vector space $L(p) \cong \mathbb{C}$. Thus at each point p we have that $\mathcal{V}^*$ gives the vector space $\mathrm{Hom}(V(p), L(p)) \cong \mathrm{Hom}(V(p), \mathbb{C})$. All we did, in passing from the first construction to the second, was to allow the vector spaces to vary along the manifold X.

The first non-trivial theorem one meets is the Serre duality theorem. We remind the reader:

Theorem 1.5. (Serre [14]**).** *Let X be a compact, connected complex manifold, of dimension $d \geq 0$. Let $\mathcal{L}$ be the line bundle Ω^d_X of holomorphic d–forms on X. Then, for every vector bundle $\mathcal{V}$ over X and every integer i in the range $0 \leq i \leq d$, one has a natural isomorphism*

$$H^i(X, \mathcal{V})^* \quad \cong \quad H^{d-i}(X, \mathcal{V}^*) \ .$$

Remark 1.6. The sheaf cohomology functors H^i constitute a procedure, which takes a vector bundle $\mathcal{V}$ on X and produces out of it finite dimensional vector spaces, that is vector bundles over the 1–point space. And Serre duality is roughly the statement that this process commutes with duality. Up to the confusing change of indices, which replaces H^i by H^{d-i}, we have what looks like the simple formula

$$H(\mathcal{V})^* \cong H(\mathcal{V}^*) \ .$$

The question people asked themselves, back in the 1950s when Serre duality was discovered, was whether there could be a relative version. Is there a version that holds for holomorphic maps $X \longrightarrow Y$?

Remark 1.7. It might help if we make this question a little more precise. It would be nice if the following wish list could be made to work:

(i) On each X we should have a method of taking a $\mathcal{V}$ to a dual object $\mathcal{V}^* = \mathcal{H}\mathrm{om}(\mathcal{V}, \mathcal{L})$.

(ii) The natural map

$$\mathcal{V} \longrightarrow \mathcal{V}^{**}$$

should be an isomorphism.

(iii) Suppose $f\colon X \longrightarrow Y$ is a proper holomorphic map. (We remind the reader: a holomorphic map is *proper* if the inverse images of compact sets are compact.) There should be a procedure that takes a $\mathcal{V}$ on X, and somehow pushes it forward to an $\mathbf{R}f_*\mathcal{V}$ on Y.

(iv) The pushforward ought to be compatible with the duality; that is, there should be a natural isomorphism

$$(\mathbf{R}f_*\mathcal{V})^* \quad \cong \quad \mathbf{R}f_*(\mathcal{V}^*) \ .$$

(v) Serre's theorem should be an immediate special case, where we take Y to be the 1–point space.

Of course some care will have to be taken; note that the functor Serre considered takes a single vector bundle $\mathcal{V}$ on X to a string of $(d+1)$ vector bundles on the 1–point space, namely all the $H^i(X,\mathcal{V}), 0 \leq i \leq d$. If we hope to achieve our wish-list (i)–(v) above, the allowed input and output of the functor $\mathbf{R}f_*$ will have to be more general than a single vector bundle.

The fact that the wish-list can be attained, at least to some degree, is the outcome of Grothendieck's duality theory.

Remark 1.8. We briefly remind the reader of the history: in his 1958 talk at the Edinborough ICM Grothendieck asserted he had a solution to this problem; see [2]. He also said that, for the time being, the necessary homological algebra framework did not yet exist. Grothendieck set this as a PhD thesis problem for Jean-Louis Verdier; in Verdier's thesis [15] derived categories were born. Derived categories grew out of this problem, it was their first application, and for a long time it remained the most important one. Once Verdier's formalism was ready Grothendieck wrote down a set of notes outlining his ideas, which Hartshorne expanded into a book [3]. A compendium to Hartshorne's book, with several tricky points worked out in more detail, appeared much later in Conrad [1].

Several comments are in order.

Remark 1.9. The case where X and Y are smooth and compact, and where Y is algebraic, is relatively easy. The hypothesis that Y is algebraic is there in order to guarantee that there are enough vector bundles on Y; it can be relaxed somewhat without essential change to the theory. But things become much more delicate if we allow singularities. Let me therefore begin by sketching what happens in the simple case; for this remark we will assume that $f\colon X \longrightarrow Y$ is a holomorphic map of smooth, compact, complex manifolds, with Y algebraic.

The allowed input for $\mathcal{V}$ is a *bounded cochain complex* of vector bundles on X. That is, our permissible $\mathcal{V}$s will be cochain complexes

$$\cdots \longrightarrow \mathcal{V}^{-2} \longrightarrow \mathcal{V}^{-1} \longrightarrow \mathcal{V}^{0} \longrightarrow \mathcal{V}^{1} \longrightarrow \mathcal{V}^{2} \longrightarrow \cdots$$

where each $\mathcal{V}^i$ is a vector bundle on X, and they vanish outside a bounded range; that is $\mathcal{V}^i = 0$ if either $i \gg 0$ or $i \ll 0$. The category we will be considering will therefore be $\mathbf{D}^b(\mathrm{Vect}/X)$, the bounded derived category of vector bundles on X. The objects are bounded complexes as above, but the reader should note that isomorphisms are isomorphisms in the derived category; many cochain complexes are isomorphic in $\mathbf{D}^b(\mathrm{Vect}/X)$.

There is a pushforward functor, which takes a bounded complex of vector bundles on X and pushes it forward to a bounded complex of vector bundles on Y, defined up to canonical isomophism in the derived category $\mathbf{D}^b(\mathrm{Vect}/Y)$. This achieves what we wished for in Remark 1.7(iii).

Next note that there is an internal Hom on the category $\mathbf{D}^b(\mathrm{Vect}/X)$; given two complexes $\mathcal{V}$ and $\mathcal{L}$ we can construct an obvious double complex $\underline{\mathcal{H}\mathrm{om}}(\mathcal{V},\mathcal{L})$; the associated single complex, obtained by forming the total complex of the double complex, is what we call $\mathcal{H}\mathrm{om}(\mathcal{V},\mathcal{L})$. Now let $\mathcal{L}=\Sigma^d\Omega^d_X$ be the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \Omega^d_X \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

that is, $\mathcal{L}^i=0$ unless $i=-d$, and $\mathcal{L}^{-d}=\Omega^d_X$ is the line bundle of holomorphic d–forms on X. We define $\mathcal{V}^*=\mathcal{H}\mathrm{om}(\mathcal{V},\Sigma^d\Omega^d_X)$.

Now our definitions are all finished. With these definitions, Grothendieck's theorem tells us that the entire wish-list of Remark 1.7(i)–(v) is fulfilled.

