GREEN VS. LEMPERT FUNCTIONS: A MINIMAL EXAMPLE

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Abstract. The Lempert function for a set of poles in a domain of $\mathbb{C}^n$ at a point z is obtained by taking a certain infimum over all analytic disks going through the poles and the point z, and majorizes the corresponding multi-pole pluricomplex Green function. Coman proved that both coincide in the case of sets of two poles in the unit ball. We give an example of a set of three poles in the unit ball where this equality fails.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$, and $a_j \in \Omega$, $j = 0, ..., N$. The pluricomplex Green function with logarithmic singularities at $S := \{a_1, \dots, a_N\}$ is defined by

$$G_S(z) := \sup \{u \in PSH(\Omega, \mathbb{R}_-) : u(z) \leq \log |z - a_j| + C_j, j = 0, ..., N\},$$

where $PSH(\Omega, \mathbb{R}_-)$ stands for the set of all negative plurisubharmonic functions in Ω. When Ω is hyperconvex, this solves the Monge-Ampère equation with right hand side equal to $\sum_{i=1}^N \delta_{a_j}$.

Pluricomplex Green functions have been studied by many authors at different levels of generality. See e.g. Demailly [3], Zahariuta [16], Lempert [10], Lelong [9], Lárusson and Sigurdsson [8].

A deep result due to Poletsky [13], see also [8], [6], is that the Green function may be computed from analytic disks:

$$(1.1)\quad G_S(z) = \inf \left\{ \sum_{\alpha : \varphi(\alpha) \in S} \log |\alpha| : \text{ such that there exists } \varphi \in \mathcal{O}(\mathbb{D}, \Omega), \varphi(0) = z \right\}.$$

2000 *Mathematics Subject Classification.* 32F07, 32F45.

Key words and phrases. pluricomplex Green function, Lempert function, analytic disks, Schwarz Lemma.

Part of this work was done when the author was a guest of the Hanoi University of Education in March 2011. He wishes to thank his colleagues there for their hospitality, in particular Do Duc Thai and Nguyen Van Trao. A detailed exposition of the results in a workshop organized in Bedlewo by colleagues from the Jagiellonian University in Cracow resulted in some streamlining of the proof. Finally, the Semester in Complex Analysis and Spectral Theory of the Centre de Recerca Matemàtica at the Universitat Autònoma de Barcelona provided the occasion to mention the result in a talk, and to put the finishing touches on the present manuscript.

However, it is tempting to pick only one $\alpha_j \in \varphi^{-1}(a_j)$, $1 \le j \le N$, which motivated the definition of Coman's Lempert function [2]:

$$\ell_S(z) := \inf \Big\{ \sum_{j=0}^{N} \log|\zeta_j| : \varphi(0) = z, \qquad \varphi(\zeta_j) = a_j, j = 1, ..., N \text{ for some } \varphi \in \mathcal{O}(\mathbb{D}, \Omega) \Big\}, \tag{1.2}$$

where $\mathbb{D}$ is the unit disc in $\mathbb{C}$.

One easily sees that $\ell_S(z) \ge G_S(z)$ without recourse to (1.1); the fact that equality holds when $N = 1$ and Ω is convex is part of Lempert's celebrated theorem, which was, in fact, the starting point for many of the notions defined above [10], see also [4]. Coman proved that equality holds when $N = 2$ and $\Omega = \mathbb{B}^2$, the unit ball of $\mathbb{C}^2$ [2]. The goal of this note is to present an example that shows that this is as far as it can go.

Theorem 1.1. *There exists a set of* 3 *points* $S \subset \mathbb{B}^2$ *such that for some* $z \in \mathbb{B}^2$, $\ell_S(z) > G_S(z)$.

Other examples in the same direction have been found in [1], [15], [12]. The interesting features of this one are that it involves no multiplicities and is minimal in the ball. Furthermore, the corresponding Green function can be recovered, up to a bounded error, by using an analytic disk with just one more pre-image than the number of points: one of the points has exactly two pre-images and each of the other two points, only one, see [11, Lemma 5.16, §5.8.2].

More specifically, the Theorem will follow from a precise calculation in the bidisk $\mathbb{D}^2$. Let $S_\varepsilon = \{(0,0), (\rho(\varepsilon), 0), (0, \varepsilon)\} \subset \mathbb{D}^2$, where $\lim_{\varepsilon\to 0} \rho(\varepsilon)/\varepsilon = 0$.

