A TYPE OF PERTURBATION OF THE HARMONIC OSCILLATOR

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Abstract. The derivative d/dx is perturbed by adding a multiple of the product by x^{-1} when it acts on odd functions. This gives rise to a new perturbed harmonic oscillator, whose study is the goal of the paper: self-adjointness, spectrum, perturbed Hermite polynomials, eigenfunction estimates and embedding results. Conjugation by powers of x on $\mathbb{R}_+$ produce a more general perturbed harmonic oscillator, which satisfies the same kind of properties.

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1. Introduction

On smooth functions of a variable x, a perturbation, D_σ, of the usual derivative, depending on a parameter $\sigma \in \mathbb{R}$, is defined by $D_\sigma = \frac{d}{dx}$ on even functions and $D_\sigma = \frac{d}{dx} - \sigma x^{-1}$ on odd functions. This gives rise to a perturbation of the

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harmonic oscillator, $J = -D_\sigma^2 + s^2x^2$, depending on σ and a usual parameter $s > 0$. The following properties are proved for J when $\sigma > -1$, which generalize well known properties of the harmonic oscillator:

- With domain the Schwartz space $\mathcal{S}$, the operator J is essentially self-adjoint in $L^2(\mathbb{R}, |x|^\sigma\, dx)$. Its spectrum is described, like in the case of the usual harmonic oscillator, by using the corresponding perturbed annihilation and creation operators.
- The eigenfunctions of J are of the form $\phi_k(x) = p_k(x)e^{-sx^2/2}$, $k \in \mathbb{N}$, for certain perturbed Hermite polinomials p_k, which form a sequence of orthogonal polynomials on $\mathbb{R}$ for the measure $e^{-sx^2}|x|^\sigma\, dx$.
- $\max_x |x|^\sigma \phi_k^2(x)$ satisfies certain upper and lower estimates, where x varies in an appropriate subset of $\mathbb{R}$.
- $\mathcal{S}$ consists of the functions $\phi \in L^2(\mathbb{R}, |x|^\sigma\, dx)$ whose "Fourier coefficients" $\int_{\mathbb{R}} \phi(x)\phi_k(x)|x|^\sigma\, dx$ are rapidly decreasing on k.

The first and second properties follow with an adaptation of the arguments used in the case of the harmonic oscillator.

To prove the estimates of $\max_x |x|^\sigma \phi_k^2(x)$, we apply the method of Bonan-Clark [1]. But the computations become more involved in this perturbed version; indeed, several cases will be considered separately, and the estimates have some significant difference in some of them.

To characterize the functions in $\mathcal{S}$ by having rapidly decreasing "Fourier coefficients" with respect to the eigenfunctions ϕ_k, we prove embedding results; in particular, a generalization of the Sobolev embedding theorem is shown. These embedding results involve perturbations of the usual norms involved in the definition of $\mathcal{S}$. Those perturbed norms give rise to a perturbation $\mathcal{S}_\sigma$ of $\mathcal{S}$. It will be shown that $\mathcal{S}_\sigma = \mathcal{S}$ as Fréchet spaces, but the proof is difficult and very indirect. A more direct proof would be desirable.

Finally, we restrict J to $\mathbb{R}_+$, and consider its conjugation by operators of multiplication by functions of the form x^a for $a \in \mathbb{R}$. This gives rise to versions of the above properties for operators of the form

$$-\frac{d^2}{dx^2} + s^2x^2 - c_1x^{-1}\frac{d}{dx} + c_2x^{-2}\ ,$$

for $s > 0$ and appropriate $c_1, c_2 \in \mathbb{R}$, acting on functions on $\mathbb{R}_+$; we get an operator of the same type if $\frac{d}{dx}x^{-1}$ is used instead of $x^{-1}\frac{d}{dx}$.

We hope to apply these results to give a new interpretation to the analysis with differential forms on pseudo-manifolds began by J. Cheeger [2, 3].

See *e.g.* [6, 7] for the study of perturbed harmonic operators satisfying other conditions.

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2. Preliminaries

Recall that the harmonic oscillator is the operator

$$H = -\frac{d^2}{dx^2} + s^2x^2 ,$$

on $C^\infty = C^\infty(\mathbb{R})$, which depends on some fixed $s > 0$ (see *e.g.* [8]). In the study of H, an important role is played by the annihilation and creation operators,

$$A = sx + \frac{d}{dx} , \quad A^* = sx - \frac{d}{dx} ,$$

which satisfy

$$H = AA^* - s = A^*A + s , \tag{1}$$

$$[H, A] = -2sA , \quad [H, A^*] = 2sA^* , \tag{2}$$

$$[A, A^*] = 2s . \tag{3}$$

Recall also that the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is the space of functions $\phi \in C^\infty$ such that

$$\|\phi\|_{\mathcal{S}^m} = \sum_{i+j\le m} \sup_x |x^i \phi^{(j)}(x)|$$

is finite for all $m \in \mathbb{N}$ (including zero[1]). This defines a sequence of norms $\|\ \|_{\mathcal{S}^m}$ on $\mathcal{S}$, which is endowed with the corresponding Fréchet topology. The Banach space completion of $\mathcal{S}$ with respect to each norm $\|\ \|_{\mathcal{S}^m}$ will be denoted by $\mathcal{S}^m$. We have $\mathcal{S}^{m+1} \subset \mathcal{S}^m$ continuously[2], and $\mathcal{S} = \bigcap_m \mathcal{S}^m$.

With domain $\mathcal{S}$, the operator H is essentially self-adjoint operator in $L^2 = L^2(\mathbb{R})$, and its spectrum consists of the eigenvalues $(2k + 1)s$ of multiplicity one for $k \in \mathbb{N}$. The corresponding normalized eigenfunctions ψ_k are inductively defined by

$$\psi_0 = s^{1/4}\pi^{-1/4}e^{-sx^2/2} , \tag{4}$$

$$\psi_k = (2ks)^{-1/2}A^*\psi_{k-1} , \quad k \ge 1 . \tag{5}$$

In this sense, A^* "creates" the spectrum of H. On the other hand, A "annihilates" it:

$$A\psi_0 = 0 , \tag{6}$$

$$A\psi_k = (2ks)^{1/2}\psi_{k-1} , \quad k \ge 1 . \tag{7}$$

Writing

$$\psi_k(x) = h_k(x)e^{-sx^2/2}$$

[1] We adopt the convention $0 \in \mathbb{N}$.

[2] Let X and Y be topological vector spaces. It is said that $X \subset Y$ continuously if X is a linear subspace of Y and the inclusion map $X \hookrightarrow Y$ is continuous. The term bounded inclusion can be similarly used when X and Y are Banach spaces.

for some functions h_k, the conditions (4) and (5) become

$$h_0 = s^{1/4}\pi^{-1/4} \,, \tag{8}$$

$$h_k = (2ks)^{-1/2}(2sxh_{k-1} - h'_{k-1}) \,, \quad k \geq 1 \,. \tag{9}$$

Hence the functions h_k are, up to normalization, the Hermite polynomials. Each h_k, and therefore ψ_k as well, is an even (respectively, odd) function just when k is even (respectively, odd). They also satisfy

$$h'_k = (2ks)^{1/2}h_{k-1} \,, \quad k \geq 1 \,. \tag{10}$$

Finally, recall that any $f \in L^2$ is in $\mathcal{S}$ if and only if its "Fourier coefficients" $\langle \psi_k, f\rangle$ are rapidly decreasing on k.

3. First perturbation of the derivative

Recall that (see *e.g.* [5, Theorem 1.1.9]), for any $\phi \in C^\infty$, there is some $\psi \in C^\infty$ such that $\phi(x) - \phi(0) = x\psi(x)$. It is given by

$$\psi(x) = \int_0^1 \phi'(tx)\, dt \,, \tag{11}$$

obtaining

$$\psi^{(m)}(x) = \int_0^1 t^m \phi^{(m+1)}(tx)\, dt \tag{12}$$

for all $m \in \mathbb{N}$; in particular,

$$\psi^{(m)}(0) = \frac{1}{m+1}\phi^{(m+1)}(0) \,. \tag{13}$$

When $\phi(0) = 0$, we may write $x^{-1}\phi$ for ψ.

Consider the decomposition $C^\infty = C^\infty_{\text{even}} \oplus C^\infty_{\text{odd}}$, as direct sum of subspaces of even and odd functions. The matrix expressions of operators on C^∞ will be considered with respect to this decomposition. Observe that $\frac{d}{dx}$ and (the operator of multiplication by) x interchange C^∞_{even} and C^∞_{odd}; *i.e.*, we can write

$$\frac{d}{dx} = \begin{pmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix} \,, \quad x = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \,.$$

Moreover $C^\infty_{\text{odd}} = x\, C^\infty_{\text{even}}$ because any odd function vanishes at zero, and therefore the operator $x^{-1} : C^\infty_{\text{odd}} \to C^\infty_{\text{even}}$ is well defined and continuous. Then, for any fixed $\sigma \in \mathbb{R}$, we can define the perturbed derivative

$$D_\sigma = \begin{pmatrix} 0 & \frac{d}{dx} + \sigma x^{-1} \\ \frac{d}{dx} & 0 \end{pmatrix} = \frac{d}{dx} + \sigma \begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix}$$

on C^∞. Let also

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \,.$$

Since

$$\left[\begin{pmatrix}0 & x^{-1}\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & x\\ x & 0\end{pmatrix}\right] = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} ,$$

we get

$$[D_\sigma, x] = 1 + \Sigma , \tag{14}$$

$$D_\sigma\Sigma + \Sigma D_\sigma = x\,\Sigma + \Sigma\, x = 0 . \tag{15}$$

For $\psi \in C^\infty$ and $\phi = x\psi$, it follows from (11) and (12) that

$$(D_\sigma^m\psi)(x) + \sigma\int_0^1 t^m(D_\sigma^m\psi)(tx)\,dt = \int_0^1 t^m(D_\sigma^{m+1}\phi)(tx)\,dt \tag{16}$$

for all $m \in \mathbb{N}$, obtaining the following version of (13):

$$(D_\sigma^{m+1}\phi)(0) = (m+1+\sigma)(D_\sigma^m\psi)(0) . \tag{17}$$

To simplify the notation, we introduce a perturbed factorial $m!_\sigma$ of each $m \in \mathbb{N}$, which is inductively defined by setting $0!_\sigma = 1$, and

$$m!_\sigma = \begin{cases}(m-1)!_\sigma m & \text{if } m \text{ is even}\\ (m-1)!_\sigma(m+\sigma) & \text{if } m \text{ is odd}\end{cases}$$

for $m > 0$. Observe that $m!_\sigma > 0$ if $\sigma > -1$, which will be the case of our interest; otherwise, $m!_\sigma$ may be ≤ 0. For $k \le m$, even when $k!_\sigma = 0$, the quotient $m!_\sigma/k!_\sigma$ can be understood as the product of the factors from the definition of $m!_\sigma$ which are not included in the definition of $k!_\sigma$.

Lemma 3.1. *For any $\phi \in C^\infty$ and $m \in \mathbb{N}$,*

$$(D_\sigma^m\phi)(0) = \frac{m!_\sigma}{m!}\phi^{(m)}(0) .$$

Proof. In the first case, suppose that $\phi \in C^\infty_{\text{even}}$. If m is odd, then both sides of this equality vanish. When m is even, we proceed by induction. For $m = 0$, this equality is obvious. Now let m be an even integer > 0 and assume that the statement holds for $m-2$. Let $\psi \in C^\infty_{\text{even}}$ such that $x\psi = \phi' = D_\sigma\phi$. Then $D_\sigma^2\phi = \phi'' + \sigma\psi$. By (13),

$$\begin{aligned}(D_\sigma^m\phi)(0) = (D_\sigma^{m-2}(\phi''+\sigma\psi))(0) &= \frac{(m-2)!_\sigma}{(m-2)!}(\phi''+\sigma\psi)^{(m-2)}(0)\\ &= \frac{(m-2)!_\sigma}{(m-2)!}(1+\frac{\sigma}{m-1})\phi^{(m)}(0) = \frac{(m-1)!_\sigma}{(m-1)!}\phi^{(m)}(0) = \frac{m!_\sigma}{m!}\phi^{(m)}(0) .\end{aligned}$$

In the second case, suppose that $\phi \in C^\infty_{\text{odd}}$. If m is even, then both sides of the equality of the statement vanish. If $m = 0$, the result is obvious. So we can assume that m is an odd integer > 0. Let $\psi \in C^\infty_{\text{even}}$ with $x\psi = \phi$. Then $D_\sigma\phi = \phi' + \sigma\psi$. By (13) and the above case,

$$(D_\sigma^m\phi)(0) = (D_\sigma^{m-1}(\phi'+\sigma\psi))(0) = \frac{(m-1)!_\sigma}{(m-1)!}(\phi'+\sigma\psi)^{(m-1)}(0)$$

$$= \frac{(m-1)!_\sigma}{(m-1)!}(1+\frac{\sigma}{m})\phi^{(m)}(0) = \frac{m!_\sigma}{m!}\phi^{(m)}(0)\ . \quad \square$$

Remark 1. Observe that (17) also follows from Lemma 3.1 and (13).

4. First perturbation of the harmonic oscillator

By using D_σ instead of $\frac{d}{dx}$, we get the perturbed harmonic oscillator

$$J = -D_\sigma^2 + s^2x^2 = H - \sigma \begin{pmatrix} x^{-1}\frac{d}{dx} & 0 \\ 0 & \frac{d}{dx}x^{-1} \end{pmatrix}$$

on C^∞. The more precise notation J_σ will be used instead of J only if necessary. Now the perturbed annihilation and creation operators are:

$$B = sx + D_\sigma = A + \sigma \begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix}\ ,$$

$$B^{*_\sigma} = sx - D_\sigma = A^* - \sigma \begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix}\ .$$

By (14) and (15),

$$J = BB^{*_\sigma} - (1+\Sigma)s = B^{*_\sigma}B + (1+\Sigma)s = \frac{1}{2}(BB^{*_\sigma} + B^{*_\sigma}B)\ , \tag{18}$$

$$[J, B] = -2sB\ , \quad [J, B^{*_\sigma}] = 2sB^{*_\sigma}\ , \tag{19}$$

$$[B, B^{*_\sigma}] = 2s(1+\Sigma)\ , \tag{20}$$

$$[J, \Sigma] = B\Sigma + \Sigma B = B^{*_\sigma}\Sigma + \Sigma B^{*_\sigma} = 0\ . \tag{21}$$

Here, (18)–(20) are perturbed versions of (1)–(3).

The above decomposition of C^∞ can be restricted to $\mathcal{S}$, giving $\mathcal{S} = \mathcal{S}_{\rm even} \oplus \mathcal{S}_{\rm odd}$. The matrix expressions of operators on $\mathcal{S}$ will be considered with respect to this decomposition. For $\phi \in C^\infty_{\rm even}$, $\psi = x^{-1}\psi$ and $i, j \in \mathbb{N}$, it follows from (12) that

$$|x^i\psi^{(j)}(x)| \le \int_0^1 t^{j-i}|(tx)^i\phi^{(j+1)}(tx)|\,dt \le \sup_{y\in\mathbb{R}} |y^i\phi^{(j+1)}(y)|$$

for all $x \in \mathbb{R}$. Thus $\|\psi\|_{\mathcal{S}^m} \le \|\phi\|_{\mathcal{S}^{m+1}}$ for all $m \in \mathbb{N}$, obtaining that $\mathcal{S}_{\rm odd} = x\,\mathcal{S}_{\rm even}$ and $x^{-1} : C^\infty_{\rm odd} \to C^\infty_{\rm even}$ restricts to a continuous operator $x^{-1} : \mathcal{S}_{\rm odd} \to \mathcal{S}_{\rm even}$. Therefore $x : \mathcal{S}_{\rm even} \to \mathcal{S}_{\rm odd}$ is an isomorphism of Fréchet spaces, and D_σ, B, B^{*_σ} and J define continuous operators on $\mathcal{S}$. From now on, consider D_σ, B, B^{*_σ} and J with domain $\mathcal{S}$, unless otherwise stated.

Let $\langle\ ,\ \rangle_\sigma$ and $\|\ \|_\sigma$ denote the scalar product and the norm of the weighted L^2 space $L^2_\sigma = L^2(\mathbb{R}, |x|^\sigma\,dx)$. Assume from now on that $\sigma > -1$, and therefore $\mathcal{S}$ is a dense subset of L^2_σ.

Lemma 4.1. *When $\mathcal{S}$ is considered as domain, $-D_\sigma$ is adjoint of D_σ in L^2_σ.*

Proof. For $\phi \in \mathcal{S}_{\text{even}}$ and $\psi \in \mathcal{S}_{\text{odd}}$,

$$\begin{aligned}\langle \frac{d}{dx}\phi, \psi\rangle_\sigma &= \int_{-\infty}^{\infty} \phi'\psi|x|^\sigma\, dx \\ &= 2\int_0^\infty \phi'\psi x^\sigma\, dx \\ &= -2\int_0^\infty \phi(\psi' x^\sigma + \psi\sigma x^{\sigma-1})\, dx \\ &= -2\int_0^\infty \phi(\psi' + \sigma x^{-1}\psi)x^\sigma\, dx \\ &= -\int_{-\infty}^{\infty} \phi(\psi' + \sigma x^{-1}\psi)|x|^\sigma\, dx \\ &= -\langle \phi, (\frac{d}{dx} + \sigma x^{-1})\psi\rangle_\sigma\ . \end{aligned}$$

□

Corollary 4.2. *When $\mathcal{S}_\sigma$ is considered as domain, $B^{*\sigma}$ is adjoint of B in L^2_σ, and J is symmetric in L^2_σ.*

Let ϕ_k be the sequence of functions in $\mathcal{S}$ inductively defined by the following versions of (4) and (5):

$$\phi_0 = s^{(\sigma+1)/4}\Gamma((\sigma+1)/2)^{-1/2}e^{-sx^2/2}\ , \tag{22}$$

$$\phi_k = \begin{cases}(2ks)^{-1/2}B^{*\sigma}\phi_{k-1} & \text{if } k \text{ is even}\\ (2(k+\sigma)s)^{-1/2}B^{*\sigma}\phi_{k-1} & \text{if } k \text{ is odd}\end{cases} \tag{23}$$

for $k \geq 1$. The following is the corresponding version of (6) and (7).

