EXISTENCE AND UNIQUENESS OF VISCOSITY SOLUTION FOR HAMILTON-JACOBI EQUATION WITH DISCONTINUOUS COEFFICIENTS DEPENDENT ON TIME

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ABSTRACT. The main result is a proof of the existence of a unique viscosity solution for Hamilton-Jacobi equation, where the hamiltonian is discontinuous with respect to variable, usually interpreted as the spatial one. Obtained generalized solution is continuous, but not necessarily differentiable.

1. INTRODUCTION

The transport equation is used in many areas of applied mathematics for modelling phenomena as age-structured population, epidemiology and transport phenomena in physics. We are particulary interested in the model of vintage capital

$$\begin{cases} u_t + u_x = -\mu(t,x)u + u_1(t,x) & (t,x) \in (0,T] \times \mathbb{R} \\ u(0,x) = f(x) & x \in \mathbb{R}. \end{cases} \tag{$*$}$$

This partial differential equation generalizes well-known dynamical description of the firm capital accumulation $u'(t) = u_1(t) - \mu u(t)$. Such a model appeared in [2], [9], [8]. Fabbri ([8], 2008) gives a solution of control problem in case of constant depreciation factor $\mu(t,x) = \mu$ in state equation. The problem is reformulated in infinite-dimensional manner and definitions and tools for showing existence of viscosity solution is appropiate to this. However, author indicates that his proof does not work in case of μ dependent on t or x. Barucci and Gozzi ([2], 2001) and Feichtinger, Prskawetz and Veliov ([9], 2004) discuss model governed by $(*)$ from economic point of view (again, depreciation factor is constant there), with assumptions assuring existence of classical solutions to the control problems.

In this paper we will take an attempt to answer (partially) the question put in [8]. We use different mathematical tools for that, following the idea proposed by Stromberg in [11], who treats $(*)$ not as a state equation but as a particular form of Hamilton-Jacobi equation. Moreover, assumptions allows some coefficients in the model to be discontinuous in variable x. Similarily to [11], the arguments used in the proofs might be applied not only to the model $(*)$, but also to equation in the generalized form.

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The line of the proof is to approximate the problem by problems with continuous coefficients, for which the classical results by Crandall, Lions and Ishii ([5]) may be applied.
Thorough this article we consider pointwise convergence, if it is not specified different.

2. Formulation of the problem

We are interested in the existence and uniqueness result for the Cauchy problem

$$u_t + H(t, x, u, Du) = 0, \qquad (t, x) \in (0, T] \times \mathbb{R}^n \tag{1}$$

$$u(0, x) = f(x), \qquad x \in \mathbb{R}^n \tag{2}$$

where discontinuity for function H in variable x is allowed as well as dependence of coefficients of equation on time variable.

The classical theory of viscosity solutions is derived under continuity assumption put on H. This is the main difficulty here to overcome. In order to formulate definitions of viscosity solution in such case, the auxiliary functions will be needed:

$$H_\bullet(t, x, r, p) = \liminf_{(t',x',r',p') \longrightarrow (t,x,r,p)} H(t', x', r', p')$$

$$H^\bullet(t, x, r, p) = \limsup_{(t',x',r',p') \longrightarrow (t,x,r,p)} H(t', x', r', p').$$

Now we are ready to explain what we understand by viscosity solution of (1).

Definition 1.
A function u is *a viscosity subsolution* of (1), if it is upper semicontinuous on $(0, T] \times \mathbb{R}^n$ and for any $\varphi \in C^1((0, T) \times \mathbb{R}^n)$, such that $u - \varphi$ assigns local maximum at $(t, x) \in (0, T] \times \mathbb{R}^n$, the following inequality is satisfied:

$$\varphi_t(t, x) + H_\bullet(t, x, u(t, x), D\varphi(t, x)) \leq 0. \tag{$*$}$$

A function u is *a viscosity supersolution* of (1), if it is lower semicontinuous on $(0, T] \times \mathbb{R}^n$ and for any $\varphi \in C^1((0, T) \times \mathbb{R}^n)$, such that $u - \varphi$ assigns local minimum at $(t, x) \in (0, T] \times \mathbb{R}^n$, the following inequality is satisfied:

$$\varphi_t(t, x) + H^\bullet(t, x, u(t, x), D\varphi(t, x)) \geq 0. \tag{$**$}$$

A function u is *a viscosity solution* of (1), if it is viscosity sub- and supersolution of (1).