Remark 1.10. Perhaps it would help if we remind the reader how (v) works; that is, we will be a little more explicit in showing that Serre's theorem really is a special case of what was done in Remark 1.9.

Let $\mathcal{V}$ be a vector bundle on X, which we also view as the cochain complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{V} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where the only non-zero term is in degree 0. If $f\colon X\longrightarrow Y$ is the projection from X to the 1–point space Y, then the formalism is such that the functor $\mathbf{R}f_*$ takes the complex $\mathcal{V}$ to the bounded complex $\mathbf{R}f_*\mathcal{V}$ below

$$0 \longrightarrow H^0(X,\mathcal{V}) \xrightarrow{\ 0\ } H^1(X,\mathcal{V}) \xrightarrow{\ 0\ } \cdots$$
$$\cdots \xrightarrow{\ 0\ } H^d(X,\mathcal{V}) \longrightarrow 0$$

here $H^i(X,\mathcal{V})$ is in degree i. Thus the dual $(\mathbf{R}f_*\mathcal{V})^*$ is the complex

$$0 \longrightarrow H^d(X,\mathcal{V})^* \xrightarrow{\ 0\ } H^{d-1}(X,\mathcal{V})^* \xrightarrow{\ 0\ } \cdots$$
$$\cdots \xrightarrow{\ 0\ } H^0(X,\mathcal{V})^* \longrightarrow 0$$

with $H^i(X,\mathcal{V})^*$ in degree $(-i)$.

Now let us look at $\mathbf{R}f_*(\mathcal{V}^*)$. By definition $\mathcal{V}^*=\mathcal{H}\mathrm{om}(\mathcal{V},\Sigma^d\Omega^d_X)$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{V}^* \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

which vanishes in degrees $\neq -d$, and where the term of degree $(-d)$ is the vector bundle $\mathcal{V}^*=\mathcal{H}\mathrm{om}(\mathcal{V},\Omega^d_X)$. The functor $\mathbf{R}f_*$ takes this to the complex of vector spaces

$$0 \longrightarrow H^0(X,\mathcal{V}^*) \xrightarrow{\ 0\ } H^1(X,\mathcal{V}^*) \xrightarrow{\ 0\ } \cdots$$
$$\cdots \xrightarrow{\ 0\ } H^d(X,\mathcal{V}) \longrightarrow 0$$

which start in degree $(-d)$; that is we have $H^i(X, \mathcal{V}^*)$ in degree $(i - d)$. The isomorphism in the derived category gives us, when we read what happens to the cohomology in degree $(-i)$, an isomorphism of finite dimensional vector spaces

$$H^i(X, \mathcal{V})^* \quad \cong \quad H^{d-i}(X, \mathcal{V}^*) \ ;$$

comparing with Theorem 1.5 we see that Serre duality follows.

Remark 1.11. Suppose now that we relax a little the hypothesis on X and Y. We still require $f \colon X \longrightarrow Y$ to be a proper holomorphic map of smooth manifolds, we continue to assume that Y has enough vector bundles, but we drop the hypothesis that X and Y be compact. What happens then?

Let us take a complex $\mathcal{V}$ of vector bundles on X. There is still a way to push forward to a sensible complex $\mathbf{R}f_*\mathcal{V}$ on Y, but it no longer has to be a bounded complex of vector bundles. On any relatively compact open subset $U \subset Y$ we obtain, upon restricting $\mathbf{R}f_*\mathcal{V}$ to the open set U, a complex isomorphic, in some suitable derived category, to a bounded complex of vector bundles. But there is no reason for the global $\mathbf{R}f_*\mathcal{V}$ to be "small" on all of Y. That is, the wish articulated in Remark 1.7(iii) runs into trouble.

In the algebraic category, where one works with noetherian schemes instead of complex analytic spaces, this problem disappears completely; the fact that a noetherian scheme is quasicompact makes all such subtleties vanish into thin air.

Remark 1.12. In Remark 1.9 we saw that the case of manifolds is straightforward; allowing X and Y to become singular makes the theory subtler. Now it is time to turn our attention to what happens in the presence of singularities.

The first observation is that we can no longer get away with looking only at cochain complexes of vector bundles; there is, for general, singular X and Y, no sensible pushforward map, taking bounded complexes of vector bundles to bounded complexes of vector bundles. We must relax our restrictions and allow cochain complexes of *coherent sheaves.* That is, our $\mathcal{V}$ can now be any object in $\mathbf{D}^b(\mathrm{Coh}/X)$.

This improves our prospects; given a holomorphic map $f \colon X \longrightarrow Y$ of (possibly singular) compact, complex analytic spaces, or else a proper morphism of noetherian schemes, there is a natural pushforward map $\mathbf{R}f_* \colon \mathbf{D}^b(\mathrm{Coh}/X) \longrightarrow \mathbf{D}^b(\mathrm{Coh}/Y)$. The functor $\mathbf{R}f_*$ takes a bounded complex of coherent sheaves on X to a bounded complex of coherent sheaves on Y, unique up to canonical isomorphism in $\mathbf{D}^b(\mathrm{Coh}/Y)$. So far we have achieved Remark 1.7(iii).

But we pay a price; our next problem is that there is no reasonable internal Hom on the derived category $\mathbf{D}^b(\mathrm{Coh}/X)$. Given two bounded complexes $\mathcal{V}$ and $\mathcal{L}$, the way to naturally form a Hom-complex, in the derived category, would be to take injective resolutions first; and the injective resolution of a bounded complex is not bounded. The complex $\mathbf{R}\mathcal{H}om(\mathcal{V}, \mathcal{L})$ makes perfect sense in some suitable unbounded derived category, but it rarely belongs to $\mathbf{D}^b(\mathrm{Coh}/X)$. This means that already parts (i) and (ii) of Remark 1.7 pose a serious challenge. We define this problem away by making it our next list of desiderata:

Definition 1.13. *A* dualizing complex $\mathcal{I} = \mathcal{I}_X$ *is an object* $\mathcal{I} \in \mathbf{D}^b(\mathrm{Coh}/X)$ *so that*

(i) *For all objects* $\mathcal{V} \in \mathbf{D}^b(\mathrm{Coh}/X)$*, the derived* Hom*–complexes* $\mathcal{V}^* = \mathbf{R}\mathcal{H}om(\mathcal{V}, \mathcal{I})$ *belong to* $\mathbf{D}^b(\mathrm{Coh}/X)$*.*

(ii) *For all objects* $\mathcal{V} \in \mathbf{D}^b(\mathrm{Coh}/X)$*, the natural maps* $\mathcal{V} \longrightarrow \mathcal{V}^{**}$ *are isomorphisms.*

Remark 1.14. In Remark 1.12 we learned how to achieve Remark 1.7(iii). Definition 1.13 delivers for us Remark 1.7(i) and (ii); if a dualizing complex exists then these two wishes come for free. Of course all this means is that we have codified our problem as a definition; it is not clear how to obtain dualizing complexes.