Lemma 4.3. *We have $B\phi_0 = 0$, and*

$$B\phi_k = \begin{cases}(2ks)^{1/2}\phi_{k-1} & \textit{if } k \textit{ is even}\\ (2(k+\sigma)s)^{1/2}\phi_{k-1} & \textit{if } k \textit{ is odd}\ .\end{cases}$$

for $k \geq 1$.

Proof. By (22),

$$B\phi_0 = s^{(\sigma+1)/4}\Gamma((\sigma+1)/2)^{-1/2}(sx + \frac{d}{dx})e^{-sx^2/2} = 0\ .$$

Next, we proceed by induction on $k \geq 1$. By (18) and (23),

$$\begin{aligned}B\phi_1 &= (2(1+\sigma)s)^{-1/2}BB^{*\sigma}\phi_0 \\ &= (2(1+\sigma)s)^{-1/2}(B^{*\sigma}B + 2(1+\Sigma)s)\phi_0 \\ &= (2(1+\sigma)s)^{-1/2}2(1+\sigma)s\phi_0) \\ &= (2(1+\sigma)s)^{1/2}\phi_0\ .\end{aligned}$$

Now, let $k \geq 2$ and suppose that the statement holds for ϕ_{k-1}. To simplify the notation, let $\nu_k = 1 - (-1)^k$. Observe that $\nu_k = \nu_{k-1} + 2(-1)^{k-1}$. Then, by (18) and (23) again,

$$\begin{aligned} B\phi_k &= ((2k + \nu_k\sigma)s)^{-1/2} BB^{*\sigma}\phi_{k-1} \\ &= ((2k + \nu_k\sigma)s)^{-1/2}(B^{*\sigma}B + 2(1 + \Sigma)s)\phi_{k-1} \\ &= ((2k + \nu_k\sigma)s)^{-1/2}((2(k - 1 + \nu_{k-1}\sigma)s)^{1/2}B^{*\sigma}\phi_{k-2} \\ &\quad + 2(1 + (-1)^{k-1}\sigma)s\phi_{k-1}) \\ &= ((2k + \nu_k\sigma)s)^{-1/2}(2(k - 1 + \nu_{k-1}\sigma)s + 2(1 + (-1)^{k-1}\sigma)s)\phi_{k-1} \\ &= ((2k + \nu_k\sigma)s)^{1/2}\phi_{k-1} . \end{aligned}$$

□

Proposition 4.4. *For each $k \in \mathbb{N}$, ϕ_k is an eigenfunction of J, normalized in L^2_σ, with corresponding eigenvalue $(2k + 1 + \sigma)s$.*

Proof. Like in the case of H, this follows by induction on k. For $k = 0$,

$$\begin{aligned} J\phi_0 = \|\psi_0\|_\sigma^{-1} J\psi_0 &= \|\psi_0\|_\sigma^{-1}(H\psi_0 - \sigma x^{-1}\,\psi_0') \\ &= \|\psi_0\|_\sigma^{-1}(1 + \sigma)s\psi_0 = (1 + \sigma)s\phi_0 , \end{aligned}$$

and $\|\phi_0\|_\sigma = 1$ because

$$\int_{-\infty}^{\infty} e^{-sx^2}|x|^\sigma\,dx = 2\int_0^\infty e^{-sx^2}x^\sigma\,dx = s^{-(\sigma+1)/2}\Gamma((\sigma + 1)/2) .$$

Now suppose that $k \geq 1$ and the result holds for ϕ_{k-1}. Let $\nu_k = 1 - (-1)^k$, like in the proof of Lemma 4.3. By (18), (19), (23) and Corollary 4.2,

$$\begin{aligned} J\phi_k &= ((2k + \nu_k\sigma)s)^{-1/2} JB^{*\sigma}\phi_{k-1} \\ &= ((2k + \nu_k\sigma)s)^{-1/2}(B^{*\sigma}J + 2sB^{*\sigma})\phi_{k-1} \\ &= ((2k + \nu_k\sigma)s)^{-1/2}((2(k - 1) + 1 + \sigma)s + 2s)B^{*\sigma}\phi_{k-1} \\ &= (2k + 1 + \sigma)s\phi_k , \\ \|\phi_k\|^2_\sigma &= ((2k + \nu_k\sigma)s)^{-1}\,\langle BB^{*\sigma}\phi_{k-1}, \phi_{k-1}\rangle_\sigma \\ &= ((2k + \nu_k\sigma)s)^{-1}\,\langle (J + (1 + \Sigma)s)\phi_{k-1}, \phi_{k-1}\rangle_\sigma \\ &= ((2k + \nu_k\sigma)s)^{-1}(2k + \sigma + (-1)^{k-1}\sigma)s\,\|\phi_{k-1}\|^2_\sigma \\ &= 1 . \end{aligned}$$

□

From (22), (23) and the definition of $B^{*\sigma}$, it follows that $\phi_k = p_k e^{-sx^2/2}$ for the sequence of perturbed Hermite polynomials p_k inductively defined by

$$p_0 = s^{(\sigma+1)/4}\Gamma((\sigma + 1)/2)^{-1/2} , \tag{24}$$

$$p_k = \begin{cases} (2ks)^{-1/2}(2sxp_{k-1} - D_\sigma p_{k-1}) & \text{if } k \text{ is even} \\ (2(k + \sigma)s)^{-1/2}(2sxp_{k-1} - D_\sigma p_{k-1}) & \text{if } k \text{ is odd ,} \end{cases} \tag{25}$$

for $k \geq 1$. Each p_k is of precise degree k, even (respectively, odd) if k is even (respectively, odd), and with positive leading coefficient. So p_k is the sequence of orthogonal polynomials associated with the measure $|x|^\sigma e^{-sx^2}\,dx$ [9]. It follows that the functions ϕ_k form a base of the linear subspace

$$\mathcal{P} = \left\{ pe^{-sx^2/2} \mid p \text{ is a polynomial} \right\} \subset \mathcal{S}\ .$$

The density of $\mathcal{P}$ in L^2_σ does not follow from the general theory of orthogonal polynomials [9, Section 3.1], and therefore a particular proof must be given like in the case of the Hermite polynomials [9, Theorem 5.7.1].

Proposition 4.5. *$\mathcal{P}$ is dense in L^2_σ.*

Proof. For each integer $j \geq 0$, let $f_j(x) = x^j e^{-sx^2/2}$. We have

$$\begin{aligned}
\|f_j\|^2_\sigma &= \int_{\infty}^{\infty} x^{2j} e^{-sx^2} |x|^\sigma\,dx \\
&= 2\int_0^\infty x^{2j+\sigma} e^{-sx^2}\,dx \\
&= s^{-1/2} \int_0^\infty y^{j+\frac{\sigma-1}{2}} e^{-y}\,dy \\
&= s^{-1/2}\Gamma(j + \frac{\sigma+1}{2}) \\
&\leq s^{-1/2}(j + \lfloor \frac{\sigma}{2} \rfloor)!\ ,
\end{aligned}$$

where we have used the substitution $y = sx^2$. Hence

$$\|(i\lambda)^j (j!)^{-1/2} f_j\|_\sigma \leq s^{-1/4} (\lfloor \frac{\sigma}{2} \rfloor !\, 2^{\lfloor \frac{\sigma}{2} \rfloor})^{1/2} (2^{1/2}|\lambda|)^j (j!)^{-1/2}$$

for each $\lambda \in \mathbb{R}$ because

$$\frac{(j + \lfloor \frac{\sigma}{2} \rfloor)!}{j!} = \lfloor \frac{\sigma}{2} \rfloor ! \binom{j + \lfloor \frac{\sigma}{2} \rfloor}{j} \leq \lfloor \frac{\sigma}{2} \rfloor !\, 2^{j + \lfloor \frac{\sigma}{2} \rfloor}\ .$$

It follows that the series

$$e^{i\lambda x - sx^2/2} = \sum_{j=0}^\infty \frac{(i\lambda)^j}{j!} f_j$$

is convergent in L^2_σ; indeed, it belongs to $\overline{\mathcal{P}}$ because $f_j \in \mathcal{P}$. Therefore any f orthogonal to $\overline{\mathcal{P}}$ in L^2_σ satisfies

$$\int_{-\infty}^\infty f(x) e^{i\lambda x - sx^2/2} |x|^\sigma\,dx = 0$$

for all $\lambda \in \mathbb{R}$, obtaining $f(x)e^{-sx^2/2}|x|^\sigma = 0$ almost everywhere with respect to dx by Plancherel's theorem. So $f = 0$ almost everywhere with respect to $|x|^\sigma\,dx$. □

The following result is a direct consequence of Propositions 4.4 and 4.5, and Corollary 4.2.

Corollary 4.6. *With domain $\mathcal{S}$, the operator J is essentially self-adjoint in L^2_σ, and its spectrum consists of the eigenvalues and eigenfunctions stated in Proposition 4.4.*

5. Basic properties of the perturbed Hermite polynomials

Let $\gamma_k > 0$ denote the leading coefficient of each p_k. By (25),

$$\gamma_k = \begin{cases} k^{-1/2}(2s)^{1/2}\gamma_{k-1} & \text{if } k \text{ is even} \\ (k+\sigma)^{-1/2}(2s)^{1/2}\gamma_{k-1} & \text{if } k \text{ is odd}\,. \end{cases} \tag{26}$$

The following is a version of (10).

Lemma 5.1. *We have $D_\sigma p_0 = 0$, and*

$$D_\sigma p_k = \begin{cases} (2ks)^{1/2}p_{k-1} & \text{if } k \text{ is even} \\ (2(k+\sigma)s)^{1/2}p_{k-1} & \text{if } k \text{ is odd}\,. \end{cases}$$

Proof. The first equality is obvious, and the second one is proved by induction on k. For $k = 1$, by (14) and (25),

$$\begin{aligned} D_\sigma p_1 &= ((2+2\sigma)s)^{-1/2}2sD_\sigma(xp_0) \\ &= ((2+2\sigma)s)^{-1/2}2s(1+\sigma)p_0 \\ &= ((2+2\sigma)s)^{1/2}p_0\,. \end{aligned}$$

Now let $k > 0$ and assume that the statement holds for $k-1$. Consider once more the simplifying notation $\nu_k = 1-(-1)^k$. Then, by (14) and (25) again,

$$\begin{aligned} D_\sigma p_k &= ((2k+\nu_k\sigma)s)^{-1/2}(2sD_\sigma(xp_{k-1}) - D_\sigma^2 p_{k-1}) \\ &= ((2k+\nu_k\sigma)s)^{-1/2}(2s(1+\Sigma)p_{k-1} \\ &\quad + ((2(k-1)+\nu_{k-1}\sigma)s)^{1/2}(2sxp_{k-2} - D_\sigma p_{k-2})) \\ &= ((2k+\nu_k\sigma)s)^{-1/2}(2s(1+(-1)^{k-1}\sigma) + (2(k-1)+\nu_{k-1}\sigma)s)p_{k-1} \\ &= ((2k+\nu_k\sigma)s)^{1/2}p_{k-1}\,. \end{aligned}$$

□

The following recursion formula follows directly from (25) and Lemma 5.1:

$$p_k = \begin{cases} k^{-1/2}\big((2s)^{1/2}xp_{k-1} - (k-1+\sigma)^{1/2}p_{k-2}\big) & \text{if } k \text{ is even} \\ (k+\sigma)^{-1/2}\big((2s)^{1/2}xp_{k-1} - (k-1)^{1/2}p_{k-2}\big) & \text{if } k \text{ is odd}\,. \end{cases} \tag{27}$$

We have $p_k(0) = 0$ if and only if k is odd, and $p_k'(0) = 0$ if and only if k is even. By (27) and induction on k,

$$p_k(0) = (-1)^{k/2}\sqrt{\frac{(k-1+\sigma)(k-3+\sigma)\cdots(1+\sigma)}{k(k-2)\cdots 2}}\,p_0 \tag{28}$$

if k is even. When k is odd, by Lemma 5.1 and (28),

$$(D_\sigma p_k)(0) = (-1)^{(k-1)/2} \sqrt{\frac{(k+\sigma)(k-2+\sigma)\cdots(1+\sigma)2s}{(k-1)(k-3)\cdots 2}}\, p_0\ ,$$

obtaining

$$p_k'(0) = \frac{(-1)^{(k-1)/2}}{1+\sigma} \sqrt{\frac{(k+\sigma)(k-2+\sigma)\cdots(1+\sigma)2s}{(k-1)(k-3)\cdots 2}}\, p_0 \tag{29}$$

by Lemma 3.1. From (27) and by induction on k, we also get

$$x^{-1}p_k = \sum_{\ell\in\{0,2,\dots,k-1\}} (-1)^{\frac{k-\ell-1}{2}} \sqrt{\frac{(k-1)(k-3)\cdots(\ell+2)2s}{(k+\sigma)(k-2+\sigma)\cdots(\ell+1+\sigma)}}\, p_\ell \tag{30}$$

if k is odd[3].

The following assertions come from the general theory of orthogonal polynomials [9, Chapter III]. All zeros of each polynomial p_k are real and of multiplicity one. Each open interval between consecutive zeros of p_k contains exactly one zero of p_{k+1}, and at least one zero of every p_ℓ with $\ell > k$. Moreover p_k has exactly $\lfloor k/2 \rfloor$ positive zeros and $\lfloor k/2 \rfloor$ negative zeros. The zeros of each p_k will be denoted $x_{k,1} > x_{k,2} > \cdots > x_{k,k}$. On each interval $(x_{k,i+1}, x_{k,i})$, the function p_{k+1}/p_k is strictly increasing, and satisfies

$$\lim_{x\to x_{k,i}^{\pm}} \frac{p_{k+1}(x)}{p_k(x)} = \mp\infty\ .$$

For every polynomial p of degree $\le k-1$, we have

$$p^2(x) \le \int_{-\infty}^{\infty} p^2(t)|t|^\sigma e^{-st^2}\, dt \cdot \sum_{\ell=0}^{k} p_\ell^2(x) \tag{31}$$

for all $x \in \mathbb{R}$. The Gauss-Jacobi formula states that there are $\lambda_{k,1}, \lambda_{k,2}, \dots, \lambda_{k,k} \in \mathbb{R}$ such that, for any polynomial p of degree $\le 2k-1$,

$$\int_{-\infty}^{\infty} p(x)|x|^\sigma e^{-sx^2}\, dx = \sum_{i=1}^{k} p(x_{k,i})\lambda_{k,i}\ . \tag{32}$$

Lemma 5.2. *We have*

$$p_k'^{\,2}(x_{k,i})\lambda_{k,i} = \begin{cases} 2s & \text{if } k \text{ is even} \\ 2s/(1+\sigma) & \text{if } k \text{ is odd}\ . \end{cases}$$

[3]As a convention, the product of an empty set of factors is 1. Thus $(k-1)(k-3)\cdots(\ell+2) = 1$ for $\ell = k-1$ in (30), and (28) and (29) also hold for $k = 0$ and $k = 1$, respectively.

Proof. This is a direct adaptation of the proof of [1, Corollary 3]. With

$$p = \frac{p_k p_{k-1}}{x - x_{k,i}}\;,$$

the formula (32) becomes

$$\frac{\gamma_k}{\gamma_{k-1}} = p_k'(x_{k,i}) p_{k-1}(x_{k,i}) \lambda_{k,i}\;,$$

and the result follows from (26) and Lemma 5.1. □

6. Estimates of the perturbed Hermite functions

To get uniform estimates of the functions ϕ_k, they are multiplied by $|x|^{\sigma/2}$, obtaining eigenfunctions of another perturbation of H.

6.1. Second perturbation of H.

By conjugation, we get another perturbed derivative,

$$E_\sigma = |x|^{\sigma/2} D_\sigma |x|^{-\sigma/2} = \begin{pmatrix} 0 & \frac{d}{dx} + \frac{\sigma}{2}x^{-1} \\ \frac{d}{dx} - \frac{\sigma}{2}x^{-1} & 0 \end{pmatrix}\;,$$

and another perturbed harmonic oscillator,

$$\begin{aligned} K = |x|^{\sigma/2} J |x|^{-\sigma/2} &= -E_\sigma^2 + s^2 x^2 \\ &= \begin{pmatrix} H + \frac{\sigma}{4}(\sigma - 2)x^{-2} & 0 \\ 0 & H + \frac{\sigma}{4}(\sigma + 2)x^{-2} \end{pmatrix}\;, \end{aligned}$$

defined on

$$|x|^{\sigma/2}\,\mathcal{S} = |x|^{\sigma/2}\,\mathcal{S}_{\text{even}} \oplus |x|^{\sigma/2}\,\mathcal{S}_{\text{odd}}\;.$$

Like in the case of J, the notation K_σ will be used instead of K only if necessary. By Corollary 4.6 and since $|x|^{\sigma/2} : L^2(\mathbb{R}, |x|^\sigma\,dx) \to L^2(\mathbb{R}, dx)$ is a unitary isomorphism, K is essentially self-adjoint in $L^2(\mathbb{R}, dx)$, and its spectrum consists of the eigenvalues $(2k + 1 + \sigma)s$, $k \in \mathbb{N}$, of multiplicity one with the corresponding eigenspaces generated by

$$\xi_k = |x|^{\sigma/2} \phi_k = p_k |x|^{\sigma/2} e^{-sx^2/2}\;.$$

Each ξ_k is C^∞ on $\mathbb{R} \setminus \{0\}$, and it is C^∞ on $\mathbb{R}$ if and only if $\sigma \in 2\mathbb{N}$. If $\sigma > 0$ or k is odd, then ξ_k is defined and continuous on $\mathbb{R}$, and $\xi_k(0) = 0$. If $\sigma < 0$ and k is even, then ξ_k is only defined on $\mathbb{R} \setminus \{0\}$; in fact, by (28),

$$\lim_{x \to 0} \xi_k(x) = (-1)^{k/2} \infty\;.$$

By Lemma 5.1 and (27),

$$\xi_k' = (p_k' + (\frac{\sigma}{2x} - sx)p_k)|x|^{\sigma/2}e^{-sx^2/2} \tag{33}$$

$$= \begin{cases} (\sqrt{2ks}p_{k-1} + (\frac{\sigma}{2x} - sx)p_k)|x|^{\sigma/2}e^{-sx^2/2} & \text{if } k \text{ is even} \\ (\sqrt{2(k+\sigma)s}p_{k-1} - (\frac{\sigma}{2x} + sx)p_k)|x|^{\sigma/2}e^{-sx^2/2} & \text{if } k \text{ is odd} \end{cases}$$

$$= \begin{cases} ((sx + \frac{\sigma}{2x})p_k - \sqrt{2(k+1+\sigma)s}p_{k+1})|x|^{\sigma/2}e^{-sx^2/2} & \text{if } k \text{ is even} \\ ((sx - \frac{\sigma}{2x})p_k - \sqrt{2(k+1)s}p_{k+1})|x|^{\sigma/2}e^{-sx^2/2} & \text{if } k \text{ is odd .} \end{cases} \tag{34}$$

By (33), (28) and (29),

$$\lim_{x\to 0^\pm} \xi_k'(x) = \begin{cases} 0 & \text{if } \sigma > 2 \text{ or } \sigma = 0 \\ \pm p_k(0) & \text{if } \sigma = 2 \\ \pm(-1)^{k/2}\infty & \text{if } 0 < \sigma < 2 \\ \mp(-1)^{k/2}\infty & \text{if } -1 < \sigma < 0 \end{cases}$$

if k is even,

$$\lim_{x\to 0} \xi_k'(x) = \begin{cases} 0 & \text{if } \sigma > 0 \\ p_k'(0) & \text{if } \sigma = 0 \\ (-1)^{(k-1)/2}\infty & \text{if } -1 < \sigma < 0 \end{cases}$$

if k is odd, and

$$\lim_{x\to 0^\pm} (\xi_k\xi_k')(x) = \begin{cases} 0 & \text{if } k \text{ is odd or } \sigma \in \{0\} \cup (1,\infty) \\ \pm p_k^2(0)/2 & \text{if } k \text{ is even and } \sigma = 1 \\ \pm\infty & \text{if } k \text{ is even and } \sigma \in (0,1) \\ \mp\infty & \text{if } k \text{ is even and } \sigma \in (-1,0) \text{ .} \end{cases} \tag{35}$$

By (34),

$$\frac{\xi_k'}{\xi_k} = \begin{cases} sx + \frac{\sigma}{2x} - \sqrt{2(k+1+\sigma)s}\frac{p_{k+1}}{p_k} & \text{if } k \text{ is even} \\ sx - \frac{\sigma}{2x} - \sqrt{2(k+1)s}\frac{p_{k+1}}{p_k} & \text{if } k \text{ is odd ,} \end{cases} \tag{36}$$

which generalizes a formula of [4] for the Hermite functions.

