ON A SUPREMUM OPERATOR

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Abstract. For a supremum operator $R\varphi(t) := \operatorname*{ess\,sup}_{y\in[t,\infty)} \Phi(y,t)\varphi(y)$ on the semi-axis with a measurable non-negative function $\Phi(x,y)$ the weighted $L_p - L_q$ boundedness on the cone of non-increasing functions is characterized.

1. Introduction

Let $\mathbb{R}_+ := [0,\infty)$. Denote $\mathfrak{M}^+$ the set of all non-negative measurable functions on $\mathbb{R}_+$ and $\mathfrak{M}^\downarrow \subset \mathfrak{M}^+$ the subset of all non-increasing functions. For a measurable non-negative function $\Phi(x,y)$, on $\{(x,y) : x \geq y \geq 0\}$ we define the supremum operator

$$R\varphi(t) := \operatorname*{ess\,sup}_{y\in[t,\infty)} \Phi(y,t)\varphi(y), \quad \varphi \in \mathfrak{M}^\downarrow. \tag{1.1}$$

The paper is devoted to the necessary and sufficient conditions for the inequality

$$\left(\int_0^\infty [R\varphi(t)]^q \, w(t)dt\right)^{1/q} \leq C \left(\int_0^\infty \varphi^p(t)v(t)dt\right)^{1/p}, \quad \varphi \in \mathfrak{M}^\downarrow \tag{1.2}$$

with non-negative locally integrable on $\mathbb{R}_+$ weight functions v and w and a constant $C \geq 0$, independent on φ, which we suppose to be the least possible.

This problem was studied in the paper by A. Gogatishvili, B. Opic and L. Pick ([1], Theorem 3.2) in a more simple case, when $\Phi(x,y)$ is independent on y and contnious with respect to x. In our work we use the technigue of the paper [1]. With different supremum operators some similar problems were studied in [2], [3], [4], [5], [6]. This area is currently developing intensively and finds many interesting applications.

In Section 2 we give some preliminaries. In particular, we extend our result ([7], Theorem 4.1). In Sections 3 and 4 the cases $0 < p \leq q < \infty$ and $0 < q < p < \infty$ of (1.2) are considered, respectively.

We use signs := and =: for determining new quantities and $\mathbb{Z}$ for the set of all integers. For positive functionals F and G we write $F \ll G$, if $F \leq cG$ with some

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positive constant c, which depends only on irrelevant parameters. $F \approx G$ means $F \ll G \ll F$ or $F = cG$. χ_E denotes the characteristic function (indicator) of a set E. Uncertainties of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be zero. $\square$ stands for the end of proof.

2. Preliminaries

We need the following simple case of ([8], Lemma 1.2).

Lemma 2.1. *Let $f \in \mathfrak{M}^{\downarrow}$. Then there exist the sequence of non-negative finitely supported integrable functions $\{g_n\} \subset \mathfrak{M}^{+}$ such, that the functions $f_n(x) := \int_x^\infty g_n(s)ds$ are increasing with respect to n for any $x > 0$ and $f(x) = \lim_{n\to\infty} \int_x^\infty g_n(y)dy$ for almost all $x > 0$.*

Let $\alpha > 0$ and let $\Phi(y,t) \ge 0$ be a measurable function with respect to both variables on the set $\{(y,t) \in \mathbb{R}^2 : y \ge t \ge 0\}$. On the cone $\mathfrak{M}^{\downarrow}$ we define the operator

$$R_\alpha\varphi(t) := \left(\int_t^\infty \Phi^\alpha(y,t)\varphi^\alpha(y)dy\right)^{1/\alpha}, \quad \varphi \in \mathfrak{M}^{\downarrow}. \tag{2.1}$$

For $p, q \in (0,\infty)$ and weight functions v and w we define

$$J_\alpha := \sup_{\varphi\in\mathfrak{M}^{\downarrow}} \frac{\left(\int_0^\infty [R_\alpha\varphi]^q w\right)^{1/q}}{\left(\int_0^\infty \varphi^p v\right)^{1/p}}. \tag{2.2}$$

In the next assertion we extend ([7], Theorem 4.1).