Dualizing complexes often exist, but not always. The classical way to build them is revolting; one constructs them globally by gluing together local bits, a very painful process in the derived category. It seems fair to say that there are mysteries about dualizing complexes that we have yet to understand properly, even today, more that forty years after the subject was introduced. I do not want to dwell on this here; it will be discussed in more detail elsewhere.

The good news is that our understanding of Remark 1.7(iv) and (v) has come a long way since the beginnings of the subject. It is now known that (iv) and (v) are basically formal; they follow from the existence of certain adjoints, and from the fact that these adjoints preserve dualizing complexes. The reader can find a modern treatment of the subject in [11], which builds on results in the earlier [9, 7].

Remark 1.15. This ends our review of the classical approach to Grothendieck duality. In Remark 1.14 we noted that today, in our modern day and age, we have come to have a relatively good grasp of the parts of theory that deal with morphisms of schemes. Strangely enough the part that has eluded us, which remains murky after all these years, is the question of the existence of dualizing complexes and of their construction.

In the last few years we have come to have a very new way of looking at dualizing complexes, and in the remainder of this survey we try to give the reader a sketch of the novel theory.

Let me immediately make a disclaimer, explaining what we *do not* plan to include here. In Remark 1.14 we noted that the current understanding, of how to go about constructing dualizing complexes, is not substantially advanced on what Grothendieck knew in the 1960s. In the previous paragraph we mentioned that we now have a new way of looking at dualizing complexes, and the reader can wonder whether this can lead to improvements on the construction methods. At the moment we do not know; the possibility is there but there are no theorems yet. In the remainder of this survey we will talk about results, not potential. We try to give the reader an overview of the new developments, as they now stand; more speculative thoughts on what we hope to achieve in the future will appear elsewhere.

2. A reminder of compact objects

The new results, whose exposition constitutes the core of this survey, rely on the formalism of compact objects in [TR5] triangulated categories. We briefly recall the key definitions.

Definition 2.1. *Let $\mathcal{T}$ be a [TR5] triangulated category; that is $\mathcal{T}$ is a triangulated category in which all small coproducts exist. We define:*

(i) *An object $k \in \mathcal{T}$ is* compact *if any map from k, to an arbitrary coproduct, factors through a finite subcoproduct.*
(ii) *The full subcategory of all compact objects in $\mathcal{T}$ will be denoted $\mathcal{T}^c$.*
(iii) *$\mathcal{T}$ is* compactly generated *if*
  (a) $\mathcal{T}^c$ is essentially small.
  (b) $\mathcal{T}^c$ generates $\mathcal{T}$.

Remark 2.2. Part (iii)(b) of Definition 2.1 might need some elaboration. The assertion that $\mathcal{T}^c$ generates $\mathcal{T}$ means that one of the following equivalent conditions holds:

(i) If $\mathcal{S} \subset \mathcal{T}$ is a triangulated subcategory closed under coproducts and containing $\mathcal{T}^c$, then $\mathcal{S} = \mathcal{T}$.
(ii) Every non-zero object $t \in \mathcal{T}$ admits a non-zero map $k \longrightarrow t$, with $k \in \mathcal{T}^c$.

3. The results of Jørgensen, Iyengar and Krause

In §1 we recalled the classical theory of Grothendieck duality. Much of what we said, in §1, was valid both in the complex analytic and in the algebraic categories. For the recent results, which I am about to start discussing, we do not know this. We have theorems valid for schemes; it may well be true that they generalize to complex analytic spaces, but at the moment we cannot prove this. It is not even entirely clear what the right analytic statements ought to be.

Notation 3.1. For this reason, from now on all our spaces will be noetherian, separated schemes. If X is a (noetherian, separated) scheme, all the sheaves we will consider on X will be quasicoherent. Thus the category Inj/X will be the category of all injective objects in the abelian category of quasicoherent sheaves over X. Similarly, the category Proj/X will be the category of projective quasicoherent sheaves, and Flat/X will denote that category of flat quasicoherent sheaves. The symbols $\mathbf{K}(\mathrm{Inj}/X)$, $\mathbf{K}(\mathrm{Proj}/X)$ and $\mathbf{K}(\mathrm{Flat}/X)$ will be the associated homotopy categories. Their objects are, respectively, cochain complexes of injective, projective or flat quasicoherent sheaves. The morphisms, in all three cases, are homotopy equivalence classes of cochain maps.

One more category will figure prominently; it is the category $\mathbf{D}^b(\mathrm{Coh}/X)$ which we met in §1. This is our notation for the bounded derived category of coherent sheaves on X. In §1 we sometimes allowed ourselves to consider coherent analytic sheaves on a complex analytic space X; from now on we only look at the algebraic framework, meaning we restrict attention to coherent algebraic sheaves over a noetherian, separated scheme X.

This fixes our conventions. In §1 we recalled the classical theory of Grothendieck duality and, in particular, of the key role played by dualizing complexes. In §2 we had a brief reminder of compactly generated triangulated categories. The reader may well wonder what the two could possibly have to do with each other. A dualizing complex $\mathcal{I}$, over a noetherian separated scheme X, is an object of $\mathbf{D}^b(\mathrm{Coh}/X)$ so that the functor

$$\mathbf{R}\mathcal{H}om(-,\mathcal{I}) \ : \ \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} \longrightarrow \mathbf{D}^b(\mathrm{Coh}/X)$$

is an equivalence of categories. Note that neither $\mathbf{D}^b(\mathrm{Coh}/X)$ nor $\mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}}$ can possibly be a [TR5] triangulated categories; arbitrary direct sums or products of coherent sheaves are not coherent. There is therefore no way that either $\mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}}$ or $\mathbf{D}^b(\mathrm{Coh}/X)$ could be compactly generated. What conceivable relevance could compactly generated categories have?