For the sake of simplicity, let

$$\bar\sigma_k = \sigma(\sigma - (-1)^k 2) \text{ .}$$

Each ξ_k satisfies

$$\xi_k'' + q_k\xi_k = 0 \text{ ,} \tag{37}$$

where

$$q_k = (2k+1+\sigma)s - s^2x^2 - \frac{\bar\sigma_k}{4}x^{-2} \text{ .}$$

6.2. **Description of q_k.** The function q_k is even, defined at least on $\mathbb{R}\setminus\{0\}$, and satisfies

$$\lim_{x\to\pm\infty} q_k(x) = -\infty \ .$$

We have

$$q_k' = -2s^2x + \frac{\bar\sigma_k}{2}x^{-3} \ ,$$

which satisfies

$$\lim_{x\to\pm\infty} q_k'(x) = \mp\infty \ .$$

We get $q_k \in C^\infty(\mathbb{R})$ if and only if $\bar\sigma_k = 0$. Otherwise, we get

$$\lim_{x\to 0} q_k(x) = \begin{cases} -\infty & \text{if } \bar\sigma_k > 0 \\ \infty & \text{if } \bar\sigma_k < 0 \ , \end{cases}$$

$$\lim_{x\to 0^\pm} q_k'(x) = \begin{cases} \pm\infty & \text{if } \bar\sigma_k > 0 \\ \mp\infty & \text{if } \bar\sigma_k < 0 \ . \end{cases}$$

We have the following cases for the zeros of q_k':

- If $\bar\sigma_k > 0$, then q_k' has two zeros, which are

$$\pm x_{\max} = \pm\sqrt{\sqrt{\bar\sigma_k}/2s} \ ,$$

  At these points, q_k reaches its maximum, which equals $c_{\max}s$ for

$$c_{\max} = 2k + 1 + \sigma - \sqrt{\bar\sigma_k} \ .$$

- If $\bar\sigma_k = 0$, then q_k' has one zero, which is 0, where q_k reaches its maximum $c_{\max}s$ as above with $c_{\max} = 2k + 1 + \sigma$.
- If $\bar\sigma_k < 0$, then $q_k' > 0$ on $\mathbb{R}_-$ and $q_k' < 0$ on $\mathbb{R}_+$.

We have the following possibilities for the zeros of q_k:

- If $\bar\sigma_k > 0$ and $c_{\max} > 0$, then q_k has four zeros, which are

$$\pm a_k = \pm\sqrt{\frac{2k+1+\sigma-\sqrt{(2k+1+\sigma)^2-\bar\sigma_k}}{2s}} \ ,$$

$$\pm b_k = \pm\sqrt{\frac{2k+1+\sigma+\sqrt{(2k+1+\sigma)^2-\bar\sigma_k}}{2s}} \ .$$

- If $\bar\sigma_k > 0$ and $c_{\max} = 0$, then q_k has two zeros, $\pm b_k = \pm a_k$, defined as above, and $q_k < 0$ elsewhere.
- If $\bar\sigma_k > 0$ and $c_{\max} < 0$, then $q_k < 0$.
- If $\bar\sigma_k < 0$, then q_k has two zeros, $\pm b_k$, defined as above.
- If $\bar\sigma_k = 0$, then q_k has two zeros, $\pm b_k$, defined as above.

If q_k has four zeros, $\pm a_k$ and $\pm b_k$, then

$$s(b_k - a_k)^2 = c_{\max} \,, \tag{38}$$

and

$$2sa_k^2 = \frac{\bar\sigma_k}{2k+1+\sigma+\sqrt{(2k+1+\sigma)^2-\bar\sigma_k}} \,,$$

obtaining

$$a_k \in O(k^{-1/2}) \tag{39}$$

as $k \to \infty$. If q_k has at least two zeros, $\pm b_k$, then

$$2s(b_k^2 - b_\ell^2) = 2 + \frac{4(k^2-\ell^2)+4(1+\sigma)(k-\ell)+\bar\sigma_\ell-\bar\sigma_k}{\sqrt{(2k+1+\sigma)^2-\bar\sigma_k}+\sqrt{(2\ell+1+\sigma)^2-\bar\sigma_\ell}}$$

for $\ell \le k$, obtaining

$$b_{k+1} - b_k \in O(k^{-1/2}) \tag{40}$$

as $k \to \infty$, and

$$b_k - b_\ell \ge C(k-\ell)k^{-1/2} \tag{41}$$

for some $C > 0$ if k and ℓ are large enough. If $\bar\sigma_k = 0$, then $sb_k^2 = c_{\max}$.

The maximal open intervals where q_k is defined and > 0 (respectively, < 0) will be called *oscillation* (respectively, *non-oscillation*) intervals of ξ_k; this terminology is justified by Lemma 6.1 bellow. We have the following possibilities for the oscillation intervals:

- If $\bar\sigma_k > 0$ and $c_{\max} > 0$, then ξ_k has two oscillation intervals, (a_k, b_k) and $(-b_k, -a_k)$, containing $x_{\max}$ and $-x_{\max}$, respectively.
- If $\bar\sigma_k > 0$ and $c_{\max} \le 0$, then ξ_k has no oscillation intervals.
- If $\bar\sigma_k < 0$, then ξ_k has two oscillation intervals, $(-b_k, 0)$ and $(0, b_k)$.
- If $\bar\sigma_k = 0$, then ξ_k has one oscillation interval, $(-b_k, b_k)$.

These conditions on $\bar\sigma_k$ and $c_{\max}$ have simple interpretations that depend on the parity of k. When k is even, we have the following:

- $\bar\sigma_k > 0$ and $c_{\max} > 0$ if and only if $k > 0$ and $\sigma \in (-1, 0) \cup (2, \infty)$, or $k = 0$ and $\sigma \in (-1/4, 0) \cup (2, \infty)$.
- $\bar\sigma_k > 0$ and $c_{\max} = 0$ if and only if $k = 0$ and $\sigma = -1/4$.
- $\bar\sigma_k > 0$ and $c_{\max} < 0$ if and only if $k = 0$ and $\sigma \in (-1, -1/4)$.
- $\bar\sigma_k < 0$ if and only if $\sigma \in (0, 2)$.
- $\bar\sigma_k = 0$ if and only if $\sigma \in \{0, 2\}$.

When k is odd, we have the following:

- $\bar\sigma_k > 0$ and $c_{\max} > 0$ if and only if $\sigma > 0$.
- $\bar\sigma_k < 0$ if and only if $\sigma < 0$.
- $\bar\sigma_k = 0$ if and only if $\sigma = 0$.

6.3. **Location of the zeros of ξ_k and ξ_k'.** In $\mathbb{R}\setminus\{0\}$, the functions ξ_k and p_k have the same zeros. Then ξ_k and ξ_k' have no common zeros by (33). The functions ξ_0 and ξ_1 have no zeros in $\mathbb{R}\setminus\{0\}$, and the two zeros $\pm x_{2,1}$ of ξ_2 are in $\mathbb{R}\setminus\{0\}$.

Lemma 6.1. *On $\mathbb{R}\setminus\{0\}$:*

(i) *the zeros of ξ_k' belong to the oscillation intervals of ξ_k;*
(ii) *if k is odd or $\sigma \geq 0$, the zeros of ξ_k belong to the oscillation intervals of ξ_k; and*
(iii) *if k is even and $\sigma < 0$, the zeros of ξ_k, possibly except $\pm x_{k,k/2}$, belong to the oscillation intervals of ξ_k.*

Proof. It is enough to consider the zeros in $\mathbb{R}_+$ because ξ_k is either even or odd. We can also assume that $\xi_k\xi_k'$ has zeros on $\mathbb{R}_+$, otherwise there is nothing to prove.

Let x_* and x^* denote the minimum and maximum of the zeros of $\xi_k\xi_k'$ in $\mathbb{R}_+$. By (37),

$$(\xi_k\xi_k')' = {\xi_k'}^2 - q_k\xi_k^2 > 0$$

on the non-oscillation intervals, and therefore $\xi_k\xi_k'$ is strictly increasing on those intervals. In particular, since $\xi_k\xi_k'$ is strictly increasing on (b_k,∞) and $(\xi_k\xi_k')(x) \to 0$ as $x\to\infty$, it follows that $x^* < b_k$. This shows the statement when there is one oscillation interval of the form $(-b_k, b_k)$. So it remains to consider the case where there is an oscillation interval of ξ_k in $\mathbb{R}_+$ of the form (a_k, b_k). This holds when k is odd and $\sigma > 0$, $k = 0$ and $\sigma \in (-1/4, 0)\cup(2,\infty)$, or $k \in 2\mathbb{Z}_+$ and $\sigma \in (-1,0)\cup(2,\infty)$.

If k is odd and $\sigma > 0$, or k is even and $\sigma\in(2,\infty)$, then $x_* \leq a_k$ because $\xi_k\xi_k'$ is strictly increasing on $(0, a_k)$ and $(\xi_k\xi_k')(x) \to 0$ as $x \to 0^+$ by (35).

Finally, assume that $k \in 2\mathbb{Z}_+$ and $\sigma \in (-1, 0)$. Then the above arguments do not work because $(\xi_k\xi_k')(x) \to -\infty$ as $x \to 0^+$ by (35). Let f be the function on $\mathbb{R}_+$ defined by $f(x) = sx + \frac{\sigma}{2x}$. We have $f(x) \to -\infty$ as $x \to 0^+$, and $f' = s - \frac{\sigma}{2x^2} > 0$ on $\mathbb{R}_+$. Moreover $\sqrt{-\sigma/2s}$ is the unique zero of f in $\mathbb{R}_+$.

If x_* is a zero of ξ_k', then p_k has no zeros in $[-x_*, x_*]$, and therefore 0 is the unique zero of p_{k+1} in this interval. So $p_{k+1}/p_k > 0$ on $(0, x_*]$. Since

$$0 = f(x_*) - \sqrt{2(k+1+\sigma)s}\,\frac{p_{k+1}(x_*)}{p_k(x_*)}$$

by (36), it follows that $f(x_*) > 0$, obtaining $x_* > \sqrt{-\sigma/2s}$. But

$$a_k^2 = \frac{2k+1+\sigma-\sqrt{(2k+1)^2+4(k+1)\sigma}}{2s} < -\frac{\sigma}{2s}$$

because $k > 1$, obtaining $x_* > a_k$.

If x_* is a zero of ξ_k (*i.e.*, $x_* = x_{k,k/2}$), then the other positive zeros of $\xi_k\xi_k'$ are greater than a_k because this function is strictly increasing in $(0, a_k)$. $\square$

In the case of Lemma 6.1-(iii), the zeros $\pm x_{k,k/2}$ of ξ_k may be in oscillation intervals, in non-oscilation intervals or in their common boundary points. For instance, for $k = 2$,

$$p_2 = \left(\sqrt{\frac{2}{1+\sigma}} s x^2 - \sqrt{\frac{1+\sigma}{2}} \right) p_0$$

by (25), obtaining

$$x_{2,1}^2 = \frac{1+\sigma}{2s} .$$

Moreover

$$a_2^2 = \frac{5 + \sigma - \sqrt{25 + 12\sigma}}{2s} .$$

So

$$x_{2,1} - a_2 = \frac{-4 + \sqrt{25 + 12\sigma}}{2s} ,$$

and therefore $\sigma > -3/4$ if and only if $x_{2,1} > a_2$. So (a_2, b_2) contains no zero of ξ_2 when $\sigma \in (-1, -3/4]$. For $k > 2$, every oscillation interval of ξ_k contains some zero of ξ_k by Lemma 6.1.

Lemma 6.2. *There exist $C_0, C_1, C_2 > 0$, depending on σ, such that, if $k \geq C_0$ and I is any oscillation interval of ξ_k, then there is some subinterval $J \subset I$ so that:*

(i) *for every $x \in J$, there exists some zero $x_{k,i}$ of ξ_k in I such that*

$$|x - x_{k,i}| \leq \frac{C_1}{\sqrt{q_k(x)}} ;$$

(ii) *each connected component of $I \setminus J$ is of length $\leq C_2 k^{-1/2}$.*

Proof. According to Section 6.2, for any $c > 0$ with $cs \in q_k(I)$, the set $I_c = I \cap q_k^{-1}([cs, \infty))$ is a subinterval of I, whose boundary in I is $I \cap q_k^{-1}(cs)$.

Claim 1. If $\text{length}(I_c) \geq 2\pi/\sqrt{cs}$, then each boundary point of I_c in I satisfies the condition of (iii) with $x_{k,i} \in I_c$ and $C_4 = 2\pi$.

Let f_c be the function on $\mathbb{R}$ defined by $f_c(x) = \sin(\sqrt{cs} x)$, whose zeros are $\ell\pi/\sqrt{cs}$ for $\ell \in \mathbb{Z}$. Since $f_c'' + cs f_c = 0$ and $cs \leq q_k$ on I_c, the zeros of ξ_k in I_c separate the zeros of f_c in I_c by Sturm's comparison theorem. If $\text{length}(I_c) \geq 2\pi/\sqrt{cs}$, then each boundary point x of I_c is at a distance $\leq 2\pi/\sqrt{cs}$ of two consecutive zeros of f_c in I_c, and there is some zero of ξ_k between them, which shows Claim 1 because $q_k(x) = cs$.

Now we have to analyze each type of oscillation interval separately, corresponding to the possibilities for $\bar{\sigma}_k$ and $c_{\max}$. When there are two oscillation intervals of ξ_k, it is enough to consider only the oscillation interval contained in $\mathbb{R}_+$ because the function ξ_k is either even or odd.

The first type of oscillation interval is of the form $I = (a_k, b_k)$, which corresponds to the conditions $\bar\sigma_k > 0$ and $c_{\max} > 0$. We have $cs \in q_k(I)$ when $0 < c \le c_{\max}$. Then $q_k^{-1}(cs)$ consists of the points

$$\pm a_{k,c} = \pm\sqrt{\frac{2k+1+\sigma-c-\sqrt{(2k+1+\sigma-c)^2-\bar\sigma_k}}{2s}}\;,$$

$$\pm b_{k,c} = \pm\sqrt{\frac{2k+1+\sigma-c+\sqrt{(2k+1+\sigma-c)^2-\bar\sigma_k}}{2s}}\;, \tag{42}$$

and we get $I_c = [a_{k,c}, b_{k,c}]$. Since

$$s(b_{k,c}-a_{k,c})^2 = c_{\max} - c\;, \tag{43}$$

we have $\operatorname{length}(I_c) \ge 2\pi/\sqrt{cs}$ if and only if $c(c_{\max}-c) \ge 4\pi^2$, which means that $c_{\max} \ge 4\pi$ and $c_- \le c \le c_+$ for

$$c_\pm = \frac{c_{\max} \pm \sqrt{c_{\max}^2 - 16\pi^2}}{2}\;.$$

Since $c_{\max} \in O(k)$ as $k \to \infty$, there is some $C_0 > 0$, depending on σ, such that $c_{\max} \ge 4\pi$ for all $k \ge C_0$. Assuming $k \ge C_0$, let $a_{k,\pm} = a_{k,c_\pm}$ and $b_{k,\pm} = b_{k,c_\pm}$, satisfying

$$a_k < a_{k,-} < a_{k,+} < b_{k,+} < b_{k,-} < b_k\;.$$

Fix any $x \in I$ and let $q_k(x) = cs$. First, $x \in [a_{k,-}, a_{k,+}] \cup [b_{k,+}, b_{k,-}]$ if and only if $\operatorname{length}(I_c) \ge 2\pi/\sqrt{cs}$, and in this case x satisfies the condition of (i) with $x_{k,i} \in I_c$ and $C_1 = 2\pi$ by Claim 1. Second, if $x \in (a_k, a_{k,-}) \cup (b_{k,-}, b_k)$, then $\operatorname{length}(I_c) < 2\pi/\sqrt{cs}$, $I_c \supset I_{c_-}$, and we already know that I_{c_-} contains some zero of ξ_k. Hence x also satisfies the condition of (i) with $C_1 = 2\pi$. And third, if $x \in (a_{k,+}, b_{k,+})$, then

$$s(b_{k,+}-a_{k,+})^2 = c_{\max} - c_+ = c_- = \frac{16\pi^2}{c_+} \le \frac{32\pi^2}{c_{\max}} \le \frac{32\pi^2}{c}$$

by (43), obtaining

$$\operatorname{length}(I_{c_+}) \le \frac{4\sqrt{2}\pi}{\sqrt{cs}}\;.$$

Since $I_c \subset I_{c_+}$ and it is already proved that I_{c_+} contains some zero of ξ_k, it follows that x also satisfies the condition of (i) with $C_1 = 4\sqrt{2}\pi$. Summarizing, (i) holds in this case with $J = I$ and $C_1 = 4\sqrt{2}\pi$ if $c_{\max} \ge 4\pi$. In this case, (ii) is obvious because $J = I$.

