CLASSIFYING BIG PROJECTIVES OVER SOME NOETHERIAN RINGS

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1. Intro

This text is a kind of manual how to use the technique of fair-sized modules for classifying non-finitely generated projective modules over certain noetherian rings. Unfortunately, the unifying theory is still missing and as we show at the end of the text, there is no hope for this technique to become such a theory. However, sometimes we can succeed.

The first part is just a classification of non-finitely generated projective modules over integral group ring of A_5. This process could be also useful in general - provided we know character tables and also Brauer character tables of a finite group, we can give an algorithm how to calculate idempotent ideals in the corresponding integral group ring. However, it is hard to say whether one can derive some general properties of the semilattice of idempotent ideals in integral group rings from this. Moreover, we also have to understand factors of the group ring modulo its idempotent ideals (in fact to finitely generated projectives over these factors), and this seems to be a very hard problem.

The second part of the text is the study of projective modules over so called generalized Weyl algebras. One can see here, that sometimes, a knowledge of a small part of finitely generated projective modules could help us to give a global picture of non-finitely generated projective modules. This part is based on the discussion I had with G. Puninski in February/March 2007. Some results from this section should appear in our join paper (relatively) soon.

The third part is coming from the attempts we did with D. Herbera in July and September 2007. We construct some examples I have never believed they could exist in semilocal noetherian rings. In the end it seems that this is not the best way how to construct examples (I mean we can do it better than it is written here). But my intention is to explain a (perhaps forgotten) construction of Small and Stafford in relation with projective modules.

The last section of the paper is just to explain why one could be rather critical to the fair sized-technique. In $U(\mathrm{sl}_2(\mathbb{C}))$ we can construct non-finitely generated projective modules that are not fair-sized.

This draft collects calculations I made during my postdoctoral stay in Centre de Recerca Matemàtica in Bellaterra supported by the grant SB 2005-0182 from Spain's ministry of science and education.

Let us very briefly recall results from [13] which will serve as a general framework. Suppose that R is a noetherian ring such that the following condition holds: If $I_1, I_2, \dots$ are ideals in R such that $I_{k+1}I_k = I_{k+1}$ for any $k \in \mathbb{N}$, then there exists $l \in \mathbb{N}$ such that $I_l = I_k$ for any $l \leq k \in \mathbb{N}$.

Recall that by Kaplansky's theorem any projective module is a direct sum of countably generated modules. The ambition of this work is to describe countably generated projective modules that are not finitely generated, so we try to describe basic blocks from which nonfinitely generated projective modules are built. In order to give a real classification we should also specify which direct sums of these blocks are isomorphic.

Let I be an ideal of R. We say that a countably generated projective module P is I-big if any countably generated projective module Q such that $\mathrm{Tr}(Q) \subseteq I$ is a factor of I (thus $R^{(\omega)}$ is R-big and, by the Eilenberg trick, any R-big projective module is isomorphic to $R^{(\omega)}$). A countably generated projective module P is said to be *fair-sized* if the set of ideals $\{I \subseteq R \mid P/PI \text{ is finitely generated}\}$ contains the smallest element (thus any finitely generated projective module is fair-sized).

Fact 1.1. *Let R be a noetherian ring satisfying (*). Then any countably generated projective module P is fair-sized. Let I be the smallest element of the set of ideals $\{I \subseteq R \mid P/PI \text{ is finitely generated}\}$. Then*

(i) *The ideal I is idempotent.*
(ii) *The projective module P is I-big.*

Fact 1.2. *Let R be a noetherian ring, I an idempotent ideal and let P,Q be countably generated I-big projective modules such that $P/PI \simeq Q/QI$. Then $P \simeq Q$.*

Fact 1.3. *Let R be a noetherian ring satisfying (*). Then for any idempotent ideal I and for any finitely generated projective module P' over R/I there exists unique countably generated projective module P such that P is I-big and $P/PI \simeq P'$.*

So at least the blocks over noetherian rings with (*) can be understood via the set of idempotent ideals and finitely generated projective modules over the corresponding factors.

Some examples of noetherian rings with (*) are the following

(i) Semilocal noetherian rings
(ii) Integral group rings of finite groups.
(iii) Universal enveloping algebras of solvable Lie algebras of finite dimension over a field of characteristic 0.

2. Integral group rings

The aim of this section is to consider techniques for finding idempotent ideals in the integral group rings of finite groups. We will use a standard local-global method that works in the setting of orders over Dedekind domains. However, we are not that ambitious right now. The core of this section is to give a classification of non-finitely generated projective modules over $\mathbb{Z}[A_5]$. The following fact explains the principles.

First let us introduce the notation we will use throughout this section. Let G be a finite group, $R = \mathbb{Z}[G], R_p = \mathbb{Z}_{(p)}[G], R_0 = \mathbb{Q}[G]$. For any prime p we have $R \subseteq R_p \subseteq R_0$. If I is an ideal of R, $I_{(p)}$ stands for the ideal in R_p generated by I and $I_{(0)}$ stands for the ideal of R_0 generated by I. That is $I_{(p)} = \mathbb{Z}_{(p)}I$, $I_{(0)} = \mathbb{Q}I$. We say that an ideal $I \subseteq R$ (or an ideal $I \subseteq R_p$) *extends to* an ideal $K \subseteq R_0$ if $K = \mathbb{Q}I$. The *augmentation ideal* of $S[G]$ is the kernel of canonical homomorphism $S[G] \to S$ given by $\sum_{g\in G} s_g g \mapsto \sum_{g\in G} s_g$ and it is denoted as $\mathrm{Aug}(S[G])$.

Fact 2.1. *Let G be a finite group and let $R = \mathbb{Z}[G]$. Then*

(i) *If I is an ideal of R, then $I_{(0)} = \mathbb{Q}I_{(p)}$ for any prime p.*
(ii) *Let I, K be ideals in R. Then $I = K$ if and only if $I_{(p)} = K_{(p)}$ for any prime p.*
(iii) *If $I \subseteq R$ is an ideal, then I is idempotent if and only if $I_{(p)}$ is idempotent for any prime p.*
(iv) *If I, K are idempotent ideals of R and p a prime not dividing $|G|$, then $I_{(p)} = K_{(p)}$ if and only if $I_{(0)} = K_{(0)}$. In this case all central idempotents of R_0 are in R_p and any idempotent ideal in R_p is generated by one of these central idempotents.*
(v) *Let e be a central idempotent of R_0 and suppose that for any prime $p \mid |G|$ we have an idempotent ideal $I_p \subseteq R_p$ such that $\mathbb{Q}I_p = eR_0$. Then there exists unique idempotent ideal $I \subseteq R$ such that $I_{(p)} = I_p$ for any $p \mid |G|$ and $I_{(p)} = eR_p$ for any $p \not| |G|$.*

Proof. The statements (i),(ii),(iii) and (v) are rather standard. The statement (iv) follows from the fact that $\mathbb{Z}_{(p)}[G]$ is a maximal $\mathbb{Z}_{(p)}$-order in $\mathbb{Q}[G]$ if and only if p does not divide $|G|$ (see [3, Proposition 27.1]). Then we could use the machinery for maximal orders.

However, we can give another proof of (iv). Let $\mathbb{Q} \subseteq F$ be a finite Galois extension of $\mathbb{Q}$ such that F is a splitting field of G. Recall that if ξ is a

complex character of a simple representation of G over F (considered as a function $\xi\colon G \to F$), then $\frac{\xi(1_G)}{|G|}(\sum_{g\in G}\xi(g^{-1})g)$ is a primitive central idempotent of $F[G]$. In order to get the set of primitive central idempotent of $\mathbb{Q}[G]$, we just consider the usual action of $\mathrm{Gal}(F:\mathbb{Q})$ on the set of primitive central idempotents of $F[G]$ and we take sums of the orbits. From this it follows that if p is a prime and $p \not| |G|$, then any central idempotent of R_0 is in R_p.

Now let I be an idempotent ideal, then $\mathbb{Q}I$ is an ideal of R_0 generated by a central idempotent e of R_0. Then $K = eR_p$ is an idempotent ideal of R_p, necessarily $I \subseteq K$ because $eI = I$. Since $\mathbb{Q}I = \mathbb{Q}K$, there exists $k \in \mathbb{N}$ such that $p^k K \subseteq I$. Since $\mathbb{Z}_p[G]$ is semisimple, for any $n \in \mathbb{N}$ idempotent ideals in $\mathbb{Z}_{p^n}[G]$ are generated by central idempotents. Moreover if K' is an idempotent ideal of $\mathbb{Z}_{p^{2n}}[G]$, then $p^n K'$ is an essential submodule of K'. Now let $\pi\colon R_p \to \mathbb{Z}_{p^{2k}}[G]$ be the canonical projection. Then $p^k\pi(K) \subseteq \pi(I) \subseteq \pi(K)$. Regarding the previous remarks, $\pi(I) = \pi(K)$ and since π is an epimorphism such that $\mathrm{Ker}\ \pi \subseteq J(R_p)$ (see Fact 2.3) and R_p is noetherian, $I = K$ follows. □

The following result follows also from [18].

Corollary 2.2. *Any integral group ring of a finite group satisfies (*). Moreover, there are only finitely many idempotent ideals over these rings.*

Proof. Since R is a ring of Krull dimension 1, it is enough to see that R has no descending chain of idempotent ideals. Let I be an idempotent ideal, let e be a central idempotent of R_0 such that $eR_0 = \mathbb{Q}I$. Then $I_{(p)} = eR_p$, for primes not dividing $|G|$ by Fact 2.1(iv). If p is a prime divisor of $|G|$, then we have only finitely many possibilities for $I_{(p)}$. Therefore we conclude by Fact 2.1(v). □

The proof of the corollary gives a method for a computation of idempotent ideals in R. We can proceed as follows: Take an ideal I_0 of R_0. Let P be the set of prime divisors of $|G|$. For any $p \in P$ we find the set M_p consisting of those of idempotent ideals of R_p which extend to I_0. Then there is a bijective correspondence between idempotent ideals of R extending to I_0 and the set $\prod_{p\in P} M_p$.

Thus we are left to work in localizations, which are semilocal:

Fact 2.3. *The natural homomorphism $\pi_p\colon R_p \to \mathbb{Z}_p[G]$, is a local morphism for any p prime. In particular $pR_p \subseteq J(R_p)$ and R_p is a semilocal ring.*

In order to find the number of different simple modules over R_p we use the following results proved by Berman and Witt (see [3, Theorem 21.5, Theorem 21.25])

Fact 2.4. *Suppose that G is a finite group of exponent m.*
(i) *Let $\sim$ be a relation on G given by $g \sim h$ if g is conjugate to h^t for some $t \in \mathbb{N}, (t, m) = 1$. Then the number of simple $\mathbb{Q}[G]$-modules equals to $|G/\sim|$.*
(ii) *Let p be a prime, and $G_{p'}$ the set of p-regular elements of G. On the set $G_{p'}$ consider the equivalence $g \sim h$ if g is conjugate to h^{p^j} for some $j \in \mathbb{N}_0$. Then the number of simple $\mathbb{Z}_p[G]$-modules equals to $|G_{p'}/\sim|$.*

Let us demonstrate the method in case $G = A_5$, that is the alternating group on 5 elements. The usual question "Why A_5?" has a simple answer. By a result of Swan non-finitely generated projective modules over integral group rings of finite solvable groups are free. Therefore there are no proper idempotent ideals, (a direct proof of this was given by Roggenkamp [17]). On the other hand, it is known that if G contains a perfect normal subgroup H, that is $[H, H] = H$ and $H \trianglelefteq G$, then the augmentation ideal of H is idempotent. So it is tempting to think that all idempotent ideals of R are exactly augmentation ideals of perfect normal subgroups of G. Then the projective modules over R would be induced from finitely generated projective modules over $\mathbb{Z}[G/H]$, where H varies the set of perfect normal subgroups of G. So A_5 is the easiest group where this conjecture could fail. Unfortunately, we will see that there indeed exists an idempotent ideal that is not the augmentation ideal of a perfect normal subgroup. So the structure of projective modules over integral group rings seems to be more complicated.

Throughout the next paragraphs we suppose that $G = A_5$. The conjugacy classes of G are the following: c_1 - the conjugacy class of identity; c_2 - permutations that are product of two 2-cycles (the conjugacy class of $(1, 2)(3, 4)$); c_3 - all 3-cycles; c_5 - the conjugacy class of $(1, 2, 3, 4, 5)$ and c_5' - the conjugacy class of $(1, 3, 5, 2, 4)$.

First let us recall what we know about semisimple ring R_0. The primitive central idempotents of R_0 are $e_1 = \frac{1}{60}\sum_{g\in G} g$, $e_3 = \frac{1}{20}(6 - 2\sum_{g\in c_2} g + \sum_{g\in c_5\cup c_5'} g)$, $e_2 = \frac{1}{15}(4 + \sum_{g\in c_3} g - \sum_{g\in c_5\cup c_5'} g)$, $e_5 = \frac{1}{12}(5 + \sum_{g\in c_2} g - \sum_{g\in c_3} g)$. Let T_1, T_3, T_2, T_5 be corresponding simple modules (e_i corresponds to T_i). Their dimensions over $\mathbb{Q}$ are $1, 6, 4, 5$.

We need calculate idempotent ideals over R_2, R_3, R_5. Let us denote $S_i = \mathbb{Z}_i[A_5]$, for $i = 2, 3, 5$. By Fact 2.3, we get that any simple S_i-module can be considered as a simple R_i-module and there are no other simple R_i-modules than these. Therefore one can use Fact 2.4 to calculate the number of simple

R_i-modules (the other possibility is to look into Brauer character tables of A_5). We get that each of R_2, R_3, R_5 has exactly three non-isomorphic simple modules. Recall that idempotent ideals in a semilocal ring are determined by their simple factors. We call a ring T *semi-semiperfect* if for any simple T-module M there exists a positive integer n such that M^n has a projective cover. The next lemma describes the distribution of idempotent ideals in R_i, for $i \in \{2,3,5\}$. In the following proofs we will use I_i exclusively as a shortcut for $\mathrm{Aug}(R_i)$.

Lemma 2.5. *Let $i \in \{2,3,5\}$. The ring R_i has exactly 3 minimal idempotent ideals and any idempotent ideal of R_i is a sum of minimal idempotent ideals. Moreover, R_i is semi-semiperfect and any idempotent ideal of R_i is a trace ideal of a finitely generated projective module. Finally, two minimal idempotent idempotent ideals are described as follows: If I_i is the augmentation ideal of R_i, then e_iR_i, $(1-e_i)I_i$ are minimal idempotent ideals of R_i.*

Proof. We give the proof for $i = 5$, remaining cases are similar. The augmentation ideal $I_5 \subseteq R_5$ is idempotent, since A_5 is perfect, and $e_5 \in R_5$. Therefore also e_5R_5 and $(1-e_5)I_5$ are idempotent ideals. Let M_1, M_2, M_3 be the representative set of simple R_5-modules and suppose that M_1 is the module induced from the trivial representation of S_5. Obviously $M_1I_5 = 0$ so M_1 is not a factor of I_5 and, since I_5 has to have at least two simple factors (it contains two different nontrivial idempotent ideals), M_2, M_3 are both factors of I_5. Suppose we choose the notation such that M_2 is the unique simple factor of $(1-e_5)I_5$ and M_3 is the unique simple factor of e_5R_5.

Obviously e_5R_5 is the trace ideal of the projective module e_5R_5. Let $g = (1,2)(3,4)$, then the idempotent $e' = (1-e_5)(1-\frac{1}{2}(1+g))$ gives a projective R_5-module $P' = e'R_5$ of the trace ideal $(1-e_5)I_5$, it follows $P'/P'J(R_5) = M_2^k$, for some $k \in \mathbb{N}$ (it is necessary to check that $P' \neq 0$, bellow we calculate $\mathbb{Z}_{(5)}$-rank of P' via Hattori-Stallings map).

On the other hand the projective module $P = (1-e_5)R_5$ has the radical factor $P/PJ(R_5) = M_1 \oplus M_2^l$. Therefore P'^l splits in P^k, that is there exists a projective module Q such that $P^k = P'^l \oplus Q$. Since $Q/QJ(R_5) \simeq M_1^k$, it follows that $\mathrm{Tr}(Q)$ is an idempotent ideal such that M_1 is its only simple factor.

So we have proved that finitely generated projective modules Q, P', e_5R_5 are projective covers of convenient finite powers of M_1, M_2, M_3. Therefore $\mathrm{Tr}(Q), \mathrm{Tr}(P')$ and $\mathrm{Tr}(e_5R_5)$ are the minimal idempotent ideals of R_5. $\square$

The previous lemma gives a picture of idempotent ideals in R_2, R_3 and R_5, but in order to understand the idempotent ideals of R, we have to make the calculations from the proof precisely. However, now we know enough to understand the idempotent ideals inside the augmentation ideal of R.

Lemma 2.6. *Let I be the idempotent ideal of R. Then 0 and $\mathrm{Aug}(R)$ are the only idempotent ideals of R contained in $\mathrm{Aug}(R)$.*

Proof. Put $I = \mathrm{Aug}(R)$ and let $0 \neq K$ be an idempotent ideal of R contained in I. Then also $K_{(i)}$ is a non-zero idempotent ideal of R_i contained in I_i, hence, by Lemma 2.5, $\mathbb{Q}K_{(i)}$ is either e_iR_0, $(e_2+e_3+e_5-e_i)R_0$ or $I_{(0)} = (e_2+e_3+e_5)R_0$. Now $\mathbb{Q}K_{(2)} = \mathbb{Q}K_{(3)} = \mathbb{Q}K_{(5)} = \mathbb{Q}K$, an easy inspection gives that the only possibility is $K_{(i)} = I_i$ for any $i \in \{2,3,5\}$. Therefore $K = I$ by Fact 2.1(v). □

For any $i \in \{2,3,5\}$ let K_i be the (unique) minimal idempotent ideal of R_i which is not contained in the augmentation ideal of R_i. In order to classify idempotent ideals in R that are not contained in the augmentation ideal of R we have to calculate $\mathbb{Q}K_2, \mathbb{Q}K_3$ and $\mathbb{Q}K_5$. Let us formulate an auxiliary general result which is probably well known.

Lemma 2.7. *Let $\varphi\colon S \to T$ be a ring homomorphism. If P is a projective S-module of trace ideal I, then $P \otimes_S T$ is a projective T-module of the trace ideal $T\varphi(I)T$.*

Proof. Let X be a set and let $\pi\colon S^{(X)} \to S^{(X)}$ be an idempotent endomorphism of $S^{(X)}$ such that $\pi(S^{(X)}) \simeq P$. If π is expressed as a column finite idempotent matrix A (with respect to the canonical basis), then $\varphi(A)$ is an idempotent matrix corresponding to the endomorphism $\pi'\colon T^{(X)} \to T^{(X)}$ such that $P \otimes_S T \simeq \pi'(T^{(X)})$. Now $\mathrm{Tr}(P)$ (resp. $\mathrm{Tr}(P \otimes_S T)$) is an ideal generated by the entries of A (resp. $\varphi(A)$). □

Fact 2.8. *Let S be a commutative local ring and let H be a finite group. Suppose that $e = \sum_{h\in H} s_h h$ is an idempotent of $S[H]$. The module $eS[H]$ is free when considered as an S-module. Moreover, $|H|s_1 = n.1_S$, where $n \in \mathbb{N}_0$ is the rank of the free S-module $eS[H]$.*

Proof. Let us recall some basic facts about Hattori-Stallings map (for details on this fascinating topic see [6]). Let T be a ring, $T/[T,T]$ be the group that is a factor of the additive group of T modulo $[T,T] = \langle\{t_1t_2 - t_2t_1 \mid t_1, t_2 \in T\}\rangle_{(T,+)}$. Then there exists a map $r\colon K_0(T) \to T/[T,T]$ given as follows. Let P be a finitely generated projective module over T, A some idempotent matrix representing P. Then $r([P]) := \mathrm{Tr}(A) + [T,T]$ (here $\mathrm{Tr}(A)$ is the sum of diagonal entries of A).