Until 2005 there was no known connection. Let me now begin reviewing the recent articles which changed that. We start with two theorems, both from 2005:

Theorem 3.2. (Krause [6]**)** *Suppose that X is a scheme (noetherian and separated as always).*[1] *Then the category $\mathbf{K}(\mathrm{Inj}/X)$ is compactly generated. Furthermore, there is a natural equivalence*

$$\mathbf{K}(\mathrm{Inj}/X)^c \quad \cong \quad \mathbf{D}^b(\mathrm{Coh}/X)\ .$$

Theorem 3.3. (Jørgensen [5]**)** *Let X be a (notherian, separated)* AFFINE *scheme.*[2] *Then the category $\mathbf{K}(\mathrm{Proj}/X)$ is compactly generated. Furthermore, there is a natural equivalence*

$$\mathbf{K}(\mathrm{Proj}/X)^c \quad \cong \quad \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}}\ .$$

Remark 3.4. The statement of Theorem 3.3 highlighted the hypothesis I don't like, namely that X be affine. We will return to it later.

Remark 3.5. Assume X is an affine (noetherian, separated) scheme, and therefore both theorems apply to X. Let $\mathcal{I}$ be a dualizing complex and, replacing

[1]Krause's result is more general than what we state here. We only care about the scheme version.

[2]Jørgensen's theorem [5, Theorem 2.4] imposes a further hypothesis on X, a condition which turns out to be unnecessary. See [10, Facts 2.8(iii)] for an assertion which covers Theorem 3.3, and [10, Remark 2.10] for a comparison with Jørgensen's result.

by an injective resolution if necessary, assume $\mathcal{I}$ is a bounded-below complex of injectives. Taken together, what we know so far gives us a diagram

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} & \xrightarrow{\mathbf{R}\mathcal{H}om(-,\mathcal{I})} & \mathbf{D}^b(\mathrm{Coh}/X) \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Psi} \\ \mathbf{K}(\mathrm{Proj}/X) & & \mathbf{K}(\mathrm{Inj}/X) \end{array} \qquad (\dagger)$$

In this diagram the functors Φ and Ψ are fully faithful. They are the embeddings given by Theorems 3.3 and 3.2: Φ (resp. Ψ) embeds $\mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}}$ (resp. $\mathbf{D}^b(\mathrm{Coh}/X)$) as the subcategory of compact objects in $\mathbf{K}(\mathrm{Proj}/X)$ (resp. $\mathbf{K}(\mathrm{Inj}/X)$). The dualizing complex $\mathcal{I}$ gives us an equivalence of categories $\mathbf{R}\mathcal{H}om(-,\mathcal{I})\colon \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} \longrightarrow \mathbf{D}^b(\mathrm{Coh}/X)$, and it is natural to wonder whether it can be extended, from the subcategories of compact objects, to the entire compactly generated categories. The answer is Yes; from the 2006 article by Iyengar and Krause [4] we know that the diagram (†) above can be completed to a commutative square

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} & \xrightarrow{\mathbf{R}\mathcal{H}om(-,\mathcal{I})} & \mathbf{D}^b(\mathrm{Coh}/X) \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Psi} \\ \mathbf{K}(\mathrm{Proj}/X) & \xrightarrow[\mathcal{I}\otimes -]{} & \mathbf{K}(\mathrm{Inj}/X) \end{array} \qquad (\dagger\dagger)$$

In other words the functor $\mathcal{I}\otimes -$, which takes an object $\mathcal{S} \in \mathbf{K}(\mathrm{Proj}/X)$ to the object $\mathcal{I}\otimes\mathcal{S}$ in $\mathbf{K}(\mathrm{Inj}/X)$, makes the square (††) commute (up to a canonical natural isomorphism). It follows easily, from the general theory of compactly generated triangulated categories, that the functor $-\otimes\mathcal{I}\colon \mathbf{K}(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Inj}/X)$ must be an equivalence of categories.

Remark 3.6. We have outlined some of the results of three recent articles: Krause [6], Jørgensen [5] and Iyengar–Krause [4]. I would like to emphasize that these three lovely articles contain much that we have not covered; the theorems we mentioned admit generalizations, and they have many interesting applications we have not touched. This survey makes no attempt to give the sharpest known theorems, or to discuss known or potential ways to use the theory.

Remark 3.7. In Remark 3.4 we noted that Jørgensen's Theorem 3.3 has the undesirable hypothesis that X should be affine. Everything that followed theorefore had the same restrictive hypothesis. Perhaps I should explain why this hypothesis is such a headache.

If we plan to use the formalism to study dualizing complexes, and to deduce information about Grothendieck duality, then it is a royal pain to have to assume all our schemes affine. Grothendieck duality is a statement about proper morphisms of schemes; we reviewed this in §1. The only proper morphisms between *affine* schemes are the finite maps; if our theory is going to be limited to affine schemes we will miss all the interesting geometry.

The paper by Iyengar and Krause [4] contained several intriguing results, with at least some suggestion that they had the germ of a method to handle the non-affine case. For the sake of brevity let me confine myself to the one which works, as we now know.

Jørgensen's Theorem 3.3 tells us that the category $\mathbf{K}(\text{Proj}/X)$ is compactly generated. The inclusion $j_! \colon \mathbf{K}(\text{Proj}/X) \longrightarrow \mathbf{K}(\text{Flat}/X)$ is a coproduct-preserving triangulated functor whose source is a compactly generated category $\mathbf{K}(\text{Proj}/X)$; formal nonsense, about compactly generated categories, guarantees that $j_!$ must have a right adjoint $j^* \colon \mathbf{K}(\text{Flat}/X) \longrightarrow \mathbf{K}(\text{Proj}/X)$. Iyengar and Krause noted the existence of this adjoint and made clever use of it, showing it to be quite a handy functor to work with. The observation I made, inspired by Iyengar and Krause's manuscript, was the following. The functor $j_!$ is obviously a fully faithful triangulated functor, and, by the above, it possesses a right adjoint j^*; some further general abstraction informs us that the right adjoint j^* must be a Verdier quotient map. Perhaps we should remind the reader.

Reminder 3.8. Let $\mathcal{S}$ and $\mathcal{T}$ be triangulated categories, and assume that $j_! \colon \mathcal{S} \longrightarrow \mathcal{T}$ is a triangulated functor possessing a right adjoint j^*. Define $\mathcal{S}^\perp$ to be the kernel of j^*; it is the full subcategory of all objects $t \in \mathcal{T}$ with $j^* t = 0$. It is easy to see that the objects of $\mathcal{S}^\perp$ can be characterized by the formula

$$\text{Ob}(\mathcal{S}^\perp) \quad = \quad \{t \in \mathcal{T} \mid \text{Hom}(j_! s, t) = 0 \text{ for all } s \in \mathcal{S}\} \ ;$$

that is $\mathcal{S}^\perp$ is the (right) orthogonal of $\mathcal{S}$ with respect to the Hom–pairing. The functor $j^* \colon \mathcal{T} \longrightarrow \mathcal{S}$ kills the subcategory $\mathcal{S}^\perp \subset \mathcal{T}$, and therefore there is a canonical factorization through the Verdier quotient; the functor j^* can be written, uniquely, as a composite

$$\mathcal{T} \xrightarrow{\ \pi\ } \mathcal{T}/\mathcal{S}^\perp \xrightarrow{\ \alpha\ } \mathcal{S} \ .$$

The general nonsense fact that was quoted above is that, as long as the functor $j_!$ is fully faithful, the functor α must be an equivalence of categories.