The second type of oscillation interval is of the form $I = (0, b_k)$, which corresponds to the condition $\bar\sigma_k < 0$. Now, $cs \in q_k(I)$ for any $c > 0$, the set $q_k^{-1}(cs)$ consists of the points $\pm b_{k,c}$, defined like in (42), and we have $I_c = (0, b_{k,c}]$. The equality $cs = q_k(2\pi/\sqrt{cs})$ holds when

$$(2k+1+\sigma)^2 - \bar\sigma_k - 16\pi^2 > 0 \tag{44}$$

and c is

$$c_\pm = 8\pi^2 \frac{2k+1+\sigma \pm \sqrt{(2k+1+\sigma)^2 - \bar\sigma_k - 16\pi^2}}{\bar\sigma_k - 16\pi^2} \ .$$

Assuming (44), we have $\mathrm{length}(I_c) \ge 2\pi/\sqrt{cs}$ if and only if $c_- \le c \le c_+$. Let $b_{k,\pm} = b_{k,c_\pm}$, satisfying $0 < b_{k,+} < b_{k,-} < b_k$.

Fix any $x \in I$ and let $q_k(x) = cs$. First, $x \in [b_{k,+}, b_{k,-}]$ if and only if $\mathrm{length}(I_c) \ge 2\pi/\sqrt{cs}$, and in this case x satisfies the condition of (i) with $x_{k,i} \in I_c$ and $C_1 = 2\pi$ by Claim 1. And second, if $x \in (b_{k,-}, b_k)$, then $\mathrm{length}(I_c) < 2\pi/\sqrt{cs}$, $I_c \supset I_{c_-}$, and we already know that I_{c_-} contains some zero of ξ_k. Hence x also satisfies the condition of (i) with $C_1 = 2\pi$. So, when (44) is true, (i) holds with $J = [b_{k,+}, b_k)$ and $C_1 = 2\pi$.

Notice that $c_+ \in O(k)$ as $k \to \infty$. Then there are some $C_0, C_2 > 0$, depending on σ, such that, if $k \ge C_0$, then (44) holds and $sb_{k,+}^2 = 4\pi^2/c_+ \le C_2 k^{-1}$, showing (ii) in this case.

The third and final type of oscillation interval is $I = (-b_k, b_k)$, which corresponds to the condition $\bar\sigma_k = 0$. We have $cs \in q_k(I)$ when $0 < c \le c_{\max}$. Then $q_k^{-1}(cs)$ consists of the points $\pm b_{k,c}$, defined like in (42), and we get $I_c = [-b_{k,c}, b_{k,c}]$. Since

$$sb_{k,c}^2 = c_{\max} - c \ , \tag{45}$$

we have $\mathrm{length}(I_c) \ge 2\pi/\sqrt{cs}$ if and only if $c(c_{\max} - c) \ge \pi^2$, which means that $c_{\max} \ge \pi$ and $c_- \le c \le c_+$ for

$$c_\pm = \frac{c_{\max} \pm \sqrt{c_{\max}^2 - 4\pi^2}}{2} \ .$$

Since $c_{\max} \in O(k)$ as $k \to \infty$, there is some $C_0 > 0$, depending on σ, such that $c_{\max} \ge 4\pi$ for all $k \ge C_0$. Assuming $k \ge C_0$, let $b_{k,\pm} = b_{k,c_\pm}$, which satisfy $0 < b_{k,+} < b_{k,-} < b_k$..

Fix any $x \in I$ and let $q_k(x) = cs$. First, $b_{k,+} \le |x| \le b_{k,-}$ if and only if $\mathrm{length}(I_c) \ge 2\pi/\sqrt{cs}$, and in this case x satisfies the condition of (i) with $x_{k,i} \in I_c$ and $C_1 = 2\pi$ by Claim 1. Second, if $|x| > b_{k,-}$, then $\mathrm{length}(I_c) < 2\pi/\sqrt{cs}$, $I_c \supset I_{c_-}$, and we already know that I_{c_-} contains some zero of ξ_k. Hence x also satisfies the condition of (i) with $C_1 = 2\pi$. And third, if $|x| < b_{k,+}$, then

$$sb_{k,+}^2 = c_{\max} - c_+ = c_- = \frac{4\pi^2}{c_+} \le \frac{8\pi^2}{c_{\max}} \le \frac{8\pi^2}{c}$$

by (45), obtaining

$$\mathrm{length}(I_{c_+}) \le \frac{\sqrt{2}\pi}{\sqrt{cs}} \ .$$

Since $I_c \subset I_{c_+}$ and it is already proved that I_{c_+} contains some zero of ξ_k, it follows that x also satisfies the condition of (i) with $C_1 = \sqrt{2}\pi$. Summarizing, (i) holds in this case with $J = I$ and $C_1 = 2\pi$. In this case, (ii) is also obvious because $J = I$. □

Lemma 6.3. *There exist $C_0', C_1', C_2' > 0$, depending on σ and s, such that, if $k \geq C_0'$ and I is any oscillation interval of ξ_k, then there is some subinterval $J' \subset I$ so that:*

(i) *$q_k \geq C_1' k^{1/3}$ on J'; and*

(ii) *each connected component of $I \setminus J'$ is of length $\leq C_2' k^{-1/6}$.*

Proof. We use the notation of the proof of Lemma 6.2. The same type of argument can be used for all types of oscillation intervals. Thus, *e.g.*, suppose that I is of the type $(0, b_k)$. Since $b_k \in O(k^{1/2})$ as $k \to \infty$, we have $b_k' = b_k - k^{-1/6} \in I$ for k large enough, and

$$q_k(b_k') = -s^2(k^{-1/3} - 2b_k k^{-1/6}) - \bar{\sigma}_k((b_k - k^{-1/6})^{-2} - b_k^{-2}) \in O(k^{1/3})$$

as $k \to \infty$. So there are $C_0', C_1' > 0$, depending on σ and s, such that $b_k' \in I$ and $c' = q_k(b_k') \geq C_1' k^{1/3}$ for $k \geq C_0'$. Then (i) and (ii) hold with $J' = I_{c'} = (0, b_k']$. □

Corollary 6.4. *There exist $C_0'', C_1'' > 0$, depending on σ and s, such that, if $k \geq C_0''$ and I is any oscillation interval of ξ_k, then, for each $x \in I$, there exists some zero $x_{k,i}$ of ξ_k in I so that*

$$|x - x_{k,i}| \leq C_1'' k^{-1/6} .$$

Proof. With the notation of Lemmas 6.2 and 6.3, let $C_0'' = \max\{C_0, C_0'\}$ and $C_2'' = \max\{C_2, C_2'\}$. Assume $k \geq C_0''$ and consider the subinterval $J'' = J \cap J' \subset I$. By Lemmas 6.2-(ii) and 6.3-(ii), each connected component of $I \setminus J''$ is of length $\leq C_2'' k^{-1/6}$. Then, for each $x \in I$, there is some $x'' \in J''$ such that $|x - x''| \leq C_2'' k^{-1/6}$. By Lemmas 6.2-(i) and 6.3-(i), there is some zero $x_{k,i}$ of ξ_k in I such that

$$|x'' - x_{k,i}| = \frac{C_1}{\sqrt{q_k(x'')}} \leq \frac{C_1}{\sqrt{C_1'}} k^{-1/6} .$$

Hence

$$|x - x_{k,i}| \leq \left(C_2'' + C_1/\sqrt{C_1'}\right) k^{-1/6} .$$ □

6.4. **Estimates of ξ_k.**

Lemma 6.5. *Let I be an oscillation interval of ξ_k, let $x \in I$ and let $x_{k,i}$ be a zero of ξ_k in I. Then*

$$\xi_k^2(x) \leq \begin{cases} \frac{8s}{3}|x - x_{k,i}| & \text{if } k \text{ is even} \\ \frac{8s}{3(1+\sigma)}|x - x_{k,i}| & \text{if } k \text{ is odd} . \end{cases}$$

Proof. We can assume that there are no zeros of ξ_k between x and $x_{k,i}$. For the sake of simplicity, suppose also that $x_{k,i} < x$ and $\xi_k > 0$ on $(x_{k,i}, x)$; the other cases are analogous. The key observation of [1] is that then the graph of ξ_k on $[x_{k,i}, x]$ is concave down, and therefore

$$\frac{1}{2}\xi_k(x)(x - x_{k,i}) \leq \int_{x_{k,i}}^{x} \xi_k(t)\, dt .$$

By Schwartz's inequality and (32), it follows that

$$\left(\frac{1}{2}\xi_k(x)(x-x_{k,i})\right)^2 \le \left(\int_{-\infty}^{\infty} \frac{p_k^2(t)|t|^\sigma e^{-st^2}}{(t-x_{k,i})^2}\,dt\right)\left(\int_{x_{k,i}}^{x} (t-x_{k,i})^2\,dt\right)$$
$$= {p'_k}^2(x_{k,i})\lambda_{k,i}\frac{(x-x_{k,i})^3}{3}\,,$$

and the result follows by Lemma 5.2. □

With the notation of Lemma 6.2, for each $k \ge C_0$, let $\widehat{I}_k$ denote the union of the oscillation intervals of ξ_k, and let $\widehat{J}_k \subset \widehat{I}_k$ denote the union of the corresponding subintervals J defined in the proof of Lemma 6.2. More precisely:

- if $\bar{\sigma}_k > 0$ and $c_{\max} > 0$, then $\widehat{J}_k = \widehat{I}_k = (-a_k, -b_k) \cup (a_k, b_k)$;
- if $\bar{\sigma}_k < 0$, then $\widehat{I}_k = (-b_k, 0) \cup (0, b_k)$ and $\widehat{J}_k = (-b_k, b_{k,+}] \cup [b_{k,+}, b_k)$; and
- if $\bar{\sigma}_k = 0$, then $\widehat{J}_k = \widehat{I}_k = (-b_k, b_k)$.

If $k < C_0$, we also use the notation $\widehat{J}_k = \widehat{I}_k$ for the union of the oscillation intervals, which may be empty if there are no oscillation intervals.

Theorem 6.6. *There exist $C, C', C'' > 0$, depending on σ and s, such that, for $k \ge 1$:*

(i) $\xi_k^2(x) \le C/\sqrt{q_k(x)}$ *for all* $x \in \widehat{J}_k$ *;*
(ii) *if k is odd or $\sigma \ge 0$, then* $\xi_k^2(x) \le C' k^{-1/6}$ *for all* $x \in \mathbb{R}$ *; and*
(iii) *if k is even and $\sigma < 0$, then* $\xi_k^2(x) \le C'' k^{-1/6}$ *if* $|x| \ge x_{k,k/2}$.

Proof. Part (i) follows from Lemmas 6.2 and 6.5.

In any case, $\xi_k(x) \to 0$ as $x \to \infty$. If moreover k is odd or $\sigma \ge 0$, then ξ_k is continuous on $\mathbb{R}$. Thus ξ_k^2 is bounded and reaches its maximum at some point $\bar{x} \in \mathbb{R}$. Since $\xi_k(0) = 0$ (if $\bar{\sigma}_k \ne 0$) or $0 \in \widehat{I}_k$ (if $\bar{\sigma}_k = 0$), it follows from Lemma 6.1 that $\bar{x} \in \widehat{I}_k$. Then (ii) follows by Corollary 6.4 and Lemma 6.5.

If k is even and $\sigma < 0$, then ξ_k is not defined at 0 and $\xi_k^2(x) \to \infty$ as $x \to 0$. So we can only conclude as above that the restriction of ξ_k^2 to the set defined by $|x| \ge x_{k,k/2}$ is bounded, and reaches its maximum at some point $\bar{x}$ of this set. Then $\bar{x} \in \widehat{I}_k$ by Lemma 6.1, and therefore (iii) holds by Corollary 6.4 and Lemma 6.5. □

Consider the case $\sigma < 0$ and k even, when Theorem 6.6 does not provide any estimate of ξ_k^2 around zero. According to Section 5, the function $p_k^2(x)$ on the region $|x| \le x_{k,k/2}$ reaches its maximum at $x = 0$, and moreover $p_k^2(0) < p_0^2$ by (28). Hence $\phi_k^2(x) < p_0^2$ for $|x| \le x_{k,k/2}$, which complements Theorem 6.6-(iii). On the other hand, $\phi_k^2(x) \le \xi_k^2(x)$ for $|x| \le 1$. Moreover $x_{k,k/2} \le 1$ for k large enough by Corollary 6.4 since $a_k \to 0$ as $k \to \infty$. So the following result follows from Theorem 6.6-(iii).

Corollary 6.7. *Suppose that $\sigma < 0$. There exist $C''' > 0$, depending on σ and s, such that $\phi_k^2(x) \leq C'''$ for all k even and all $x \in \mathbb{R}$.*

Theorem 6.8. *There exist $C^{(IV)}, C^{(V)} > 0$, depending on σ and s, such that, for $k \geq 1$:*

(i) $\max_{x\in\mathbb{R}} \xi_k^2(x) \geq C^{(IV)} k^{-1/6}$*; and,*
(ii) *if k is even and $\sigma < 0$, then* $\max_{|x|\geq x_{k,k/2}} \xi_k^2(x) \leq C^{(V)} k^{-1/6}$.

The following lemmas will be used in the proof of Theorem 6.8.

Lemma 6.9. *There is some $F > 0$ such that, for $k \geq 1$ and $x \geq b_{k+1}$,*

$$\xi_k(x) \leq \frac{F k^{-5/12}}{(x - b_k)^2} .$$

Proof. Let $x_0 \in (x_{k,1}, b_k)$ such that $\xi_k'(x_0) = 0$. Since

$$\xi_k'(x) = \int_{x_0}^{x} \xi_k''(t)\, dt$$

and $\xi_k'(x) < 0$ for $x > b_k$, we get

$$\int_{x_0}^{x} q_k(t)\xi_k(t)\, dt > 0$$

for $x > b_k$. Because $\xi_k(x) > 0$ for $x > x_0$, $q_k(x) > 0$ for $x_0 < x < b_k$ and $q_k(x) < 0$ for $x > b_k$, it follows that

$$\int_{x_0}^{b_k} q_k(t)\xi_k(t)\, dt > -\int_{b_k}^{x} q_k(t)\xi_k(t)\, dt . \tag{46}$$

According to Corollary 6.4 and Theorem 6.6-(ii),(iii), for $k \geq C_0''$ and with $\bar{C} = \max\{C', C''\}$, we get

$$\begin{aligned}
\int_{x_0}^{b_k} q_k(t)\xi_k(t)\, dt &\leq \bar{C}^{1/2} k^{-1/12} \int_{x_0}^{b_k} q_k(t)\, dt \\
&= \bar{C}^{1/2} k^{-1/12} \bigg((2k + 1 + \sigma) s (b_k - x_0) \\
&\quad - \frac{s^2}{3}(b_k^3 - x_0^3) + \frac{\bar{\sigma}_k}{4}(b_k^{-1} - x_0^{-1}) \bigg) \\
&\leq \bar{C}^{1/2} k^{-1/12} \bigg((2k + 1 + \sigma) s C_1'' k^{-1/6} \\
&\quad - \frac{s^2}{3}(b_k^3 - (b_k - C_1'' k^{-1/6})^3) + \frac{|\bar{\sigma}_k| C_1'' k^{-1/6}}{4 b_k (b_k - C_1'' k^{-1/6})} \bigg) \\
&\leq \bar{C}^{1/2} k^{-1/12} \bigg((2k + 1 + \sigma) s C_1'' k^{-1/6}
\end{aligned}$$

$$- s^2\left(C_1''b_k^2k^{-1/6} - C_1''^2b_kk^{-1/3} - \frac{C_1''^3k^{-1/2}}{3}\right)$$
$$+ \frac{|\bar\sigma_k|C_1''k^{-1/6}}{4b_k(b_k - C_1''k^{-1/6})}\Bigg) .$$

Since

$$2k + 1 + \sigma - sb_k^2 = \frac{\bar\sigma_k}{4sb_k^2} ,$$

there is some $F_0 > 0$ such that

$$\int_{x_0}^{b_k} q_k(t)\xi_k(t)\,dt \le F_0k^{1/12} \tag{47}$$

for all $k \in \mathbb{N}$.