If we want to talk about a viscosity solution of the Cauchy problem (1)-(2), the additional conditions $u(0, x) \leq f(x)$, $u(0, x) \geq f(x)$ and $u(0, x) = f(x)$ respectively are required for $x \in \mathbb{R}^n$.

In the following we will base on the assumptions:

(A1) There exists $C > 0$, such that for $x, p, p' \in \mathbb{R}^n$, $r, r' \in \mathbb{R}$, $t, t' \in (0, T]$:

$$|H(t, x, r, p) - H(t', x, r', p')| \leq C(1 + |x|)(|t - t'| + |r - r'| + |p - p'|)$$

and $x \longrightarrow H(t, x, r, 0)$ is continuous for any $t \in (0, T]$ and $r \in \mathbb{R}$.

(A2) For any $\rho > 0$ there exist $\Phi \in C^1(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}_+)$ such that $\Phi(x,y) = 0 \Longleftrightarrow x = y$, and $G \in C(\mathbb{R} \times \mathbb{R}; \mathbb{R}_+)$ such that $G(0,0) = 0$, and:

$$H_\bullet(t,x,r,\lambda D_x\Phi(x,y)) - H^\bullet(t,y,r,-\lambda D_y\Phi(x,y)) \geq -G(\lambda\Phi(x,y), |x-y|)$$

for $t \in (0,T]$, $x, y \in B(0,\rho)$, $r \in \mathbb{R}$ and $\lambda > 0$.

(A3) There exist a function $\alpha\colon [0,\infty) \longrightarrow (0,\infty)$ – continuous and nondecreasing, satisfying condition $\int_1^\infty \frac{ds}{\alpha(s)} = \infty$ such that

$$\forall t \in (0,T], x,p \in \mathbb{R}^n, r \in \mathbb{R}, \lambda \geq 0 \quad H_\bullet(t,x,r,p+\lambda x) - H_\bullet(t,x,r,p) \geq -\lambda\alpha(|x|)|x|.$$

(A4) For $t \in [0,T]$, $x, p \in \mathbb{R}^n$, $r, r' \in \mathbb{R}$

$$H^\bullet(t,x,r,p) \geq H^\bullet(t,x,r',p) \qquad \text{if} \quad r \geq r'.$$

Theorem 1.
Under assumptions (A1), (A2), (A3), (A4) and if the initial function f is continuous there exists a unique viscosity solution of (1)-(2).

The proof of the theorem will be based on the four lemmas. We will present them in the next section.

3. Proof of the main result

In this part of paper we prove existence and uniqueness of solution of the problem (1). They are based on the four lemmas. The first two of them describe properties of the infimal convolutions of H, necessary for further approximation. For $\varepsilon \in (0, 1/C)$, $(t,x,u,p) \in (0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ let:

$$H_\varepsilon(t,x,r,p) := \inf_{y\in\mathbb{R}^n} \left(H(t,y,r,p) - H(t,y,r,0) + \frac{|p||x-y|}{\varepsilon} \right) + H(t,x,r,0)$$

$$H^\varepsilon(t,x,r,p) := \sup_{y\in\mathbb{R}^n} \left(H(t,y,r,p) - H(t,y,r,0) - \frac{|p||x-y|}{\varepsilon} \right) + H(t,x,r,0).$$

The following proposition is known in the literature, compare [3].

Proposition 1.
Let $\varphi\colon \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lipschitz function with respect to the second variable having the first one fixed – let's denote it as x, $x \in \mathbb{R}^m$. Assume that $\varphi(x,p) \geq -C|x|$ for some positive C. Let

$$\varphi_\varepsilon(x,p) := \inf_{y\in\mathbb{R}^m} \left(\varphi(y,p) + \frac{|p||x-y|}{\varepsilon} \right).$$

Then

$$\varphi_\varepsilon \nearrow \varphi_\bullet \text{ while } \varepsilon \searrow 0,$$

where $\varphi_\bullet(x,p) := \liminf_{y\longrightarrow x} \varphi(y,p)$.