Since S is a local ring, $K_0(S) \simeq \mathbb{Z}$. As S is commutative, r is a correctly defined map from $K_0(S) \to S$. It follows that $\operatorname{Im} r \subseteq \mathbb{Z}1_S$. Now look at $S[H]$ as a free S-module of rank $|H|$. The left multiplication by e gives an idempotent endomorphism α of this S-module whose image is $eS[H]$. Now calculate $r([eS[H]])$. Consider the matrix of α with respect to basis $\{h \mid h \in H\}$. Then all diagonal entries of this matrix will be equal to s_1. Therefore $|H|.s_1 = n.1_S$, where n is the rank of the free S-module $eS[H]$. $\square$

Now we can continue in $\mathbb{Z}[A_5]$. In the following proofs I_i is again the augmentation ideal of R_i and $S_i = \mathbb{Z}_i[A_5]$ for any $i \in \{2, 3, 5\}$.

Lemma 2.9. *Let K_5 be the minimal idempotent ideal not contained in* $\operatorname{Aug}(R_5)$. *Then $\mathbb{Q}K_5 = (e_1 + e_2)R_0$.*

Proof. Let M_1, M_2, M_3 be the simple R_5-modules such that M_1 is the unique simple factor of K_5, M_2 is the unique simple factor of $(1 - e_5)I_5$ and M_3 is the unique simple factor of e_5R_5. Let $g = (1, 2)(3, 4)$, $e' = (1 - e_5)(1 - \frac{1}{2}(1 + g))$ gives a projective R_5-module $P' = e'R_5$ of the trace ideal $(1 - e_5)I_5$, so it follows $P'/P'J(R_5) = M_2^k$, $k \in \mathbb{N}$. Moreover, if $P = (1 - e_5)R_5$, then $P/PJ(R_5) \simeq M_1 \oplus M_2^l$ for some $l \in \mathbb{N}$ (recall the multiplicity of M_1 is one in $S_5/J(S_5)$. We want to find k and l. The integer l is given by the multiplicity of M_2 in $S_5/J(S_5)$. Any simple S_5-module is absolutely simple, therefore l equals to dimension of the non-trivial simple representation which is annihilated by e_5. By [20, page 200], $l = 3$. Obviously, P' is a direct summand of P, and $k \in \{1, 2, 3\}$ follows. Using Fact 2.8 we have $\mathbb{Z}_{(5)}$-rank of P equals 35 and $\mathbb{Z}_{(5)}$-rank of P' equals 20. So if $k = 1$, then P'^3 is a direct summand of P, which is not possible. Further look at S_5-module $P'/P'5R_5$. This is a vector space over $\mathbb{Z}_5$ of dimension 20. If $k = 3$, then $P'/P'5R_5 \simeq M^3$, where M is an S_5-module which is a projective cover of M_2 if M_2 is considered as a simple S_5-module. Since 3 does not divide 20, this is also impossible. Therefore $k = 2$.

As we explained in the proof of Lemma 2.5, K_5 is given as a trace of Q, where Q is a projective module by the relation $Q \oplus P'^3 \simeq P^2$. By Lemma 2.7, $\mathbb{Q}K_5 = \operatorname{Tr}(Q \otimes_{R_5} R_0)$. The module $Q \otimes_{R_5} R_0$ has $\mathbb{Q}$-dimension 10 and contains the trivial representation of R_0 with multiplicity 2. The only possibility (regarding $\mathbb{Q}$-dimension of simple R_0-modules) is $T_1^2 \oplus T_2^2$. $\square$

Lemma 2.10. *Let K_3 be the minimal idempotent ideal of R_3 that is not contained in* $\operatorname{Aug}(R_3)$. *Then $\mathbb{Q}K_3 = e_1R_0 + e_5R_0$.*

Proof. Put $e = 1 - e_3$, $g = (1, 2)(3, 4)$ and $h = (1, 2, 3, 4, 5)$. These elements of G give idempotents $e' = e(1 - \frac{1}{2}(1 + g))$ and $f' = e(1 - \frac{1}{5}(1 + h + h^2 +$

$h^3 + h^4)$). Let $P' = e'R_3$, $P'' = f'R_3$ and $P = eR_3$. Let M_1, M_2, M_3 be the simple R_3-modules such that M_1 is the unique simple factor of K_3, M_2 is the unique simple factor of eI_5 and M_3 is the unique simple factor of e_3R_3. Again we want to find $k, l \in \mathbb{N}$ such that $P/PJ(R_3) \simeq M_1 \oplus M_2^l$ and $P'/P'J(R_5) \simeq M_2^k$.

Let us consider a module M over S_3 given by an obvious action of A_5 on the vector space $\{(z_1, \dots, z_5) \in \mathbb{Z}_3^5 \mid z_1 + \dots + z_5 = 0\}$ (that is if $x \in A_5$, then $(z_1, \dots, z_5)x^{-1} = (z_{x(1)}, \dots, z_{x(5)})$). The module M can be considered as an absolutely simple representation of A_5 over $\mathbb{Z}_3$ and its dimension is 4. Now consider M as an R_3-module via the canonical epimorphism $\pi\colon R_3 \to S_3$. Then M is a simple R_3-module annihilated by e_3, therefore $M \simeq M_2$. It follows that the multiplicity of M_2 if $R_3/J(R_3)$ is 4, therefore $l = 4$.

Since P' is a direct summand of P, $k \in \{1, 2, 3, 4\}$. Using Fact 2.8, we get $\dim_{\mathbb{Z}_3} P/P(3R_3) = 42$, $\dim_{\mathbb{Z}_3} P'/P'(3R_3) = 18$, $\dim_{\mathbb{Z}_3} P''/P''(3R_3) = 36$. Now the only simple factor of P' and P'' is M_2 therefore $P'' \simeq P'^2$. Thus P'^2 is a direct summand of P, therefore $l \in \{1, 2\}$. If k, was 1, then P'^3 would be a direct summand of P and it is not possible, since $42 < 3 * 18$. Therefore $k = 2$ and there exists Q such that $P \simeq P'^2 \oplus Q$. Semisimple module $\mathbb{Q}Q$ has $\mathbb{Q}$-dimension 6 and the multiplicity of T_1 in $\mathbb{Q}Q$ is 1. The only possibility is $\mathbb{Q}Q = T_1 \oplus T_5$. Hence $\mathbb{Q}\mathrm{Tr}(Q) = e_1R_0 + e_5R_0$. □

Lemma 2.11. *Let K_2 be the minimal idempotent ideal of R_2 that is not contained in* $\mathrm{Aug}(R_2)$. *Then $\mathbb{Q}K_3 = e_1R_0 + e_3R_0 + e_5R_0$.*

Proof. Let M_1, M_2, M_3 be the simple R_2-modules such that M_1 is the simple factor of K_2, M_2 is the simple factor of $(1 - e_2)I_2$ and M_3 is the simple factor of e_2R_2. Let $e = (1 - e_2)$, $e' = e(1 - \frac{1}{3}(1 + g + g^2))$, where $g = (1, 2, 3)$. Put $P = eR_2$, $P' = e'R_2$. As above, we need $k, l \in \mathbb{N}$ given by $P/PJ(R_2) \simeq M_1 \oplus M_2^l$ and $P'/P'J(R_2) \simeq M_2^k$. Let F be a field given by adjoining a primitive fifteenth root of one to $\mathbb{Z}_2$. By [20, page 200], the ring $F \otimes S_2/J(S_2)$ has two 2-dimensional simple modules and they are annihilated by e_2 (it is because they appear as a composition factors of a representation that is annihilated by e_2). Therefore $F \otimes M_2$ is a direct sum of these two representations. Thus the $\mathbb{Z}_2$-dimension of M_2 is 4 but the multiplicity of M_2 in $S_2/J(S_2)$ is 2. It follows $l = 2$.

Using Fact 2.8 we get that $\mathbb{Z}_{(2)}$-rank of P is 44 and $\mathbb{Z}_{(2)}$-rank of P' is 32. Therefore P'^2 cannot be a direct summand of P and $k = 2$ follows. Then $P \simeq P' \oplus Q$ for some Q and $K_2 = \mathrm{Tr}(Q)$. By Lemma 2.7, $\mathbb{Q}K_2 = \mathrm{Tr}(Q \otimes_{R_2} R_0)$. Observe that $Q \otimes_{R_2} R_0$ has $\mathbb{Q}$-dimension 12 and contains T_1 with multiplicity 1. The only way, how to write 11 in multiples of 6 and 5 is $11 = 6 + 5$. Therefore $Q \otimes_{R_2} R_0 \simeq T_1 \oplus T_5 \oplus T_3$ and $\mathbb{Q}K_2 = (e_1 + e_3 + e_5)R_0$. □

Now we can finish the classification of idempotent ideals in $\mathbb{Z}[A_5]$.

Proposition 2.12. *The idempotent ideals in $R = \mathbb{Z}[A_5]$ are the following $0, \mathrm{Aug}(R), X, R$, where $\mathbb{Q}X = \mathbb{Q}[A_5]$.*

Proof. The idempotent ideals contained in $\mathrm{Aug}(R)$ were classified in Lemma 2.5. Let K be the idempotent ideal of R not contained in $\mathrm{Aug}(R)$. Then for any $i \in \{2,3,5\}$ $K_{(i)}$ is an idempotent ideal of R_i not contained in $\mathrm{Aug}(R_i)$. By Lemma 2.9 we have $e_2 \in K_{(0)}$, by Lemma 2.10 we have $e_5 \in K_{(0)}$ and by Lemma 2.11, we have $e_3 \in K_{(0)}$. It follows that $K_{(0)} = \mathbb{Q}[A_5]$.

If L is an idempotent ideal of R_5 such that $\mathbb{Q}L = \mathbb{Q}[A_5]$, then $L = R_5$ by Lemma 2.5 and Lemma 2.9. Similarly, if L is an idempotent ideal of R_3 such that $\mathbb{Q}L = \mathbb{Q}[A_5]$, then $L = R_3$ by Lemma 2.5 and Lemma 2.10. But if L is an idempotent ideal of R_2 such that $\mathbb{Q}L = \mathbb{Q}[A_5]$, then either $L = R_2$ or $L = K_2 + e_2R_2$ by Lemma 2.5 and Lemma 2.11. Therefore there exists an idempotent ideal $X \subseteq R$ such that $X_{(2)} = K_2 + e_2R_2$, $X_{(3)} = R_3$ and $X_{(5)} = R_5$. $\square$

Finally, we can classify non-finitely generated projective modules over $\mathbb{Z}[A_5]$.

Theorem 2.13. *The projective modules over $R = \mathbb{Z}[A_5]$ are the following: Let $I = \mathrm{Aug}(R)$ and let X be the other non-trivial idempotent ideal of R. Let B_I be the unique I-big projective R-module of the trace I, let B_X be the unique X-big projective module of the trace X. Apart from these, there is an X-big projective module P such that P/PX is the unique indecomposable projective module over R/X. Then*

(i) *Any countably generated projective module over R that is not free has a unique decomposition as a sum of $Q \oplus F$, where $Q \in \{B_X, B_I, P\}$ and F is a finitely generated free module.*
(ii) $B_X \oplus B_I \simeq R^{(\omega)}$, $B_I \oplus P \simeq R^{(\omega)}$.
(iii) $P \oplus B_X \simeq P$, $P \oplus P \simeq R \oplus B_I$.

Proof. Let M be a countably generated projective module over R. Since R has (*), there exists the least ideal K such that M/MK is finitely generated. If $K = 0$, M is finitely generated. If $K = R$, then M is R-big and hence free. If $K = I$, then since $R/I \simeq \mathbb{Z}$, $M/MI \simeq \mathbb{Z}^n$ for some $n \in \mathbb{N}_0$. Since $N = B_I \oplus R^n$ is a countably generated projective module such that I is the smallest ideal of the set $\{I \text{ ideal of } R \mid N/NI\}$ is finitely generated and $N/NI \simeq M/MI$, by [13], we have $M \simeq N$.

The remaining case is $X = K$. Recall that $X_{(p)} = R_p$ for any prime different from 2. It follows that there exists $k \in \mathbb{N}$ such that $2^k \in X$. Now $R/X \simeq (R/2^kR)/(X/2^kR) \simeq (R_2/2^kR_2)/(X_{(2)}/2^kR_2)$. Let $S = \mathbb{Z}_{2^k}[A_5]$,

let $\pi\colon R_2 \to S$ be the canonical epimorphism and let $X' = \pi(X)$. From the proof of Lemma 2.11 we know that $S/J(S) \simeq M_1 \oplus M_2^2 \oplus M_3^n$ for some $n \in \mathbb{N}$ (in fact $n = 4$ but we do not need this) and the M_1, M_3 are the simple factors of X'. Now $S/J(S)/(X' + J(S))/J(S) \simeq (S/X')/(J(S/X')) \simeq \mathrm{M}_2(\mathrm{End}_S(M_2))$. It follows that R/X is a homogeneous semilocal ring with the indecomposable projective module P' satisfying $P'^2 \simeq R/X$. The module P' gives a unique countably generated projective module P such that P is X-big and $P/PX \simeq P'$. Since $P' \oplus P' \simeq R/X$, we get $P \oplus P \simeq B_X \oplus R$. The relation $B_X \oplus P$ holds because P is X-big.

It remains to prove relations of (ii). Since a direct sum of an X-big module and of an I-big module is R-big, this relations follows immediately. $\square$

Remark 2.14. Let us explain, how one can use the decomposition map from the Cartan-Brauer triangle for group ring of a finite group G over the ring $S = \mathbb{Z}[e^{2\pi i/\exp(G)}]$, where $\exp(G)$ is the least common multiple of orders of elements of G. Let P be a prime of S and let S_P be the corresponding localization of S at P. By [3, Exercise 6.16], $S_P[G]$ is semiperfect, therefore minimal idempotent ideals of $S_P[G]$ are exactly two-sided ideals generated by primitive idempotents. If $K \subseteq S_P[G]$ is a minimal idempotent ideal and Q is the quotient field of S, then $QK \subseteq Q[G]$ can be computed by [3, Theorem 18.26(i)]. Let us demonstrate what's going on in case $G = A_5$, $S = e^{2\pi i/30}$. The tables of characters and Brauer characters over $2, 3, 5$ for sufficiently large fields are given in [20] as follows: Take $\xi = e^{2\pi i/5}$.

Characteristic 0:

rep	c_1	c_2	c_3	c_5	c_5'
τ	1	1	1	1	1
α_3	3	-1	0	$1+\xi+\xi^4$	$1+\xi^2+\xi^3$
α_3'	3	-1	0	$1+\xi^2+\xi^3$	$1+\xi+\xi^4$
α_4	4	0	1	-1	-1
α_5	5	1	-1	0	0

Characteristic 2:

rep	c_1	c_3	c_5	c_5'
τ	1	1	1	1
$\overline{\alpha_2}$	2	-1	$\xi+\xi^4$	$\xi^2+\xi^3$
$\overline{\alpha_2'}$	2	-1	$\xi^2+\xi^3$	$\xi+\xi^4$
α_4	4	1	-1	-1

Characteristic 3:

rep	c_1	c_2	c_5	c_5'
τ	1	1	1	1
α_3	3	-1	$1+\xi+\xi^4$	$1+\xi^2+\xi^3$
α_3'	3	-1	$1+\xi^2+\xi^3$	$1+\xi+\xi^4$
α_4	4	0	-1	-1

Characteristic 5:

rep	c_1	c_2	c_3
τ	1	1	1
α_3	3	-1	0
α_5	5	1	-1

Let us consider these tables as matrices:

$$M_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 0 & 1+\xi+\xi^4 & 1+\xi^2+\xi^3 \\ 3 & -1 & 0 & 1+\xi^2+\xi^3 & 1+\xi+\xi^4 \\ 4 & 0 & 1 & -1 & -1 \\ 5 & 1 & -1 & 0 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & \xi+\xi^4 & \xi^2+\xi^3 \\ 2 & -1 & \xi^2+\xi^3 & \xi+\xi^4 \\ 4 & 1 & -1 & -1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & 1+\xi+\xi^4 & 1+\xi^2+\xi^3 \\ 3 & -1 & 1+\xi^2+\xi^3 & 1+\xi+\xi^4 \\ 4 & 0 & -1 & -1 \end{pmatrix}, M_5 = \begin{pmatrix} 1 & 1 & 1 \\ 3 & -1 & 0 \\ 5 & 1 & -1 \end{pmatrix}$$

Looking at the table over characteristic 0, we have 5 simple $Q[G]$-modules. The primitive central idempotents corresponding to $\tau, \alpha_3, \alpha_3', \alpha_4, \alpha_5$ are denoted by e_1, e_3, e_3', e_2, e_5.

We need find constants for decomposition map from Cartan-Brauer triangle (see [3, page 427]). Let us summarize some basic facts. Let $T_1, \dots T_n$ be the representative set of simple modules over $Q[G]$, and let $T_1', \dots, T_m'$ be the representative set of simple modules over $S/P[G]$ (this is a field of positive characteristic which is sufficiently large for G by [3, Corollary 17.2]). The Grothendieck group $G_0(Q[G])$ can be considered as a free abelian group with basis $T_1, \dots, T_n$ and Grothendieck group $G_0(S/P[G])$ can be considered as a free abelian group with basis $T_1', \dots, T_m'$ by [3, Proposition 16.6]. The decomposition map $d\colon G_0(Q[G]) \to G_0(S_P/P[G])$ is calculated as follows: Take a simple representation $T \in \{T_1, \dots, T_n\}$ over Q of dimension l and take a full S_P-lattice in the corresponding $Q[G]$-module T. This is a finitely generated torsion free S_P-module, therefore T is free as an S_P-module. It means there exists a basis of T such that its corresponding representation $\varphi\colon G \to M_l(Q)$ has its image in $M_l(S_P)$. Consider $\varphi\colon G \to M_l(S_P)$ and compose it with $\pi\colon M_l(S_P) \to M_l(S/P)$. Then we get a S/P representation induced from φ, or an $S/P[G]$-module. Now the corresponding module T' over $S/P[G]$ may depend on the choice of the S_P-lattice in T but, by [3, Proposition 16.16], the factors in its composition series are independent of this choice. Then we define $d(T) = \sum_{i=1}^m n_i T_i'$, where n_i is the multiplicity of T_i' in the module T'.