Remark 3.9. When one works with these functors it is often handy to note that we know a quasi-inverse for α. The composite

$$\mathcal{S} \xrightarrow{\ j_!\ } \mathcal{T} \xrightarrow{\ \pi\ } \mathcal{T}/\mathcal{S}^\perp \xrightarrow{\ \alpha\ } \mathcal{S}$$

is naturally isomorphic to the identity, and hence $\pi j_!$ must be a quasi-inverse for α. The map $\pi j_!$ is more explicit than α, and this facilitates certain computations.

Remark 3.10. Now we return to the special case of the fully faithful functor $j_! \colon \mathbf{K}(\text{Proj}/X) \longrightarrow \mathbf{K}(\text{Flat}/X)$, with right adjoint j^*. From the general, abstract result of Remark 3.8 we learn that j^* can be identified with the projection to the

Verdier quotient

$$\pi : \mathbf{K}(\mathrm{Flat}/X) \longrightarrow \frac{\mathbf{K}(\mathrm{Flat}/X)}{\mathbf{K}(\mathrm{Proj}/X)^{\perp}} .$$

This much is abstract nonsense; if we want to actually make use of it we need to figure out concretely what is the kernel of j^*. We must try to find a manageable description of the objects in $\mathbf{K}(\mathrm{Proj}/X)^{\perp}$. I set about doing this, and produced several more approachable characterizations; Let me give two of these below.

Facts 3.11. Let X be a (noetherian, separated) affine scheme, and let $\mathcal{P}$ be an object of $\mathbf{K}(\mathrm{Flat}/X)$. It is proved in [10] that the following are equivalent:

(i) $\mathcal{P}$ belongs to $\mathbf{K}(\mathrm{Proj}/X)^{\perp}$.

(ii) $\mathcal{P}$ is an acyclic cochain complex

$$\cdots \longrightarrow \mathcal{P}^{-2} \xrightarrow{\partial^{-2}} \mathcal{P}^{-1} \xrightarrow{\partial^{-1}} \mathcal{P}^{0} \xrightarrow{\partial^{0}} \mathcal{P}^{1} \xrightarrow{\partial^{1}} \mathcal{P}^{2} \longrightarrow \cdots$$

where, for each $i \in \mathbb{Z}$, the image of the map $\partial^i \colon \mathcal{P}^i \longrightarrow \mathcal{P}^{i+1}$ is a sheaf of flat, quasicoherent $\mathcal{O}_X$–modules.

(iii) For any object $\mathcal{Q} \in \mathbf{K}(\mathrm{Qcoh}/X)$, the tensor product $\mathcal{P} \otimes \mathcal{Q}$ is acyclic.

Remark 3.12. We learned, in Facts 3.11, that (i), (ii) and (iii) are equivalent as long as X is affine. What about the non-affine case? For example, what can I say about $\mathbb{P}^1$?

If $X = \mathbb{P}^1$ one can show easily that the only projective, quasicoherent sheaf is zero. This means that everything is orthogonal to the projectives; we have that $\mathbf{K}(\mathrm{Proj}/\mathbb{P}^1)^{\perp}$ is equal to all of $\mathbf{K}(\mathrm{Flat}/\mathbb{P}^1)$. It is equally clear that not every object in $\mathbf{K}(\mathrm{Flat}/\mathbb{P}^1)$ satisfies (ii); not every cochain complex of flat quasicoherent sheaves need be acyclic. As the special case of $X = \mathbb{P}^1$ illustrates, (i) and (ii) need not be equivalent on a general, non-affine X. It turns out that (ii) and (iii) are equivalent for every X; this is not quite so obvious, but may be found in Murfet's thesis.

The question I asked my student, Daniel Murfet, was whether the theory works better if we start with (ii) as the right generalization of $\mathbf{K}(\mathrm{Proj}/X)^{\perp}$ to the non-affine case. The answer turns out to be Yes; in the remainder of this survey I will sketch some of the results in Murfet's PhD thesis [8].

4. What Murfet proves

The starting point of Murfet's thesis is to use one of the equivalent characterizations in Facts 3.11 (ii) and (iii); we remind the reader that they are equivalent even for non-affine X. We therefore define:

Definition 4.1. *Let X be a noetherian, separated scheme. The full subcategory $\mathcal{E}(X) \subset \mathbf{K}(\mathrm{Flat}/X)$ has for its objects the cochain complexes satisfying the criterion of Fact 3.11(iii); that is an object $\mathcal{P} \in \mathbf{K}(\mathrm{Flat}/X)$ belongs to the subcategory $\mathcal{E}(X)$ if and only if, for all $\mathcal{Q} \in \mathbf{K}(\mathrm{Qcoh}/X)$, the tensor product $\mathcal{P} \otimes \mathcal{Q}$ is acyclic.*

And now follows the key definition:

Definition 4.2. *The* mock homotopy category of projectives, *denoted* $\mathbf{K}_m(\mathrm{Proj}/X)$*, is the Verdier quotient* $\mathbf{K}(\mathrm{Flat}/X)/\mathcal{E}(X)$*. The Verdier quotient map will be denoted as*

$$j^*\colon \mathbf{K}(\mathrm{Flat}/X) \longrightarrow \mathbf{K}_m(\mathrm{Proj}/X) \quad = \quad \frac{\mathbf{K}(\mathrm{Flat}/X)}{\mathcal{E}(X)} \,.$$

Remark 4.3. In §3 we learned that, if X is affine, then the functor j^* has a left adjoint $j_!$, and this left adjoint may be canonically identified with the natural inclusion $j_!\colon \mathbf{K}(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Flat}/X)$. Among other things this means that, as long as X is affine, the mock homotopy category of projectives $\mathbf{K}_m(\mathrm{Proj}/X)$ canonically agrees with the usual homotopy category of projectives $\mathbf{K}(\mathrm{Proj}/X)$.