Proposition 1.2. *There exist* $C_1 > 0$ *and, for any* $\delta \in (0, \frac{1}{4})$, $C_2 = C_2(\delta) > 0$, *and* $\varepsilon_0 > 0$ *such that for any* $z = (z_1, z_2) \in \mathbb{D}^2$ *with*

$$\frac{1}{2}|z_2|^{3/2} \le |z_1| \le |z_2|^{3/2}, \tag{1.3}$$

and for any ε *with* $|\varepsilon| < \varepsilon_0$, *then*

$$G_{S_\varepsilon}(z) \le 2\log|z_2| + C_1, \tag{1.4}$$

$$\ell_{S_\varepsilon}(z) \ge (2 - \delta)\log|z_2| - C_2. \tag{1.5}$$

Proof of Theorem 1.1. If U, V are domains, and $S \subset U \subset V$, then the definitions of the Green and Lempert functions imply that $G_S^U(z) \ge G_S^V(z)$, $\ell_S^U(z) \ge \ell_S^V(z)$. So, using the fact that $\mathbb{B}^2 \subset \mathbb{D}^2$, we have, when z verifies (1.3),

$$\ell_{S_\varepsilon}^{\mathbb{B}^2}(z) \ge \ell_{S_\varepsilon}^{\mathbb{D}^2}(z) \ge (2 - \delta)\log|z_2| - C_2.$$

Using the fact that $\frac{\sqrt{2}}{2}\mathbb{D}^2 \subset \mathbb{B}^2$ and the invariance of the Green function under biholomorphic mappings,

$$G^{\mathbb{B}^2}_{S_\varepsilon}(z) \le G^{\frac{\sqrt{2}}{2}\mathbb{D}^2}_{S_\varepsilon}(z) = G^{\mathbb{D}^2}_{\sqrt{2}S_\varepsilon}(\sqrt{2}z) \le 2\log|z_2| + \log 2 + C_1.$$

The last inequality follows from the fact that when z verifies (1.3), $\sqrt{2}z$ also does, and $\sqrt{2}S_\varepsilon$ has the same form as S_ε, so we can apply (1.4).

Comparing the last two estimates, we see that for $|z_2|$ small enough, $G^{\mathbb{B}^2}_{S_\varepsilon}(z) < \ell^{\mathbb{B}^2}_{S_\varepsilon}(z)$. □

Open Questions.

1. This example is minimal in the ball, in terms of number of poles; what is the situation for the bidisk? Are the Green and Lempert functions equal when one takes two poles, not lying on a line parallel to the coordinate axes? Do they at least have the same order of singularity as one pole tends to the other?

2. What is the precise order of the singularity of the limit as $\varepsilon \to 0$ of the Lempert function in this case? Looking at the available analytic disks that give the correct order of the singularity of the limit of the Green function, one finds $\frac{3}{2}\log|z_2|$, so one would hope that the Proposition can still be proved at least for $\delta < \frac{1}{2}$.

3. Do the analytic disks from [11] yield the Green function itself, without any bounded error term?

4. More generally, when one is given a finite number of points in a given bounded (hyperconvex) domain, is there a bound on the number of pre-images required to attain the Green function in the Poletsky formula? For instance, is 4 the largest possible number of pre-images required when looking at 3 points in the ball?

Acknowledgements.

I wish to thank Nguyen Van Trao for useful discussions on this topic.

2. Computations

Proof of Proposition 1.2. The upper bound (1.4) follows from [11, Lemma 5.9, §5.5], see also [11, Lemma 5.16, §5.8.2] for the definition of the relevant analytic disk.

The proof of (1.5) will follow the methods and notations of [14]. We will make repeated use of the involutive automorphisms of the unit disk given by $\phi_a(\zeta) := \frac{a-\zeta}{1-\bar{a}\zeta}$ for $a \in \mathbb{D}$, which exchange 0 and a. We will denote the invariant (pseudohyperbolic) distance between points of the unit disk by

$$d_G(a,b) := |\phi_a(b)| = |\phi_b(a)|.$$

Write $\rho(\varepsilon) = \varepsilon s(\varepsilon)$ with $\lim_{\varepsilon\to 0} s(\varepsilon) = 0$. We will only consider points z such that $|z_1| = |z_2|^{3/2}$.