So d can be considered as an $m \times n$ matrix over $\mathbb{N}_0$. The character χ of T has entries in S, as S is integrally closed. The character of T' is then given by $\pi\chi$. Then we need to calculate Brauer character of T' and express it as

an integral combination of Brauer characters of simple representations. The definition of Brauer characters as in [3] depends on the choice of suitable primitive root of 1. It is not obvious whether one should really take care of it, as we do.

Let us compute the map d for various primes of S. It is enough to do it for primes over $2, 3$ and 5. Take P to be a prime of S such that $P \cap \mathbb{Z} = 2\mathbb{Z}$. The Brauer character table mod 2 is with respect to 15-th root of 1 chosen $\omega = e^{2\pi i/15}$ and ξ means ω^3. The characters of simple representation over Q are expressed in rows of $M_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 0 & 1+\omega^3+\omega^{12} & 1+\omega^6+\omega^9 \\ 3 & -1 & 0 & 1+\omega^6+\omega^9 & 1+\omega^3+\omega^{12} \\ 4 & 0 & 1 & -1 & -1 \\ 5 & 1 & -1 & 0 & 0 \end{pmatrix}$.
Therefore if we delete the second column of M_0, we get a matrix M_0' which (in rows) contains Brauer characters of representations over S/P given by reductions from simple representations over Q. In order to calculate the decomposition map for this so called 2-modular system $(S_P, Q, S/P)$ we need to solve the system of linear equations $D_2 M_2 = M_0'$. Then D_2^T is the matrix of d with respect to bases given above. So $D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.
Now use [3, Theorem 18.26(i)]: Each column of D_2 corresponds to one minimal idempotent ideal and the extension of this ideal to $Q[G]$ is given by non-zero entries in the row. Therefore if I_1, I_2, I_3, I_4 are the minimal idempotent ideals of $S_P[G]$, then (maybe after renumbering) $QI_1 = (e_1 + e_3 + e_3' + e_5)Q[G], QI_2 = (e_3 + e_5)Q[G], QI_3 = (e_3' + e_5)Q[G], QI_4 = e_2 Q[G]$.

Similarly for primes over 3, we let M_0'' be M_0 with the third column deleted. The decomposition map is then given by matrix D_3 satisfying $D_3 M_3 = M_0''$, therefore $D_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ and minimal idempotent ideals of $S_P[G]$ extend to ideals $(e_1 + e_5)Q[G], e_3 Q[G], e_3' Q[G], (e_2 + e_5)Q[G] \subseteq Q[G]$.

Finally, for primes over 5, let M_0''' be the matrix M_0 without the fourth and the fifth column. As a solution of $D_5M_5 = M_0'''$ we get $D_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In this case minimal idempotent ideals over $S_P[G]$ extend to $(e_1+e_2)Q[G]$, $(e_3+e_3'+e_2)Q[G], e_5Q[G]$.

Now we can calculate the idempotent ideals of $S[G]$. First we determine set of possible extensions of idempotent ideals in $S[G]$ to $Q[G]$, let M denotes this set. Then M has to be closed under the action of $\mathrm{Gal}(Q:\mathbb{Q})$ on the subsets of $Q[G]$. Therefore, $I \in M$ if and only if there are primes $P_2, P_3, P_5 \subseteq S$, $P_i \cap \mathbb{Z} = i\mathbb{Z}$ and idempotent ideals $I_2 \subseteq S_{P_2}[G], I_3 \subseteq S_{P_3}[G], I_5 \subseteq S_{P_5}[G]$ such that $QI_i = I$ for any $i \in \{2,3,5\}$.

Suppose that K is a nonzero idempotent ideal of $S[G]$. If K is contained in $\mathrm{Aug}(S[G])$, then $e_3 \in QK \Rightarrow e_5 \in QK \Rightarrow e_2 \Rightarrow e_3'$ (consider extension of K to S_{P_2}, S_{P_3} and S_{P_5}, where P_i is a prime over $i \in \{2,3,5\}$) and $QK = (e_3+e_3'+e_2+e_5)Q[G]$. Similarly $e_3' \in QK$ implies $QK = (e_3+e_3'+e_2+e_5)Q[G]$. If $e_2 \in QK$, then $e_3 \in KQ$ (modulo 5), and $QK = (e_3+e_3'+e_2+e_5)Q[G]$ again. Finally, if $e_5 \in QK$, then $e_3 \in QK$ (modulo 2), and $QK = (e_3+e_3'+e_2+e_5)Q[G]$. In any case, $QK = \mathrm{Aug}S[G]$. It follows that $S_{P_i}K = \mathrm{Aug}(S_{P_i}[G])$ and $K = \mathrm{Aug}(S[G])$.

Further suppose that K is a idempotent ideal of $S[G]$ not contained in $\mathrm{Aug}(S[G])$. Then $QK = Q[G]$ follows in a similar way as above. Then $S_{P_3}K = S_{P_3}[G]$ and $S_{P_5}K = S_{P_5}[G]$. On the other hand, if I_1, I_2, I_3, I_4 are minimal idempotent ideals such that $QI_1 = (e_1+e_3+e_3'+e_5)Q[G], QI_2 = (e_3+e_5)Q[G], QI_3 = (e_3'+e_5)Q[G], QI_4 = e_2Q[G]$. Then $S_{P_2}K \in \{I_1+I_4, I_1+I_4+I_2, I_1+I_3+I_4, Q[G]\}$.

But in order to give the number of idempotent ideals in $S[G]$, we have to know the prime factorization of $2S$ over S. By [3, Theorem 4.40], $2S$ is a product of two different primes. Therefore the number of idempotent ideals over $S[G]$ is $2^4+2=18$.

Observe that it is not enough to have the picture of idempotent ideals, in fact we need factors. On the other hand, suppose we know the idempotent ideals of $S[G]$, where $S = \mathbb{Z}[\omega]$, $\omega = e^{2\pi i/\exp(G)}$. Then it is possible to calculate idempotent ideals in $\mathbb{Z}[G]$. Let $m = \varphi(\exp(G))$, then $S[G]$ is a free $\mathbb{Z}[G]$-module with basis $B = \{1, \omega, \dots, \omega^{m-1}\}$. If I is an idempotent ideal of $S[G]$, then the ideal consists of all elements of $\mathbb{Z}[G]$ which occur as components in expression of elements of I in basis B is an idempotent ideal in $\mathbb{Z}[G]$ and all idempotent ideals of $\mathbb{Z}[G]$ are of this form.

From now on forget $\mathbb{Z}[A_5]$ and let us try to say at least something about $R_p = \mathbb{Z}_p[G]$ if G is a finite group. We will need the following result (see [4, Corollary 38.19])

Fact 2.15. *Let S be a commutative ring, let $A = K_0(S[G])$ and B be the subgroup of A generated by classes of projective modules induced from cyclic subgroups of G. Then $|G|^2 A/B = 0$.*

Let ω be a primitive $|G|$-th root of unit, let $S' = \mathbb{Z}[\omega]$ and let P be a prime over p. Put $S = S'_{(P)}$ and $R = S[G]$. Then we know from [3] that R is semiperfect which implies there exists a set $P_1, \dots, P_k$ of finitely generated indecomposable projective modules such that any projective module is isomorphic to exactly one of $\oplus_{i=1}^k P_i^{(\kappa_i)}$ ($\kappa_1, \dots, \kappa_k$ are cardinals). Therefore the set of minimal idempotent ideals of R is $\{\mathrm{Tr}(P_1), \dots, \mathrm{Tr}(P_k)\}$ and any idempotent ideal of R is a sum of minimal idempotent ideals. The point is, that $\mathrm{Tr}(P_i) = \mathrm{Tr}(P_i^n)$ for any $n \in \mathbb{N}$ and $1 \leq i \leq k$. And Fact 2.15 says that P_i^n can be computed from projective modules induced from the cyclic subgroups.

Proposition 2.16. *Let G be a finite group and let p be a prime. Then any idempotent ideal of $\mathbb{Z}_{(p)}[G]$ is a trace of a finitely generated projective module.*

Proof. Let K be an nonzero idempotent ideal of $\mathbb{Z}_{(p)}[G]$. We consider $L = SK$, where S is as above. Since $S[G]$ is semiperfect, $L = \mathrm{Tr}(Q) = \mathrm{Tr}(Q^l)$, where Q is a projective module generated by an idempotent of $S[G]$ and $l \in \mathbb{N}$. By Fact 2.15, there exist projective modules $M_1, \dots, M_m, N_1, \dots, N_n$ all induced from projective modules induced form group rings over cyclic subgroups of G such that $Q^l \oplus (\oplus_{j=1}^m M_j) \simeq \oplus_{j=1}^n N_j$. If follows that we can suppose that any N_i (or M_i) is of the form $eS[G]$, where e is an idempotent of $S[H]$, for some cyclic subgroup H of G.

Now let $\mathcal{G}$ be the Galois group $\mathrm{Gal}(\mathbb{Q}[\omega] : \mathbb{Q})$. For any $\gamma \in \mathcal{G}$ there is a canonical automorphism $\varphi_\gamma \in \mathrm{Aut}(S[G])$ and if e is an idempotent generating M_i (or N_i), we put M_i^γ (or N_i^γ) to be the projective module generated by $\varphi_\gamma(e)$. Let f be the idempotent generating Q, and let Q^γ be the projective module generated by idempotent $\varphi_\gamma(f)$. Then

$$(\oplus_{\gamma\in\mathcal{G}}(Q^\gamma)^l) \oplus (\oplus_{j=1}^m \oplus_{\gamma\in\mathcal{G}} M_j^\gamma) \simeq \oplus_{j=1}^n \oplus_{\gamma\in\mathcal{G}} N_j^\gamma.$$

Observe that the modules M_i, N_i are induced from group rings over cyclic (hence commutative) groups, so they are expressed from (ordinary) characters of cyclic subgroups. It follows $\oplus_{\gamma\in\mathcal{G}} M_j^\gamma$ and $\oplus_{\gamma\in\mathcal{G}} N_j^\gamma$ are induced from $\mathbb{Z}_{(p)}[G]$. Therefore also $P = \oplus_{\gamma\in\mathcal{G}}(Q^\gamma)^l$ is induced from $\mathbb{Z}_{(p)}[G]$. Since L is closed under any $\gamma \in \mathcal{G}$, it follows that $L = \mathrm{Tr}(P)$. Thus we have proved

that there exists a finitely generated projective module P' over $\mathbb{Z}_{(p)}[G]$ such that $S\mathrm{Tr}(P') = L = SK$. Since idempotent ideals over $\mathbb{Z}_{(p)}[G]$ are determined by their simple factors, we conclude $\mathrm{Tr}(P') = K$. $\square$

3. Generalized Weyl algebras

Let k be an algebraically closed field of characteristic 0. Fix a nonconstant polynomial $a \in \mathsf{k}[H]$. Generalized Weyl algebra $A(a)$ is given as a free k-algebra on X, Y, H satisfying relations $YX = a(H), XY = a(H-1), HY = Y(H-1), HX = X(H+1)$). If $b \in \mathsf{k}[H]$ we write $b(H)$ if we need stress that b is a polynomial in H and $b(q(H))$ is the image of b in the endomorphism of $\mathsf{k}[H]$ given by $H \mapsto q(H)$. So if $b(H)$ is any polynomial, then the relations $b(H)Y = Yb(H-1), b(H)X = Xb(H+1)$ are satisfied in $A(a)$. It may be confusing whether for example $q(H+1)$, where $q \in \mathsf{k}[H]$ means a multiple of q and $(H+1)$ or the image of q in substitution $H \mapsto H+1$. If it is written like this, it will always mean the later. The multiple would be written as $q \cdot (H+1)$ or $q(H)(H+1)$. In this section we want to compare finitely and non finitely generated projective modules. In the first subsection we investigate homogeneous left ideals and give a kind of criterion when a homogeneous left ideal is projective. We think that there are much more finitely generated projective modules over these algebras then sums of projective left homogeneous ideals, however we do not have an example right now. On the other hand, the non-finitely generated projective modules are easy to describe as we show in subsection B.

A. Homogeneous left ideals

The aim of this subsection is to give a criterion of projectivity for homogeneous left ideals in generalized Weyl algebras. Fix a nonconstant polynomial $a(H) \in \mathsf{k}[H]$ and let A denote $A(a)$. A homogeneous left ideal I is a left ideal in A generated by elements of the form $p_i(H)X^i, p_{-i}(H)Y^i$, where $p_i, p_{-i} \in \mathsf{k}[H]$ and $i \geq 0$. We can multiply I on the right by a suitable power of X and we get a homogeneous left ideal of A isomorphic to I which is generated by elements of the form $p_i(H)X^i, i \geq 0$. Observe that homogeneous left ideals have the following property: if $\sum_{i\geq 0} p_i(H)X^i + \sum_{i>0} p_{-i}(H)Y^i \in I$ then for any $p_i(H)X^i \in I$ and $p_{-i}(H)Y^i \in I$ for every $i \in \mathbb{N}_0$. It motivates the following definition: If $S \subseteq A$, we say that S is a *homogeneous set* if $\sum_{i\geq 0} p_i(H)X^i + \sum_{i>0} p_{-i}(H)Y^i \in S$ implies $p_i(H)X^i \in S$ and $p_{-i}(H)Y^i \in S$.

For any $i \in \mathbb{Z}$ let K_i be the ideal of $\mathsf{k}[H]$ given by:

(i) If $i \geq 0$, let $K_i = \{p \in \mathsf{k}[H] \mid pX^i \in I\}$.
(ii) If $i < 0$, let $K_i = \{p \in \mathsf{k}[H] \mid pY^i \in I\}$.

Then $I = (\oplus_{i\geq 0} K_i X^i) \oplus (\oplus_{i>0} K_{-i} Y^i)$. For any $i \in \mathbb{Z}$ there exists $c_i \in k[H]$ such that $K_i = c_i(H)\mathsf{k}[H]$. We may suppose that all c_i's are monic, so they are determined uniquely.

Our first task is to find the relations between c_i's. Since we suppose that I has generators of the form $p(H)X^l$, it follows that $c_{-i}(H) = c_0(H+i)$ for any $i \in \mathbb{N}$. Now $Xc_0(H) = c_0(H-1)X$, therefore $c_1|c_0(H-1)$. On the other hand $Yc_1(H)X = c_1(H+1)a(H)$, thus $c_0(H)|c_1(H+1)a(H)$. Let $x_1(H) \in \mathsf{k}[H]$ be such that $c_0(H-1) = x_1(H)c_1(H)$, then $x_1(H+1)c_1(H+1)|c_1(H+1)a(H)$ and, consequently, $x_1(H)$ divides $a(H-1)$. Thus we have a relation $c_0(H-1) = x_1(H)c_1(H)$, where x_1 is a (monic) divisor of $a(H-1)$. By the same arguments we have $c_k(H-1) = x_{k+1}(H)c_{k+1}(H)$, where x_{k+1} is some monic divisor of $a(H-1)$. Observe that $\deg(c_{k+1}(H)) \leq \deg(c_k(H))$ for any $k \in \mathbb{N}_0$. Of course, the equality holds for almost all k, therefore only finitely many many x_k's are different from 1. By direct calculations we get the relation $c_l(H)x_l(H)x_{l-1}(H-1)\cdots x_1(H-l+1) = c_0(H-l)$ (or, equivalently $c_l(H+l)x_l(H+l)x_{l-1}(H+l-1)\cdots x_1(H+1) = c_0(H)$). It follows, that $x_l(H+l)x_{l-1}(H+l-1)\cdots x_1(H+1)$ divides $c_0(H)$. This gives us an alternative description of homogeneous left ideals of A.

Lemma 3.1. *Let $x_1(H), \ldots, x_l(H)$ be monic divisors of $a(H-1)$ and let $c_0(H)$ be a multiple of $x_l(H+l)x_{l-1}(H+l-1)\cdots x_1(H+1)$. Put $c_{-i} = c_0(H+i)$ for any $i \in \mathbb{N}$, $c_i(H) = \frac{c_0(H-i)}{x_i(H)x_{i-1}(H-1)\cdots x_1(H-i+1)}$ for any $1 \leq i \leq l$ and $c_{l+i}(H) = c_l(H-i)$ for any $i \in \mathbb{N}$. Then $(\oplus_{i\geq 0} c_i(H)X^i) \oplus (\oplus_{i>0} c_{-i}(H)Y^i)$ is a homogeneous left ideal of A. Furthermore, any homogeneous left ideal is isomorphic to a left ideal given by this construction.*

Remark 3.2. In the notation of the previous lemma, we see that the corresponding homogeneous ideal is generated by $c_i X^i$, where $0 \leq i \leq l$. But some of these generators can be redundant, for example we can leave out the subset $\{c_i X^i \mid 1 \leq i \leq j, x_i = 1\}$.

Remark 3.3. Suppose that $c_0 = x_1(H+1)\cdots x_l(H+l)d(H)$. Then $c_{-i} = x_1(H+1+i)\cdots x_l(H+l+i)d(H+i), i \geq 0$, $c_i(H) = \frac{x_1(H+1-i)\cdots x_l(H+l-i)}{x_i(H)x_{i-1}(H-1)\cdots x_1(H-i+1)} d(H-i), 1 \leq i \leq l$ and $c_i(H) = d(H-i), i \geq l$. Observe that we can cancel $d(H)$ on the right and we obtain an ideal $I' = (\oplus_{i\geq 0} c'_i(H)X^i) \oplus (\oplus c'_{-i}(H)Y^i)$ where $c_i(H) = x_{i+1}(H+1)\cdots x_l(H+l-i)$ for $0 \leq i < l$, $c_i(H) = 1$ for $i \geq l$ and $c_{-i}(H) = c_0(H+i)$ for $i \geq 0$.

Question: Could we classify isomorphism classes of homogeneous left ideals? What about stable isomorphisms?

Let us fix some $l \geq 1$, $x_1, \ldots, x_l$ monic divisors of $a(H-1)$ and $c_0(H)$, a multiple of $x_l(H+l)x_{l-1}(H+l-1)\cdots x_1(H+1)$. For brevity, put $y_i(H) =$

$x_i(H)x_{i-1}(H-1)\cdots x_1(H-i+1)$ for any $1 \le i \le l$, so $c_i(H)y_i(H) = c_0(H-i)$ and let I be the left ideal generated by $c_i(H)X^i$, $0 \le i \le l$.

Our next task is to determine homomorphisms from I to ${}_AA$. Recall that A is a noetherian domain, therefore it has (left and right) classical ring of quotients, which is a skew field Q. Any element of the set $\mathrm{Hom}_A(I, {}_AA)$ is therefore realized as a right multiplication by a convenient element of Q: Suppose that $\varphi\colon I \to {}_AQ$ is a homomorphism of left A-modules. Observe that for any $1 \le i \le l$ there exists $r \in A$ such that $rc_i(H)X^i = s(H)c_0(H)$ (take $r = Y^i$). Then $\varphi(c_iX^i) = r^{-1}s\varphi(c_0) = r^{-1}sc_0c_0^{-1}\varphi(c_0) = c_iX^ic_0^{-1}\varphi(c_0)$.