Proof.
The monotonicity of the sequence $(\varphi_\varepsilon)_\varepsilon$ comes stright from the definition of those functions. For the proof of convergence let's write the following:

$$\begin{aligned}\varphi_\bullet(x,p) &= \liminf_{y\longrightarrow x}\varphi(y,p) = \sup_{\varepsilon>0}\left(\inf_{y\in B(x,\varepsilon)}\varphi(y,p)\right)\\ &= \lim_{\varepsilon\longrightarrow 0^+}\left(\inf_{y\in\mathbb{R}^n}\left(\varphi(y,p)+\tfrac{|x-y||p|}{\varepsilon}\right)\right) = \lim_{\varepsilon\longrightarrow 0^+}\varphi_\varepsilon(x,p). \qquad \square\end{aligned}$$

The analogous result is obvious for $\varphi^\bullet(x,p) := \limsup_{y\longrightarrow x}\varphi(y,p)$ (which is nonincerasing sequence of functions), taking into account the relations $\varphi^\varepsilon = -(-\varphi)_\varepsilon$ and $\varphi^\bullet = -(-\varphi)_\bullet$.

Lemma 2.
Assume $(A1)$. *Then*

(i) *for any* $(t,x,r,p)\in(0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$: $H_\bullet(t,x,r,p) = \liminf_{x'\longrightarrow x}H(t,x',r,p)$ *and* $H^\bullet(t,x,r,p) = \limsup_{x'\longrightarrow x}H(t,x',r,p)$

(ii) *for* $\varepsilon\in(0,1/C)$ *the functions* H_ε, H^ε *are continuous on* $(0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$, *i.e.:*

$$|H_\varepsilon(t,x,r,p)-H_\varepsilon(t,y,r,p)| \le \frac{|p||x-y|}{\varepsilon} + |H(t,x,r,0)-H(t,y,r,0)|$$

$$|H_\varepsilon(t,x,r,p)-H_\varepsilon(t',x,r',p')| \le D_\varepsilon(1+|x|)(|t-t'|+|r-r'|+|p-p'|)$$

for $t,t'\in(0,T]$, $r,r'\in\mathbb{R}$, $x,y,p,p'\in\mathbb{R}^n$, $D_\varepsilon = \frac{C(3+C\varepsilon)}{1-C\varepsilon}$ *(analog. for* H^ε*).*

(iii) $H_\varepsilon \nearrow H_\bullet$ *and* $H^\varepsilon \searrow H^\bullet$ *when* $\varepsilon\searrow 0$.

The proof will be formulated only for the case H_ε and $H_\bullet$ as for the other function is analogous.

Proof.
Basing on the definitions of our functions and assumption $(A1)$, we obtain:

$$H_\bullet(t,x,r,p) = \liminf_{(t',x',r',p')\longrightarrow(t,x,r,p)} H(t',x',r',p') \le \liminf_{x'\longrightarrow x}H(t,x',r,p)$$

$$= \liminf_{(t',x',r',p')\longrightarrow(t,x,r,p)}\{H(t',x',r',p')+H(t,x',r,p)-H(t',x',r',p')\}$$

$$\le \liminf_{(t',x',r',p')\longrightarrow(t,x,r,p)}\{H(t',x',r',p')+C(1+|x'|)(|t-t'| + |r-r'|+|p-p'|)\}$$

$$= H_\bullet(t,x,r,p).$$

The Lipschitz continuity of H_ε in x is a result of evaluation of the difference:

$$\begin{aligned}&[H_\varepsilon(t,x,r,p)-H(t,x,r,0)]-[H_\varepsilon(t,y,r,p)-H(t,y,r,0)]=\\ &\qquad=\inf_{\xi\in\mathbb{R}^n}\left(H(t,\xi,r,p)-H(t,\xi,r,0)+\frac{|p||x-\xi|}{\varepsilon}\right)\\ &\qquad\quad-\inf_{\xi\in\mathbb{R}^n}\left(H(t,\xi,r,p)-H(t,\xi,r,0)+\frac{|p||y-\xi|}{\varepsilon}\right)\\ &\qquad\le-\inf_{\xi\in\mathbb{R}^n}\left(\frac{|p|}{\varepsilon}\left(|y-\xi|+|x-\xi|\right)\right)\le\frac{|p||x-y|}{\varepsilon}\end{aligned}$$

with use of some usual properties for infima and the triangle inequality. The same calculus with inversed roles of x and y completes the proof of this part.