We will assume that the conclusion fails. That is, for any $\delta \in (0, \frac{1}{4})$, there exist arbitrarily small values of $|z_2| = \max(|z_1|, |z_2|)$, and $|\varepsilon|$ such that

$$\ell_{S_\varepsilon}(z) < (2-\delta) \log |z_2|. \tag{2.1}$$

After applying, for each analytic disk, an automorphism of the disk which exchanges the pre-image of $(0,0)$ and 0, the assumption means that there exists a holomorphic map φ from $\mathbb{D}$ to $\mathbb{D}^2$ and points $\zeta_j \in \mathbb{D}$, depending on z and ε, satisfying the conditions

$$\begin{cases} \varphi(0) = (0,0) \\ \varphi(\zeta_1) = (\varepsilon s(\varepsilon), 0) \\ \varphi(\zeta_2) = (0, \varepsilon) \\ \varphi(\zeta_0) = (z_1, z_2) \end{cases} \tag{2.2}$$

with

$$\log |\zeta_0| + \log |\phi_{\zeta_0}(\zeta_1)| + \log |\phi_{\zeta_0}(\zeta_2)| \le (2-\delta) \log |z_2|. \tag{2.3}$$

The interpolation conditions in (2.2) are equivalent to the existence of two holomorphic functions h_1, h_2 from $\mathbb{D}$ to itself such that

$$\varphi(\zeta) = \left(\zeta \phi_{\zeta_2}(\zeta) h_1(\zeta), \zeta \phi_{\zeta_1}(\zeta) h_2(\zeta)\right),$$

such that furthermore

$$h_1(\zeta_1) = \frac{\varepsilon s(\varepsilon)}{\zeta_1 \phi_{\zeta_2}(\zeta_1)} =: w_1, \tag{2.4}$$

$$h_1(\zeta_0) = \frac{z_1}{\zeta_0 \phi_{\zeta_2}(\zeta_0)} =: w_2, \tag{2.5}$$

$$h_2(\zeta_2) = \frac{\varepsilon}{\zeta_2 \phi_{\zeta_1}(\zeta_2)} =: w_4, \tag{2.6}$$

$$h_2(\zeta_0) = \frac{z_2}{\zeta_0 \phi_{\zeta_1}(\zeta_0)} =: w_3. \tag{2.7}$$

By the invariant Schwarz Lemma, the existence of a holomorphic function h_1 mapping $\mathbb{D}$ to itself and satisfying (2.4) and (2.5) is equivalent to

$$|w_1| < 1, |w_2| < 1, \text{ and } d_G(w_1, w_2) < d_G(\zeta_1, \zeta_0) = |\phi_{\zeta_1}(\zeta_0)|. \tag{2.8}$$

In the same way, the existence of h_2 is equivalent to

$$|w_3| < 1, |w_4| < 1, \text{ and } d_G(w_3, w_4) < d_G(\zeta_2, \zeta_0) = |\phi_{\zeta_2}(\zeta_0)|. \tag{2.9}$$

As in [14], we start by remarking that (2.3) can be rewritten as

$$\begin{aligned}-\log|w_2|-\log|w_3| &= \log\left|\frac{\zeta_0\phi_{\zeta_1}(\zeta_0)}{z_2}\right|+\log\left|\frac{\zeta_0\phi_{\zeta_0}(\zeta_2)}{z_1}\right| \\ &\le \log|\zeta_0|+(2-\delta)\log|z_2|-\log|z_1|-\log|z_2| \\ &\le \log|\zeta_0|-\left(\frac{1}{2}+\delta\right)\log|z_2|+\log 2,\end{aligned} \tag{2.10}$$

by (1.3). We can rewrite this in a more symmetric fashion:

$$\log\frac{1}{|w_2|}+\log\frac{1}{|w_3|}+\log\frac{1}{|\zeta_0|}\le\left(\frac{1}{2}+\delta\right)\log\frac{1}{|z_2|}+\log 2. \tag{2.11}$$

Since all terms are positive by (2.8), (2.9), each of the terms on the left hand side is bounded by the right hand side.

We will proceed as follows: we have used the contradiction hypothesis to prove that $|\zeta_0|$ and $|w_3|$ are relatively big. We will prove that $|\phi_{\zeta_2}(\zeta_0)|$ has to be relatively small, which by (2.9) forces $|w_4|$ to be roughly as large as $|w_3|$. This then allows us to bound $|\phi_{\zeta_1}(\zeta_2)|$ by a quantity which becomes as small as desired when ε can be made small, hence allows us to bound $|\phi_{\zeta_1}(\zeta_0)|$ by the triangle inequality.

The final contradiction will concern $w_2=\frac{z_1}{\zeta_0\phi_{\zeta_2}(\zeta_0)}$. On the one hand, (2.11) guarantees that it is not too small; but an explicit computation of the quotient w_1/w_4 shows that w_1 must be small, and by (2.8) and the estimate on $|\phi_{\zeta_1}(\zeta_0)|$, $|w_2|$ is small as well.