Therefore in order to describe elements of $\mathrm{Hom}_A(I, {}_AA)$ it is enough to find all $q \in Q$ such that $Iq \subseteq A$ or, equivalently, $c_iX^iq \in A$ for any $0 \le i \le l$. In other words we ask for which $\gamma \in A$ there are $\gamma_1, \dots, \gamma_l \in A$ such that $c_0^{-1}\gamma = X^{-i}c_i^{-1}\gamma_i$ (this is an equality in Q). Let S be the set of all $\gamma \in A$ having this property. Then there is an obvious bijection between $\mathrm{Hom}_A(I, {}_AA)$ and $c_0^{-1}S$. Thus we need find the set S.

Now $c_0^{-1}\gamma = X^{-i}c_i^{-1}\gamma_i$ if and only if $c_0(H-i)^{-1}X^i\gamma = c_i(H)^{-1}\gamma_i$. Since $c_0(H-i)^{-1} = c_i(H)^{-1}y_i(H)^{-1}$, we get $X^i\gamma = y_i\gamma_i$. Therefore $\gamma \in S$ if and only if $X^i\gamma \in y_iA$ for any $1 \le i \le l$. This means that S is a $\mathsf{k}[H]$-submodule of ${}_{\mathsf{k}[H]}A$ and it is also a homogeneous set. Therefore we need find polynomials $d_j(H), j \in \mathbb{Z}$ such that $p(H)X^j \in S$ if and only if $d_j|p$ for $j \ge 0$ and $pY^j \in S$ if and only if $d_{-j}|p$ for $j > 0$.

The X-part is quite easy since $X^ip(H)X^j = p(H-i)X^{i+j}$, hence $p(H)X^j \in S$ if and only if $y_i(H)|p(H-i)$ for any $1 \le i \le l$ if and only if $y_i(H+i)|p(H)$ for any $1 \le i \le l$. Observe that for any $1 \le i < l$, $y_{i+1}(H+i+1) = y_i(H+i)x_{i+1}(H+i+1)$. It follows that $d_i = y_l(H+l)$ for any $i \ge 0$.

The Y-part is more complicated. Consider $X^ip(H)Y^j$. If $j \ge i$, this equals $p(H-i)a(H-i)\cdots a(H-1)Y^{j-k}$. Observe that $y_i(H) = x_i(H)\cdots x_1(H-i+1)$ divides $a(H-1)\cdots a(H-i)$. Therefore, in this case $y_i|X^ip(H)Y^j$. Now suppose that $i = j+k$, where $1 \le k$. Then $X^ip(H)Y^j = p(H-i)a(H-j-k)\cdots a(H-1-k)X^k$. For any $1 \le m \le l$ let $z_m(H)$ be the unique polynomial satisfying $z_m(H)x_m(H) = a(H-1)$. If $i \le l$, then $y_i|X^ip(H)Y^j$ is equivalent to $y_i(H+i)|p(H)a(H)\cdots a(H-j-1)$ and this is equivalent to $x_{j+1}(H+j+1)\cdots x_i(H+i)|p(H)z_1(H+1)\cdots z_j(H+j)$. And this is satisfied if and only if $p(H)$ is a multiple of $\frac{x_{j+1}(H+1)\cdots x_i(H+i)}{\gcd(z_1(H+1)\cdots z_j(H+j), x_{j+1}(H+j+1)\cdots x_i(H+i))}$.

For $1 \le j < l$, k may vary $1, \dots, l-j$ we get a formula

$d_{-j} =$

$$\mathrm{lcm}_{1\le k\le l-j}\left(\frac{x_{j+1}(H+j+1)\cdots x_{j+k}(H+j+k)}{\gcd(z_1(H+1)\cdots z_j(H+j), x_{j+1}(H+j+1)\cdots x_{j+k}(H+j+k))}\right).$$

This least common multiple is of course $\frac{x_{j+1}(H+j+1)\cdots x_l(H+l)}{\gcd(z_1(H+1)\cdots z_j(H+j), x_{j+1}(H+j+1)\cdots x_l(H+l))}$.

Let us summarize our calculations in

Proposition 3.4. *Let I be a left homogeneous ideal given by $c_0, x_1, \dots, x_l$ as in the previous lemma. Homomorphisms of $\mathrm{Hom}_A(I, {}_AA)$ are realized by right multiplication of elements in $c_0^{-1}S$, where $S = (\oplus_{i\ge 0} d_i(H)X^i) \oplus (\oplus_{i>0} d_{-i}(H)Y^i)$. The polynomials $d_i, i \in \mathbb{Z}$ are given by*

$$d_i = x_1(H+1)\cdots x_l(H+l), i \ge 0,$$

$$d_{-i} = \frac{x_{i+1}(H+i+1)\cdots x_l(H+l)}{\gcd(z_1(H+1)\cdots z_i(H+i), x_{i+1}(H+i+1)\cdots x_l(H+l))}, 1 \le i < l,$$

$$d_{-i} = 1, i \ge l.$$

Now we can describe when a left homogeneous ideal given as in lemma is projective. In order to see this observe that a left ideal I is projective if and only if there exist $n \in \mathbb{N}$ and $f\colon I \to A^n$, $g\colon A^n \to I$ such that $gf = 1_I$. Suppose that I is a homogeneous left ideal of A given as in lemma. Now, f is realized as a right multiplication by a row $(q_1, \dots, q_n)$, where $q_i \in c_0^{-1}S$ and S is the set described in the proposition. Elements of $\mathrm{Hom}_A(A^n, I)$ are given as a right multiplication by column $(i_1, \dots, i_n)$, where $i_1, \dots, i_n \in I$. If $I \ne 0$, then $gf = 1_I$ if and only if $\sum_{j=1}^n q_j i_j = 1$ (this is an equality in Q). Furthermore, any q_j is a sum of elements of the form $c_0(H)^{-1}p(H)d_k(H)X^k$ or $c_0(H)^{-1}p(H)d_{-k}(H)Y^k$ and similarly any i_j is of the form $p(H)c_k(H)X^k$ or $p(H)c_{-k}(H)Y^k$. Therefore if we look at $\sum_{j=1}^n q_j i_j = 1$ after expanding q_j, i_j as a sum of homogeneous coordinates, we see that there exist $q'_1, \dots, q'_m, i'_1, \dots, i'_m$ such that $\sum_{j=1} q'_j i'_j = 1$ and for any $1 \le j \le m$ either $q'_j = c_0^{-1}p_j(H)d_{k_j}X^{k_j}$ and $i'_j = p_j(H)'c_{-k_j}Y^{k_j}$ or $q'_j = c_0^{-1}p_j(H)d_{-k_j}Y^{k_j}$ and $i'_j = p_j(H)'c_{k_j}X^{k_j}$. Further, observe that $c_0^{-1}d_{k_j}X^{k_j}c_{-k_j}Y^{k_j} = c_0^{-1}c_{-k_j}(H-k_j)d_{k_j}X^{k_j}Y^{k_j} \in k[H]$ and $c_0^{-1}d_{-k_j}Y^{k_j}c_{k_j}X^{k_j} = c_0^{-1}d_{-k_j}c_{k_j}(H+k_j)Y^{k_j}X^{k_j} \in \mathsf{k}[H]$. The first relation is satisfied because $c_{-k_j}(H) = c_0(H+k_j)$. The second relation is obvious if $k_j \ge l$, since $c_0 | Y^{k_j}c_{k_j}(H)X^{k_j} \in \mathsf{k}[H]$. If $1 \le k_j \le l$, we need prove $y_{k_j}(H+k_j) | d_{-k_j}Y^{k_j}X^{k_j}$, but recall that $y_{k_j}(H) | X^{k_j}d_{-k_j}(H)Y^{k_j}, 1 \le k_j \le l$ are conditions defining d_{-k_j}.

Therefore, if I is projective, then the polynomials $c_0^{-1}d_{-i}Y^ic_iX^i$, $c_0^{-1}d_iX^ic_{-i}Y^i, i \in \mathbb{N}_0$ generate $\mathsf{k}[H]$. Observe that $c_0^{-1}d_iX^ic_{-i}Y^i$ is always a multiple of $d_0 = y_l(H+l) = x_1(H+1)\cdots x_l(H+l)$ and that $c_0^{-1}d_{-i}Y^ic_iX^i$

is a multiple of $c_0^{-1}d_{-l}Y^lc_lX^l = z_1(H+1)\cdots z_l(H+l)$ whenever $i \geq l$. Therefore if I projective, then $\gcd_{0\leq i\leq l}(c_0^{-1}d_{-i}Y^ic_iX^i) = 1$.

Now suppose that $\gcd_{0\leq i\leq l}(c_0^{-1}d_{-i}Y^ic_iX^i) = 1$, that is there are polynomials $p_0(H), \cdots, p_l(H) \in \mathsf{k}[H]$ such that $\sum_{i=0}^{l} c_0^{-1}d_{-i}Y^ic_iX^ip_i(H) = 1$. Then the row $(c_0^{-1}d_0, c_0^{-1}d_{-1}Y, \cdots, c_0^{-1}d_{-l}Y^l)$ gives a homomorphism $f\colon I \to A^{l+1}$ and the column $(p_0(H)c_0(H), p_1(H-1)c_1(H)X, \cdots, p_l(H-l)c_l(H)X^l)$ gives a homomorphism $g\colon A^n \to I$ such that $gf = 1_I$.

Finally notice that $c_0^{-1}d_{-i}Y^ic_iX^i = d_{-i}z_1(H+1)\cdots z_i(H+i)$ for any $1 \leq i \leq l$. This completes the proof of the criterion of projectivity for homogeneous left ideals.

Proposition 3.5. *Let I be a left homogeneous ideal given as in Lemma 3.1. For any $1 \leq i \leq l$, put $z_i(H) = a(H-1)/x_i(H)$. Then I is projective if and only if the greatest common divisor of $l+1$ polynomials $\frac{z_1(H+1)\cdots z_i(H+i)x_{i+1}(H+i+1)\cdots x_l(H+l)}{\gcd(z_1(H+1)\cdots z_i(H+i), x_{i+1}(H+i+1)\cdots x_l(H+l))}, 0 \leq i \leq l$ is 1.*

Remark 3.6. Suppose that all roots of $a(H)$ are simple and $a(H)$ contains no comparable roots. In this case the polynomials $x_1(H+1)\cdots x_l(H+l)$ and $z_1(H+1)\cdots z_l(H+l)$ are always co-prime and all left homogeneous ideals are projective. Of course, this is no surprise because under this assumption A is hereditary.

Example 3.7 Suppose that $a(H) = H^2$. Fix some $l \in \mathbb{N}$ and $x_1, \ldots, x_l$ some monic divisors of $(H-1)^2$. Thus for any $1 \leq i \leq l$, $x_i\{1, (H-1), (H-1)^2\}$. Suppose that $x_i = (H-1)$ for some $i \in \mathbb{N}$, then also $z_i = (H-1)$. Then $(H+i-1)$ is a divisor of all polynomials $\frac{z_1(H+1)\cdots z_j(H+j)x_{j+1}(H+j+1)\cdots x_l(H+l)}{\gcd(z_1(H+1)\cdots z_j(H+j), x_{j+1}(H+j+1)\cdots x_l(H+l))}$, therefore the corresponding left homogeneous ideal is not projective. On the other hand, suppose that $x_i \in \{1, (H-1)^2\}$ for any $1 \leq i \leq l$. Then $x_1(H+1)\cdots x_l(H+l)$ and $z_1(H+1)\cdots z_l(H+l)$ are co-prime. In this case the corresponding homogeneous left ideal is projective.
Question: Could we calculate the image of the Hattori-Stallings map of $A(H^2)$ when restricted to homogeneous left ideals?

B. Non-finitely generated projective modules

Remark 3.8. *Let R be a left and right noetherian ring satisfying (*). Then any projective module is a direct sum of finitely generated modules if and only if any idempotent ideal is a trace of a finitely generated projective module and if I is an idempotent ideal and P is a finitely generated projective module over R/I then there exists a finitely generated projective module Q such that $Q/QI \simeq P$.*

Proof. Let M be a countably generated projective module over R. Recall that the isomorphic type of M is determined by a pair $(I, M/MI)$, where I

is the smallest ideal such that M/MI is finitely generated. It appears that I is idempotent. So if I is a trace of a finitely generated projective module P_1 and P_2 is a finitely generated projective module such that $P_2/P_2I \simeq M/MI$. Then $P_1^{(\omega)} \oplus P_2 \simeq M$.

On the other hand, suppose that there exists an idempotent ideal I that is not a trace of a finitely generated projective module. Then there exists a countably generated projective module Q of the trace I. If $Q \simeq \oplus_{i\in\mathbb{N}} P_i$, with P_i finitely generated, then $I = \cup_{n\in\mathbb{N}} \mathrm{Tr}(\oplus_{i=1}^n P_i)$, a contradiction to R left and right noetherian.

Finally suppose that there exits an idempotent ideal I and a finitely generated projective R/I-module P such that P is not isomorphic to $P'/P'I$ for P' finitely generated projective module. Then we have a countably generated projective module Q that corresponds to (I, P). If $Q = \oplus_{i\in\mathbb{N}} Q_i$ then $Q_iI \neq Q_i$ for only finitely many i's because P has to be finitely generated. But then $P' = \oplus_{i\in\mathbb{N}, Q_iI\neq Q_i} Q_i$ is a finitely generated module such that $P'/P'I \simeq Q/QI$, a contradiction. □

We say that roots λ, μ of $a(H)$ are *comparable* if $\lambda - \mu \in \mathbb{Z}$.

Fact 3.9. *Let $k \in \mathbb{N}$ be such that for any pair λ, μ of comparable roots of $a(H)$ the inequality $|\lambda - \mu| < k$ holds. Then X^{k-1}, Y^{k-1} belong to any nonzero ideal of A. In particular there exists the least non-zero two sided ideal of $I_0 \subseteq A$. Moreover, $I_0^2 = I_0$ and A/I_0 is a finite dimensional* **k**-*algebra.*

Proof. Let I be a nonzero ideal of A. By [16], any two-sided ideal of A is a homogeneous set, therefore I contains a nonzero polynomial $c(H)$. We can suppose $c(H)$ of degree at least one, otherwise $I = A$. Let k' be a positive integer such that $c(H - k')$ and $c(H)$ are co-prime. Then $c(H - k')X^{k'}, c(H)X^{k'} \in I$ and consequently, $X^{k'} \in I$. Similarly, we get $Y^{k'} \in I$ since $c(H + k'), c(H)$ are also co-prime. Then we have $X^{k'}Y^{k'}, Y^{k'}X^{k'} \in I$. Let $d(H) = \gcd(X^{k'}Y^{k'}, Y^{k'}X^{k'}) \in I$. Observe that if λ is a root of $d(H)$, then there exist λ', λ'' roots of $a(H)$ and nonnegative integers $1 \leq i \leq k', 0 \leq j \leq k' - 1$ such that $\lambda = \lambda' + i = \lambda'' - j$. From our assumption it follows that $|\lambda - \mu| < k - 1$ for any comparable roots of $d(H)$. Now repeat the previous argument with $d(H)$ instead of $c(H)$. It follows $X^{k-1}, Y^{k-1} \in I$. □

Observe that if a has no comparable roots, then A is simple. In this case any non-finitely generated projective module is free by Bass' uniformly big

projectives. If a has comparable roots, there is always at least one non-trivial idempotent ideal I_0. Regarding Remark 3.8 we need check that I_0 is a trace ideal of a finitely generated projective module.

In order to get some information about I_0 let us consider equivalence classes of comparable roots, that is let $S \subseteq \mathbf{k}$ be the set of the roots of $a(H)$, $\sim$ the relation on S given by $\lambda \sim \mu$ if and only if $\lambda - \mu \in \mathbb{Z}$. Each block B of $\sim$ which has at least two elements contains $x_B, y_B \in S$ such that for any $z \in B$ the relations $x_B \leq z \leq y_B$ hold. Let C denote the set of all blocks of S having at least two elements and let $T \subseteq \mathbf{k}$ be given by $\cup_{B \in C}\{x_B + 1, \cdots, y_B\}$.

As we know I_0 is a homogeneous set, therefore there exist monic polynomials $c_i \in \mathbf{k}[H], i \in \mathbb{Z}$ such that $I_0 = (\oplus_{i \in \mathbb{N}_0} \mathbf{k}[H] c_i X^i) \oplus (\oplus_{i \in \mathbb{N}} \mathbf{k}[H] c_{-i} Y^i)$. If k is the maximum of $\{y_B - x_B \mid B \in C\}$, then I_0 is as a two sided ideal generated by X^k. Therefore c_0 is the greatest common divisor of polynomials $Y^\alpha X^k Y^{k-\alpha}, 0 \leq \alpha \leq k$. Looking at the roots of $Y^k X^k$ and $X^k Y^k$, we have that all roots are elements of T. On the other hand it is easy to see that that any $c \in T$ is a root of $Y^\alpha X^k Y^{k-\alpha}$ for any $0 \leq \alpha \leq k$. For any $c \in T$ let $L_c = \{d \in S \mid c - d \in \mathbb{N}\}$, $R_c = \{d \in S \mid d - c \in \mathbb{N}_0\}$ and let $m_c \in \mathbb{N}_0$ be the multiplicity of $H - c$ in $a(H)$. It is easy to verify that the multiplicity of $H - c$ in c_0 is given by $k_c = \min(\sum_{d \in R_c} m_d, \sum_{d \in L_c} m_d)$. This gives $c_0(H) = \prod_{c \in T} (H - c)^{k_c}$.

After this introduction we can prove

Lemma 3.10. *Let A be a generalized Weyl algebra. Then there exists a homogeneous left ideal P that is projective and* $\mathrm{Tr}(P)$ *is the least ideal of A.*

Proof. We shall use the notation established above. If T is empty, A is simple and we can put $P = {}_A A$. Suppose that T is non-empty, thus $k > 0$. For any $c \in T$ let $n_c = \sum_{d \in R_c} m_d$ and let $q(H) = \prod_{c \in T} (H - c)^{n_c}$. We claim that the left ideal $P = Aq(H) + AX^k$ is projective.

Let Q be a classical quotient ring of A, $f \colon A \to_A Q^{|T|+1}$ given by a line $(1, \dots, Y^k (H - c - k)^{-n_c}, \dots)$ (each $c \in T$ gives an entry). Since $(H - c)^{n_c} Y^k = Y^k (H - c - k)^{n_c}$ and $X^k Y^k = Y^k X^k (H - k)$ (this means a substitution $H \mapsto H - k$), we see $f(P) \subseteq A^{|T|+1}$. Thus we can consider f as a homomorphism from P to $A^{|T|+1}$. Further, let $g' \colon A^{|T|+1} \to P$ be a column given by $(q(H), X^k, \dots, X^k)$. As the factor $H - c$ has multiplicity n_c in $Y^k X^k$, we get gf is given as a sum of $|T| + 1$ elements in $\mathbf{k}[H]$ having the greatest common divisor 1. Therefore there exists $p(H), p_c(H) \in \mathbf{k}[H], c \in T$ such that $q(H)p(H) + \sum_{c \in T} Y^k (H - c - k)^{-n_c} X^k p_c(H) = 1$. Then we define $g \colon A^{|T|+1} \to P$ by a column $(q(H)p(H), \dots, X^k p_c(H), \dots)$. Then $gf = 1_P$, therefore P is a direct summand of $A^{|T|+1}$.