Let's note now that thanks to $(A1)$ we have

$$H_\varepsilon(t,x,r,p)-H(t,x,r,0)\le C(1+|x|)|p|.$$

On the other hand

$$C(1+|x|)|p|\ge H(t,\xi,r,p)-H(t,\xi,r,0)+\frac{|p||x-\xi|}{\varepsilon}\ge$$

$$\ge-C(1+|\xi|)|p|+\frac{|p||x-\xi|}{\varepsilon}\ge-C(1+|x|+|x-\xi|)|p|+\frac{|p||x-\xi|}{\varepsilon}$$

so the infimum $H_\varepsilon(t,x,p)-H(t,x,0)$ *is not* assigned at the point $\xi\in\mathbb{R}^n$ satifying

$$C(1+|x|)|p|<-C(1+|x|+|x-\xi|)|p|+\frac{|p||x-\xi|}{\varepsilon}$$

or equivalently, taking $p\neq 0$

$$|x-\xi|>\frac{2C\varepsilon}{1-\varepsilon C}(1+|x|). \tag{3}$$

The stated boundary with respect to (t,p) is a result of the following (let $x\in\mathbb{R}^n$ be fixed):

$$\begin{aligned}&H_\varepsilon(t,x,r,p)-H_\varepsilon(t',x,t',p')=\\ &=\inf_{\xi\in\mathbb{R}^n}\left\{H(t,\xi,r,p)-H(t,\xi,r,0)+\frac{|x-\xi||p|}{\varepsilon}\right\}+H(t,x,r,0)\\ &\qquad-\inf_{\xi\in\mathbb{R}^n}\left\{H(t',\xi,r',p')-H(t',\xi,r',0)+\frac{|x-\xi||p'|}{\varepsilon}\right\}-H(t',x,r',0)\end{aligned}$$

$$\le \inf_{\xi\in\mathbb{R}^n}\Big\{ -[H(t,\xi,r,p)-H(t',\xi,r',p')]+[H(t',\xi,r',0)-H(t,\xi,r,0)]$$
$$+\frac{|p'|-|p|}{\varepsilon}|x-\xi|\Big\}+(H(t,x,r,0)-H(t',x,r',0))$$
$$\le \inf_{\xi\in\mathbb{R}^n}\Big\{C(1+|\xi|)(|p-p'|+|r-r'|+|t-t'|)+C(1+|\xi|)(|r-r'|+|t-t'|)$$
$$+\frac{|p-p'|}{\varepsilon}|x-\xi|\Big\}+C(1+|x|)|t-t'|$$

Hence, recalling the condition (3):

$$H_\varepsilon(t,x,r,p)-H_\varepsilon(t',x,r',p')\le C\cdot\frac{3+\varepsilon C}{1-\varepsilon C}(1+|x|)\left(|p-p'|+|r-r'|+|t-t'|\right).$$

The repeat of the evaluations above for the inversed roles of (t,r,p) and (t',r',p') finishes the proof of the assertion (ii).

The last assertion is obvious in view of proposition 1 (apply it to the function $\varphi(y,\tilde p)\equiv\varphi(t,y,r,p)=H(t,y,r,p)-H(t,y,r,0)$, $\tilde p=(t,r,p)$). □

Lemma 3 (Comparison Principle)**.**
Assume (A2)*,* (A3) *and* (A4)*. Let u i v be sub- and supersolution of* (1)*, respectively. Then*

$$\sup_{[0,T]\times\mathbb{R}^n}(u-v)^+=\sup_{\mathbb{R}^n}(u(0,\cdot)-v(0,\cdot))^+.$$

Let's note that if u and v are considered to be sub- and supersolutions of (1)–(2), the assertion of this lemma might be reformulated as $u\le v$, what is standard formulation of comparison result.

Proof.
The inequality $\sup\limits_{[0,T]\times\mathbb{R}^n}(u-v)^+\ge\sup\limits_{\mathbb{R}^n}(u(0,\cdot)-v(0,\cdot))^+$ is obvious.