On the other hand,

$$-\int_{b_k}^{x} q_k(t)\xi_k(t)\,dt \ge -\xi_k(x)\int_{b_k}^{x} q_k(t)\,dt .$$

With the substitution $u = t - b_k$, we get

$$q_k(t) = -s^2u(u + 2b_k) + \frac{\bar\sigma_k}{4b_k^2} - \frac{\bar\sigma_k}{4}(u + b_k)^{-2} ,$$

giving

$$\begin{aligned}
-\xi_k(x)\int_{b_k}^{x} q_k(t)\,dt &= \xi_k(x)\Bigg(s^2\left(\frac{1}{3}(x - b_k)^3 + b_k(x - b_k)^2\right) \\
&\quad - \frac{\bar\sigma_k}{4b_k^2}(x - b_k) - \frac{\bar\sigma_k}{4}(x^{-1} - b_k^{-1})\Bigg) \\
&\ge \xi_k(x)\left(s^2b_k(x - b_k)^2 - \frac{|\bar\sigma_k|}{4b_k^2}(x - b_k) - \frac{|\bar\sigma_k|}{4}b_k^{-1}\right) \\
&\ge \xi_k(x)\left(\left(s^2b_k - \frac{|\bar\sigma_k|}{4b_k^2(b_{k+1} - b_k)}\right)(x - b_k)^2 - \frac{|\bar\sigma_k|}{4}b_k^{-1}\right)
\end{aligned}$$

for $x \ge b_{k+1}$. By (40), it follows that there is some $F_1 > 0$ such that

$$-\int_{b_k}^{x} q_k(t)\xi_k(t)\,dt \ge F_1\xi_k(x)k^{1/2}(x - b_k)^2 \tag{48}$$

for all k and $x \ge b_{k+1}$. Now the result follows from (46)–(48). □

Lemma 6.10. *For each $\epsilon > 0$, there is some $G > 0$ such that, for all $k \in \mathbb{N}$,*

$$\max_{|x - x_{k,1}| \le \epsilon k^{-1/6}} \sum_{\ell=0}^{k-1} \xi_\ell^2(x) \le Gk^{1/6} .$$

Proof. Take any $x \in \mathbb{R}$ such that $|x - x_{k,1}| \le \epsilon k^{-1/6}$. By Corollary 6.4,

$$|x - b_k| \le |x - x_{k,1}| + |x_{k,1} - b_k| \le (\epsilon + C_1'')k^{-1/6} \tag{49}$$

for $k \ge C_0''$. In particular, $b_4 < x$ if k is large enough. With this assumption, let $\ell_0, \ell_1, \ell_2 \in \mathbb{N}$ satisfying $0 < \ell_0 < \ell_1 < \ell_2 - 1$, where ℓ_0 and ℓ_1 will be determined later, and ℓ_2 is the maximum of the naturals $\ell < k$ with $b_{\ell'} \le x$ for all $\ell' \le \ell$. Let

$$f_\pm(t) = \sqrt{2t + 1 + \sigma \pm 1}$$

for $t \ge 1$. We have

$$f_\pm(\ell) - \sqrt{s} b_\ell = \frac{\pm 4(2\ell + 1 + \sigma) + 4 + \bar{\sigma}_\ell}{2\big(2\ell + 1 + \sigma \pm 2 - \sqrt{(2\ell + 1 + \sigma)^2 - \bar{\sigma}_\ell}\big)\big(f_\pm(\ell) + \sqrt{s} b_\ell\big)}$$

for $\ell \in \mathbb{Z}_+$. So, assuming that k is large enough, we can fix ℓ_0, independently of k and x, so that

$$f_-(\ell) < \sqrt{s} b_\ell < f_+(\ell)$$

for all $\ell \ge \ell_0$. We have $f_+(\ell_1) < f_-(\ell_2)$ because $\ell_1 < \ell_2 - 1$. Moreover observe that

$$f_+'(t) = (2t + 2 + \sigma)^{-1/2} > 0 \ ,$$
$$f_+''(t) = -(2t + 2 + \sigma)^{-3/2} < 0$$

for all $t \ge 1$. Then, by Lemma 6.9,

$$\begin{aligned}\sum_{\ell=\ell_0}^{\ell_1-1} \xi_\ell^2(x) &\le \sum_{\ell=\ell_0}^{\ell_1-1} \frac{F^2 \ell^{-5/6}}{(x - b_\ell)^4} \le F^2 \sum_{\ell=\ell_0}^{\ell_1-1} \frac{\ell^{-5/6}}{(b_{\ell_2} - b_\ell)^4} \\ &\le F^2 \sqrt{s} \sum_{\ell=\ell_0}^{\ell_1-1} \frac{\ell^{-5/6}}{(f_-(\ell_2) - f_+(\ell))^4} \le F^2 \sqrt{s} \int_{\ell_0}^{\ell_1} \frac{t^{-5/6}\,dt}{(f_-(\ell_2) - f_+(t))^4} \ ,\end{aligned}$$

and, after integrating by parts four times, we get

$$\begin{aligned}\int_{\ell_0}^{\ell_1} \frac{t^{-5/6}\,dt}{(f_-(\ell_2) - f_+(t))^4} &\le \frac{\ell_1^{-5/6} {f_+'}^{-1}(\ell_1)}{3(f_-(\ell_2) - f_+(\ell_1))^3} + \frac{5\ell_1^{-11/6} {f_+'}^{-2}(\ell_1)}{36\,(f_-(\ell_2) - f_+(\ell_1))^2} \\ &\quad + \frac{55\,\ell_1^{-17/6} {f_+'}^{-3}(\ell_1)}{216\,(f_-(\ell_2) - f_+(\ell_1))} + \frac{935}{1296} \ell_1^{-23/6} {f_+'}^{-4}(\ell_1) \ln(f_-(\ell_2)) \\ &\quad + \frac{21505}{7776} \ln(f_-(\ell_2)) \int_{\ell_0}^{\ell_1} t^{-29/6} {f_+'}^{-4}(t)\,dt \ .\end{aligned}$$

Therefore, since $f_+'(t) \in O(t^{-1/2})$ as $t \to \infty$, there exists some $G_1 > 0$, independent of k and x, such that

$$\sum_{\ell=\ell_0}^{\ell_1-1} \xi_\ell^2(x) \le G_1 \bigg(\frac{\ell_1^{-1/3}}{(f_-(\ell_2) - f_+(\ell_1))^3} + \frac{\ell_1^{-5/6}}{(f_-(\ell_2) - f_+(\ell_1))^2}$$

$$+\frac{\ell_1^{-4/3}}{f_-(\ell_2)-f_+(\ell_1)}+\ell_1^{-11/6}\ln(f_-(\ell_2))+\ln(f_-(\ell_2))\Bigg) .$$

We have

$$\ell_1^{-11/6}\ln(f_-(\ell_2))+\ln(f_-(\ell_2))\leq \ell_2^{1/6}$$

for k large enough. Then $\sum_{\ell=1}^{\ell_0-1}\xi_\ell^2(x)$ has an upper bound of the type of the statement if ℓ_1 satisfies

$$\max\left\{\frac{\ell_1^{-1/3}}{(f_-(\ell_2)-f_+(\ell_1))^3},\frac{\ell_1^{-5/6}}{(f_-(\ell_2)-f_+(\ell_1))^2},\frac{\ell_1^{-4/3}}{f_-(\ell_2)-f_+(\ell_1)}\right\}\leq \ell_2^{1/6} . \tag{50}$$

On the other hand, according to Theorem 6.6-(ii),(iii),

$$\sum_{\ell_1}^{\ell_2}\xi_\ell^2(x)\leq \bar{C}\sum_{\ell_1}^{\ell_2}\ell^{-1/6}\leq \bar{C}\int_{\ell_1}^{\ell_2}y^{-1/6}\,dy=\frac{6\bar{C}}{5}(\ell_2^{5/6}-\ell_1^{5/6}) ,$$

where $\bar{C}=\max\{C',C''\}$. Then $\sum_{\ell=\ell_1}^{\ell_2}\xi_\ell^2(x)$ has an upper bound of the type of the statement if

$$\ell_2^{5/6}-\ell_1^{5/6}\leq G_2\ell_2^{1/6}$$

for some $G_2>0$, independent of k and x, which is equivalent to

$$\ell_1\geq \ell_2\big(1-G_2\ell_2^{-2/3}\big)^{6/5} . \tag{51}$$

Thus we must check the compatibility of (50) with (51) for some ℓ_1 and G_2. By (51) and since, for each $G_2,\delta>0$, we have $G_2\ell_2^{-2/3}\leq \ell_2^{-\frac{2}{3}+\delta}$ for k large enough, we can replace (50) with

$$\max\left\{\frac{\ell_2^{-1/3}\big(1-\ell_2^{-\frac{2}{3}+\delta}\big)^{-2/5}}{(f_-(\ell_2)-f_+(\ell_1))^3},\frac{\ell_2^{-5/6}\big(1-\ell_2^{-\frac{2}{3}+\delta}\big)^{-1}}{(f_-(\ell_2)-f_+(\ell_1))^2},\right.$$
$$\left.\frac{\ell_2^{-4/3}\big(1-\ell_2^{-\frac{2}{3}+\delta}\big)^{-8/5}}{f_-(\ell_2)-f_+(\ell_1)}\right\}\leq \ell_2^{1/6}$$

for some $\delta>0$, which is equivalent to

$$\ell_1\leq\frac{1}{2}\left(\sqrt{2\ell_2+\sigma}-\ell_2^a\big(1-\ell_2^{-\frac{2}{3}+\delta}\big)^b\right)^2-\frac{2+\sigma}{2}$$

for

$$(a,b)\in\{(-1/6,-2/15),(-1/2,-1/2),(-3/2,-8/5)\} .$$

Thus the compatibility of (50) with (51) holds if there is some $G_2,\delta>0$ such that

$$\ell_2\big(1-G_2\ell_2^{-2/3}\big)^{6/5}\leq\frac{1}{2}\left(\sqrt{2\ell_2+\sigma}-\ell_2^a\big(1-\ell_2^{-\frac{2}{3}+\delta}\big)^b\right)^2-\frac{4+\sigma}{2} ,$$

which is equivalent to

$$G_2 \geq \ell_2^{2/3}\left(1-\left(\frac{1}{2}\left(\sqrt{2+\sigma\ell_2^{-1}}-\ell_2^{a-\frac{1}{2}}\big(1-\ell_2^{-\frac{2}{3}+\delta}\big)^b\right)^2-\frac{4+\sigma}{2}\ell_2^{-1}\right)^{5/6}\right).$$

There is some $G_2 > 0$ satisfying this condition because the l'Hôspital rule shows that, for δ small enough, each function

$$t^{2/3}\left(1-\left(\frac{1}{2}\left(\sqrt{2+\sigma t^{-1}}-t^{a-\frac{1}{2}}\big(1-t^{-\frac{2}{3}+\delta}\big)^b\right)^2-\frac{4+\sigma}{2}t^{-1}\right)^{5/6}\right)$$

is convergent in $\mathbb{R}$ as $t \to \infty$.

Now, if $\ell_2 < k-1$, let ℓ_3 denote minimum integer $\ell < k$ such that $b_{\ell'} > x$ for all $\ell' \geq \ell$. Also, let $\bar\sigma_{\min}$ and $\bar\sigma_{\max}$ denote the minimum and maximum values of $\bar\sigma_\ell$ for $\ell \in \mathbb{N}$. Then

$$\sqrt{\frac{2(\ell_3-1)+1+\sigma+\sqrt{(2(\ell_3-1)+1+\sigma)^2+\bar\sigma_{\min}}}{2s}} \leq x < \sqrt{\frac{2(\ell_2+1)+1+\sigma+\sqrt{(2(\ell_2+1)+1+\sigma)^2+\bar\sigma_{\max}}}{2s}}\,,$$

obtaining

$$2(\ell_3-\ell_2)-4 < \sqrt{(2(\ell_2+1)+1+\sigma)^2+\bar\sigma_{\max}}-\sqrt{(2(\ell_3-1)+1+\sigma)^2+\bar\sigma_{\min}}\,.$$

If $\ell_3 > \ell_2+1$, it follows that

$$(2(\ell_2+1)+1+\sigma)^2+\bar\sigma_{\max} > (2(\ell_3-1)+1+\sigma)^2+\bar\sigma_{\min}\,,$$

giving

$$\sqrt{\bar\sigma_{\max}-\bar\sigma_{\min}} > \sqrt{(2(\ell_2+1)+1+\sigma)^2-(2(\ell_3-1)+1+\sigma)^2} \geq 2(\ell_3-\ell_2)-4\,.$$

Therefore $\sum_{\ell=\ell_2+1}^{\ell_3}\xi^2(x)$ has an upper bound of the type of the statement by Theorem 6.6-(ii),(iii).

Let

$$h(t)=(2t+1+\sigma)s-s^2x^2-\frac{\bar\sigma_{\max}}{4}x^{-2}$$

for $t \geq 0$. According to Theorem 6.6-(i), if $\ell_3 < k-1$, then

$$\sum_{\ell=\ell_3+1}^{k-1}\xi_\ell^2(x) \leq C\sum_{\ell=\ell_3+1}^{k-1}\frac{1}{\sqrt{q_\ell(x)}} \leq C\sum_{\ell=\ell_3+1}^{k-1}\frac{1}{\sqrt{h(\ell)}} \leq C\int_{\ell_3}^{k-1}\frac{dt}{\sqrt{h(t)}} = \frac{C}{2s}\big(\sqrt{h(k-1)}-\sqrt{h(\ell_3)}\big) \leq \frac{C}{2s}\sqrt{2(k-1-\ell_3)}\,.$$

Hence $\sum_{\ell=\ell_3+1}^{k-1}\xi_\ell^2(x)$ also has an upper bound like in the statement because, by (41), (40) and (49), there is some $G_3, G_4 > 0$ such that

$$G_3(k-1-\ell_3)k^{-1/2} \le b_{k-1} - b_{\ell_3} \le b_{k-1} - x \le G_4 k^{-1/6} . \quad \square$$

Proof of Theorem 6.8. By (32),

$$1 = \int_{-\infty}^{\infty} \left(\frac{p_k(x)}{x - x_{k,1}}\right)^2 \frac{|x|^\sigma e^{-sx^2}}{{p'_k}^2(x_{k,1})\lambda_{k,1}}\, dx .$$

Thus, by (31) and Lemma 6.10,

$$\int_{|x-x_{k,1}|\le\epsilon k^{-1/6}} \left(\frac{p_k(x)}{x - x_{k,1}}\right)^2 \frac{|x|^\sigma e^{-sx^2}}{{p'_k}^2(x_{k,1})\lambda_{k,1}}\, dx$$
$$\le \int_{|x-x_{k,1}|\le\epsilon k^{-1/6}} \sum_{\ell=0}^{k-1} \xi_\ell^2(x)\, dx \le 2\epsilon k^{-1/6} \max_{|x-x_{k,1}|\le\epsilon k^{-1/6}} \sum_{\ell=0}^{k-1} \xi_\ell^2(x) \le 2\epsilon G$$

for any $\epsilon > 0$. It follows that

$$\int_{|x-x_{k,1}|\ge\epsilon k^{-1/6}} \left(\frac{p_k(x)}{x - x_{k,1}}\right)^2 \frac{|x|^\sigma e^{-sx^2}}{{p'_k}^2(x_{k,1})\lambda_{k,1}}\, dx \ge \frac{1}{2} \tag{52}$$

when $\epsilon \le \frac{1}{4G}$, which implies part (i).

When k is even and $\sigma < 0$, either $0 < x_{k,k/2} < a_k$, or $|x_{k,k/2} - a_k| \le C''_1 k^{-1/6}$ for k large enough according to Corollary 6.4. Moreover $|x_{k,1} - b_k| \le C''_1 k^{-1/6}$ for k large enough by Corollary 6.4 as well. So, by (39) and (38), there are some $C_0, C_1 > 0$, independent of k, such that

$$x_{k,k/2} \le a_k + C''_1 k^{-1/6} \le C_0 k^{-1/2} ,$$
$$x_{k,1} - x_{k,k/2} \ge b_k - a_k - 2C''_1 k^{-1/6} = \sqrt{\frac{c_{\max}}{s}} - 2C''_1 k^{-1/6} \ge C_1 k^{1/2}$$

On the other hand, by (28), there is some $C_2 > 0$, independent of k, such that $\xi_k^2(x) \le C_2|x|^\sigma$ for $|x| \le x_{k,k/2}$. Therefore

$$\int_{|x|\le x_{k,k/2}} \frac{\xi_k^2(x)\, dx}{(x - x_{k,1})^2} \le \frac{C_2}{(x_{k,1} - x_{k,k/2})^2} \int_{|x|\le x_{k,k/2}} |x|^\sigma\, dx$$
$$= \frac{2C_2 x_{k,k/2}^{\sigma+1}}{(\sigma+1)(x_{k,1} - x_{k,k/2})^2} \le \frac{2C_2 C_0^{\sigma+1}}{(\sigma+1)C_1^2} k^{-\frac{\sigma+3}{2}} < \frac{2C_2 C_0^{\sigma+1}}{(\sigma+1)C_1^2} k^{-1} .$$

This inequality and (52) imply part (ii). $\square$

7. Perturbed Schwartz space

We introduce a perturbed version $\mathcal{S}_\sigma$ of $\mathcal{S}$. It will be shown that $\mathcal{S}_\sigma = \mathcal{S}$ after all, but the relevance of this new definition to study J will become clear in the next section; in particular, the norms used to define $\mathcal{S}_\sigma$ will be appropriate to show embedding results, like a version of the Sobolev embedding theorem. Since $\mathcal{S}_\sigma$ must contain the functions ϕ_k, Theorem 6.6 and Corollary 6.7 indicate that different definitions must be given for $\sigma \geq 0$ and $\sigma < 0$.

When $\sigma \geq 0$, for any $\phi \in C^\infty$ and $m \in \mathbb{N}$, let

$$\|\phi\|_{\mathcal{S}^m_\sigma} = \sum_{i+j\leq m} \sup_x |x|^{\sigma/2} |x^i D^j_\sigma \phi(x)| \,. \tag{53}$$

This defines a norm $\| \ \|_{\mathcal{S}^m_\sigma}$ on the linear space of functions $\phi \in C^\infty$ with $\|\phi\|_{\mathcal{S}^m_\sigma} < \infty$, and let $\mathcal{S}^m_\sigma$ denote the corresponding Banach space completion. There is a canonical inclusion $\mathcal{S}^{m+1}_\sigma \subset \mathcal{S}^m_\sigma$, and the perturbed Schwartz space is defined as $\mathcal{S}_\sigma = \bigcap_m \mathcal{S}^m_\sigma$, endowed with the corresponding Fréchet topology. In particular, $\mathcal{S}_0$ is the usual Schwartz space $\mathcal{S}$. Like in the case of $\mathcal{S}$, there are direct sum decompositions into subspaces of even and odd functions, $\mathcal{S}^m_\sigma = \mathcal{S}^m_{\sigma,\text{even}} \oplus \mathcal{S}^m_{\sigma,\text{odd}}$ for each $m \in \mathbb{N}$, and $\mathcal{S}_\sigma = \mathcal{S}_{\sigma,\text{even}} \oplus \mathcal{S}_{\sigma,\text{odd}}$.