It remains to compute the trace ideal of P. It is the ideal generated by entries of the matrix corresponding to $gf\colon A^{|T|+1} \to A^{|T|+1}$. Observe that $q(H) \in I_0$. Let us have a look at $X^kY^k(H-c-k)^{-n_c}$. This is a factor of X^kY^k where we canceled the term $H-c-k$.led the term $H-c-k$. As $c+k \notin T$, $X^kY^k \in I_0$, it follows $X^kY^k(H-c-k)^{-n_c} \in I_0$ from the expression for c_0 given above. It follows that the trace ideal of P is contained in I_0, therefore $\mathrm{Tr}(P) = I_0$. □

Question: Could we prove the lemma using the criterion for projectivity from the previous section?

Suppose that A is not simple. Suppose that $J(A)$, the Jacobson radical of A, is not zero. Then $I_0 \subseteq J(A)$, so $J(A)$ contains a polynomial $p \in \mathbf{k}[H]$ of degree at least one. But $1-p$ is not invertible in A, a contradiction. Therefore $J(A) = 0$ if A is not simple. The following lemma is an analogy of (radikalfaktorlemma) for noetherian rings that have the least non-zero ideal.

Lemma 3.11. *Let R be a left and right noetherian and suppose that the set of nonzero ideals of R contains the least element I. If P, Q are countably but not finitely generated projective modules such that $P/PI \simeq Q/QI$, then $P \simeq Q$.*

Proof. If R is simple, then $P \simeq Q \simeq R^{(\omega)}$ by [1]. In general any countably generated projective module is either finitely generated or I-big. Hence, our assumption gives that P and Q are I-big. Then $P \simeq Q$ by [13, Lemma 2.5]. □

Suppose that $I_0 \neq A$. Then A/I_0 is a $\mathbf{k}$-algebra of finite dimension. Hence the structure of projective modules over A/I_0 is easy: If $S_1, \dots, S_n$ is a representative set of simple modules over A/I_0, there exist A/I_0-modules $P_1, \dots, P_n$ such that P_i is a projective cover of S_i for any $1 \leq i \leq n$. Then any projective module over A/I_0 can be uniquely written as a sum of copies of $P_1, \dots, P_n$.

Suppose that we can find finitely generated projective A-modules $P'_1, \dots,$ P'_n such that $P_i \simeq P_i/P_iI_0$. If P'_0 is any finitely generated projective module of trace ideal I_0, then $Q_i = P'_i \oplus P_0'^{(\omega)}$ is a countably but not finitely generated projective module such that $Q_i/Q_iI_0 \simeq P_i$. Suppose that Q is a countably generated projective module, then there are (unique) $0 \leq n_i \leq \omega$ such that $Q/QI_0 \simeq \oplus_{i=1}^n P_i^{(n_i)}$ and, by Lemma 3.11, $Q \simeq P_0^{(\omega)} \oplus_{i=1}^n P_i'^{(n_i)}$.

The previous paragraph explains our strategy. We need find finitely generated projective modules $P'_1, \dots, P'_n$ such that $P'_i/P'_iI_0 \simeq P_i$. Let J be an ideal such that $J/I_0 = J(A/I_0)$. Then it is enough to check that for a given

simple A/J-module S there exists a finitely generated projective A-module P such that $P/JP \simeq S$. Therefore we proceed as follows: We find the ideal J, and a set of simple A/J-modules containing a representative set of simple A/J-modules (note that the difference between simple A/I_0-modules and simple A/J-modules is only formal). Then for any simple module of this set we construct the corresponding finitely generated projective module.

In the following we keep the notation introduced above. Let J be the two-sided ideal generated by $r(H) = \prod_{c\in T}(H-c)$.

Lemma 3.12. *The ideal J annihilates any simple A/I_0-module. In particular, there exists $m \in \mathbb{N}$ such that $J^m \subseteq I_0$ and $J/I_0 \subseteq J(A/I_0)$.*

Proof. Since simple A/I_0-modules are simple A-modules annihilated by I_0, it follows that simple A/I_0-modules coincide with simple A-modules of finite dimension. Simple A-modules of finite dimension were described in (Gena's notes Proposition 1.28, official reference ??) as follows: Let $\lambda < \lambda'$ be comparable roots of $a(H)$ and let $n_i = \lambda' - \lambda$. Then $A/(AY^{n_i} + A(H - \lambda') + AX)$ is a finite dimensional simple module and any simple A-module of finite dimension is isomorphic to a module of this form. Observe that $r(H) = \prod_{c\in T}(H-c)$ annihilates $A/(AY^{n_i} + A(H-\lambda') + AX)$ because $\lambda' - i \in T$ for any $0 \le i < n_i$. Since $r(H)$ annihilates any simple A/I_0-module, the element $r(H) + I_0$ belongs to $J(A/I_0)$ and $J/I_0 \subseteq J(A/I_0)$. Since A/I_0 is artinian, it also follows $J^m \subseteq I_0$ for some $m \in \mathbb{N}$. $\square$

Let $d_i \in \mathbf{k}[H], i \in \mathbb{Z}$ such that $(\oplus_{i\in\mathbb{N}_0}\mathbf{k}[H]d_i(H)X^i)\oplus(\oplus_{i\in\mathbb{N}}\mathbf{k}[H]d_{-i}(H)Y^i) = J$. Since $d_0 = r$ has simple roots, for any $i \in \mathbb{N}$, $d_i(H)$ is a polynomial having exactly roots $x_B + 1 + i, \dots, y_B$ and each of them of multiplicity one, where B varies the set C. Similarly, d_{-i} has exactly roots $x_B + 1, \dots, y_B - i$, for any $B \in C$ and any of them of multiplicity 1. (Of course, $X^k, Y^k \in J$.) Now let $c \in T, i \in \mathbb{Z}$ be arbitrary. Let us define $e_i \in \mathbf{k}[H]$ by $e_i = \frac{d_i}{(H-c-i)}$ if $c+i$ is a root of d_i and $e_i = d_i$ otherwise. The left ideal $J_c = (\oplus_{i\in\mathbb{N}_0}\mathbf{k}[H]e_i)X^i \oplus (\oplus_{i\in\mathbb{N}}\mathbf{k}[H]e_{-i}Y^i)$ contains J and one can check that J_c/J is a simple left A-module. Since $\sum_{c\in T} J_c = A$ (left ideal $\sum_{c\in T} J_c$ contains 1), A/J is semisimple. Thus we have

Lemma 3.13. *The Jacobson radical of A/I_0 is generated as a two-sided ideal by $\prod_{c\in T}(H-c)$.*

Finally, the last step is to prove the following:

Lemma 3.14. *Let $c \in T$ and let J_c be the left ideal defined above. Then there exists a projective module P_c over A such that $P_c/JP_c \simeq J_c/J$. Moreover, the module P_c can be chosen as a homogeneous left ideal.*

Proof. Fix any $c \in T$. Put $q(H) = \prod_{c' \in T, c' \neq c}(H - c')^{n_{c'}}$ and let P_c be the left ideal generated by $q(H), X^k$. Let $f\colon A \to {}_AQ^{|T|}$ be given by a line $(1, \dots, Y^k(H - c' - k)^{-n_{c'}}, \dots)$ (for each $c \neq c' \in T$ one entry). In the same manner as in Lemma 3.10 we see $f(P_c) \subseteq A^{|T|}$. Since the greatest common divisor of polynomials $q(H), Y^k(H - c' - k)^{-n_{c'}}X^k, c \neq c' \in T$ is 1, there are $p(H) \in \mathbf{k}[H]$ and $p_{c'}(H) \in \mathbf{k}[H], c \neq c' \in T$ such that $q(H)p(H) + \sum_{c \neq c' \in T} Y^k(H - c' - k)^{-n_{c'}}X^k p_{c'}(H) = 1$. (Observe that $H - c$ is a factor of $Y^k(H - c' - k)^{-n_{c'}}X^k$ for any $c \neq c' \in T$, so c is not a root of $p(H)$). Therefore if $g\colon A^{|T|} \to P_c$ is a homomorphism given by a column $(q(H)p(H), \dots, p_{c'}(H - k)X^k,)$, then $gf = 1$. It follows that P_c is projective.

It remains to compute P_c/JP_c. One possible way how to do it is to look at the idempotent matrix $m = fg \in M_{|T|}(A)$ modulo J. The resulting idempotent matrix $\overline{m}$ gives a projective module over A/J that is isomorphic to P_c/JP_c. As remarked in the proof of Lemma 3.10, $p_{c'}(H)X^kY^k(H - c' - k)^{-n'_c} \in I_0$, therefore $\overline{m}$ has nonzero only the first row. In particular, the top left corner of $\overline{m}$, that is the element $q(H)p(H) + J$ is an idempotent in A/J which generate P_c/P_cJ. Since c is the only element that is not a root of $q(H)p(H)$, $q(H)p(H) \in J_c \setminus J$, therefore $q(H)p(H) + J$ is a nonzero idempotent of A/J contained in J_c/J. Therefore $P_c/JP_c \simeq J_c/J$. $\square$

Let us summarize calculations of this section in the following theorem

Theorem 3.15. *Let A be a generalized Weyl algebra then any non-finitely generated projective A-module is a direct sum of homogeneous left ideals.*

4. Construction of semilocal rings satisfying some requirements on the structure of projective modules

A. Two constructions as a motivation

In this section we discuss a method of addition of idempotent ideals described in Small and Stafford [19]. Suppose that we have a semilocal noetherian ring R. Let X be a set of simple modules over R. Our goal is to find a noetherian ring S such that $R/J(R) \simeq S/J(S)$ and if X' is a set of simple S-modules canonically corresponding to simple R-modules in X, then there exists an idempotent ideal $I \subseteq S$ such that a simple S-module M is a factor of I if and only if M is isomorphic to a module in X'. Moreover, we require the same picture of finitely generated projective modules over R and S.

The extension of S will be realized as a subring of a noetherian ring $M_n(R)$, therefore we need some easily verifiable condition which will imply

that S is again noetherian. The following one was suggested by Dolors Herbera.

Lemma 4.1. *Let $T \subseteq R$ be rings and let K be an ideal in R such that $K = K \cap T$ and R/K is finitely generated as a left T/K-module. If R is a left noetherian ring and T/K is also left noetherian, then T is left noetherian and since ${}_TR$ is finitely generated, R is a left noetherian T-module.*

Proof. First observe that if $x_1, \dots, x_n$ are such that $x_1 + K, \dots, x_n + K$ generate R/K as left T/K module, then $Tx_1 + \cdots + Tx_n + T.1 = R$, so R is indeed a finitely generated left T-module. Further, let I be a left ideal of T. Then KI is a left ideal of R, therefore there exist $y_1, \dots, y_m \in KI \subseteq I$ such that $KI = Ry_1 + \cdots + Ry_m$. Then $K^2I = Ky_1 + \cdots + Ky_m \subseteq Ty_1 + \cdots + Ty_m \subseteq I$. Therefore $KI/K^2I, RI/KI$ are finitely generated left R/K-modules, so they are finitely generated left T/K-modules. As T/K is left noetherian, I/KI, it is also a finitely generated left T-module. Now $KI/K^2I, I/KI$ are finitely generated left T-modules, therefore I/K^2I is a finitely generated T-module. Finally K^2I is contained in a finitely generated submodule of I which concludes the proof. □

Let us consider the following problem. Let T be a semilocal noetherian ring such that $T/J(T)$ is a product of two skew-fields. That is, the representative set of simple T-modules has two elements and S_1, S_2, say, and $T/J(T) \simeq S_1 \oplus S_2$ as T-modules. What can we say about countably generated projective modules? Let us distinguish two cases. If there exists a finitely generated indecomposable projective module P of dimension $(0, n)$, $n \in \mathbb{N}$, then there exists also finitely generated indecomposable projective module Q of dimension $(n, 0)$. An easy consideration of structure of full submonoids of $\mathbb{N}_0^2$ shows that two possibilities may happen:

(i) If $n = 1$, then P, Q are the only indecomposable projective modules over T. Moreover, both of them have local endomorphism ring, therefore any projective module over T is isomorphic to a unique module of the form $P^{(\kappa)} \oplus Q^{(\lambda)}$.

(ii) If $n > 1$, P, Q, T_T are the only indecomposable projective modules. They satisfy the relation $P \oplus Q \simeq T_T^n$. Moreover, any projective module is written as a direct sum of copies of P, Q, T but not uniquely.

By a result of Wiegand [22] we know that both (i),(ii) may happen over semilocal noetherian rings (see also bellow). Thus it remains to understand the case when no such P exists or, equivalently, when all finitely generated projective modules are free. In theory 3 things may happen here

(a) All projective modules are free.

(b) Finitely generated projective modules are free and there exists a projective module P of dimension $(\infty, 0)$ and any projective module is given as a sum of copies of T_T and P.
(c) Finitely generated projective modules are free, but there exist projective modules P and Q of dimensions $(\infty, 0)$ and $(0, \infty)$. In this case any projective module is a direct sum of copies of P, Q, T_T.

It remains to check whether all three cases may happen.

Suppose that R is a commutative semilocal noetherian ring such that M_1, M_2 are its only maximal ideals, $M = J(R) = M_1 M_2$, and the rings R/M_1 and R/M_2 are isomorphic. Further, suppose that there are no proper idempotent ideals in R. This means that any such an R realizes the case (a). For example, take a commutative field k and let R be a localization of $\mathsf{k}[x]$ at two different primes $x\mathsf{k}[x]$ and $(x-1)\mathsf{k}[x]$.

Now, let us follow the construction of Small and Stafford: Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ given by $c \in M$, $b \in M_1$, $a, d \in R$, but a, d are related as follows: $R/M \simeq R/M_1 \times R/M_2$ and fixing and isomorphism we can suppose that $R/M_1 = R/M_2 = F$. Now we require if $a + M = (x, y) \in F \times F$, then $d + M = (x, x) \in F \times F$. We can verify directly that S is a ring. Let us compute its Jacobson radical: Consider $I = \begin{pmatrix} M & M_1 \\ M & M \end{pmatrix}$. We can verify directly that elements of $1 - I$ are (right) invertible, thus $I \subseteq J(S)$. On the other hand $S/I \simeq R/M$, thus S has two maximal (right) ideals, and $I = J(S)$. The isomorphism of S/I and R/M can be written explicitly as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + M$.

In order to prove that S is (left and right) noetherian, note that $I \subseteq S$ is an ideal in $R' = \begin{pmatrix} R & M_1 \\ M & R \end{pmatrix}$ which is a noetherian ring (a finitely generated algebra over a noetherian ring R). The embedding $S/I \subseteq R'/I$ can be imagined as follows: $R'/I \simeq R/M \times R/M$ and S/I is a subring $\{((x, y), (x, x)) \mid x, y \in F\}$. Thus F is contained in R'/I by a map $x \mapsto ((x, x), (x, x))$. This image is contained in S/I, the assumptions of Lemma 4.1 are satisfied, and S is noetherian.

Next we investigate maximal ideals of S. Put $P = \begin{pmatrix} M_1 & M_1 \\ M & M \end{pmatrix}$. Obviously this is one of the maximal ideals. The other one is given as $J(S) + SxS$, where $x = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, $u + M = (1, 0)$, observe that $uR + M = M_2$.

By direct calculation, the condition $\cap_{i\in\mathbb{N}} M_1^i = 0$ implies $\cap_{i\in\mathbb{N}} P^i = 0$. On the other hand $Qx \supseteq \begin{pmatrix} 0 & M_1 \\ 0 & M \end{pmatrix}$, $xQ \supseteq \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}$, $J(S)J(S) \supseteq \begin{pmatrix} MM_1 & 0 \\ 0 & 0 \end{pmatrix}$. Finally, our remark implies, that for any $v \in M_2$, Q contains an element $\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}$, therefore also $Q^2 \supseteq \begin{pmatrix} MM_2 & 0 \\ 0 & 0 \end{pmatrix}$. Since we supposed $M_1 + M_2 = R$, $M = MM_1 + MM_2$. Thus Q^2 contains $J(S)$ as a proper submodule. Therefore, the only possibility is that $Q^2 = Q$.

The structure of projective modules over S is clear: Since P does not contain an idempotent ideal, finitely generated projective modules are free. On the other hand there exists an (infinitely generated) projective module X of the trace ideal Q. Therefore projective modules over S are described by (b).

Before we continue, we point out the following properties: The rings S/P and S/Q are isomorphic (because $S/J(S) \simeq R/M$) and $J(S) = PQ+J(S)P$.

Now we can continue: Suppose we have a semilocal noetherian ring S, with exactly 2 maximal ideals P, Q. Suppose that Q is idempotent, $S/P \simeq S/Q$ as rings, $J(S) = PQ+J(S)P$. The construction of the ring B is similar as above: The elements of B are matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $b \in P$, $c \in Q$, $a, d \in S$ are glued: Again $S/J(S)$ can be imagined as $T \times T$ (the first T is modulo Q, the second one modulo P), and if $a + J(S) = (x, y)$, then $d = (x, x)$. The calculation of the Jacobson radical of B is similar as above, we get $J(B) = \begin{pmatrix} J(S) & P \\ Q & J(S) \end{pmatrix}$. We can apply Lemma 4.1 as above to extension $B \subseteq \begin{pmatrix} S & P \\ Q & S \end{pmatrix}$ in order to deduce that B is noetherian.

There is one obvious idempotent ideal, this is the two-sided ideal generated by $\begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$. Let us have a look for the other one: Let P' be the two-sided ideal generated by $J(B)$ and x, where $x = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, $u+J(S) = (1, 0)$. Similarly as above, $P'^2 \supseteq \begin{pmatrix} PQ & P \\ Q & J(S) \end{pmatrix}$. Again, for any $v \in P$ there exits v' such that $\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix} \in P'$, therefore also $\begin{pmatrix} J(S)P & 0 \\ 0 & 0 \end{pmatrix}$ is contained in P'^2. Therefore P'^2 contains $J(B)$, and it is not contained there. Thus P' is an idempotent ideal.

It remains to verify that all finitely generated modules over B are free. But if it is not the case, there is a finitely generated projective module of

the trace ideal Q'. Now embed B into full matrix ring of 2×2 matrices over S. The corresponding idempotent matrix after change of rings gives a finitely generated projective module over S having the trace ideal Q. This is impossible by our construction. Projective modules over B are described by (c).

Let us summarize what we have done so far:

Proposition 4.2. *Let R be a semilocal noetherian ring such that $R/J(R)$ is a product of two skew fields. Then the structure of projective R modules is described (i),(ii),(a),(b) or (c) and all possibilities may happen.*

Remark 4.3. Let us have a look at idempotent ideals of ring B constructed above. One of them is a maximal ideal, but the other is not. Is it possible to have a semilocal noetherian ring R such that $R/J(R)$ is a product of two skew-fields, finitely generated projective modules over R is free, and both maximal ideals of R are idempotent?

Let us have a look at some non-noetherian application. In [5] we mentioned a problem whether there exists a semilocal ring R such that $R/J(R)$ is a product of two skew fields, that is $R/J(R) \simeq S_1 \oplus S_2$ as R-modules, where S_1 and S_2 are simple modules over R, such that the structure of projective modules is described as follows:

(i) All finitely generated projective modules over R are free.
(ii) There exists a projective module P such that $P/PJ(R) \simeq S_1$.
(iii) There exists a projective module Q such that $Q/QJ(R) \simeq S_2^{(\infty)}$.