For the proof of the inverse one, let's consider the function

$$F(t,x)=t+\int_1^{(1+|x|^2)^{\frac12}}\frac{ds}{\alpha(s)}$$

and the family of compacts (with $\beta>0$):

$$S_\beta:=\{(t,x)\in[0,T]\times\mathbb{R}^n : F(t,x)\le\beta\}.$$

Let $g\colon(0,\infty)\longrightarrow(0,\infty)$ be a nondecreasing function such that

$$e^{g(\beta)}>\max\{u(t,x)-v(t,x):(t,x)\in S_\beta\} \tag{4}$$

and

$$\psi_\beta(t,x):=e^{g(\beta)(1+F(t,x)-\beta)}. \tag{5}$$

The functions ψ_β are smooth solutions to differetial inequality for $(t,x)\in(0,T]\times\mathbb{R}^n$

$$\psi_t(t,x)-\alpha(|x|)|D\psi(t,x)|\geq 0. \tag{6}$$

Taking into account that $\lim_{\beta\longrightarrow\infty}\psi_\beta(t,x)=0$, for completion of the proof it's enough to show that

$$u(t,x)-v(t,x)-\psi_\beta(t,x)\leq\sup_{\mathbb{R}^n}(u(0,\cdot)-v(0,\cdot))^+\qquad\forall\beta>0,\forall(t,x)\in S_\beta.$$

Let's assume inversely, that for some $\beta>0$ (fixed from here) there exists a constant $c>0$ and point $(t,x)\in(0,T]\times\mathbb{R}^n$, for which

$$u(t,x)-v(t,x)-\psi_\beta(t,x)-ct>\sup_{\mathbb{R}^n}(u(0,\cdot)-v(0,\cdot))^+. \tag{7}$$

Define the penalty function, corresponding to the left side of the inequality above:

$$(t,s,x,y)\longrightarrow u(t,x)-v(s,y)-(\psi_\beta(t,x)+ct+j\Phi(x,y))$$

for $(t,x)\in S_\beta$ and $(s,y)\in S_\beta$. Having j fixed, each such function assigns maximum at some point (t_j,s_j,x_j,y_j), as a difference of upper semicontinuous function $u-v$ and the one of class C^1 on the compact span of variables.

We may assume that the sequence $((t_j,s_j,x_j,y_j))_j$ converges to the point of the form $(\bar t,\bar t,\bar x,\bar x)$. Indeed, let $M_j=\sup_{S_\beta\times S_\beta}[u(t,x)-v(s,y)-(\psi_\beta(t,x)+ct+j\Phi(x,y))]$, it's finite. It's nonincreasing sequence and denote the limit of (t_j,s_j,x_j,y_j) as $(\bar t,\bar s,\bar x,\bar y)$. Then

$$\lim_{j\longrightarrow\infty}\delta_j:=\lim_{j\longrightarrow\infty}\{M_j-[u(t_j,x_j)-v(s_j,y_j)-(\psi_\beta(t_j,x_j)+ct_j+j\Phi(x_j,y_j))]\}=0.$$

We have

$$\begin{aligned}M_{j/2}&\geq u(t_j,x_j)-v(s_j,y_j)-\left(\psi_\beta(t_j,x_j)+ct_j+\frac{j}{2}\Phi(x_j,y_j)\right)\\&\geq[u(t_j,x_j)-v(s_j,y_j)-(\psi_\beta(t_j,x_j)+ct_j+j\Phi(x_j,y_j))]+\frac{j}{2}\Phi(x_j,y_j)\\&\geq M_j-\delta_j+\frac{j}{2}\Phi(x_j,y_j)\end{aligned}$$

and therefore

$$2(M_{j/2}-M_j+\delta_j)\geq j\Phi(x_j,y_j).$$

Recalling the regularity of Φ we see that $\lim_{j\longrightarrow\infty}j\Phi(x_j,y_j)=0$ and $\lim_{j\longrightarrow\infty}\Phi(x_j,y_j)=0$, which gives $\bar x=\bar y$ in view of properties of Φ. Moreover, the point $(\bar t,\bar x)$ also maximizes the left side of (7), without lost of generality we may set $\bar t=\bar s$.