When $\sigma < 0$, the spaces of even and odd functions are considered separately. Let

$$\|\phi\|_{\mathcal{S}^m_\sigma} = \sum_{i+j\leq m,\ i+j \text{ even}} \sup_x |x^i (D^j_\sigma \phi)(x)| + \sum_{i+j\leq m,\ i+j \text{ odd}} \sup_{x\neq 0} |x|^{\sigma/2} |x^i (D^j_\sigma \phi)(x)| \tag{54}$$

for $\phi \in C^\infty_{\text{even}}$, and let

$$\|\phi\|_{\mathcal{S}^m_\sigma} = \sum_{i+j\leq m,\ i+j \text{ even}} \sup_{x\neq 0} |x|^{\sigma/2} |x^i (D^j_\sigma \phi)(x)| + \sum_{i+j\leq m,\ i+j \text{ odd}} \sup_x |x^i \, (D^j_\sigma \phi)(x)| \tag{55}$$

for $\phi \in C^\infty_{\text{odd}}$. These expressions define a norm $\| \ \|_{\mathcal{S}^m_\sigma}$ on the linear spaces of functions ϕ in C^∞_{odd} and C^∞_{even} with $\|\phi\|_{\mathcal{S}^m_\sigma} < \infty$. The corresponding Banach space completions will be denoted by $\mathcal{S}^m_{\sigma,\text{odd}}$ and $\mathcal{S}^m_{\sigma,\text{even}}$. Let $\mathcal{S}^m_\sigma = \mathcal{S}^m_{\sigma,\text{even}} \oplus \mathcal{S}^m_{\sigma,\text{odd}}$, which is also a Banach space by considering *e.g.* the norm, also denoted by $\| \ \|_{\mathcal{S}^m_\sigma}$, defined by the maximum of the norms on both components. There are canonical inclusions $\mathcal{S}^{m+1}_\sigma \subset \mathcal{S}^m_\sigma$, and let $\mathcal{S}_\sigma = \bigcap_m \mathcal{S}^m_\sigma$, endowed with the corresponding Fréchet topologies. We have $\mathcal{S}_\sigma = \mathcal{S}_{\sigma,\text{even}} \oplus \mathcal{S}_{\sigma,\text{odd}}$ for $\mathcal{S}_{\sigma,\text{even}} = \bigcap_m \mathcal{S}^m_{\sigma,\text{even}}$ and $\mathcal{S}_{\sigma,\text{odd}} = \bigcap_m \mathcal{S}^m_{\sigma,\text{odd}}$.

From these definitions, it easily follows that $\mathcal{S}_\sigma$ consists of functions which are C^∞ on $\mathbb{R}\setminus\{0\}$ but *a priori* possibly not even defined at zero, and $\mathcal{S}_\sigma^m\cap C^\infty$ is dense in $\mathcal{S}_\sigma^m$ for all m; thus $\mathcal{S}_\sigma\cap C^\infty$ is dense in $\mathcal{S}_\sigma$.

Obviously, Σ defines a bounded operator on each $\mathcal{S}_\sigma^m$. It is also easy to see that D_σ defines a bounded operator $\mathcal{S}_\sigma^{m+1}\to\mathcal{S}_\sigma^m$ for any m; notice that, when $\sigma<0$, the role played by the parity of $i+j$ fits well to prove this property. Similarly, x defines a bounded operator $\mathcal{S}_\sigma^{m+1}\to\mathcal{S}_\sigma^m$ for any m because

$$[D_\sigma^j,x]=\begin{cases}jD_\sigma^{j-1} & \text{if } j \text{ is even}\\ (j+\Sigma)D_\sigma^{j-1} & \text{if } j \text{ is odd}\end{cases}$$

by (14) and (15). So B and B^{*_σ} define bounded operators $\mathcal{S}_\sigma^{m+1}\to\mathcal{S}_\sigma^m$ too, and J defines a bounded operator $\mathcal{S}_\sigma^{m+2}\to\mathcal{S}_\sigma^m$. Therefore D_σ, x, Σ, B, B^{*_σ} and J define continuous operators on $\mathcal{S}_\sigma$. When these operators are considered with domain $\mathcal{S}_\sigma$ instead of $\mathcal{S}$, the equations (14)–(21) hold as well. Moreover the operators D_σ and x interchange $\mathcal{S}_{\sigma,\mathrm{even}}$ and $\mathcal{S}_{\sigma,\mathrm{odd}}$.

Proposition 7.1. *$\mathcal{S}_\sigma=\mathcal{S}$ as Fréchet spaces.*

In order to prove Proposition 7.1, we introduce an intermediate weakly perturbed Schwartz space $\mathcal{S}_{w,\sigma}$. Like $\mathcal{S}_\sigma$, it is defined as a Fréchet space of the form $\mathcal{S}_{w,\sigma}=\bigcap_m\mathcal{S}_{w,\sigma}^m$, where each $\mathcal{S}_{w,\sigma}^m$ is the Banach space defined like $\mathcal{S}_\sigma^m$ by using $\frac{d}{dx}$ instead of D_σ in the right hand sides of (53)–(55). The notation $\|\ \|_{\mathcal{S}_{w,\sigma}^m}$ will be used for the norm of $\mathcal{S}_{w,\sigma}^m$. As before, $\mathcal{S}_{w,\sigma}$ consists of functions which are C^∞ on $\mathbb{R}\setminus\{0\}$ but *a priori* possibly not even defined at zero, $\mathcal{S}_{w,\sigma}\cap C^\infty$ is dense in $\mathcal{S}_{w,\sigma}$, and there is a canonical decomposition $\mathcal{S}_{w,\sigma}=\mathcal{S}_{w,\sigma,\mathrm{even}}\oplus\mathcal{S}_{w,\sigma,\mathrm{odd}}$ given by the subspaces of even and odd functions, and $\frac{d}{dx}$ and x define continuous operators on $\mathcal{S}_{w,\sigma}$, which interchange $\mathcal{S}_{w,\sigma,\mathrm{even}}$ and $\mathcal{S}_{w,\sigma,\mathrm{odd}}$.

Lemma 7.2. *$\mathcal{S}=\mathcal{S}_{w,\sigma}$ as Fréchet spaces.*

Proof of Lemma 7.2 when $\sigma\geq0$. This follows from the following assertions.

Claim 2. $\mathcal{S}^{m+\lceil\sigma/2\rceil}\subset\mathcal{S}_{w,\sigma}^m$ continuously for each $m\in\mathbb{N}$.

Claim 3. For each $m\in\mathbb{N}$, there is some $m'\in\mathbb{N}$ such that $\mathcal{S}_{w,\sigma}^{m'}\subset\mathcal{S}^m$ continuously.

To prove Claim 2, let $\phi\in\mathcal{S}$. For all i and j, we have

$$|x|^{\sigma/2}|x^i\phi^{(j)}(x)|\leq|x^{i+\lceil\sigma/2\rceil}\phi^{(j)}(x)|$$

for $|x|\geq1$, and

$$|x|^{\sigma/2}|x^i\phi^{(j)}(x)|\leq|x^i\phi^{(j)}(x)|$$

for $|x|\leq1$. So

$$\|\phi\|_{\mathcal{S}_{w,\sigma}^m}\leq\|\phi\|_{\mathcal{S}^{m+\lceil\sigma/2\rceil}}$$

for all m.

To prove Claim 3, let $\phi \in \mathcal{S}_{w,\sigma}$. For all i and j,

$$|x^i\phi^{(j)}(x)| \le |x|^{\sigma/2}|x^i\phi^{(j)}(x)| \tag{56}$$

for $|x| \ge 1$. It remains to prove an inequality of this type for $|x| \le 1$, which is the only difficult part of the proof. It will be a consequence of the following assertion.

Claim 4. For each $n \in \mathbb{N}$, there are finite families of real numbers, $c^n_{a,b}$, $d^n_{k,\ell}$ and $e^n_{u,v}$, where the indices a, b, k, ℓ, u and v run in finite subsets of $\mathbb{N}$, such that all indices k are $\ge n$ and

$$\phi(x) = \sum_{a,b} c^n_{a,b}x^a\phi^{(b)}(1) + \sum_{k,\ell} d^n_{k,\ell}x^k\phi^{(\ell)}(x) + \sum_{u,v} e^n_{u,v}x^u \int_x^1 t^n\phi^{(v)}(t)\,dt$$

for all $\phi \in C^\infty$.

Assuming that Claim 4 is true, the proof of Claim 3 can be completed as follows. Let $\phi \in \mathcal{S}_{w,\sigma}$ and set $n = \lceil\sigma/2\rceil$. For $|x| \le 1$, according to Claim 4,

$$\begin{aligned}|\phi(x)| &\le \sum_{a,b}|c^n_{a,b}||\phi^{(b)}(1)| + \sum_{k,\ell}|d^n_{k,\ell}||x^k\phi^{(\ell)}(x)| \\ &\quad + \sum_{u,v}|e^n_{u,v}|2\max_{|t|\le 1}|t^n\phi^{(v)}(t)| \\ &\le \sum_{i,j}|c^n_{a,b}||\phi^{(b)}(1)| + \sum_{k,\ell}|d^n_{k,\ell}||x|^{\sigma/2}|\phi^{(\ell)}(x)| \\ &\quad + \sum_{u,v}|e^n_{u,v}|2\max_{|t|\le 1}|t|^{\sigma/2}|\phi^{(v)}(t)|\ .\end{aligned}$$

Let $m, i, j \in \mathbb{N}$ with $i + j \le m$. By applying the above inequality to the function $x^i\phi^{(j)}$, and expressing each derivative $(x^i\phi^{(j)})^{(r)}$ as a linear combination of functions of the form $x^p\phi^{(q)}$ with $p + q \le i + j + r$, it follows that there is some $C \ge 1$, depending only on σ and m, such that

$$|x^i\phi^{(j)}(x)| \le C\,\|\phi\|_{\mathcal{S}^{i+j+M}_{w,\sigma}} \tag{57}$$

for $|x| \le 1$, where M is the maximum of the indices b, ℓ and v. By (56) and (57),

$$\|\phi\|_{\mathcal{S}^m} \le C\,\|\phi\|_{\mathcal{S}^{m'}_{w,\sigma}}$$

with $m' = m + M$.

Now, let us prove Claim 4. By induction on n and using integration by parts, it is easy to prove that

$$\int_x^1 t^n\phi^{(n+1)}(t)\,dt = \sum_{r=0}^n(-1)^{n-r}\frac{n!}{r!}\left(\phi^{(r)}(1) - x^r\phi^{(r)}(x)\right)\ . \tag{58}$$

This shows directly Claim 4 for $n \in \{0, 1\}$. Proceeding by induction, let $n > 1$ and assume that Claim 4 holds for $n-1$. By (58), it is enough to find appropriate expressions of $x^r\phi^{(r)}(x)$ for $0<r<n$. For that purpose, apply Claim 4 for $n-1$ to each function $\phi^{(r)}$, and multiply the resulting equality by x^r to get

$$x^r\phi^{(r)}(x)=\sum_{a,b}c^{n-1}_{a,b}x^{r+a}\phi^{(r+b)}(1)+\sum_{k,\ell}d^{n-1}_{k,\ell}x^{r+k}\phi^{(r+\ell)}(x)+\sum_{u,v}e^{n-1}_{u,v}x^{r+u}\int_x^1 t^{n-1}\phi^{(r+v)}(t)\,dt\ ,$$

where a, b, k, ℓ, u and v run in finite subsets of $\mathbb{N}$ with $k\geq n-1$. In this expression, the exponents $r+k$ are $\geq n$, and therefore it only remains to rise the exponent of t by a unit in the integrals of the last sum. Once more, integration by parts makes the job:

$$\int_x^1 t^n\phi^{(r+v+1)}(t)\,dt=\phi^{(r+v)}(1)-x^n\phi^{(r+v)}(x)-n\int_x^1 t^{n-1}\phi^{(r+v)}\,dt\ . \qquad \square$$

Remark 2. In Claim 4, let $M=M_n$ denote the maximum of the indices b, ℓ and v. Notice that its proof shows that M_n is reached with the indices v, and satisfies $M_0=1$ and $M_n=M_{n-1}+n$ for $n>0$; *i.e.*, $M_n=1+\frac{n(n+1)}{2}$. So we can take

$$m'=m+1+\frac{\lceil\sigma/2\rceil\left(\lceil\sigma/2\rceil+1\right)}{2}$$

in Claim 3.

Proof of Lemma 7.2 when $\sigma<0$. Like in the above case, this is a consequence of the following assertions.

Claim 5. $\mathcal{S}^{m+1}_{w,\sigma}\subset\mathcal{S}^m$ continuously for each $m\in\mathbb{N}$.

Claim 6. $\mathcal{S}^{m+2}\subset\mathcal{S}^m_{w,\sigma}$ continuously for each $m\in\mathbb{N}$.

To prove Claim 5, let $i,j\in\mathbb{N}$ such that $i+j\leq m$. Since

$$|x^if^{(j)}(x)|\leq\begin{cases}|x|^{\sigma/2}|x^i\phi^{(j)}(x)| & \text{if } 0<|x|\leq 1\\ |x|^{\sigma/2}|x^{i+1}\phi^{(j)}(x)| & \text{if } |x|\geq 1\ .\end{cases}$$

for any $\phi\in C^\infty$, we get $\|\phi\|_{\mathcal{S}^m}\leq\|\phi\|_{\mathcal{S}^{m+1}_{w,\sigma}}$.

Claim 6 is proved by induction on m. We have $\|\ \|_{\mathcal{S}^0_{w,\sigma}}=\|\ \|_{\mathcal{S}^0}$ on C^∞_{even}. On the other hand, for $\phi\in C^\infty_{\text{odd}}$ and $\psi=x^{-1}\phi\in C^\infty_{\text{even}}$, we get

$$|x|^{\sigma/2}|\phi(x)|\leq\begin{cases}|\psi(x)| & \text{if } 0<|x|\leq 1\\ |\phi(x)| & \text{if } |x|\geq 1\ .\end{cases}$$

So, by (12),

$$\|\phi\|_{\mathcal{S}^0_{w,\sigma}}\leq\max\{\|\phi\|_{\mathcal{S}^0},\|\psi\|_{\mathcal{S}^0}\}\leq\|\phi\|_{\mathcal{S}^1}\ .$$

Now, assume that $m > 0$ and that Claim 6 holds for $m - 1$. Let $i, j \in \mathbb{N}$ such that $i + j \leq m$, and let $\phi \in \mathcal{S}_{\text{even}}$. If $i = 0$ and j is odd, then $\phi^{(j)} \in \mathcal{S}_{\text{odd}}$. Thus there is some $\psi \in \mathcal{S}_{\text{even}}$ such that $\phi^{(j)} = x\psi$, obtaining

$$|x|^{\sigma/2}|\phi^{(j)}(x)| \leq \begin{cases} |\psi(x)| & \text{if } 0 < |x| \leq 1 \\ |\phi^{(j)}(x)| & \text{if } |x| \geq 1 \ . \end{cases}$$

If $i + j$ is odd and $i > 0$, then

$$|x|^{\sigma/2}|x^i\phi^{(j)}(x)| \leq \begin{cases} |x^{i-1}\phi^{(j)}(x)| & \text{if } 0 < |x| \leq 1 \\ |x^i\phi^{(j)}(x)| & \text{if } |x| \geq 1 \ . \end{cases}$$

Hence, by (12), there is some $C > 0$, independent of f, such that

$$\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \leq C \max\{\|\phi\|_{\mathcal{S}^m}, \|\psi\|_{\mathcal{S}^0}\} \leq C \max\big\{\|\phi\|_{\mathcal{S}^m}, \|\phi^{(j)}\|_{\mathcal{S}^1}\big\} \leq C\,\|\phi\|_{\mathcal{S}^{m+1}} \ .$$

Finally, let $\phi \in \mathcal{S}_{\text{odd}}$. There is some $\psi \in \mathcal{S}_{\text{even}}$ such that $\phi = x\psi$. If i is even $j = 0$, then

$$|x|^{\sigma/2}|x^i\phi(x)| \leq \begin{cases} |x^i\psi(x)| & \text{if } 0 < |x| \leq 1 \\ |x^i\phi(x)| & \text{if } |x| \geq 1 \ . \end{cases}$$

If $i + j$ is even and $j > 0$, then

$$|x|^{\sigma/2}|x^i\phi^{(j)}(x)| \leq \begin{cases} |x^i\psi^{(j)}(x)| + j|x|^{\sigma/2}|x^i\psi^{(j-1)}(x)| & \text{if } 0 < |x| \leq 1 \\ |x^{i+1}\psi^{(j)}(x)| + j|x|^{\sigma/2}|x^i\psi^{(j-1)}(x)| & \text{if } |x| \geq 1 \end{cases}$$

because

$$[\frac{d^j}{dx^j}, x] = j\frac{d^{j-1}}{dx^{j-1}} \ .$$

Therefore, by (12) and the induction hypothesis, there are some $C', C'' > 0$, independent of f, such that

$$\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \leq C' \max\big\{\|\phi\|_{\mathcal{S}^m}, \|\psi\|_{\mathcal{S}^{m+1}} + \|\psi\|_{\mathcal{S}^{m-1}_{w,\sigma}}\big\} \leq C''\,\|\phi\|_{\mathcal{S}^{m+2}} \ . \qquad \square$$

From Lemma 7.2, it follows that $x^{-1} : C^\infty_{\text{odd}} \to C^\infty_{\text{even}}$ defines a continuous operator $\mathcal{S}_{w,\sigma,\text{odd}} \to \mathcal{S}_{w,\sigma,\text{even}}$.

Lemma 7.3. *$\mathcal{S}_{w,\sigma} \subset \mathcal{S}_\sigma$ continuously.*

Proof. The result follows from the following assertion, which is shown by induction on m.

Claim 7. For each $m \in \mathbb{N}$, there is some $m_1 \in \mathbb{N}$ such that $\mathcal{S}^{m_1}_{w,\sigma} \subset \mathcal{S}^m_\sigma$ continuously.

We obviously have $\mathcal{S}^0_{w,\sigma} = \mathcal{S}^0 = \mathcal{S}^0_\sigma$ as Fréchet spaces. Now, take any $m > 0$, and assume that the result holds for $m - 1$.