In fact, an easy analysis shows that if R is a semilocal ring with $R/J(R) \simeq S_1 \oplus S_2$ such that (i), (ii) holds, then there are only two possibilities. If (iii) holds, any projective module isomorphic to a direct sum of copies of R, P, Q. If (iii) does not hold, then any projective module over R is a direct sum of copies of R, P.

The first construction of R with (i),(ii) was given by Gerasimov and Sakhaev in [10] and in [5] we proved that for this construction (iii) does not hold. Let us show, how one can use the construction of Small and Stafford to "add" the module Q. We stay with the notation of [5]. The ring R_Σ is in fact a k-algebra, where k is a commutative field and $R_\Sigma/J(R_\Sigma) \simeq \mathsf{k} \times \mathsf{k}$ (as rings). For details of the construction see [10], for a brief summary, see [5]. By [5, Theorem 6.8], this ring satisfies (i),(ii) but not (iii). From the knowledge on R_Σ we will need the following. There are important elements $x, y \in R_\Sigma$ (canonically given by the construction) such that $(y-1)R_\Sigma$ and $R_\Sigma(x-1)$ are two-sided ideals of R_Σ. Moreover, these are the only maximal (left, right, two-sided) ideals of R_Σ.

Recall that by [5, Fact 4.1], there exists a homomorphism $\overline{\alpha}\colon R_\Sigma \to k \oplus k$ such that $\operatorname{Ker} \overline{\alpha} = \operatorname{Jac}(R_\Sigma)$, $\overline{\alpha}(x) = (0,1)$ and $\overline{\alpha}(y) = (1,0)$, of course

$\overline{\alpha}(1) = (1,1)$. If $a, d \in R_\Sigma$, we write that $a \sim d$ if there exist $t_1, t_2 \in k$ such that $\overline{\alpha}(a) = (t_1, t_2)$ and $\overline{\alpha}(d) = (t_1, t_1)$ (thus the relation $\sim$ is not symmetric).

Now let us define a ring T, which is a subring of $M_2(R_\Sigma)$:

$$T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b \in \mathrm{Jac(R_\Sigma), a, c, d} \in \mathrm{R_\Sigma}, a \sim d \right\}.$$

Observe that

$$J = \begin{pmatrix} \mathrm{Jac}(R_\Sigma) & \mathrm{Jac}(R_\Sigma) \\ R_\Sigma & \mathrm{Jac}(R_\Sigma) \end{pmatrix} \subseteq T$$

is an ideal of T. Moreover, one can verify that any element of $1 + J$ is invertible in T. It follows $J \subseteq \mathrm{Jac}(T)$. On the other hand there is a homomorphism of rings $\beta\colon T \to R_\Sigma/\mathrm{Jac}(R_\Sigma)$ given by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \overline{\alpha}(a).$$

Since $\ker(\beta) = J$ and $R_\Sigma/\mathrm{Jac}(R_\Sigma)$ is semisimple, $J = \mathrm{Jac}(R_\Sigma)$. Therefore T is a semilocal ring such that $T/\mathrm{Jac}(T) \simeq R_\Sigma/\mathrm{Jac}(R_\Sigma)$.

In particular, T has exactly 2 maximal (left, right, two-sided) ideals. Let us denote $L = R_\Sigma(x-1)$ and let L' be the set of all matrices in T having the top left corner in L. Obviously, L' is a maximal ideal in T.

Let us show that L' is a finitely generated left T-module. Observe that L' contains the elements

$$u = \begin{pmatrix} (1-x) & 0 \\ 0 & 1 \end{pmatrix}, v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It follows

$$\begin{pmatrix} \mathrm{Jac}(R_\Sigma) & \mathrm{Jac}(R_\Sigma) \\ \mathrm{Jac}(R_\Sigma) & \mathrm{Jac}(R_\Sigma) \end{pmatrix} \subseteq Tu + Tv$$

by direct calculations. Also since $vu = \begin{pmatrix} 0 & 0 \\ (1-x) & 0 \end{pmatrix}$ and $1+\mathrm{Jac}(R_\Sigma), (1-x)+\mathrm{Jac}(R_\Sigma)$ is a basis of $R_\Sigma/\mathrm{Jac}(R_\Sigma)$, we see that $\mathrm{Jac}(T) \subsetneq Tu+Tv$. Since $Tu + Tv \subseteq L'$, the only possibility is $L' = Tu + Tv$.

Further we need prove that L' is idempotent. It is easy to see that $\mathrm{Jac}(R_\Sigma) \subseteq L'^2$. Obviously $u^2 \in L'^2 \setminus \mathrm{Jac}(R_\Sigma)$, so the only possibility is $L'^2 = L'$.

Now we have enough information to classify projective right modules over T. Put

$$I' = \begin{pmatrix} (y-1)R_\Sigma & \mathrm{Jac}(R_\Sigma) \\ R_\Sigma & \mathrm{Jac}(R_\Sigma) \end{pmatrix}.$$

Observe that I', L' are the maximal ideals of T and as in [5] we write $\dim(P) = (\dim_k(P/PL'), \dim_k(P/PI'))$, where P is a projective module

over T. We use the same kind of arguments we used for the classification of projective modules over R_Σ:

Take a sequence of nonzero elements $r_1, r_2, \cdots \in \langle x \rangle$ satisfying $r_{i+1}r_i = r_i$. Put $s_i = \begin{pmatrix} r_i & 0 \\ 0 & 0 \end{pmatrix}$ and observe that s_i are elements of T satisfying $s_{i+1}s_i = s_i$. Therefore the module $P' = \cup_{i=1}^{\infty} s_i T$ is projective. Moreover, since the R_Σ-module $\cup_{i=1}^{\infty} r_i R_\Sigma$ is not finitely generated, the module P' is not finitely generated. By arguments similar to those in [5, Remark 6.1], $\dim(P') = (1, 0)$. And it follows as in Corollary [5, Corollary 6.3] that any finitely generated projective T-module is free. Finally, L' is an idempotent ideal finitely generated on the left. By the result of Whitehead [21, Corollary 2.7], there exists a countably generated projective T-module Q' of the trace ideal L'. Since $Q'L' = Q'$, $\dim(Q') = (0, n)$ for some $0 < n \leq \omega$. The relation $P'^{(n)} \oplus Q'^{(n)} = T^{(n)}$ implies $n = \omega$, because P' is not finitely generated. Now it is immediate to see that any projective T-module is a direct sum of copies of T, P', Q'.

B. K_0 of a semilocal ring revisited

Suppose that we want to turn the construction above into a general process of adding idempotent ideals without changing radical factor of the ring. Suppose that $R/J(R) \simeq M_{n_1}(F_1) \times M_{n_2}(F_2)$, where F_1, F_2 are skew-fields. For the identification along diagonal like above we would need some homomorphisms between $M_{n_1}(F_1)$ and $M_{n_2}(F_2)$, but in general there may be no such a homomorphism. We will change our strategy a bit, but first let us construct rings that will be corner stones for our construction. We also show that using pull-backs of these rings we can represent all possible direct-sum decompositions difficulties for finitely generated projective modules. In the following lemma the ring $R = S[x, \alpha]$, where $\alpha \in \mathrm{Aut}(S)$, is a ring of twisted polynomials, that is $xs = \alpha(s)x$ for any $s \in S$.

Lemma 4.4. *Let $T \subseteq S$ be a Galois extension of fields such that the corresponding Galois group is cyclic of order n. Suppose that T is infinite. Let α be a T-automorphism of S that is a generator of the Galois group. Then for any $k \in \mathbb{N}$ the polynomial ring $R = S[x, \alpha]$ has a semisimple factor isomorphic to $M_n(T)^k$.*

Proof. Let s be an element of S such that $\{\alpha^i(s) \mid 0 \leq i < n\}$ is a T-basis of S. Observe that the element $e = x^n - 1$ commutes with $S \cup \{x\}$, so it is a central element of R. We claim that eR is a maximal ideal of R. That is for any $f \in R \setminus eR$, $RfR + eR = R$. In order to see this, observe that $RfR + eR$ always contains a polynomial that is of order less than n and has a non-zero constant term. Let p be such a polynomial of non-zero degree d, then $sp - \frac{s}{\alpha^d(s)}ps$ is again an element $RfR + eR$ of degree at most $d - 1$ and

non-zero constant term. In this way we get that $eR + RfR$ contains a unit. Observe that the same argument applies to elements $e_t = x^n - t$ where t is any nonzero $t \in T$. Since R/e_tR is a simple ring of finite dimension over T, it is a simple artinian ring.

Fix $0 \neq t \in T$ and let us try to calculate $R/(x^n - t^n)R$. Write $x^n - t^n$ as $p(x)(x-t)$ and observe that the R-modules $R/(x-t)R$ and $p(x)R/e_{t^n}R$ are isomorphic. Moreover, the endomorphism ring of $R/(x-t)R$ is calculated as an idealizer of $x - t$ that is the set $\{r \in R \mid r(x-t) \in (x-t)R\}$ modulo $(x-t)R$. The idealizer is given by $s + (x-t)R$, where $s \in S$ such that $s(x-t) \in (x-t)R$, that is $\frac{st}{\alpha^{-1}(s)} = t$. This happens if and only if $s \in T$. Thus the endomorphism rings of $R/e_{t^n}R$ is always $M_n(T)$.

Further let us investigate when $R/(x-t)R$ is isomorphic to $R/(x-t')R$ for $t, t' \neq 0$. Again, homomorphisms are given by elements $r \in R$ such that $r(x-t)R \subseteq (x-t')R$. The considered modules are isomorphic if and only if there exists $r \in S$ such that $r(x-t) \in (x-t')R$ that is $\frac{r}{\alpha^{-1}(r)}t = t'$. That is r has to satisfy relation $q\alpha(r) = r$ for some $q \in T$. Suppose $r = c_0 s + c_1\alpha(s) + \cdots + c_{n-1}\alpha^{n-1}(s)$, $c_0, \ldots, c_{n-1} \in T$. Comparing the equality $q\alpha(r) = r$ expressed in normal basis we get $c_{i+1} = qc_i$ for $i = 0, \ldots, n-2$ and $c_0 = qc_{n-1}$. Observe that $c_0 \neq 0$ and also $c_0 q^n = c_0$. That is $q^n = 1$. Since q is the ratio of t and t', there is only finitely many possibilities for q. Therefore for a fixed $t \in T$ there are only finitely many possibilities for t'.

Now since we suppose T to be infinite, there are non-zero elements $t_1, \ldots, t_k \in T$ such that $R/(x-t_i)R$ are pairwise different simple modules. Let $I_i = e_{t_i^n}R$, then $I_1, \ldots, I_k$ are pairwise different maximal ideals. Then $R/\cap I_i \simeq R/I_1 \times \cdots \times R/I_k \simeq M_n(T)^k$. □

Lemma 4.5. *Let $n, k \in \mathbb{N}$. Then there exists a noetherian semilocal $\mathbb{Q}$-algebra R such that $R/J(R) \simeq M_n(\mathbb{Q})^k$ and such the projective modules over R are free.*

Proof. By a result of Shafarevic any finite solvable group can be found as a Galois group of a Galois extension of $\mathbb{Q}$ (well, for cyclic groups the proof is much easier). Let $\mathbb{Q} \subseteq S$ be a Galois extension of fields with the corresponding Galois group cyclic of order n with generator α. Then there exists an onto homomorphism $S[x, \alpha] \to M_n(\mathbb{Q})^k$. As $S[x, \alpha]$ is a(n non-commutative) principal ideal domain, we can use [8, Proposition 4.1] to get a semilocal principal ideal domain R with radical factor $M_n(\mathbb{Q})^k$. This is again a $\mathbb{Q}$-algebra. The fact that projective modules are free is a consequence of R is a principal ideal domain. □

Now let us recall a result due to Facchini and Herbera [7]. Let R be a semilocal ring and such that $R/J(R) \simeq S_1^{m_1} \oplus \cdots \oplus S_k^{m_k}$, where $m_1, \ldots, m_k \in \mathbb{N}$ and $\{S_1, \ldots, S_k\}$ is a representative set of simple R-modules. If P is a finitely generated projective module, there are unique $x_1, \ldots, x_k \in \mathbb{N}_0$ such that $P/PJ(R) \simeq S_1^{x_1} \oplus \cdots \oplus S_k^{x_k}$. In this case we say that P is of dimension $(x_1, \ldots, x_k)$. Obviously finitely generated projective R-modules are isomorphic if and only if they are of the same dimension.

Let $A \subseteq \mathbb{N}_0^k$ be the set of all possible dimensions of finitely generated projective R-modules. Then A is a full subsemigroup of $\mathbb{N}_0^k$, that is if $(a_1, \ldots, a_k), (b_1, \ldots, b_k) \in A$ and $a_i \leq b_i$ for any $1 \leq i \leq k$, then also $(b_1 - a_1, \ldots, b_k - a_k) \in A$. But the main result of [7] claims that the converse is also true: Given any $k, m_1, \ldots, m_k \in \mathbb{N}$ and $A \subseteq \mathbb{N}_0^k$ a full subsemigroup such that $(m_1, \ldots, m_k) \in A$. Then there exists a semilocal ring R with a representative set of simple modules $S_1, \ldots, S_k$ such that $R/J(R) \simeq S_1^{m_1} \oplus \cdots \oplus S_k^{m_k}$ (as modules) and A is exactly the set of all dimensions of all finitely generated projective R-modules.

The original proof in [7] used quite good knowledge of universal localization. The resulting semilocal rings were hereditary but probably not noetherian in general. Later, Wiegand [22] proved that it is possible to represent any full subsemigroup also by a semilocal noetherian ring. His construction is quite indirect (in fact a corollary of a related problem). Keeping the spirit of Wiegand's paper, we present another construction using the rings from Lemma 4.5. The advantage of this approach is that we get also some information of non-finitely generated projective modules. The disadvantage is that in general we cannot get rid of some projective modules that are not direct sums of finitely generated modules. (Of course, this is not the case of hereditary rings constructed by Facchini and Herbera.) As we want to deal with non-finitely generated projective modules, we define a dimension of a countably generated projective modules over a semilocal ring R with a representative set of simple modules $\{S_1, \ldots, S_k\}$. If P is a countably generated projective module over such a ring, there are unique $x_i \in \mathbb{N}_0^*$ such that $P/PJ(R) \simeq S_1^{(x_1)} \oplus \cdots \oplus S_k^{(x_k)}$. We say that P is of dimension $(x_1, \ldots, x_k)$. Again countably generated projective modules over R are isomorphic if and only if they are of the same dimension ([14]).

The following fact is a corollary of Milnor's results on projective modules over pull-backs.

Fact 4.6. *Let S be a semisimple ring and let $\varphi\colon R \to S$ be a homomorphism such that $J(R) = \operatorname{Ker} \varphi$ and $\varphi(R) = S$. Suppose that $T \subseteq S$ is also a semisimple ring, with $\{S_1, \ldots, S_n\}$ being representative set of simple modules over S. Let $\nu\colon T \to S$ be the inclusion. Let U, $\alpha\colon U \to R$, $\beta\colon U \to T$ be the pullback of maps φ and ν. Then*

(i) *The homomorphism* β *is an onto map and* $\operatorname{Ker}\beta = J(U)$. *In particular,* U *is a semilocal ring such that* $U/J(U) \simeq T$.

(ii) *Suppose that* $\{M_1, \dots, M_k\}$ *is a representative set of simple* T*-modules. Then there is a matrix* $A = (a_{i,j})_{i,j=1}^{k,n}$ *with entries in* $\mathbb{N}_0$ *given by* $M_i \otimes_T S \simeq \oplus_{j=1}^n S_j^{a_{i,j}}$. *Further, let* $(m_1, \dots, m_k) \in \mathbb{N}_0^{*k}$. *Then there exists a (countably generated) projective module over* U *of dimension* $(m_1, \dots, m_k)$ *if and only if there exists a (countably generated) projective module over* R *of dimension* $(m_1, \dots, m_k)A$.

(iii) *Suppose that any projective* R*-module* P *is finitely generated if and only if* $P/PJ(R)$ *is finitely generated (for example if* R *is right noetherian). Then the similar property holds for* U*, that is a projective* U*-module* P *is finitely generated if and only if* $P/PJ(U)$ *is finitely generated.*

Remark 4.7. The dimension of a projective module over U here has the following meaning. We say that a countably generated projective module P over U has dimension $(m_1, \dots, m_k) \in \mathbb{N}_0^{*k}$ if $P \otimes_U T \simeq \oplus_{i=1}^k M_i^{(m_i)}$. Since $U/J(U) \simeq T$, the difference from the usual notion of dimension is only formal.

Proof. (i) As always, write U as a subring of $R \times T$ given by $U = \{(r,t) \mid \varphi(r) = \nu(t)\}$. The homomorphisms α and β are then just projections of $R \times T$ on R and T restricted to U. Observe that since φ is an onto map, β is also an onto map. Further, $I = \operatorname{Ker}\beta = \{(j,0) \mid j \in J(R)\}$. It is easy to see that elements of $1 - I$ are invertible in U, therefore $I \subseteq J(U)$. On the other hand, $U/I \simeq T$ is semisimple, therefore $I = J(U)$.

(ii) This is just a specialization of general Milnor's results [12, Theorem 2.1,Theorem 2.2, Theorem 2.3] which says that any projective module P over R and any projective module Q over T such that S-modules $P \otimes_R S$ and $Q \otimes_T S$ are isomorphic, then there exists a projective module M over U such that $P \simeq M \otimes_U R$ and $Q \simeq M \otimes_U T$. Moreover, any projective U-module is isomorphic to such an M if we take convenient modules P, Q.

Since T is semisimple, any T-module is projective. Moreover, since $\operatorname{Ker}\beta = J(U)$, we have that projective U-modules M and N are isomorphic if and only if (semisimple) T-modules $M \otimes_U T$ and $N \otimes_U T$ are isomorphic. In other words in order to describe possible dimensions for countably generated projective modules over U, we have to know when for a T-module Q there exists a projective R-module P such that $P \otimes_R S \simeq Q \otimes_T S$. Our statement is just a formalization of this fact.

Finally observe that if P is a projective U-module such that $P/PJ(U)$ is countably generated, then P is countably generated. This is because $P/PJ(U) \oplus (U/J(U))^{(\omega)} \simeq U/J(U)^{(\omega)}$ and therefore $P \oplus U^{(\omega)} \simeq U^{(\omega)}$.

(iii) This follows from [12, Theorem 2.1] which also says that (in the above notation) M is finitely generated over U whenever P is finitely generated over R and Q is finitely generated over T. $\square$

I did not have time to search for the following lemma in the literature. The statement appeared as an exercise in [2, page 290].

Obviously, it can be done in more general way, the proof is included just for the sake of completeness.