Hence $(\bar t,\bar x)\in S_\beta$ and according to the definition of S_β, $F(\bar t,\bar x)\leq\beta$. Taking into account the fact of realizing maximum of left side of (7) at $(\bar t,\bar x)$ and properties of

ψ_β it's easy to see that $\bar{t} > 0$ and more strict boundary holds: $F(\bar{t}, \bar{x}) < \beta$. Therefore for sufficiently large j at points $(t_j, x_j), (s_j, y_j)$ F also attains a value strictly less than β. Summarizing, we have upper semicontinuous function $u = u(t, x)$, supersolution of (1) and smooth function $\varphi(t, x) = v(s, y) + \psi_\beta(t, x) + ct + j\Phi(x, y)$ such that $u - \varphi$ assigns maximum at $(t_j, x_j) \in (0, T] \times \mathbb{R}^n$. Then by definition the following inequality holds:

$$c + (\psi_\beta)_t(t_j, x_j) + H_\bullet(t_j, x_j, u(t_j, x_j), D\psi_\beta(t_j, x_j) + jD_x\Phi(x_j, y_j)) \leq 0.$$

Analogously, for subsolution $v = v(s, y)$ and $\varphi(s, y) = u(t, x) - \psi_\beta(t, x) - ct - j\Phi(x, y)$ we derive

$$H^\bullet(s_j, y_j, v(s_j, y_j), -jD_y\Phi(x_j, y_j)) \geq 0.$$

Finally,

$$\begin{aligned} c + (\psi_\beta)_t(t_j, x_j) + H_\bullet(t_j, x_j, u(t_j, x_j), D\psi_\beta(t_j, x_j) + jD_x\Phi(x_j, y_j)) \\ - H^\bullet(s_j, y_j, v(s_j, y_j), -jD_y\Phi(x_j, y_j)) \leq 0 \end{aligned} \tag{8}$$

Thanks to the choice of ψ_β,

$$D\psi = \frac{\psi \cdot g(\beta)}{\sqrt{1 + |x|^2}\alpha(\sqrt{1 + |x|^2})} \cdot x,$$

so taking $\lambda = \lambda(t, x) = \frac{\psi \cdot g(\beta)}{\sqrt{1+|x|^2}\alpha(\sqrt{1+|x|^2})} \geq 0$ in the assumption $(A3)$ and using (6) we obtain

$$c + H_\bullet(t_j, x_j, u(t_j, x_j), jD_x\Phi(x_j, y_j)) - H^\bullet(s_j, y_j, v(s_j, y_j), -jD_y\Phi(x_j, y_j)) \leq 0.$$

Notice that thanks to hypothesis (7) for all j

$$\begin{aligned} u(t_j, x_j) - v(s_j, y_j) &\geq \max_{S_\beta \times S_\beta} [u(t, x) - v(s, y) - (\psi_\beta(t, x) + ct + j\Phi(x, y))] \\ &\geq \max_{S_\beta} [u(t, x) - v(t, x) - (\psi_\beta(t, x) + ct)] > 0 \end{aligned} \tag{9}$$

Finally, through $(A2)$, $(A4)$, (9) and regularity of $H^\bullet$:

$$c - G(j\Phi(x_j, y_j), |x_j - y_j|) - C(1 + |y_j|)|t_j - s_j| \leq 0,$$

what in the limit gives a contradiction with the choice of $c > 0$.

So, we have that for any $\beta > 0$ and $(t, x) \in S_\beta$

$$u(t, x) - v(t, x) - \psi_\beta(t, x) \leq \sup_{\mathbb{R}^n}(u(0, \cdot) - v(0, \cdot))^+.$$

Tending to the limit with β and taking supremum on the domain, we complete the proof. □

The classical theory of viscosity solutions in view of regularity of infimal convolutions H_ε and H^ε stated in lemma 2 assures existence and uniqueness of the continuous viscosity solutions for $u_t + H_\varepsilon(t, x, u, Du) = 0$ and $u_t + H^\varepsilon(t, x, u, Du) = 0$ with initial condition $u(0, x) = f(x)$. Denote them as u_ε and u^ε respectively. The next lemma states the monotone properties of those sequences and, what is more important, proves that their pointwise limits are super- and subsolutions of the original problem $u_t + H(t, x, u, Du) = 0$.

However, such assertion requires a bit stronger convergence arguments, namely the epi-convergence (known in the literature also as gamma-convergence) of the sequences of approximating solutions to assure convergence of a sequence of extrema of the functions u_ε to an extremum of the limit function.