Lemma 2.2. *Let $0 < p \le \min\{\alpha, q\} < \infty$ and $V(t) := \int_0^t v(s)ds$. Then $J_\alpha = \mathbb{A}_\alpha$, where*

$$\mathbb{A}_\alpha := \sup_{t>0} \left(\int_0^t \left(\int_s^t \Phi^\alpha(y,s)dy\right)^{q/\alpha} w(s)ds\right)^{1/q} V^{-1/p}(t).$$

Proof. Let $t > 0$ and $\varphi_t(s) := \chi_{[0,t]}(s)$. Then

$$R_\alpha\varphi_t(s) = \chi_{[0,t]}(s)\left(\int_s^t \Phi^\alpha(y,s)dy\right)^{1/\alpha}$$

and

$$J_\alpha \ge \left(\int_0^t \left(\int_s^t \Phi^\alpha(y,s)dy\right)^{q/\alpha} w(s)ds\right)^{1/q} V^{-1/p}(t).$$

Hence, $J_\alpha \ge \mathbb{A}_\alpha$.

Conversely, we make in J_α the change of variable $\varphi^\alpha \to \varphi$. Then

$$J_\alpha^q = \sup_{\varphi\in\mathfrak{M}\downarrow} \frac{\int_0^\infty \left(\int_t^\infty \Phi^\alpha(y,t)\varphi(y)dy\right)^{q/\alpha} w(t)dt}{\left(\int_0^\infty \varphi^{p/\alpha}v\right)^{q/p}}.$$

Let $\mathbb{A}_\alpha < \infty$ and suppose first for $\varphi \in \mathfrak{M}^{\downarrow}$ the representation $\varphi(y) = \int_y^\infty h(s)ds$ holds with a non-negative finitely supported integrable function h. Then

$$\varphi(y) = \left(\frac{p}{\alpha}\right)^{\alpha/p} \left(\int_y^\infty \left(\int_s^\infty h \right)^{\frac{p}{\alpha}-1} h(s)ds \right)^{\alpha/p}. \tag{2.3}$$

Using this representation and Minkowskii's inequality we find

$$\begin{aligned}
&\int_t^\infty \Phi^\alpha(y,t)\varphi(y)dy \\
&= \left(\frac{p}{\alpha}\right)^{\alpha/p} \int_t^\infty \Phi^\alpha(y,t) \left(\int_y^\infty \left(\int_s^\infty h \right)^{\frac{p}{\alpha}-1} h(s)ds \right)^{\alpha/p} dy \\
&\le \left(\frac{p}{\alpha}\right)^{\alpha/p} \left(\int_t^\infty \left(\int_s^\infty h \right)^{\frac{p}{\alpha}-1} h(s)ds \left(\int_t^s \Phi^\alpha(y,t)dy \right)^{p/\alpha} ds \right)^{\alpha/p}.
\end{aligned}$$

Now, again applying Minkowskii's inequality, we obtain

$$\begin{aligned}
&\int_0^\infty \left(\int_t^\infty \Phi^\alpha(y,t)\varphi(y)dy \right)^{q/\alpha} w(t)dt \\
&\le \left(\frac{p}{\alpha}\right)^{q/p} \int_0^\infty \left(\int_t^\infty \left(\int_s^\infty h \right)^{\frac{p}{\alpha}-1} h(s) \left(\int_t^s \Phi^\alpha(y,t)dy \right)^{p/\alpha} ds \right)^{q/p} w(t)dt \\
&\le \left(\frac{p}{\alpha}\right)^{q/p} \left(\int_0^\infty \left(\int_s^\infty h \right)^{\frac{p}{\alpha}-1} h(s) \left(\int_0^s \left(\int_t^s \Phi^\alpha(y,t)dy \right)^{q/\alpha} w(t)dt \right)^{p/q} ds \right)^{q/p} \\
&\le \mathbb{A}_\alpha^q \left(\frac{p}{\alpha}\right)^{q/p} \left(\int_0^\infty \left(\int_s^\infty h \right)^{\frac{p}{\alpha}-1} h(s)V(s)ds \right)^{q/p} \\
&= \mathbb{A}_\alpha^q \left(\int_0^\infty \varphi^{\frac{p}{\alpha}} v \right)^{q/p}.
\end{aligned}$$

The proof of the upper bound $J_\alpha \le \mathbb{A}_\alpha$ now follows by Lemma 2.1 and by the Theorem on Monotone convergence. □