Lemma 4.8. *Let $A \subseteq \mathbb{N}_0^k$ be a full affine subsemigroup such that A contains an archimedean element of $\mathbb{N}_0^k$. Then there exist a finite set I and relations $r_i(x_1, \dots, x_k), i \in I$ such that $(x_1, \dots, x_k) \in \mathbb{N}_0^k$ if and only if $r_i(x_1, \dots, x_k)$ for any $i \in I$. Moreover, r_i's are of the form $r_i(x_1, \dots, x_k) \equiv \sum_{j=1}^k x_j z_j \in m\mathbb{Z}$, where $z_1, \dots, z_k, m$ are integers depending on i.*

Proof. Recall that A has a composition series of ideals $0 = I_0 \subseteq I_1 \subseteq \dots \subseteq I_m = A$ (here ideals are given by set of elements having supports contained in a fixed set, we suppose there is no other ideal between I_j and I_{j+1}, for details see for example [15]). After convenient changes of coordinates we can suppose that $\mathrm{Supp}(I_j) = \{1, \dots, l_j\}$, $1 \le l_1 < l_2 < \dots < l_m = k$. Any ideal I_j also generates a subgroup B_j in $\mathbb{Z}^{l_j}$, further any I_j contains an archimedean element of $\mathbb{N}_0^{l_j}$. Therefore we may prove the claim by induction on j. Relations that we are going to construct will be of the form $r_i(x_1, \dots, x_k) \equiv \sum_{j=1}^k x_j z_j \in m\mathbb{Z}$, where $z_1, \dots, z_n \in \mathbb{Q}$ and $m \in \mathbb{Z}$. But it is not difficult to clear the denominators and get the relations in the required form.

Suppose that $j = 1$. Then B_1 is a cyclic subgroup generated by $(y_1, \dots, y_{l_1})$, all y_i's are nonzero. Then $(x_1, \dots, x_{l_1}) \in \mathbb{Z}^{l_1}$ belongs to B_l if and only if $y_{l_1} | x_{l_1}$ and $x_{l_1} y_{l'} = x_{l'} y_{l_1}$ for any $1 \le l' < l_1$. These l_1 relations determine B_1.

Suppose that for some $j < l$ we have a finite set I and relations $r_i(x_1, \dots, x_{l_j})$ of the prescribed form such that any $(x_1, \dots, x_{l_j}) \in \mathbb{Z}^{l_j}$ is an element of B_j if and only if $r_i(x_1, \dots, x_{l_j})$ for any $i \in I$. We know there exists an element $(y_1, \dots, y_{l_{j+1}}) \in \mathbb{N}^k$ such that $B_{j+1} = \mathbb{Z}(y_1, \dots, y_{l_{j+1}}) + B_j$. Thus an element of $(x_1, \dots, x_{l_{j+1}}) \in B_{l+1}$ satisfies the following relations $y_{l_{j+1}} | x_{l_{j+1}}$, $x_{l'} y_{l_{j+1}} = x_{l_{j+1}} y_{l'}$ for any $l_j < l' < l_{j+1}$. And since $(x_1, \dots, x_{l_{j+1}}) - \frac{x_{l_{j+1}}}{y_{l_{l+1}}}(y_1, \dots, y_{l_{j+1}})$ (respectively its projection on the first l_j components) is an element of B_j, the relations $r_i((x_1, \dots, x_{l_j}) - \frac{x_{l_{j+1}}}{y_{l_{l+1}}}(y_1, \dots, y_{l_j}))$ (observe we put integers into the arguments of r_i). We define I' to be the set of following relations (all in variables $x_1, \dots, x_{l_{j+1}}$): $y_{l_{j+1}} | x_{l_{j+1}}$, $x_{l'} y_{l_{j+1}} = x_{l_{j+1}} y_{l'}$ for any $l_j < l' < l_{j+1}$ and for any $i \in I$ we

add $r_i((x_1 - \frac{x_{l_{j+1}}}{y_{l_{j+1}}} y_1, \dots, x_{l_j} - \frac{x_{l_{j+1}}}{y_{l_{j+1}}} y_{l_j}))$ (after expansion, these relations will be of the required form). It remains to show that any $(x_1, \dots, x_{l_{j+1}}) \in \mathbb{Z}^{l_{j+1}}$ satisfying all defined relations, then $(x_1, \dots, x_{l_{j+1}}) \in B_{j+1}$. First of all there exists $z \in \mathbb{Z}$ such that $x_{l_{j+1}} = z y_{l_{j+1}}$. Then $(x_1, \dots, x_{l_{j+1}}) - z(y_1, \dots, y_{l_{j+1}})$ has only the first l_j components nonzero. Furthermore, $r_i((x_1 - zy_1, \dots, x_{l_j} - zy_{l_j}))$, therefore $(x_1, \dots, x_{l_{j+1}}) - z(y_1, \dots, y_{l_{j+1}}) \in B_j$ and, consequently, $(x_1, \dots, x_{l_{j+1}}) \in B_{j+1}$.

Now the set of relations constructed of B_k (after clearing denominators, if necessary) is the desired set of relations. It is because of $A = B_k \cap \mathbb{N}_0^k$, which is a general property of full affine semigroups. $\square$

Now it is more or less obvious how could we give an alternative proof for the result of Facchini and Herbera [7].

Proposition 4.9. *Let $A \subseteq \mathbb{N}_0^k$ be a full affine subsemigroup of $\mathbb{N}_0^k$ and let $(n_1, \dots, n_k) \in A$ be an archimedean element of $\mathbb{N}_0^k$. Then there exists a semilocal noetherian $\mathbb{Q}$-algebra R such that $R/J(R) \simeq M_{n_1}(\mathbb{Q}) \times \dots \times M_{n_k}(\mathbb{Q})$. Moreover, if S_i is the simple module corresponding to the i-th homogeneous component of $R/J(R)$, for any $(x_1, \dots, x_k) \in \mathbb{N}_0^k$ we have $(x_1, \dots, x_k) \in A$ if and only if there exits a projective R-module P such that $P/PJ(R) \simeq S_1^{x_1} \oplus \dots \oplus S_k^{x_k}$.*

Proof. First of all we use Lemma 4.8 to construct relations $r_i(x_1, \dots, x_k), i \in I$ of the described form. The relations are of two kinds.

First, suppose $r_i(x_1, \dots, x_k)$ is of the form $\sum_{i=1}^k x_i z_i \in m\mathbb{Z}$, $m, z_1, \dots, z_k \in \mathbb{Z}$ and $m \neq 0$. We can suppose that $m \in \mathbb{N}$ and after adding sufficient multiples of m to z_i's we can also suppose that $z_i \in \mathbb{N}$. For any $i \in I$ such that r_i is of this form we construct a semilocal $\mathbb{Q}$ algebra R_i such that $R_i/J(R_i) \simeq M_{n_1}(\mathbb{Q}) \times \dots \times M_{n_k}(\mathbb{Q})$ and if S_j is the simple module corresponding to j-th homogeneous component of $R/J(R)$, then there exists a finitely generated projective R_i-module P such that $P/PJ(R_i) \simeq \oplus_{j=1}^k S_j^{m_j}$ if and only if $m | \sum_{j=1}^k m_i z_i$. Let $ml = \sum_{j=1}^k n_i z_i$. Let Y be a noetherian $\mathbb{Q}$-algebra such that $Y/J(Y) \simeq M_m(\mathbb{Q})$ and such that any projective Y-module is free. Further, let $X = M_l(Y)$, therefore Y is a $\mathbb{Q}$-algebra such that $X/J(X) \simeq M_{ml}(\mathbb{Q})$. Observe that there is only one simple X-module S and all finitely generated projective X-modules are P^a, $a \in \mathbb{N}_0$, where $P/PJ(X) \simeq S^m$. Let $\varphi\colon X \to M_{ml}(\mathbb{Q})$ be an epimorphism with the kernel $J(X)$. Further, let $T = M_{n_1}(\mathbb{Q}) \times \dots \times M_{n_k}(\mathbb{Q})$. Since $ml = \sum_{j=1}^k n_i z_i$, there exists a canonical embedding $\nu\colon T \to M_{ml}(\mathbb{Q})$ (the components $M_{n_j}(\mathbb{Q})$ are put along diagonal of $M_{lm}(\mathbb{Q})$, $M_{n_j}(\mathbb{Q})$ is placed z_j-times). Now let $R_i, \alpha\colon R_i \to X, \beta\colon X \to T$ be the pullback of

maps φ, ν. Then a direct application of Fact 4.6 gives that R_i has the desired structure for projective modules. Finally, put $\beta_i := \beta$. Observe that $\beta_i \colon R_i \to M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$ is an epimorphism and Ker $\beta = J(R_i)$.

Now let us treat the case when r_i is a relation of the form $\sum_{j=1}^{k} x_j z_j = 0$, for $z_1, \ldots, z_k \in \mathbb{Z}$. Suppose that $j_1, \ldots, j_l, j'_1, \ldots, j'_{l'}$ are integers such that $j_1, \ldots, j_l$ are indexing positive z_j's and $j'_1, \ldots, j'_{l'}$ are indices of negative z_j's. It may happen that $\{j_1, \ldots, j_l, j'_1, \ldots, j'_{l'}\} \neq \{1, \ldots, k\}$, let $\{j''_1, \ldots, j''_{l''}\} = \{1, \ldots, k\} \setminus \{j_1, \ldots, j_l, j'_1, \ldots, j'_{l'}\}$. So the relation can be written as $\sum_{a=1}^{l} z_{j_a} x_{j_a} = \sum_{a=1}^{l'} z_{j'_a} x_{j'_a}$. Let $y' = \sum_{a=1}^{l} z_{j_a} n_{j_a} = \sum_{a=1}^{l'} z_{j'_a} n_{j'_a}$ and let $y = y' + \sum_{a=1}^{l''} n_{j''_a}$.

Let Y be a noetherian $\mathbb{Q}$-algebra such that any projective Y-module is free and $Y/J(Y) \simeq \mathbb{Q} \times \mathbb{Q}$. Further, let $X = M_y(\mathbb{Q})$ and let $\varphi \colon X \to M_y(\mathbb{Q}) \times M_y(\mathbb{Q})$. Hence there are two non-isomorphic simple X-modules S_1, S_2 and there exists a projective X-module P such that $P/PJ(X) \simeq S_1 \oplus S_2$ and any finitely generated projective module over X is isomorphic to P^a. Now let $T = M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$ and we construct $\nu \colon T \to M_y(\mathbb{Q}) \times M_y(\mathbb{Q})$ as follows. Take $j \in \{1, \ldots, k\}$. If $j = j_a$ for some $1 \leq a \leq l$, we place the j-th component $M_{n_j}(\mathbb{Q})$ into the first copy of $M_y(\mathbb{Q})$ along the diagonal z_j-times. If $j = j'_a$ for some $1 \leq a \leq l'$, we place the j-th component $M_{n_j}(\mathbb{Q})$ into the second copy of $M_y(\mathbb{Q})$ along the diagonal z_j-times. If $j = j''_a$ for some $1 \leq a \leq l''$, we place the j-th component $M_{n_j}(\mathbb{Q})$ into both copies of $M_y(\mathbb{Q})$ along the diagonal once. Let R_i, α, β be the pullback of the maps φ, ν. Then $\beta \colon R_i \to M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$ is an epimorphism such that $\varphi(R_i) = J(R_i)$. Then a direct application of Fact 4.6 gives that R_i has the desired structure for projective modules. That is, if $(m_1, \ldots, m_k) \in \mathbb{N}_0^k$ then there exists a finitely generated projective R_i-module P such that $P \otimes_{R_i} T \simeq \oplus_{j=1}^{k} S_j^{m_j}$ if and only if $r_i(m_1, \ldots, m_k)$. Again, put $\beta_i := \beta$.

Therefore for any $i \in I$ we have a ring R_i and an epimorphism $\beta_i \colon R_i \to M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$ such that Ker$\beta_i = J(R_i)$. Let $R, (\psi_i \colon R \to R_i, i \in I)$ be a pullback of homomorphisms $\beta_i \colon R_i \to M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q}), i \in I$. A direct calculation shows that R is again a noetherian $\mathbb{Q}$-algebra and the homomorphism $\psi = \beta_i \psi_i$ (for some/any $i \in I$) is an epimorphism of R onto $M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$ such that Ker $\psi = J(R)$. Further, application of [12, Theorem 2.1] gives that for any $(m_1, \ldots, m_k) \in \mathbb{N}_0^k$ there exits a projective R-module P such that $P \otimes_R T \simeq \oplus S_i^{m_i}$ if and only if $r_i(m_1, \ldots, m_k)$ for any $i \in I$ if and only if $(m_1, \ldots, m_k) \in A$. This concludes the proof. $\square$

C. A variant of Small and Stafford's construction as a pull-back

In this subsection we show how to "add" idempotent ideals without changing the semigroup of finitely generated projective modules. Our approach is as follows: We start with a semilocal ring R which will be a $\mathbb{Q}$-algebra and $R/J(R)$ is a product of matrices over $\mathbb{Q}$. As we know from subsection B. all semigroups of projective modules can be realized over such rings (and, moreover, R can be chosen also noetherian). First we investigate projective modules over a ring U that is a subring of $M_n(R)$ for some $n \in \mathbb{N}$. Finally, we construct a new ring R' as a pull-back of U and a convenient semi-simple ring. Results of [12] then help us to understand projective modules over R'.

Now let us continue with more details. Let R be a semilocal ring such that $R/J(R)$ is a product of matrices over $\mathbb{Q}$. Let T be given $T = J(R)+\mathbb{Q} \subseteq R$, observe that T is a local ring. Moreover, an application of Lemma 4.1 with $K = J(R)$ gives that T is noetherian whenever R is noetherian and R/K is finitely generated over T on both sides. The later condition is satisfied whenever $R/J(R)$ is a product of matrices over $\mathbb{Q}$. Fix some $m \in \mathbb{N}$ and let $S = M_m(T)$, therefore S is a homogeneous semilocal $\mathbb{Q}$-algebra. Take P, Q, ideals of R such that $QP \subseteq J(R)$. We define the ring U to be the subring of $M_{m+1}(R)$ given by matrices of the form $\begin{pmatrix} R & P^m \\ Q^m & S \end{pmatrix}$. Here Q^m means a column of length m with entries in Q, P^m means a row of length m with entries in P. Now, if R is noetherian and $R/J(R)$ is of finite dimension over $\mathbb{Q}$, then U is finitely generated on both sides over a noetherian ring T, therefore U is noetherian. Moreover, ideals P, Q are finitely generated (on both sides) over T.

From now on, we suppose that R is also noetherian and $R/J(R)$ is a product of matrices over $\mathbb{Q}$. Under this assumption, the description of countably generated projective modules reduces to searching for idempotent ideals of U and for each idempotent ideal $I \subseteq U$ also finitely generated projective U/I-modules. In our approach we suppose we have this knowledge over the ring R.

First, let us describe idempotent ideals of U.

Lemma 4.10. *Let U be the ring described above. If I is an idempotent ideal of U, then either $I = \begin{pmatrix} Z & P^m \\ Q^m & S \end{pmatrix}$, where Z is an ideal of R containing PQ and Z/PQ is an idempotent ideal of R/PQ or there exists an idempotent ideal Z over R such that I is generated by $\begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix}$ (here 0's are zero matrices of appropriate sizes), that is $I = \begin{pmatrix} Z & (ZP)^m \\ (QZ)^m & M_m(QZP) \end{pmatrix}$.*

Proof. Let I be an idempotent ideal of U. Suppose that I contains an element that is not in the set $\begin{pmatrix} R & P^m \\ Q^m & J(S) \end{pmatrix}$. Then I contains $\begin{pmatrix} 0 & P^m \\ Q^m & S \end{pmatrix}$. Let Z be the ideal of R given by the elements which occur in the top left corner of elements in I. Then Z contains PQ and Z/PQ is an idempotent ideal of R/PQ. Therefore $I = \begin{pmatrix} Z & P^m \\ Q^m & S \end{pmatrix}$. It is easily verified that any ideal of this form is idempotent.

Next, suppose that I is contained in $\begin{pmatrix} R & P^m \\ Q^m & J(S) \end{pmatrix}$. Then there exist an ideal X in T, bimodules ${}_RP'_T \subseteq P$, ${}_TQ'_R \subseteq Q$ and an ideal Z in R such that I is given by all elements of the set $\begin{pmatrix} Z & P'^m \\ Q'^m & M_m(X) \end{pmatrix}$. Since I is an ideal, it is necessary to have $QZ \subseteq Q'$, $ZP \subseteq P'$, $Q'P' \subseteq X$. Moreover, since I is idempotent, we have $Z = Z^2 + P'Q'$, $P' = ZP' + P'X = ZP + P'X$, $Q' = Q'Z + XQ' = QZ + XQ'$, $X = Q'P' + X^2$. Now, P_T is finitely generated, T is noetherian, therefore P'_T is also finitely generated and $P' = ZP$ by Nakayama lemma. Similarly, $Q' = QZ$. Moreover, $X = Q'P'$. Observe, that $P'Q'$ is contained in Z^2 and therefore Z is an idempotent ideal. Then I has to be an ideal generated by $\begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix}$, where Z is an idempotent ideal. Of course, any ideal of this form is idempotent. $\square$

Therefore idempotent ideals of U are divided into two classes. If $I = \begin{pmatrix} Z & P^m \\ Q^m & S \end{pmatrix}$, where $PQ \subseteq Z$ and Z is idempotent modulo PQ, then $U/I \simeq R/Z$. Thus I-big projective modules over U are described by Z/PQ-big projective modules over R/PQ. It seems that we cannot say more in general, however, recall we suppose good knowledge of R.

In the opposite case, that is when $I = \begin{pmatrix} Z & (ZP)^m \\ (QZ)^m & M_m(QZP) \end{pmatrix}$, where Z is an idempotent ideal, we have to describe finitely generated projective modules over the factor $U/I \simeq \begin{pmatrix} R/Z & (P/ZP)^m \\ (Q/QZ)^m & M_m(T/QZP) \end{pmatrix}$. For $Z = 0$ we will get description of finitely generated projective modules over U.

Let us move to a more general setting. Let R_0 be a semilocal $\mathbb{Q}$-algebra, T_0 a local $\mathbb{Q}$-algebra, P an $R_0 - T_0$ bimodule finitely generated on both sides and Q_0 a $T_0 - R_0$ bimodule finitely generated on both sides. As usually, suppose we have homomorphisms of bimodules $\alpha\colon P_0 \otimes_{T_0} Q_0 \to R_0$ and $\beta\colon Q_0 \otimes_{R_0} P_0 \to T_0$. We shall write $\alpha(p \otimes q)$ as pq and $\beta(q \otimes p)$ as qp. Furthermore, we suppose that $P_0Q_0 \subseteq J(R_0)$ and $Q_0P_0 \subseteq J(T_0)$. (So in

our setting $R_0 := R/Z$, $P_0 = P/ZP$, $Q_0 = Q/QZ$ and $T_0 = T/QZP$.) Let $S_0 = M_m(T_0)$ and $U_0 = \begin{pmatrix} R_0 & P_0^m \\ Q_0^m & S_0 \end{pmatrix}$, the multiplication in U_0 is given as ordinary matrix multiplication regarding the meaning of $pq, qp, p \in P_0, q \in Q_0$ explained above.

Let us have a look how one could classify finitely generated projective modules over U_0. The ideal $I = \begin{pmatrix} J(R_0) & P_0^m \\ Q_0^m & J(S_0) \end{pmatrix}$ is a two-sided ideal such that $1 - i$ is invertible for any $i \in I$. Thus $I \subseteq J(U_0)$ and on the other hand $U_0/I \simeq R_0/J(R_0) \times S_0/J(S_0)$, which is a semisimple ring, therefore $I = J(U_0)$.