We recall some basic facts on epi- and hypo-convergence (see [7], [10]) used in the proof of lemma 4. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the epigraph of f is defined by

$$epi f = \{(x, \alpha) \colon \forall x \in X f(x) \leq \alpha\}.$$

For any sequence $(f_n)_{n\in\mathbb{N}}$ *the lower epi-limit* $\text{e-}\liminf_{n\longrightarrow\infty} f_n$ is the function having as its epigraph the outer limit of the sequence of sets $epi f_n$, i.e.

$$f = \text{e-}\liminf_{n\longrightarrow\infty} f_n \Longleftrightarrow \text{epi} f = \limsup_{n\longrightarrow\infty}(\text{epi} f_n);$$

the upper epi-limit $\text{e-}\limsup_{n\longrightarrow\infty} f_n$ is the function having as its epigraph the inner limit of the sequence of sets $epi f_n$, i.e.

$$f = \text{e-}\limsup_{n\longrightarrow\infty} f_n \Longleftrightarrow \text{epi} f = \liminf_{n\longrightarrow\infty}(\text{epi} f_n);$$

the epi-limit $\text{e-}\lim_{n\longrightarrow\infty} f_n$ exists iff both the lower and the upper epi-limits exist and equal.

The hypo-convergence might be defined in a parallel way as a limit of corresponding hypographs or simply through epi-convergence by

$$f = \text{h-}\lim_{n\longrightarrow\infty} f_n \Longleftrightarrow -f = \text{e-}\lim_{n\longrightarrow\infty}(-f_n).$$

In view of the obvious correspondence we discuss in the following the facts for epi-convergence only. The epi-convergence has a series of very nice properties. One of them is that for any monotone sequence the epi-limit exists. In case of nonincreasing sequence of functions $\text{e-}\lim_{n\longrightarrow\infty} f_n = cl(\inf_{n\in\mathbb{N}} f_n)$ and if the sequence is nondecreasing, $\text{e-}\lim_{n\longrightarrow\infty} f_n = \sup_{n\in\mathbb{N}}(cl f_n)$. The pointwise and epi-limits do not need to coincide even if both exist. However, they are equal in case of monotone sequence. Finally, for an epi-convergent sequence of functions the inclusion

$$\limsup_{n\to\infty}(argmin f_n) \subset argmin f$$

holds. That means that every cluster point of a sequence $(x_n)_n$, where x_n is optimal for f_n, is optimal for f. Unfortunately, in general it does not guarantee that *any* optimum of f might be approximated by the subsequence of this kind. We overcome this difficulty in the proof by proper choice of neighbourhood of considering minimum.

Let's move to the next step in proof now.

Lemma 4.
Let all taken in the previous lemmas assumptions be satisfied. Then:

1) *if* $0 < \delta < \varepsilon < \frac{1}{C}$*, then* $u^\varepsilon \le u^\delta \le u_\delta \le u_\varepsilon$*.*
2) $u^\varepsilon \nearrow u^0$ *i* u^0 *is viscosity supersolution of* $u_t + H(t,x,u,Du) = 0$ *in* $(0,T] \times \mathbb{R}^n$*,* $u(0,x) = f(x)$ *in* $\mathbb{R}^n$*.*
3) $u_\varepsilon \searrow u_0$ *i* u_0 *is viscosity subsolution of* $u_t+H(t,x,u,Du)=0$ *in* $(0,T]\times\mathbb{R}^n$*,* $u(0,x) = f(x)$ *in* $\mathbb{R}^n$*.*
4) $u^0 \le u_0$*.*

Proof.
Stright by definition we have $H_\delta \ge H_\varepsilon$ for $0 < \delta < \varepsilon < \frac{1}{C}$. Let $\varphi \in C^1((0,T]\times\mathbb{R}^n)$ such that $u_\delta - \varphi$ assigns local maximum at (t_δ, x_δ). u_δ is viscosity solution of $u_t + H_\delta(t,x,u,Du) = 0$, so

$$\varphi_t(t_\delta,x_\delta) + H_\varepsilon(t_\delta,x_\delta,u_\delta,D\varphi(t_\delta,x_\delta)) \le \varphi_t(t_\delta,x_\delta) + H_\delta(t_\delta,x_\delta,u_\delta,D\varphi(t_\delta,x_\delta)) = 0,$$

for any smooth φ such that $u_\delta - \varphi$ assigns maximum at (t,x). In particular u_δ is viscosity subsolution of $u_t + H_\varepsilon(t,x,u,Du) = 0$ and satisfies initial condition. By lemma 3 for $u = u_\delta$ i $v = u_\varepsilon$ we have $u_\delta \le u_\varepsilon$.