The first result we want to mention is

Theorem 4.4. *The functor* $j^*\colon \mathbf{K}(\mathrm{Flat}/X) \longrightarrow \mathbf{K}_m(\mathrm{Proj}/X)$ *has a right adjoint* $j_*\colon \mathbf{K}_m(\mathrm{Proj}/X) \longrightarrow \mathbf{K}(\mathrm{Flat}/X)$*.*

Remark 4.5. Note that Theorem 4.4 is already non-trivial in the affine case. In Remark 4.3 we noted that, for X affine, the functor j^* has a *left* adjoint $j_!$, and this left adjoint is induced by the obvious embedding of projectives into flats. The existence of a *right* adjoint j_* is certainly not immediate. In my article [12] the existence is proved for X affine, and Murfet [8] shows us how to deduce the general case from the affine statement.

It turns out that this is really key to Murfet's entire approach; the existence of the adjoint j_* permits one to reduce many global statements to affine ones. It goes without saying that, when I proved the results in [12], I had no idea that this would turn out to be a key lemma. I was interested in it for other reasons, related to some intriguing theorems in Iyengar and Krause [4]. We will not discuss them in this article.

Corollary 4.6. *The category* $\mathbf{K}_m(\mathrm{Proj}/X)$ *has small* Hom*–sets.*

Proof. We have a Verdier quotient map $j^*\colon \mathbf{K}(\mathrm{Flat}/X) \longrightarrow \mathbf{K}_m(\mathrm{Proj}/X)$, and Theorem 4.4 provides us with a right adjoint j_*. The right adjoint of a Verdier quotient map is always fully faithful, by general nonsense of the same type as we mentioned in Reminder 3.8. Thus j_* embeds $\mathbf{K}_m(\mathrm{Proj}/X)$ as a full subcategory of $\mathbf{K}(\mathrm{Flat}/X)$, which obviously has small Hom–sets. ☐

Remark 4.7. If X is affine we have two fully faithful embeddings $j_!$, j_* of the category $\mathbf{K}(\mathrm{Proj}/X) \cong \mathbf{K}_m(\mathrm{Proj}/X)$ into $\mathbf{K}(\mathrm{Flat}/X)$. The functor $j_!$ is the obvious inclusion, while j_* is much more mysterious.

When X is general, that is not necessarily affine, Theorem 4.4 shows that j_* still makes perfect sense; the right adjoint exists unconditionally, for every X. It turns out that the left adjoint usually doesn't exist; only the mysterious embedding extends to the case of arbitrary X.

Murfet's next pivotal result is

Theorem 4.8. *The category* $\mathbf{K}_m(\mathrm{Proj}/X)$ *is compactly generated. Furthermore, an object* $\mathcal{Q} \in \mathbf{K}_m(\mathrm{Proj}/X)$ *is compact if and only if, for every affine open subset* $U \subset X$*, the restriction of* $\mathcal{Q}$ *to the category* $\mathbf{K}_m(\mathrm{Proj}/U) \cong \mathbf{K}(\mathrm{Proj}/U)$ *is compact.*

Remark 4.9. For the benefit of the experts let me say a tiny bit about the way Murfet proves Theorem 4.8.

We begin with the easy part: if $U \subset X$ is any open subset, and if $g\colon U \longrightarrow X$ is the inclusion, then there are a couple of obvious functors connecting $\mathbf{K}_m(\mathrm{Proj}/U)$ with $\mathbf{K}_m(\mathrm{Proj}/X)$. There is the restriction $g^*\colon \mathbf{K}_m(\mathrm{Proj}/X) \longrightarrow \mathbf{K}_m(\mathrm{Proj}/U)$, and it has a right adjoint g_*. The right adjoint can be given quite explicitly and one sees, almost immediately, that it respects coproducts. From [9, Theorem 5.1] it follows that the restriction map g^* takes compacts to compacts. It is slightly less obvious, but still quite formal, to see that if $\mathcal{Q}$ is locally compact then it is compact. The non-trivial part is to see the compact generation; it is not clear how to glue together non-zero compacts on open affine sets to obtain global compacts.

Choose a finite cover of X by open affine subsets U_i; that is $X = \cup_{i=1}^n U_i$, with U_i open and affine. For every subset $J \subset \{1, 2, \ldots, n\}$ let $U_J = \cap_{i\in J} U_i$. We have an open immersion $g_J\colon U_J \longrightarrow X$, and the previous paragraph informs us that there is a restriction functor $g_J^*\colon \mathbf{K}_m(\mathrm{Proj}/X) \longrightarrow \mathbf{K}_m(\mathrm{Proj}/U_J)$ with a coproduct-preserving right adjoint. Let $\mathcal{I}(J)$ be the kernel of the functor g_J^*; it is the full subcategory $\mathcal{I}(J) \subset \mathbf{K}_m(\mathrm{Proj}/X)$ of all objects annihilated by the functor g_J^*. What Murfet actually proves is that the $\mathcal{I}(J)$ satisfy the technical conditions of Rouquier. We remind the reader: this means

(i) Let $J \subset \{1, 2, \ldots, n\}$ be any subset, and let $j \in \{1, 2, \ldots, n\}$ be any element. Then the Verdier quotient $\frac{\mathcal{I}(J\cup\{j\})}{\mathcal{I}(J)}$ is compactly generated.

(ii) The categories $\mathcal{I}(J)$ are "transversal"; this means that any map from an object $x \in \mathcal{I}(J)$ to an object $z \in \mathcal{I}(J')$ factors through an object $y \in \mathcal{I}(J\cup J')$.

At this point Rouquier's general theory kicks in, as elaborated in [13]; the compact generation asserted in Theorem 4.8 becomes an immediate consequence.

Remark 4.10. Theorem 4.8 tells us that $\mathbf{K}_m(\mathrm{Proj}/X)$ is compactly generated, and from [9, Theorem 3.1] we learn that it must satisfy Brown representability. Next we will use this.