Observe that $U_0/J(U_0) \simeq R_0/J(R_0) \times S_0/J(S_0)$. If M is a projective module over U, we write $M/MJ(U_0)$ as a pair $M/MM_1 \oplus M/MM_2$, where

$$M_1 = \begin{pmatrix} J(R_0) & P_0^m \\ Q_0^m & S_0 \end{pmatrix}, M_2 = \begin{pmatrix} R_0 & P_0^m \\ Q_0^m & J(S_0) \end{pmatrix}$$

(Observe that $M_1 \cap M_2 = J(U_0)$ and $U_0/M_1 \simeq R_0/J(R_0)$, $U/M_2 \simeq S_0/J(S_0)$)

Observe, that there exists a projective module M' over U such that $M'/M'M_1 = 0$ and $M'/M'M_2$ is a simple $S_0/J(S_0)$-module. It follows that any projective U-module is a direct sum of M'^k and M'', where $M'' = M''M_2$. We claim that any such an M'' is "induced" from a finitely generated projective module over R_0. That is there exists a finitely generated projective module M''' over R_0 such that $M'''/M'''J(R_0) \simeq M''/M''M_1$, where the module on the right is considered as an $R_0/J(R_0)$-module. Let $A \in M_n(U_0)$ be an idempotent matrix representing M''. Since $M'' = M''M_2$, the entries are elements of M_2 (so in the S_0-part of entries occur only elements of $J(S_0)$). Observe, that $M_n(U_0)$ is isomorphic to ring of matrices written in the block form as $\begin{pmatrix} B & C \\ D & E \end{pmatrix}$, where B is an $n \times n$ matrix over R_0, C an $n \times mn$ matrix over P_0, D an $mn \times n$ matrix over Q_0 and D is an $mn \times mn$ matrix with entries in S_0 (we can consider $M_n(U_0)$ as an $n(m+1) \times n(m+1)$ matrix of symbols, then we make some permutation on rows and the same permutation of columns). Let us denote this ring X. Using this isomorphism, we get an idempotent element $B' \in X$ as an image of B. In particular, $B' = \begin{pmatrix} B_0 & C \\ D & E \end{pmatrix}$, where B_0 is an $n \times n$ matrix over R_0 given by rows and columns of B with number $1, m+2, \ldots, (m+1)(n-1)+1$, C is an $n \times mn$ matrix with entries in P_0, D is an $mn \times n$ matrix with entries in Q_0 and E is an $mn \times mn$-matrix with entries in $J(T_0)$. Now B' is idempotent, therefore $B_0 = B_0^2 + CD, D = DB_0 + ED$,

and $C = B_0C + CE$. Therefore $(1_{mn} - E)D = DB_0$ and $C(1_{mn} - E) = B_0C$. Since entries of E are in $J(T_0)$ we can calculate $F = (1 - E)^{-1}$ over T_0. Thus $B_0 = B_0^2 + B_0CF^2DB_0 = B_0(1 + CF^2D)B_0$. Thus we have an idempotent matrix $B_0' = B_0(1 + CF^2D)$. Observe that B_0 and B_0' are the same modulo $J(R_0)$. It follows that $B_0'R_0^n$ is the desired projective R_0-module.

Let us summarize what we have proved

Lemma 4.11. *Let $I \subseteq U$ be an idempotent ideal of the form $\begin{pmatrix} Z & ZP \\ QZ & QZP \end{pmatrix}$. The structure of finitely generated projective modules over $U_0 = U/I$ is given as follows. The ring $U_0/J(U_0)$ is isomorphic to $R/Z/J(R/Z) \times S/J(S)$. Let $S_1, \dots, S_n$ be a representative set of simple R/Z-modules, and S_{n+1} the unique simple S-module. Then for any $m_1, \dots, m_{n+1} \in \mathbb{N}_0$ there exists a finitely generated projective U_0-module M such that $M/MJ(U_0) \simeq \oplus_{i=1}^{n+1} S_i^{m_i}$ if and only if there exists a finitely generated projective R/Z-module M' such that $M'/M'J(R/Z) \simeq \oplus_{i=1}^{n} S_i^{m_i}$.*

Taking $Z = 0$ in the lemma above we get the picture for finitely generated projective U-modules. In general, finitely generated projective U/I-modules can be understand from finitely generated projective modules over some factors of R.

Let us continue with our notation, that is R is a semilocal noetherian $\mathbb{Q}$-algebra and such that $R/J(R) \simeq M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$. The conditions on ideals $P, Q \subseteq R$ gives that $P + J(R)/J(R)$ and $Q + J(R)/J(R)$ give a decomposition of $R/J(R)$. Suppose that homogeneous components of $R/J(R)$ are ordered such that (under identification of $R/J(R)$ and $M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$ we have $P + J(R)/J(R) = M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_l}(\mathbb{Q}) \times 0 \times \cdots \times 0$ and $Q + J(R)/J(R) = 0 \times \cdots \times 0 \times M_{n_{l+1}}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$ for some $1 < l < k$. Now put $m = n_1 + \cdots + n_l$ and consider the ring $U = \begin{pmatrix} R & P^m \\ Q^m & S \end{pmatrix}$, where $S = M_m(T)$ and $T = J(R) + \mathbb{Q}$.

Now let us consider the following diagram: As we know $U/J(U) \simeq R/J(R) \times S/J(S)$, so there exists an onto homomorphism $\varphi\colon U \to M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q}) \times M_m(\mathbb{Q})$ such that Ker $\varphi = J(U)$. Further, let $V = M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$. Since $m = n_1 + \cdots + n_l$, there is an obvious homomorphism of $h\colon V \to M_m(\mathbb{Q})$ (put the first l homogeneous components along the diagonal of $M_m(\mathbb{Q})$). Therefore there exists a monomorphism $\nu\colon V \to M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q}) \times M_m(\mathbb{Q})$ given by $\nu = (\mathrm{id}, h)$, where id is the identity on V. Now let $R', \alpha\colon R \to U, \beta\colon R' \to V$ be a pullback of φ and ν. Now β is an onto map such that Ker $\beta = J(R')$. Hence $R/J(R) \simeq R'/J(R')$ and, by Lemma 4.11 with $Z = 0$ and by Fact 4.6, we get $V(R) \simeq V(R')$ canonically. So the semigroups of finitely generated projective modules over R and R' are the same.

But there are changes if we look on non-finitely generated projective modules. Observe that $\alpha\colon R' \to U$ is a monomorphism. As a subring of U, R' is given by matrices of the form $\begin{pmatrix} r & p \\ q & s \end{pmatrix}$, where r, p, q, s are matrices of appropriate sizes and, moreover, $\pi(r+J(R)) = s+J(S)$, where $\pi\colon M_{n_1}(\mathbb{Q})\times \cdots \times M_{n_k}(\mathbb{Q}) \to M_{n_1}(\mathbb{Q})\times\cdots\times M_{n_l}(\mathbb{Q})$ is the projection (again we consider identifications $R/J(R) \simeq M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q})$ and $S/J(S) \simeq M_m(\mathbb{Q})$). When R' is considered as a subring of U, then the set $P' \subseteq R'$ given by matrices of R' that have the top left corner in P is an idempotent ideal of R' (it is not hard to verify, bellow we will discuss the structure of idempotent ideals of R' in general). Therefore R' has always a projective module of dimension $(\infty, \dots, \infty, 0, \dots, 0)$, that is the first l components are ∞ and the others are 0. This explains why we can look at this construction as on a process how to add non-finitely generated projective modules without changing the semigroup of finitely generated ones.

Let $S_1, \dots, S_k$ be a representative set of simple V-modules such that S_i corresponds to the component $M_{n_i}(\mathbb{Q})$. Similarly, let $S'_1, \dots, S'_k, S'_{k+1}$ be a representative set of simple modules of $M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q}) \times M_m(\mathbb{Q})$, where S'_i corresponds to the i-th homogeneous component. Since R' is a (left) noetherian ring, the idempotent ideals are given as traces of countably generated projective modules. Since R' is semilocal, we are left to question for which $(m_1, \dots, m_k) \in \{0, \infty\}^k$ there exists a projective R' module M such that $M \otimes_{R'} V \simeq \oplus_{i=1}^k S_i^{(m_i)}$. Similar correspondence works also over R and U.

We know that if Z is an idempotent ideal over R, then it induces an idempotent ideal in U. Suppose that $S''_1, \dots, S''_k$ are simple R-modules such that when $R/J(R)$ is identified with $M_{n_1}(\mathbb{Q})\times\cdots\times M_{n_k}(\mathbb{Q})$, S''_i corresponds to the i-th component. If $\{i_1, \dots, i_j\} \subseteq \{1, \dots, k\}$ such that S''_i is a factor of Z if and only if $i \in \{i_1, \dots, i_j\}$, then the corresponding projective R-module M is given by $M/MJ(R) \simeq \oplus_{i=1}^k S_i^{''(t_i)}$, where $t_i = \infty$ if $i \in \{i_1, \dots, i_j\}$ and $t_i = 0$ otherwise. Similarly, the ideal $\begin{pmatrix} Z & ZP \\ QZ & QZP \end{pmatrix} \subseteq U$ gives a projective module M' such that $M' \otimes_U (M_{n_1}(\mathbb{Q})\times\cdots\times M_{n_k}(\mathbb{Q})\times M_m(\mathbb{Q})) \simeq \oplus_{i=1}^k S_i^{'(t_i)}$ (just consider simple factors of the idempotent ideal of U). Thus projective module M' over U is induced from R'' if and only if $\{j_1, \dots, j_l\} \subseteq \{l+1, \dots, k\}$ as we can see directly from Fact 4.6.

Now let I be an idempotent ideal of U of the form $I = \begin{pmatrix} Z & P \\ Q & S \end{pmatrix}$, where $PQ \subseteq Z$ and Z is idempotent modulo PQ. Let $S''_{i_1}, \dots, S''_{i_j}$ be simple factors of Z as above (but observe, Z is not idempotent in general, so the corresponding projective module may not exist over R). Then the projective

U-module M' which corresponds to S is given by $M' \otimes_U (M_{n_1}(\mathbb{Q}) \times \cdots \times M_{n_k}(\mathbb{Q}) \times M_m(\mathbb{Q})) \simeq \oplus_{i=1}^{k+1} S_i'^{(t_i)}$, where $t_i = \infty$ if $i \in \{i_1, \ldots, i_j\}$ or $i = k+1$, and $t_i = 0$ otherwise. Applying Fact 4.6 again, we see that there exists a projective module M over R' such that $M \otimes_{R'} U \simeq M'$ if and only if Z is not contained in $Q + J(R)$.

We have proved

Lemma 4.12. *Suppose that $R' \subseteq U$. Then*

(i) *Let Z be an idempotent ideal of R such that $Z + J(R) \subseteq Q + J(R)$. Then $\begin{pmatrix} Z & ZP \\ QZ & QZP \end{pmatrix} \subseteq R'$ is an idempotent ideal of R'.*

(ii) *Let Z be an ideal of R such that $PQ \subseteq Z \not\subseteq Q$ and Z is idempotent modulo PQ then the set $\begin{pmatrix} Z & P^m \\ Q^m & S \end{pmatrix} \cap R'$ is an idempotent ideal of R'.*

Moreover, all idempotent ideals of R' are described by (i) and (ii).

5. No hope in $U(\mathrm{sl}_2(\mathbb{C}))$

So far we have considered examples of noetherian rings with (*), so we could apply our machinery. It is well known, that there are noetherian rings that do not satisfy (*). However we can formulate another question: Are there rings having projective modules that are not fair-sized?

Let us recall a result proved in [11]

Fact 5.1. *Let R be a ring and let I be an ideal such that R/I^2 is semisimple. Then I is idempotent.*

Proof. Suppose that $I^2 \neq I$. Look at the semisimple ring R/I^2. The ideal I/I^2 is nonzero, and nilpotent. This cannot happen in a semisimple ring. □

Remark 5.2. *If L is a semisimple Lie algebra of finite dimension over $\mathbb{C}$, $U(L)$ stands for its universal enveloping algebra. It is well known that for any ideal I of $U(L)$ of finite codimesion the ring R/I is semisimple. Moreover, if I is of finite codimension, so is I^2. The previous fact gives that $I = I^2$ for any ideal in $U(L)$ of finite codimension.*

First of all let us consider a general construction. Take a noetherian ring R such that there exists a chain of ideals $I_1 \supsetneq I_2 \supsetneq \cdots$ such that for any $i \in \mathbb{N}$ the factor R/I_i is semi-simple. Further suppose that $I_{k+1} = I_{k+1}I_k$ for any $k \in \mathbb{N}$. For any $i \in \mathbb{N}$ take a finite set $G_i \subseteq I_i$ such that $RG_i = I_i$. Now for any $i, j \in \mathbb{N}$ let $e_{i,j} \in R$ be the element such that $e_{i,j} + I_j$ is the

central idempotent of R/I_j generating I_i/I_j. Observe that if $i > j > k$, then $e_{i,k} + I_j = e_{i,j} + I_j$.

First of all let us consider a $1 + |G_2| \times 1$ matrix over R given as a column $A_1 = \begin{pmatrix} e_{1,3} \\ G_2 \end{pmatrix}$ (here G_2 means elements of G_2 written in the column). Next we find a $1 + |G_2| \times 1 + |G_2|$ matrix B_1 over R such that $B_1 A_1 = A_1$. We can do this: The first column of C_1 is simply A_1, the remaining columns of A_1 have their entries in I_3 (recall that $I_3 I_2 = I_3$, thus $I_3 = I_3 G_2$, further $ie_{1,3} - i \in I_3$ for any $i \in I_1$.) Finally, we put A_2 to be a matrix with $1 + |G_2|$ columns of the form

$$\begin{pmatrix} e_{1,4} & 0 & \cdots & 0 \\ G_3 & 0 & \cdots & 0 \\ 0 & e_{2,4} & \cdots & 0 \\ 0 & G_3 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & e_{2,4} \\ 0 & 0 & \cdots & G_3 \end{pmatrix}$$

Observe that there exists a matrix C_1 such that $B_1 = C_1 A_2$. The matrix C_1 is a $1 + |G_2| \times m$ matrix, where m is the number of columns of A_2. We can group the columns of C_1 into blocks $C_1 = (D_1, D_2, \ldots, D_l)$, where any D_i is a $1 + |G_2| \times 1 + |G_3|$ matrix over R. The first column of D_i is just the i-th column of B_1 and the remaining columns of D_i are convenient elements of I_4.

We can continue inductively. The matrix A_k has blocks placed inside independently

$$\begin{pmatrix} X_1 & 0 & \cdots 0 \\ 0 & X_2 & \cdots 0 \\ & & \vdots \\ 0 & 0 & \cdots X_n \end{pmatrix}$$

each block X_i is a column of the form $\begin{pmatrix} e_{n_i,k+2} \\ G_{k+1} \end{pmatrix}$, where the integer n_i depends on i but always we have $n_i \leq k$. For each block we can do the same as we did above, that is we find Y_i such that $Y_i X_i = X_i$, where Y_i is a $|G_{k+1}| + 1 \times |G_{k+1}| + 1$ matrix, the first column of Y_i is just X_i and the remaining entries of Y_i are some convenient elements of I_{k+2}. We put B_k

to be the square matrix

$$\begin{pmatrix} Y_1 & 0 & \cdots 0 \\ 0 & Y_2 & \cdots 0 \\ & & \vdots \\ 0 & 0 & \cdots Y_n \end{pmatrix}$$

Then we find Z_i, T_i such that $Y_i = T_i Z_i$. The matrix Z_i is of the form

$$\begin{pmatrix} e_{n_i,k+3} & 0 & \cdots & 0 \\ G_{k+2} & 0 & \cdots & 0 \\ 0 & e_{k+1,k+3} & \cdots & 0 \\ 0 & G_{k+2} & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & e_{k+1,k+3} \\ 0 & 0 & \cdots & G_{k+2} \end{pmatrix}$$

The matrix T_i is of size $r(Y_i) \times r(Z_i)$, where $r(X)$ is the number of the rows in matrix X. Again, we imagine T_i as a matrix with blocks $(D_1, \ldots, D_{r(Y_i)})$, where the first column of D_j is the corresponding column of Y_i and the remaining entries are convenient elements of I_{k+3}. Then we put A_{k+1} to be the matrix with blocks

$$\begin{pmatrix} Z_1 & 0 & \cdots 0 \\ 0 & Z_2 & \cdots 0 \\ & & \vdots \\ 0 & 0 & \cdots Z_n \end{pmatrix}$$

Further, put

$$C_k = \begin{pmatrix} T_1 & 0 & \cdots 0 \\ 0 & T_2 & \cdots 0 \\ & & \vdots \\ 0 & 0 & \cdots T_n \end{pmatrix}$$

Obviously, $C_k A_{k+1} A_k = B_k A_k = A_k$

So if we consider integers $m_1 = 1$ and $m_k := r(A_{k-1})$ for $k \geq 2$, then the matrix A_k gives a homomorphism $f_k \colon R^{m_k} \to R^{m_{k+1}}$. The direct limit of the chain $f_1, f_2, \ldots$ is a projective module P since there are homomorphisms $g_k \colon R^{m_{k+2}} \to R^{m_{k+1}}$ such that $g_k f_{k+1} f_k = f_k$.

In general the module P given in this construction is not fair-sized. That means that the set of ideals $\{I \subseteq R \mid P/PI \text{ is finitely generated } \}$ does not contain the smallest element. Suppose that the sequence of ideals we used in the construction has the additional property $\cap_{i \in \mathbb{N}} I_i = 0$. Then in order to prove that P is not fair-sized, it is enough to show

(i) P is not finitely generated
(ii) P/PI_i is finitely generated for any $i \in \mathbb{N}$.

For $R = U(\mathrm{sl}_2(\mathbb{C}))$ the first statement can be guaranteed by a condition $I_1 \neq R$ as any finitely generated nonzero projective module over R is a generator.

From the construction the following is obvious: For any $k < l$ the number of entries of A_l equal to $e_{k,l+1}$ is the same as the number of entries of A_k equal to $e_{k,k+1}$. In order to calculate P/PI_i, apply $- \otimes_R R/I_i$ to the direct system defining P. The matrix of $f_k \otimes_R R/R_i$ consists of zeros and idempotents of R/I_i placed independently. An easy inspection gives that $P/PI_i \simeq \oplus_{j=1}^{i-1}(I_j/I_i)^{n_j}$, where n_j is the number of entries of A_j equal to $e_{j,j+2}$. Hence P/PI_i is a finitely generated module.

As in $U(\mathrm{sl}_2(\mathbb{C}))$ we can find a sequence of ideals $I_1 \supsetneq I_2 \supsetneq \cdots$ such that the dimension of $U(\mathrm{sl}_2(\mathbb{C}))/I_i$ is finite (so I_i is idempotent), $U(\mathrm{sl}_2(\mathbb{C})) \neq I_1$ and $\cap_{i\in\mathbb{N}} I_i = 0$, there are countably generated projective modules that are not fair-sized.

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