For $\phi \in C^\infty_{\text{odd}}$, $i + j \leq m$ with $j > 0$ and $x \in \mathbb{R}$, we have

$$|x^iD^j_\sigma\phi(x)| \leq |x^iD^{j-1}_\sigma\phi'(x)| + \sigma|x^iD^{j-1}_\sigma x^{-1}\phi(x)| \ ,$$

So there is some $C > 0$, independent of ϕ, such that

$$\|\phi\|_{\mathcal{S}_\sigma^m} \le C\big(\|\phi'\|_{\mathcal{S}_\sigma^{m-1}} + \|x^{-1}\phi\|_{\mathcal{S}_\sigma^{m-1}}\big) \ ,$$

By the induction hypothesis, it follows that there are some $C' > 0$ and some $m_0 \in \mathbb{N}$, independent of ϕ, so that

$$\|\phi\|_{\mathcal{S}_\sigma^m} \le C'\big(\|\phi'\|_{\mathcal{S}_{w,\sigma}^{m_0}} + \|x^{-1}\phi\|_{\mathcal{S}_{w,\sigma}^{m_0}}\big) \ .$$

Since x^{-1} and $\frac{d}{dx}$ define continuous operators on $\mathcal{S}_{w,\sigma,\mathrm{odd}} \to \mathcal{S}_{w,\sigma,\mathrm{even}}$, we get that there is some $C'' > 0$ and some $m_1 \in \mathbb{N}$, independent of ϕ, such that

$$\|\phi\|_{\mathcal{S}_\sigma^m} \le C'' \, \|\phi\|_{\mathcal{S}_{w,\sigma}^{m_1}} \ .$$

For $\phi \in C^\infty_{\mathrm{even}}$, and i, j and x as above, we have

$$|x^i D_\sigma^j \phi(x)| = |x^i D_\sigma^{j-1} \phi'(x)| \ .$$

So we similarly get

$$\|\phi\|_{\mathcal{S}_\sigma^m} \le C \, \|\phi'\|_{\mathcal{S}_\sigma^{m-1}} \le C' \, \|\phi'\|_{\mathcal{S}_{w,\sigma}^{m_0}} \le C'' \, \|\phi\|_{\mathcal{S}_{w,\sigma}^{m_1}} \ . \qquad \square$$

To complete the proof of Proposition 7.1, we use the following lemma that will be proved in the next section in a rather indirect way.

Lemma 7.4. *The operator $x^{-1} : C^\infty_{\mathrm{odd}} \to C^\infty_{\mathrm{even}}$ defines a continuous operator $x^{-1} : \mathcal{S}_{\sigma,\mathrm{odd}} \to \mathcal{S}_{\sigma,\mathrm{even}}$.*

The following result can be proved like Lemma 7.3 by using Lemma 7.4.

Lemma 7.5. *$\mathcal{S}_\sigma \subset \mathcal{S}_{w,\sigma}$ continuously.*

After showing Lemma 7.4, Proposition 7.1 will follow from Lemmas 7.2, 7.3 and 7.5.

8. Perturbed Sobolev spaces

Since Lemma 7.4 is not proved yet, we still do not know that $\mathcal{S}_\sigma = \mathcal{S}$ as Fréchet spaces. We only know that $\mathcal{S} \subset \mathcal{S}_\sigma$ by Lemmas 7.2 and 7.3. Observe also that $\mathcal{S}_\sigma \subset L^2_\sigma$. When D_σ is considered with domain $\mathcal{S}_\sigma$ instead of $\mathcal{S}$, the proof of Lemma 4.1 works exactly the same to get the following.

Lemma 8.1. *With domain $\mathcal{S}_\sigma$, $-D_\sigma$ is adjoint of D_σ in L^2_σ.*

Corollary 8.2. *With domain $\mathcal{S}_\sigma$, $B^{*\sigma}$ is adjoint of B in L^2_σ, and J is symmetric in L^2_σ.*

Corollary 8.3. *With domain $\mathcal{S}_\sigma$, J is essentially self-adjoint in L^2_σ, and it has the same self-adjoint extension as J with domain $\mathcal{S}$.*

Proof. To be precise, let J and $\widetilde{J}$ denote the operators defined by J with domains $\mathcal{S}$ and $\mathcal{S}_\sigma$, respectively, and let $\overline{J}$ denote the closure of J in L^2_σ, which is its self-adjoint extension. Since $\mathcal{S} \subset \mathcal{S}_\sigma$ and by Corollary 8.2, we have $J \subset \widetilde{J} \subset \overline{J}$, and the result follows. $\square$

For each $m \in \mathbb{N}$, let W^m_σ be the Hilbert space completion of $\mathcal{S}$ with respect to the scalar product $\langle\ ,\ \rangle_{W^m_\sigma}$ defined by

$$\langle \phi, \psi \rangle_{W^m_\sigma} = \langle (1+J)^m \phi, \psi \rangle_\sigma \ .$$

The corresponding norm will be denoted by $\|\ \|_{W^m_\sigma}$, whose equivalence class is independent of the parameter s used to define J. In particular, $W^0_\sigma = L^2_\sigma$. As usual, $W^{m'}_\sigma \subset W^m_\sigma$ when $m' > m$, and let $W^\infty_\sigma = \bigcap_m W^m_\sigma$, which is endowed with the induced Fréchet topology. Once more, there are direct sum decompositions into subspaces of even and odd (generalized) functions, $W^m_\sigma = W^m_{\sigma,\mathrm{even}} \oplus W^m_{\sigma,\mathrm{odd}}$ and $W^\infty_\sigma = W^\infty_{\sigma,\mathrm{even}} \oplus W^\infty_{\sigma,\mathrm{odd}}$.

By Corollary 4.6, the space W^m_σ can be defined for any real number m because $(1+J)^m$ is given by the spectral theorem. Also, according to Corollary 8.3, $\mathcal{S}_\sigma$ could be used instead of $\mathcal{S}$ in the definition of each W^m_σ.

Obviously, J defines a bounded operator $W^{m+2}_\sigma \to W^m_\sigma$ for each $m \geq 0$, and therefore a continuous operator on W^∞_σ. Moreover, by (21), Σ defines a bounded operator on each W^m_σ, and therefore a continuous operators on W^∞_σ.

Lemma 8.4. *B and $B^{*\sigma}$ define bounded operators $W^{m+1}_\sigma \to W^m_\sigma$ for each m.*

Proof. This follows by induction on m. For $m = 0$, by (18) and Corollary 4.2, for each $\phi \in \mathcal{S}$,

$$\|B\phi\|^2_\sigma = \|B^{*\sigma}\phi\|^2_\sigma = \langle B^{*\sigma} B\phi, \phi \rangle_\sigma = \langle (J - (1+\Sigma)s)\phi, \phi \rangle_\sigma \leq C_0 \|\phi\|^2_{W^1_\sigma}$$

for some $C_0 > 0$ independent of ϕ. It follows that B and $B^{*\sigma}$ define bounded operators $W^1_\sigma \to L^2_\sigma$.

Now take $m > 0$ and assume that there are some $C_{m-1}, C'_{m-1} > 0$ so that

$$\|B\phi\|^2_{W^{m-1}_\sigma} \leq C_{m-1} \|\phi\|^2_{W^m_\sigma} \ , \quad \|B^{*\sigma}\phi\|^2_{W^{m-1}_\sigma} \leq C'_{m-1} \|\phi\|^2_{W^m_\sigma}$$

for all $\phi \in \mathcal{S}$. Then, by (19),

$$\begin{aligned}
\|B\phi\|^2_{W^m_\sigma} &= \langle (1+J)B\phi, B\phi \rangle_{W^{m-1}_\sigma} \\
&= \|B\phi\|^2_{W^{m-1}_\sigma} + \langle JB\phi, B\phi \rangle_{W^{m-1}_\sigma} \\
&= (1-2s) \|B\phi\|^2_{W^{m-1}_\sigma} + \langle BJ\phi, B\phi \rangle_{W^{m-1}_\sigma} \\
&\leq (1-2s) \|B\phi\|^2_{W^{m-1}_\sigma} + \|BJ\phi\|_{W^{m-1}_\sigma} \|B\phi\|_{W^{m-1}_\sigma} \\
&\leq C_{m-1}((1-2s) \|\phi\|^2_{W^{m-1}_\sigma} + \|J\phi\|_{W^m_\sigma} \|\phi\|_{W^m_\sigma}) \\
&\leq C_m \|\phi\|^2_{W^{m+1}_\sigma}
\end{aligned}$$

for some $C_m > 0$ independent of ϕ. Similarly,

$$\|B^{*\sigma}\phi\|^2_{W^m_\sigma} \leq C'_m \|\phi\|^2_{W^{m+1}_\sigma}$$

for some $C'_m > 0$ independent of ϕ. □

Remark 3. $B^{*\sigma}$ is not adjoint of B in W^m_σ for $m > 0$.

J and Σ preserve $W^m_{\sigma,\mathrm{even}}$ and $W^m_{\sigma,\mathrm{odd}}$ for each m, whilst B and $B^{*\sigma}$ interchange these subspaces.

The motivation of our tour through perturbed Schwartz spaces is the following embedding results; the second one is a version of the Sobolev embedding theorem.

Proposition 8.5. *For integers $m, m' \geq 0$, if $m' - m > 1/2$, then $\mathcal{S}^{m'}_\sigma \subset W^m_\sigma$ continuously.*

Proposition 8.6. *If $m' - m > 1$, then $W^{m'}_\sigma \subset \mathcal{S}^m_\sigma$ continuously.*

Corollary 8.7. *$\mathcal{S}_\sigma = W^\infty_\sigma$ as Fréchet spaces for $\sigma \geq 0$.*

For each non-commutative polynomial p, the continuous operators $p(B, B^{*\sigma})$ and $p(B^{*\sigma}, B)$ on $\mathcal{S}_\sigma$ are adjoint from each other in L^2_σ by Corollary 8.2. Thus $p(B, B^{*\sigma})$ is symmetric in L^2_σ if and only if $p(B, B^{*\sigma}) = p(B^{*\sigma}, B)$; in this case, we can assume that p is symmetric. The following lemma will be used in the proof of Proposition 8.5

Lemma 8.8. *For each non-negative integer m, we have*

$$(1 + J)^m = \sum_a q_a(B^{*\sigma}, B) q_a(B, B^{*\sigma})$$

for some finite family of homogeneous non-commutative polynomials q_a of degree $\leq m$.

Proof. The result follows easily from the following assertions.

Claim 8. If m is even, then $J^m = g_m(B, B^{*\sigma})^2$ for some symmetric homogeneous non-commutative polynomial g_m of degree m.

Claim 9. If m is odd, then

$$J^m = g_{m,1}(B^{*\sigma}, B) g_{m,1}(B, B^{*\sigma}) + g_{m,2}(B^{*\sigma}, B) g_{m,2}(B, B^{*\sigma})$$

for some homogeneous non-commutative polynomials $g_{m,1}$ and $g_{m,2}$ of degree m.

If m is even, then $J^{m/2} = g_m(B, B^{*\sigma})$ for some symmetric homogeneous non-commutative polynomial g_m of degree $\leq m$ by (18) and Corollary 4.2. So $J^m = g_m(B, B^{*\sigma})^2$, showing Claim 8.

If m is odd, then write $J^{\lfloor m/2 \rfloor} = f_m(B, B^{*\sigma})$ as above for some symmetric homogeneous non-commutative polynomial f_m of degree $\leq m-1$. Then, by (18),

$$J^m = \frac{1}{2} f_m(B, B^{*\sigma})(BB^{*\sigma} + B^{*\sigma}B) f_m(B, B^{*\sigma}) \; .$$

Thus Claim 9 follows with

$$g_{m,1}(B, B^{*\sigma}) = \frac{1}{\sqrt{2}} B^{*\sigma} f_m(B, B^{*\sigma}) \; , \quad g_{m,2}(B, B^{*\sigma}) = \frac{1}{\sqrt{2}} B f_m(B, B^{*\sigma}) \; . \quad \square$$

Proof of Proposition 8.5 when $\sigma \geq 0$. By the definitions of B and $B^{*\sigma}$, for each non-commutative polynomial p of degree $\leq m'$, there is some $C_p > 0$ such that $|x|^{\sigma/2}|(p(x, B, B^{*\sigma})\phi|$ is uniformly bounded by $C_p \|\phi\|_{\mathcal{S}_\sigma^{m'}}$ for all $\phi \in \mathcal{S}_\sigma$. Write

$$(1 + J)^m = \sum_a q_a(B^{*\sigma}, B)q_a(B, B^{*\sigma})$$

according to Lemma 8.8, and let

$$\bar{q}_a(x, B, B^{*\sigma}) = x^{m'-m}q_a(B, B^{*\sigma}) \ .$$

Then, for each $\phi \in \mathcal{S}_\sigma$,

$$\begin{aligned}
\|\phi\|^2_{W^m_\sigma} &= \sum_a \langle q_a(B, B^{*\sigma})\phi, q_a(B, B^{*\sigma})\phi\rangle_\sigma \\
&\leq 2\sum_a \int_0^\infty |(q_a(B, B^{*\sigma})\phi)(x)|^2 x^\sigma \, dx \\
&\leq 2\sum_a \left(C^2_{q_a} + C^2_{\bar{q}_a} \int_1^\infty x^{-2(m'-m)} \, dx\right) \|\phi\|^2_{\mathcal{S}^{m'}_\sigma} \ ,
\end{aligned}$$

where the integral is finite because $-2(m' - m) < -1$. □

Proof of Proposition 8.5 when $\sigma < 0$. Now, for each homogeneous non-commutative polynomial p of degree $d \leq m'$, there is some $C_p > 0$ such that:

- $|(p(x, B, B^{*\sigma})\phi|$ is uniformly bounded by $C_p \|\phi\|_{\mathcal{S}^{m'}_{\sigma,\mathrm{even}}}$ for all $\phi \in \mathcal{S}_{\sigma,\mathrm{even}}$ if d is even, and by $C_p \|\phi\|_{\mathcal{S}^{m'}_{\sigma,\mathrm{odd}}}$ for all $\phi \in \mathcal{S}_{\sigma,\mathrm{odd}}$ if d is odd;
- $|x|^{\sigma/2}|(p(x, B, B^{*\sigma})\phi|$ is uniformly bounded by $C_p \|\phi\|_{\mathcal{S}^{m'}_{\sigma,\mathrm{odd}}}$ for all $\phi \in \mathcal{S}_{\sigma,\mathrm{odd}}$ if d is even, and by $C_p \|\phi\|_{\mathcal{S}^{m'}_{\sigma,\mathrm{even}}}$ for all $\phi \in \mathcal{S}_{\sigma,\mathrm{even}}$ if d is odd.

With the notation of Lemma 8.8, let d_a denote the degree of each homogenous non-commutative polynomial q_a, and let $\bar{q}_a(x, B, B^{*\sigma})$ be defined like in the above case. Then, as above,

$$\begin{aligned}
\|\phi\|^2_{W^m_\sigma} \leq 2 &\sum_{a \text{ with } d_a \text{ even}} \left(C^2_{q_a} \int_0^1 x^\sigma \, d\sigma + C^2_{\bar{q}_a} \int_1^\infty x^{-2(m'-m)+\sigma} \, dx\right) \|\phi\|^2_{\mathcal{S}^{m'}_{\sigma,\mathrm{even}}} \\
&+ 2 \sum_{a \text{ with } d_a \text{ odd}} \left(C^2_{q_a} + C^2_{\bar{q}_a} \int_1^\infty x^{-2(m'-m)} \, dx\right) \|\phi\|^2_{\mathcal{S}^{m'}_{\sigma,\mathrm{even}}}
\end{aligned}$$

for $\phi \in \mathcal{S}_{\sigma,\mathrm{even}}$, and

$$\begin{aligned}
\|\phi\|^2_{W^m_\sigma} \leq 2 &\sum_{a \text{ with } d_a \text{ even}} \left(C^2_{q_a} + C^2_{\bar{q}_a} \int_1^\infty x^{-2(m'-m)} \, dx\right) \|\phi\|^2_{\mathcal{S}^{m'}_{\sigma,\mathrm{odd}}} \\
&+ 2 \sum_{a \text{ with } d_a \text{ odd}} \left(C^2_{q_a} \int_0^1 x^\sigma \, d\sigma + C^2_{\bar{q}_a} \int_1^\infty x^{-2(m'-m)+\sigma} \, dx\right) \|\phi\|^2_{\mathcal{S}^{m'}_{\sigma,\mathrm{odd}}}
\end{aligned}$$

for $\phi \in \mathcal{S}_{\sigma,\mathrm{even}}$, where the integrals are finite because $-1 < \sigma < 0$ and $-2(m'-m) < -1$. □

Let $\mathcal{C}$ denote the space of rapidly decreasing sequences of real numbers. Recall that a sequence $c = (c_k) \in \mathbb{R}^{\mathbb{N}}$ is rapidly decreasing if

$$\|c\|_{\mathcal{C}_m} = \sup_k |c_k|(1+k)^m$$

is finite for all $m \geq 0$, and these expressions define norms $\| \ \|_{\mathcal{C}_m}$ on $\mathcal{C}$. Let $\mathcal{C}_m$ denote the completion of $\mathcal{C}$ with respect to $\| \ \|_{\mathcal{C}_m}$, which consists of the sequences $c \in \mathbb{R}^{\mathbb{N}}$ with $\|c\|_{\mathcal{C}_m} < \infty$. So $\mathcal{C} = \bigcap_m \mathcal{C}_m$ with the induced Fréchet topology. Let also ℓ^2_m denote the Hilbert space completion of $\mathcal{C}$ with respect to the scalar product $\langle \ , \ \rangle_{\ell^2_m}$ defined by

$$\langle c, c' \rangle_{\ell^2_m} = \sum_k c_k c'_k (1+k)^m$$

for $c = (c_k)$ and $c' = (c'_k)$. The corresponding norm will be denoted by $\| \ \|_{\ell^2_m}$. Thus ℓ^2_m is a weighted version of ℓ^2; in particular, $\ell^2_0 = \ell^2$. Let $\ell^2_\infty = \bigcap_m \ell^2_m$ with the corresponding Fréchet topology.