Starting from simple inequalities $H^\delta \le H^\varepsilon$ and $H^\delta \ge H_\delta$ in analogous way we obtain $u^\delta \ge u^\varepsilon$ and $u^\delta \le u_\delta$ respectively.

Hence we have two sequences: for the equation $u_t + H_\varepsilon(t,x,u,Du) = 0$ there is nonincreasing sequence of viscosity subsolutions $(u_\varepsilon)_\varepsilon$ bounded from below and for $u_t + H^\varepsilon(t,x,u,Du) = 0$ there is nondecreasing sequence of viscosity supersolutions $(u^\varepsilon)_\varepsilon$ bounded from above. Moreover, each of those functions u_ε and u^ε is continuous and real-valued. There exist pointwise limits of those sequences, $u^0 := \lim_{\varepsilon\to 0} u^\varepsilon$ – lower semicontinuous and $u_0 := \lim_{\varepsilon\to 0} u_\varepsilon$ – upper semicontinuous, and in addition $u^0 \le u_0$.

In view of monotone convergence of the sequences $(u_\varepsilon)_\varepsilon$ and $(H_\varepsilon)_\varepsilon$, $(u_\varepsilon)_\varepsilon$ hypo-converges to its pointwise limit u_0 and $(H_\varepsilon)_\varepsilon$ epi-converges to $H_\bullet$. Thanks to that u^0 is a natural candidate for a subsolution of $u_t + H(t,x,u,Du) = 0$. Indeed, let $\varphi \in C^1((0,T] \times \mathbb{R}^n)$ such that $u_0 - \varphi$ assigns local maximum at $(t_0,x_0) \in (0,T] \times \mathbb{R}^n$. Choose a compact neighbourhood of this point, $E \subset (0,T] \times \mathbb{R}^n$, in such way that (t_0,x_0) is the only strict maximum:

$$\forall (t,x) \in E \setminus \{(t_0,x_0)\} : \quad u_0(t,x) - \varphi(t,x) > u_0(t_0,x_0) - \varphi(t_0,x_0).$$

For each $\varepsilon > 0$ we denote by $(t_\varepsilon, x_\varepsilon)$ a maximum of $u_\varepsilon - \varphi$ in E. In view of hypoconvergence of $(u_\varepsilon)_\varepsilon$, each cluster point of such sequence belongs to $argmin(u_0 - \varphi)$, and the choice of the set E guarantees convergence of $(t_\varepsilon, x_\varepsilon)$

to (t_0, x_0). Therefore the inequality

$$\varphi_t(t_\varepsilon, x_\varepsilon) + H_\varepsilon(t_\varepsilon, x_\varepsilon, u_\varepsilon(t_\varepsilon, x_\varepsilon), D\varphi(t_\varepsilon, x_\varepsilon)) \leq 0$$

in the limit with $\varepsilon \to 0$ gives

$$\varphi_t(t_0, x_0) + H_\bullet(t_0, x_0, u_0, (t_0, x_0), D\varphi(t_0, x_0)) \leq 0$$

thanks to epi-convergece of $(H_\varepsilon)_\varepsilon$, which proves the assertion 2) of the lemma. Proceeding in analogous way with the sequences $(u^\varepsilon)_\varepsilon$ and $(H^\varepsilon)_\varepsilon$, we complete the proof. □

Proof of the main theorem. The lemma 4 proves existence of the subsolution u_0 and the supersolution u^0 of (1)–(2), which in addition satisfy $u^0 \leq u_0$. On the other hand, by the comparison theorem (lemma 3) for those functions the inequality $u^0 \geq u_0$ holds. That guarantees the existence of the viscosity solution for (1)–(2). The uniqueness is implied by lemma 3. Suppose that u_1 and u_2 are viscosity solutions for our problem. It's enough to consider derived in the lemma inequality once for the choice $u = u_1$, $v = u_2$ and later for inverse choice of u and v. □

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