On the category $\mathbf{K}(\mathrm{Flat}/X)$ there is an obvious tensor product; the tensor product of two complexes of flat, quasicoherent sheaves is a complex of flat, quasicoherent sheaves. The subcategory $\mathcal{E}(X) \subset \mathbf{K}(\mathrm{Flat}/X)$ is obviously a tensor ideal, and hence there is an induced tensor product on the quotient $\mathbf{K}_m(\mathrm{Proj}/X) = \mathbf{K}(\mathrm{Flat}/X)/\mathcal{E}(X)$. Let $\mathcal{G}$ be an object of $\mathbf{K}_m(\mathrm{Proj}/X)$; the functor $-\otimes\mathcal{G}$, which takes an object $\mathcal{F} \in \mathbf{K}_m(\mathrm{Proj}/X)$ to the tensor product $\mathcal{F}\otimes\mathcal{G}$, is a map

$$-\otimes\mathcal{G}\ :\ \mathbf{K}_m(\mathrm{Proj}/X) \longrightarrow \mathbf{K}_m(\mathrm{Proj}/X)\ .$$

It is a coproduct-preserving triangulated functor $-\otimes \mathcal{G}$ from the compactly generated triangulated category $\mathbf{K}_m(\mathrm{Proj}/X)$ to itself; Brown representability, more precisely [9, Theorem 4.1], tells us that there must be a right adjoint. There is a functor

$$\mathcal{H}\mathrm{om}(\mathcal{G},-) \ : \ \mathbf{K}_m(\mathrm{Proj}/X) \longrightarrow \mathbf{K}_m(\mathrm{Proj}/X)$$

right adjoint to $-\otimes \mathcal{G}$. The category $\mathbf{K}_m(\mathrm{Proj}/X)$ becomes a closed monoidal category; there is a tensor product and an internal Hom, adjoint to each other.

Remark 4.10 produced for us the internal Hom–functor on the category $\mathbf{K}_m(\mathrm{Proj}/X)$. Since we pulled it out of a hat, by waving the magic wand of Brown representability, it is not immediately clear how to compute with it.

Given a noetherian, separated scheme X, and an open subset $U \subset X$, we can wonder how the internal Hom on X compares to the internal Hom on U. By this I mean the following. Let $\mathcal{F}, \mathcal{G}$ be two objects in $\mathbf{K}_m(\mathrm{Proj}/X)$. We can form $\mathcal{H}\mathrm{om}_X(\mathcal{F},\mathcal{G})$, which also belongs to $\mathbf{K}_m(\mathrm{Proj}/X)$. If $g\colon U \longrightarrow X$ is the inclusion we can look at the three restrictions

$$g^*\mathcal{F}, \quad g^*\mathcal{G} \quad \text{and} \quad g^*\mathcal{H}\mathrm{om}_X(\mathcal{F},\mathcal{G}) \ ,$$

all of which are objects in $\mathbf{K}_m(\mathrm{Proj}/U)$. It becomes interesting to wonder how $g^*\mathcal{H}\mathrm{om}_X(\mathcal{F},\mathcal{G})$ might compare with $\mathcal{H}\mathrm{om}_U(g^*\mathcal{F}, g^*\mathcal{G})$. General nonsense easily provides us with a comparison map

$$\varphi \ : \ g^*\mathcal{H}\mathrm{om}_X(\mathcal{F},\mathcal{G}) \longrightarrow \mathcal{H}\mathrm{om}_U(g^*\mathcal{F}, g^*\mathcal{G}) \ ;$$

we could reasonably wonder whether the map φ is an isomorphism. This would be particularly useful in the case where $U = \mathrm{Spec}(R)$ is affine, since in the category $\mathbf{K}_m(\mathrm{Proj}/U) \cong \mathbf{K}(R\text{–}\mathrm{Proj})$ we know quite well how to compute $\mathcal{H}\mathrm{om}_U(g^*\mathcal{F}, g^*\mathcal{G})$. Sadly φ need not always be an isomorphism; but it is if we impose some conditions on $\mathcal{F}$ and $\mathcal{G}$. The relevant theorem of Murfet is

Theorem 4.11. *Suppose that X is a (noetherian, separated) scheme and $U \subset X$ is an open subset. Let g stand for the inclusion $g\colon U \longrightarrow X$. Assume $\mathcal{F}$ and $\mathcal{G}$ are two objects in $\mathbf{K}_m(\mathrm{Proj}/X)$ so that*

(i) *$\mathcal{F}$ is locally isomorphic to a complex of vector bundles. That is there is a cover of X by open affine subsets $V_i = \mathrm{Spec}(R_i)$, and on each V_i the restriction of $\mathcal{F}$ is isomorphic, in $\mathbf{K}_m(\mathrm{Proj}/V_i) \cong \mathbf{K}(R_i\text{–}\mathrm{Proj})$, to a complex of finitely generated projective modules.*

(ii) *The complex $\mathcal{G}$ is bounded.*

Then the natural map

$$\varphi \ : \ g^*\mathcal{H}\mathrm{om}_X(\mathcal{F},\mathcal{G}) \longrightarrow \mathcal{H}\mathrm{om}_U(g^*\mathcal{F}, g^*\mathcal{G}) \ ;$$

is an isomorphism.

Remark 4.12. We could now consider the following composite functor

$$\begin{array}{ccc} \mathbf{K}_m(\mathrm{Proj}/X)^c & \xrightarrow{\mathcal{H}\mathrm{om}(-,\mathcal{O}_X)} & \mathbf{K}_m(\mathrm{Proj}/X)^{\mathrm{op}} \\ & & \downarrow{\scriptstyle \pi} \\ & & \mathbf{D}(\mathrm{Qcoh}/X)^{\mathrm{op}} \end{array}$$

That is, the functor takes a compact object $\mathcal{F} \in \mathbf{K}_m(\mathrm{Proj}/X)$ first to the object $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_X) \in \mathbf{K}_m(\mathrm{Proj}/X)$, and then views this complex in $\mathbf{K}_m(\mathrm{Proj}/X)$ as belonging to $\mathbf{D}(\mathrm{Qcoh}/X)$; the functor π has the effect of inverting all homology isomorphisms, not only the ones whose mapping cones are in the category $\mathcal{E}(X)$ of Definition 4.1. A local computation permits us to show that the image of this composite functor lies in $\mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}}$; that is the object $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_X)$ is quasi-isomorphic to a bounded complex of coherent sheaves. The reader should note that $\mathcal{O}_X$ is certainly bounded, while $\mathcal{F}$ is assumed compact, hence is locally compact, and on open affines Jørgensen's Theorem 3.3 tells us exactly what the compacts are. The two technical conditions of Theorem 4.11 are satisfied, and we can compute locally. Anyway we deduce a commutative diagram

$$\begin{array}{ccc} \mathbf{K}_m(\mathrm{Proj}/X)^c & \xrightarrow{\mathcal{H}\mathrm{om}(-,\mathcal{O}_X)} & \mathbf{K}_m(\mathrm{Proj}/X)^{\mathrm{op}} \\ {\scriptstyle \Phi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} & \longrightarrow & \mathbf{D}(\mathrm{Qcoh}/X)^{\mathrm{op}} \end{array},$$

and Murfet's next theorem is

Theorem 4.13. *The functor* $\Phi' \colon \mathbf{K}_m(\mathrm{Proj}/X)^c \longrightarrow \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}}$ *of Remark 4.12 is an equivalence of categories.*

Remark 4.14. Theorem 4.13 gives a generalization of Jørgensen's Theorem 3.3 to the global case, where X is not assumed affine. In fact Murfet even explicitly constructs a fully faithful functor $\Phi \colon \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} \longrightarrow \mathbf{K}_m(\mathrm{Proj}/X)$ whose essential image is the subcategory of compacts $\mathbf{K}_m(\mathrm{Proj}/X)^c \subset \mathbf{K}_m(\mathrm{Proj}/X)$. This Φ is a quasi-inverse for the functor Φ' which we produced in Remark 4.12.