A sequence $c = (c_k)$ will be called even (respectively, odd) if $c_k = 0$ for all odd (respectively, even) k. We get the following direct sum decompositions into subspaces of even and odd sequences:

$$\mathcal{C}_m = \mathcal{C}_{m,\mathrm{even}} \oplus \mathcal{C}_{m,\mathrm{odd}} , \qquad \mathcal{C} = \mathcal{C}_{\mathrm{even}} \oplus \mathcal{C}_{\mathrm{odd}} ,$$
$$\ell^2_m = \ell^2_{m,\mathrm{even}} \oplus \ell^2_{m,\mathrm{odd}} , \qquad \ell^2_\infty = \ell^2_{\infty,\mathrm{even}} \oplus \ell^2_{\infty,\mathrm{odd}} .$$

Lemma 8.9. *$\ell^2_{2m} \subset \mathcal{C}_m$ and $\mathcal{C}_{m'} \subset \ell^2_m$ continuously if $2m' - m > 1$.*

Proof. It is easy to see that

$$\|c\|_{\mathcal{C}_m} \leq \|c\|_{\ell^2_{2m}} , \quad \|c\|_{\ell^2_m} \leq \|c\|_{\mathcal{C}_{m'}} \Big(\sum_k (1+k)^{m-2m'} \Big)^{1/2}$$

for any $c \in \mathcal{C}$, where the last series is convergent because $m - 2m' < -1$. □

Corollary 8.10. *$\ell^2_\infty = \mathcal{C}$ as Fréchet spaces.*

By Corollary 4.6, the "Fourier coefficients" mapping $\phi \mapsto (\langle \phi_k, \phi \rangle_\sigma)$ defines a quasi-isometry $W^m_\sigma \to \ell^2_m$ for all m, and therefore an isomorphism $W^\infty_\sigma \to \mathcal{C}$ of Fréchet espaces. Notice that the "Fourier coefficients" mapping can be restricted to the even and odd subspaces.

Corollary 8.11. *Any $\phi \in L^2_\sigma$ is in $\mathcal{S}_\sigma$ if and only if its "Fourier coefficients" $\langle \phi_k, \phi \rangle_\sigma$ are rapidly degreasing on k.*

Proof. By Corollary 8.7, the "Fourier coefficients" mapping defines an isomorphism $\mathcal{S}_\sigma \to \mathcal{C}$ of Fréchet spaces. □

There is also a version of the Rellich theorem stated as follows.

Proposition 8.12. *The operator* $W_\sigma^{m'} \hookrightarrow W_\sigma^m$ *is compact for* $m' > m$.

By using the "Fourier coefficients" mapping, Proposition 8.12 follows from the following lemma (see *e.g.* [8, Theorem 5.8]).

Lemma 8.13. *The operator* $\ell^2_{m'} \hookrightarrow \ell^2_m$ *is compact for* $m' > m$.

Proof of Proposition 8.6. For $\phi \in \mathcal{S}_\sigma$, its "Fourier coefficients" $c_k = \langle \phi_k, \phi \rangle_\sigma$ form a sequence $c = (c_k)$ in $\mathcal{C}$, and

$$\sum_k |c_k|(1+k)^{m/2} \le \|c\|_{\ell^2_{m'}} \Big(\sum_k (1+k)^{m-m'} \Big)^{1/2}$$

by Cauchy-Schwartz inequality, where the last series is convergent since $m - m' < -1$. Therefore

$$\sum_k |c_k|(1+k)^{m/2} \le C\, \|\phi\|_{W_\sigma^{m'}} \tag{59}$$

for some $C > 0$ independent of ϕ.

On the other hand, for all $i, j \in \mathbb{N}$ with $i + j \le m$, there is some homogeneus non-commutative polynomial $p_{i,j}$ of degree $i + j$ such that $x^i D_\sigma^j = p_{i,j}(B, B^{*_\sigma})$. Then, by (23) and Lemma 4.3,

$$|\langle \phi_k, x^i D_\sigma^j \phi \rangle_\sigma| \le C_{i,j}(1+k)^{m/2} \sum_{|\ell - k| \le m} |c_\ell| \tag{60}$$

for some $C_{i,j} > 0$ independent of ϕ.

Now suppose that $\sigma \ge 0$. By (59), (60) and Theorem 6.6-(ii), there is some $C'_{i,j} > 0$ independent of ϕ and x so that

$$\begin{aligned} |x|^{\sigma/2}|x^i D_\sigma^j \phi(x)| &\le |x|^{\sigma/2} \sum_k |\langle \phi_k, x^i D_\sigma^j \phi \rangle_\sigma||\phi_k(x)| \\ &= \sum_k |\langle \phi_k, x^i D_\sigma^j \phi \rangle_\sigma||\xi_k(x)| \le C'_{i,j}\, \|\phi\|_{W_\sigma^{m'}} \end{aligned} \tag{61}$$

for all x. Hence $\|\phi\|_{\mathcal{S}_\sigma^m} \le C' \|\phi\|_{W_\sigma^{m'}}$ for some $C' > 0$ independent of ϕ.

Finally assume that $\sigma < 0$. By (59), (60) and Corollary 6.7, there is some $C'_{i,j} > 0$, independent of ϕ and x, so that

$$|x^i D_\sigma^j \phi(x)| \le \sum_k |\langle \phi_k, x^i D_\sigma^j \phi \rangle_\sigma||\phi_k(x)| \le C'_{i,j}\, \|\phi\|_{W_\sigma^{m'}}$$

for all x if $\phi \in \mathcal{S}_{\sigma,\mathrm{even}}$ and $i+j$ is even, or $\phi \in \mathcal{S}_{\sigma,\mathrm{odd}}$ and $i+j$ is odd. On the other hand, by (59), (60) and Theorem 6.6-(ii), there is some $C''_{i,j} > 0$, independent of ϕ and x, such that, like in (61),

$$|x|^{\sigma/2}|x^i D_\sigma^j \phi(x)| \le C''_{i,j}\, \|\phi\|_{W_\sigma^{m'}}$$

for all $x \neq 0$ if $\phi \in \mathcal{S}_{\sigma,\text{odd}}$ and $i+j$ is even, or $\phi \in \mathcal{S}_{\sigma,\text{even}}$ and $i+j$ is odd. Therefore there is some $C' > 0$ such that $\|\phi\|_{\mathcal{S}^m_{\sigma,\text{even}}} \le C'\|\phi\|_{W^{m'}_\sigma}$ for all $\phi \in \mathcal{S}_{\sigma,\text{even}}$, and $\|\phi\|_{\mathcal{S}^m_{\sigma,\text{odd}}} \le C'\|\phi\|_{W^{m'}_\sigma}$ for all $\phi \in \mathcal{S}_{\sigma,\text{odd}}$. □

Proof of Lemma 7.4. As suggested by (30), consider the mapping $c = (c_k) \mapsto d = (d_\ell)$, where c is odd and d is even with

$$d_\ell = \sum_{k\in\{\ell+1,\ell+3,\dots\}} (-1)^{\frac{k-\ell-1}{2}} \sqrt{\frac{(k-1)(k-3)\cdots(\ell+2)2s}{(k+\sigma)(k-2+\sigma)\cdots(\ell+1+\sigma)}}\, c_k$$

for ℓ even, assuming that this series is convergent. For $m'-m > 1$, it defines a bounded map $\Xi : \ell^2_{m',\text{odd}} \to \mathcal{C}_{m,\text{even}}$ because, by the Cauchy-Schwartz inequality,

$$\begin{aligned}
\|d\|_{\mathcal{C}_m} &= \sup_\ell \sum_{k\in\{\ell+1,\ell+3,\dots\}} \sqrt{\frac{(k-1)(k-3)\cdots(\ell+2)2s}{(k+\sigma)(k-2+\sigma)\cdots(\ell+1+\sigma)}}\, |c_k|(1+\ell)^m \\
&\le \sqrt{2s}\sup_\ell \sum_{k\in\{\ell+1,\ell+3,\dots\}} |c_k|(1+\ell)^m \\
&\le \sqrt{2s}\,\|c\|_{\ell^2_{m'}} \sup_\ell \Big(\sum_{k\in\{\ell+1,\ell+3,\dots\}} (1+k)^{-m'}(1+\ell)^m \Big)^{1/2} \\
&\le \sqrt{2s}\,\|c\|_{\ell^2_{m'}} \Big(\sum_k (1+k)^{m-m'} \Big)^{1/2} ,
\end{aligned}$$

where the last series is convergent since $m - m' < -1$. Then, by Propositions 8.5 and 8.6, Lemma 8.9, and using the "Fourier coefficients" mapping, we get the following composition of bounded maps:

$$\mathcal{S}^{m_1}_{\sigma,\text{odd}} \hookrightarrow W^{m_2}_{\sigma,\text{odd}} \to \ell^2_{m_2,\text{odd}} \xrightarrow{\Xi} \mathcal{C}_{m_3,\text{even}} \hookrightarrow \ell^2_{m_4,\text{even}} \to W^{m_4}_{\sigma,\text{even}} \hookrightarrow \mathcal{S}^m_{\sigma,\text{even}} ,$$

where

$$m_1 - m_2 > 1/2 , \quad m_2 - m_3 > 1 , \quad 2m_3 - m_4 > 1 , \quad m_4 - m > 1 . \tag{62}$$

By (30), this composite is an extension of the map $x^{-1} : \mathcal{S}_{\text{odd}} \to \mathcal{S}_{\text{even}}$. □

Remark 4. According to (62), it is enough to take $2m_1 \ge m + 10$ in the proof of Lemma 7.4.

Question 8.14. Is it possible to prove Lemma 7.4 without using (30) and the perturbed Sobolev spaces? It seems that (16) should be involved in a direct proof.

Since the proof of Proposition 7.1 is completed, Corollary 8.11 can be written as follows.

Corollary 8.15. *Any $\phi \in L^2_\sigma$ is in $\mathcal{S}$ if and only if its "Fourier coefficients" $\langle \phi_k, \phi\rangle_\sigma$ are rapidly degreasing on k.*

9. More general perturbations of the harmonic oscillator

More general perturbations of H can be obtained with conjugation of J by $|x|^a$ for arbitrary $a \in \mathbb{R}$, like we did in Section 6.1 for the case $a = \sigma/2$. For the sake of simplicity, we will consider the conjugations of the even and odd components of J separately, and acting on spaces of functions on $\mathbb{R}_+$. This will be also enough for the application indicated in Section 1.

Let J_{even} and J_{odd}, or more explicitly $J_{\sigma,\text{even}}$ and $J_{\sigma,\text{odd}}$, denote the restrictions of $J = J_\sigma$ to $\mathcal{S}_{\text{even}}$ and $\mathcal{S}_{\text{odd}}$, respectively. Since the function $|x|^\sigma$ is even, there is an orthogonal decomposition $L^2_\sigma = L^2_{\sigma,\text{even}} \oplus L^2_{\sigma,\text{odd}}$ as direct sum of subspaces of even and odd functions. By restriction to each of those components, we get obvious versions of Corollaries 4.6 and 8.15 for J_{even} in $L^2_{\sigma,\text{even}}$ and J_{odd} in $L^2_{\sigma,\text{odd}}$.

Let $\mathcal{S}_{\text{even},+}$ and $\mathcal{S}_{\text{odd},+}$ denote the linear subspaces of $C^\infty(\mathbb{R}_+)$ consisting of the restrictions to $\mathbb{R}_+$ of the functions in $\mathcal{S}_{\text{even}}$ and $\mathcal{S}_{\text{odd}}$, respectively. The restriction to $\mathbb{R}_+$ defines linear isomorphisms

$$\mathcal{S}_{\text{even}} \cong \mathcal{S}_{\text{even},+} \ , \quad \mathcal{S}_{\text{odd}} \cong \mathcal{S}_{\text{odd},+} \ , \tag{63}$$

and unitary isomorphisms

$$L^2_{\sigma,\text{even}} \cong L^2(\mathbb{R}_+, 2x^\sigma\, dx) \cong L^2_{\sigma,\text{odd}} \ . \tag{64}$$

Set $L^2_{\sigma,+} = L^2(\mathbb{R}_+, x^\sigma\, dx)$, whose scalar product is denoted by $\langle\ ,\ \rangle_{\sigma,+}$. The restriction of each ϕ_k to $\mathbb{R}_+$ will be denoted by $\phi_{k,+}$. Let $J_{\text{even},+}$ and $J_{\text{odd},+}$, or more explicitly $J_{\sigma,\text{even},+}$ and $J_{\sigma,\text{odd},+}$, denote the operators defined by J_{even} and J_{odd} on $\mathcal{S}_{\text{even},+}$ and $\mathcal{S}_{\text{odd},+}$ via the isomorphisms (63).

Going one step further, for any $a \in \mathbb{R}$, the operator (of multiplication by) x^a defines a unitary isomorphism $x^a : L^2_{\sigma,+} \to L^2_{\sigma-2a,+}$. Via this unitary isomorphism and (64), we get obvious versions of Corollaries 4.6 and 8.15 for the operators $x^a J_{\text{even},+} x^{-a}$ and $x^a J_{\text{odd},+} x^{-a}$ in $L^2_{\sigma-2a,+}$ with respective domains $x^a\, \mathcal{S}_{\text{even},+}$ and $x^a\, \mathcal{S}_{\text{odd},+}$. By using

$$\Big[\frac{d}{dx}, x^a\Big] = a x^{a-1} \ , \quad \Big[\frac{d^2}{dx^2}, x^a\Big] = 2a x^{a-1} \frac{d}{dx} + a(a-1) x^{a-2} \ , \tag{65}$$

it easily follows that all of those operators are of the form

$$P = H - c_1 x^{-1} \frac{d}{dx} + c_2 x^{-2}$$

for some $c_1, c_2 \in \mathbb{R}$.

Lemma 9.1. *For $a \in \mathbb{R}$, $\sigma > -1$ and P as above, we have $P = x^a J_{\sigma,\text{even},+} x^{-a}$ on $x^a\, \mathcal{S}_{\text{even},+}$ if and only if $a^2 - (1 - c_1)a - c_2 = 0$ and $\sigma = 2a + c_1$.*

Proof. By (65),

$$x^{-a} P x^a = -x^{-a} \frac{d^2}{dx^2} x^a + s x^2 - c_1 x^{-a-1} \frac{d}{dx} x^a + c_2 x^{-2} + c_3 s$$

$$= -\frac{d^2}{dx^2} - 2ax^{-1}\frac{d}{dx} - a(a-1)x^{-2} + sx^2$$
$$- c_1x^{-1}\frac{d}{dx} - c_1ax^{-2} + c_2x^{-2} + c_3s$$
$$= H - (2a+c_1)x^{-1}\frac{d}{dx} + (-a(a-1) - c_1a + c_2)x^{-2} + c_3s\ .$$

So

$$x^{-a}Px^a = H - (2a+c_1)x^{-1}\frac{d}{dx} + c_3s$$

if and only if $a^2 - (1-c_1)a - c_2 = 0$. □

Corollary 9.2. *If the conditions of Lemma 9.1 are satisfied, then P, with domain $x^a\,\mathcal{S}_{\mathrm{even},+}$, is essentially self-adjoint in $L^2_{c_1,+}$; its spectrum consists of the eigenvalues $(4k+1+\sigma)s$, for $k\in\mathbb{N}$, with multiplicity one and corresponding normalized eigenfunctions $x^a\phi_{2k,+}$; and any $\phi\in L^2_{c_1,+}$ is in $x^a\,\mathcal{S}_{\mathrm{even},+}$ if and only if its "Fourier coefficients" $\langle x^a\phi_{2k,+},\phi\rangle_{c_1,+}$ are rapidly decreasing on k.*

Remark 5. We may also consider an operator of the form

$$Q = H - c_1\frac{d}{dx}x^{-1} + c_2x^{-2}\ ,$$

but it is really of the same kind as P by (65):

$$Q = H - c_1x^{-1}\frac{d}{dx} + (c_2+c_1)x^{-2} + c_3s\ .$$

Remark 6. By using (65), it is easy to check that $J_{\sigma,\mathrm{odd},+} = xJ_{2+\sigma,\mathrm{even},+}x^{-1}$ on $\mathcal{S}_{\mathrm{odd},+} = x\,\mathcal{S}_{\mathrm{even},+}$ for all $\sigma > -1$. So no new operators are obtained by conjugating $J_{\sigma,\mathrm{odd},+}$ by powers of x.

The solutions

$$a = \frac{1-c_1 \pm \sqrt{(1-c_1)^2+4c_2}}{2} \tag{66}$$

of the polynomial equation of Lemma 9.1 are real if and only if

$$(1-c_1)^2 + 4c_2 \geq 0\ . \tag{67}$$

In this case,

$$\sigma = 2a + c_1 = 1 \pm \sqrt{(1-c_1)^2+4c_2}\ , \tag{68}$$

which is > -1 if and only if the positive square root is chosen, or if the negative square root is chosen and $(1-c_1)^2 + 4c_2 < 4$. If these conditions are fulfilled, then Corollary 9.2 can be applied to P.

Example 9.3. Suppose that

$$P = H - c_1x^{-1}\frac{d}{dx}\ ,$$

Thus P is has the form of $J_{c_1,\mathrm{even},+}$ if $c_1 > -1$; however, this inequality is not required *a priori*. Then (67) holds, and (66) means that $a \in \{0, 1 - c_1\}$.

In the case $a = 0$, we get $\sigma = c_1$ by (68). Then the condition $c_1 > -1$ is needed to apply Lemma 9.1, which simply asserts that $P = J_{c_1,\mathrm{even},+}$ on $\mathcal{S}_{\mathrm{even},+}$. So the statement of Corollary 9.2 becomes a direct consequence of Corollary 4.6 in this case.

Nevertheless, Corollary 9.2 gives new information in the case $a = 1 - c_1$. Then $\sigma = 2 - c_1$ by (68). Thus $\sigma > -1$ just when $c_1 < 3$ ($c_1 \leq -1$ is allowed!). When this inequality is satisfied, Corollary 9.2 states that P, with domain $x^{1-c_1}\,\mathcal{S}_{\mathrm{even},+}$, is also essentially self-adjoint in $L^2_{c_1,+}$; its spectrum consists of the eigenvalues $(4k + 3 - c_1)s$, for $k \in \mathbb{N}$, with multiplicity one and corresponding normalized eigenfunctions $x^{1+c_1}\phi_{2k,+}$; and its domain is characterized by the condition on the "Fourier coefficients" to be rapidly decreasing.

When $-1 < c_1 < 3$, we have got two essentially self-adjoint operators in $L^2_{c_1,+}$ defined by P, with domains $\mathcal{S}_{\mathrm{even},+}$ and $x^{1-c_1}\,\mathcal{S}_{\mathrm{even},+}$, which are equal just when $c_1 = 1$. In particular, if $c_1 = 0$, these operators are defined by H with domains $\mathcal{S}_{\mathrm{even},+}$ and $x\,\mathcal{S}_{\mathrm{even},+} = \mathcal{S}_{\mathrm{odd},+}$.

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