We provide the details. From (2.11),

$$\log|w_3|\ge\left(\frac{1}{2}+\delta\right)\log|z_2|-\log 2. \tag{2.12}$$

From (2.5) and (2.10),

$$\begin{aligned}\log|\phi_{\zeta_2}(\zeta_0)| &= \log\left|\frac{z_1}{\zeta_0}\right|-\log|w_2| \\ &\le \log\left|\frac{z_1}{\zeta_0}\right|+\log|\zeta_0|-\left(\frac{1}{2}+\delta\right)\log|z_2|+\log 2=(1-\delta)\log|z_2|+\log 2.\end{aligned} \tag{2.13}$$

Since $\delta<\frac{1}{4}$, (2.13) and (2.12) imply that for small enough values of $|z_2|$, $|\phi_{\zeta_2}(\zeta_0)|<\frac{1}{2}|w_3|$, so by (2.9) and the triangle inequality for d_G,

$$|w_4|\ge\frac{1}{2}|w_3|. \tag{2.14}$$

We now prove that both ζ_1 and ζ_2 must be close to ζ_0 and even closer to each other. First, since (2.11) implies that $\log|\zeta_0|\ge(\frac{1}{2}+\delta)\log|z_2|+\log 2$, by (2.13), for small enough values of $|z_2|$, $|\phi_{\zeta_2}(\zeta_0)|\le\frac{1}{2}|\zeta_0|$. By the triangle inequality for d_G,

$$\frac{1}{2}|\zeta_0|\le|\zeta_2|\le\frac{3}{2}|\zeta_0|. \tag{2.15}$$

On the other hand, from (2.11),

$$\log|w_3| + \log|\zeta_0| \geq \left(\frac{1}{2}+\delta\right)\log|z_2| - \log 2, \text{ i.e. } |w_3\zeta_0| \geq \frac{1}{2}|z_2|^{\delta+1/2}.$$

Therefore, applying (2.14) and (2.15),

$$|\phi_{\zeta_1}(\zeta_2)| = \left|\frac{\varepsilon}{\zeta_2 w_4}\right| \leq 4\left|\frac{\varepsilon}{\zeta_0 w_3}\right| \leq 8|\varepsilon||z_2|^{-\delta-1/2}. \tag{2.16}$$

In particular, for

$$|\varepsilon| < \frac{1}{8}|z_2|^{3/2}, \tag{2.17}$$

this implies $|\phi_{\zeta_1}(\zeta_2)| < |z_2|^{1-\delta}$, and by the triangle inequality,

$$|\phi_{\zeta_1}(\zeta_0)| < |\phi_{\zeta_2}(\zeta_0)| + |\phi_{\zeta_1}(\zeta_2)| < 3|z_2|^{1-\delta}. \tag{2.18}$$

We now establish the two (contradictory) estimates for w_2. On the one hand, (2.11) implies that

$$\log|w_2| \geq \left(\frac{1}{2}+\delta\right)\log|z_2| - \log 2, \text{ i.e. } s|w_2| \geq \frac{1}{2}|z_2|^{\delta+1/2}. \tag{2.19}$$

On the other hand,

$$\left|\frac{w_1}{w_4}\right| = \left|\frac{\varepsilon s(\varepsilon)}{\zeta_1\phi_{\zeta_2}(\zeta_1)}\frac{\zeta_2\phi_{\zeta_1}(\zeta_2)}{\varepsilon}\right| = \left|s(\varepsilon)\frac{\zeta_2}{\zeta_1}\right|.$$

By the triangle inequality for d_G, when (2.17) holds, the lower bound in (2.15) and the corollary to (2.16) imply

$$|\zeta_1| \geq |\zeta_2| - |\phi_{\zeta_1}(\zeta_2)| \geq \frac{1}{2}|\zeta_0| - |z_2|^{1-\delta} \geq \frac{1}{4}|\zeta_0|$$

for $|z_2|$ small enough, because of (2.11) again. So finally, using the upper bound in (2.15), $|\frac{w_1}{w_4}| \leq 6|s(\varepsilon)|$. We require that ε be small enough so that

$$|s(\varepsilon)| < |z_2|^{1-\delta}. \tag{2.20}$$

The triangle inequality for d_G and (2.18) imply

$$|w_2| \leq |w_1| + |\phi_{\zeta_1}(\zeta_0)| \leq 6|s(\varepsilon)| + 3|z_2|^{1-\delta} \leq 9|z_2|^{1-\delta}.$$

Finally, if we choose $|z_2|$ small enough (depending only on δ), and then ε small enough (depending on $|z_2|$), we see that this last bound contradicts (2.19). □

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