The final theorem we want to mention here, from Murfet's thesis, is

Theorem 4.15. *Let X be a (noetherian, separated) scheme, and let $\mathcal{I}$ be a bounded-below complex in $\mathbf{K}(\mathrm{Inj}/X)$. Whenever possible (see Remark 4.16 for the precise hypothesis) the following square commutes, up to canonical natural isomorphism:*

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}} & \xrightarrow{\mathbf{R}\mathcal{H}om(-,\mathcal{I})} & \mathbf{D}^b(\mathrm{Coh}/X) \\ {\scriptstyle \Phi}\big\downarrow & & \big\downarrow{\scriptstyle \Psi} \\ \mathbf{K}_m(\mathrm{Proj}/X) & \xrightarrow[\mathcal{I}\otimes -]{} & \mathbf{K}(\mathrm{Inj}/X) \end{array} \qquad (**)$$

Remark 4.16. We need to explain what the "whenever possible" might mean, in the statement of Theorem 4.15.

The top horizontal morphism is a map $\mathbf{R}\mathcal{H}om(-,\mathcal{I})$; it is a derived Hom, which usually does not take bounded complexes of coherent sheaves to bounded complexes of coherent sheaves. We already mentioned this in Remark 1.12. The more precise version of Theorem 4.15 asserts that, if $\mathcal{I}$ is a bounded-below complex of injectives so that $\mathbf{R}\mathcal{H}om(-,\mathcal{I})$ takes $\mathbf{D}^b(\mathrm{Coh}/X)$ to itself, then the square commutes.

The bottom horizontal morphism is denoted $\mathcal{I}\otimes -$, which might perhaps be confusing. The category $\mathbf{K}_m(\mathrm{Proj}/X)$ is defined to be the quotient of $\mathbf{K}(\mathrm{Flat}/X)$ by a subcategory $\mathcal{E}(X)\subset\mathbf{K}(\mathrm{Flat}/X)$. Ordinary tensor product, not derived in any sense, defines a functor $\mathcal{I}\otimes -\colon \mathbf{K}(\mathrm{Flat}/X)\longrightarrow\mathbf{K}(\mathrm{Inj}/X)$. What is being asserted, when we write the bottom arrow of the square, is that this functor annihilates $\mathcal{E}(X)$ and therefore factors through the quotient $\mathbf{K}_m(\mathrm{Proj}/X)$. That is, for every $\mathcal{S}\in\mathcal{E}(X)$ we assert that $\mathcal{I}\otimes\mathcal{S}$ is a contractible cochain complex of injectives. In the case where X is affine the proof may be found in [10, Corollary 9.7(ii)]. For the general case see Murfet [8].

The relation of Theorem 4.15 with Grothendieck duality is that the complex $\mathcal{I}$ is a dualizing complex if and only if the two horizontal morphisms in $(**)$ are equivalences. It turns out that one horizontal morphism is an equivalence if and only if they both are. More precisely we know that, if either of the two horizontal maps is an equivalence, then we are in the situation in which the diagram $(**)$ must commute, and the other horizontal functor must also be an equivalence.

Maybe we should make a comment: it is clear, from the definition of dualizing complexes, that the top horizonal morphism is an equivalence if and only if $\mathcal{I}$ is a dualizing complex. What is not so immediate is that this is equivalent to the bottom arrow being an equivalence.

Remark 4.17. The reader is invited to compare the diagram $(**)$ above with the square $(\dagger\dagger)$ of Remark 3.5. To generalize, from the affine case to the global situation, all that is necessary is to replace every $\mathbf{K}(\mathrm{Proj}/X)$ by a $\mathbf{K}_m(\mathrm{Proj}/X)$.

5. Applications

So far I have said nothing about the ways one might use the theory we have outlined. The reason is that both Murfet and Salarian will presumably be writing about their work in these proceedings, and their articles will say much more about the applications than the space available to me could possibly permit.

Let me nevertheless give one small hint, a mere indication of one of the directions in which the results have been used. We now know, following Murfet's recent work, that the category $\mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}}$ can be viewed as the subcategory of compact objects in $\mathbf{K}_m(\mathrm{Proj}/X)$. We have known for a long time that the category $\mathbf{D}^b(\mathrm{Vect}/X)\cong\mathbf{D}^b(\mathrm{Vect}/X)^{\mathrm{op}}$ can be identified with the compact objects in $\mathbf{D}(\mathrm{Qcoh}/X)$. From Murfet's work we also know that the inclusion

$$\mathbf{D}^b(\mathrm{Vect}/X)^{\mathrm{op}} \longrightarrow \mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}}$$

can be extended to a coproduct-preserving inclusion

$$\mathbf{D}(\mathrm{Qcoh}/X) \longrightarrow \mathbf{K}_m(\mathrm{Proj}/X) \ ,$$

and that, up to splitting direct summands, we have an identification

$$\left\{\frac{\mathbf{K}_m(\mathrm{Proj}/X)}{\mathbf{D}(\mathrm{Qcoh}/X)}\right\}^c \quad \cong \quad \frac{\mathbf{D}^b(\mathrm{Coh}/X)^{\mathrm{op}}}{\mathbf{D}^b(\mathrm{Vect}/X)^{\mathrm{op}}} \quad .$$

The quotient on the right is Orlov's triangulated category of singularities, and the quotient $\dfrac{\mathbf{K}_m(\mathrm{Proj}/X)}{\mathbf{D}(\mathrm{Qcoh}/X)}$ is a compactly generated triangulated category, having Orlov's category as its subcategory of compacts. This gives a new way of studying Orlov's invariant of singularities. I should mention that the affine case was already noted and applied in Iyengar and Krause [4]; Murfet's thesis shows how to prove the analogous global statements.

For further results exploiting this the reader is referred to the papers by Murfet and Salarian; there are a few preprints already which will undoubtedly appear in the near future.

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