3. The case $p \le q$

Let $\Phi(x,y) \ge 0$ be a measurable function with respect to both variables on the set $\{(x,y) \in \mathbb{R}^2 : x \ge y \ge 0\}$. Put

$$\Phi_\infty(x,y) := \operatorname*{esssup}_{s\in[y,x]} \Phi(s,y) \tag{3.1}$$

Proposition 3.1. *If* $\varphi \in \mathfrak{M}^{\downarrow}$, *then*

$$\operatorname*{esssup}_{y\in[t,\infty)} \Phi(y,t)\varphi(y) = \sup_{s\in[t,\infty)} \varphi(s)\Phi_\infty(s,t)$$

Proof. It follows from the properties of the essential supremum that

$$\operatorname*{esssup}_{y\in[t,\infty)} \Phi(y,t)\varphi(y) = \sup_{s\geq t} \operatorname*{esssup}_{y\in[t,s]} \Phi(y,t)\varphi(y)$$

$$\geq \sup_{s\geq t} \varphi(s) \operatorname*{esssup}_{y\in[t,s]} \Phi(y,t) = \sup_{s\in[t,\infty)} \varphi(s)\Phi_\infty(s,t). \tag{3.2}$$

To prove the inverse we use that $\Phi(y,t) \leq \Phi_\infty(y,t)$ for almost all $y \in [t,\infty)$. Then

$$\operatorname*{esssup}_{y\in[t,\infty)} \Phi(y,t)\varphi(y) \leq \operatorname*{esssup}_{y\in[t,\infty)} \Phi_\infty(y,t)\varphi(y) = \sup_{y\in[t,\infty)} \Phi_\infty(y,t)\varphi(y). \qquad \square$$

Now, on the cone $\mathfrak{M}^{\downarrow}$ we consider the operator

$$R\varphi(t) := \operatorname*{esssup}_{y\in[t,\infty)} \Phi(y,t)\varphi(y), \quad \varphi \in \mathfrak{M}\downarrow, \tag{3.3}$$

which might be interpreted as the extremal for the set of operators (2.1) when $\alpha \to \infty$.

It follows from Proposition 3.1 that

$$R\varphi(t) = \sup_{s\geq t} \Phi_\infty(s,t)\varphi(s). \tag{3.4}$$

Therefore, without a loss of generality, the function $\Phi(y,t)$ in the definition (3.3) we may and shall assume non-decreasing with respect to y for $y \geq t$.

Analogously with (2.2) we set

$$J := \sup_{\varphi\in\mathfrak{M}^{\downarrow}} \frac{\left(\int_0^\infty [R\varphi]^q w\right)^{1/q}}{\left(\int_0^\infty \varphi^p v\right)^{1/p}}. \tag{3.5}$$

Theorem 3.2. *Let* $0 < p \leq q < \infty$. *Then* $J = \mathbb{A}$, *where*

$$\mathbb{A} := \sup_{t>0} \left(\int_0^t \Phi_\infty^q(t,s)w(s)ds\right)^{1/q} V^{-1/p}(t) \tag{3.6}$$

or, equivalently,

$$\mathbb{A} := \sup_{t>0} \left(\int_0^t \left[\operatorname*{esssup}_{y\in[t,s]} \Phi(y,s)\right]^q w(s)ds\right)^{1/q} V^{-1/p}(t). \tag{3.7}$$

Proof. Let $t > 0$ and $\varphi_t(x) := \chi_{[0,t]}(x)$. Then

$$R\varphi_t(x) = \sup_{y\geq x} \Phi_\infty(y,x)\chi_{[0,t]}(y) = \chi_{[0,t]}(x)\Phi_\infty(t,x).$$

We have

$$J \geq \left(\int_0^t \Phi_\infty(t,s) w(s) ds \right)^{1/q} V^{-1/p}(t), \;\; t > 0.$$

Hence, $J \geq \mathbb{A}$.

On the right hand side of (3.5) we make the change of variable $\varphi^q \to \varphi$. Then, using (3.4), we find

$$J^q = \sup_{\varphi \in \mathfrak{M}^\downarrow} \frac{\int_0^\infty \left[\sup_{y \geq t} \Phi_\infty^q(y,t)\varphi(y)dy\right] w(t)dt}{\left(\int_0^\infty \varphi^{p/q} v\right)^{q/p}}.$$

Applying representation (2.3) $\alpha = q$, we obtain

$$\begin{aligned}
\sup_{y \geq t} \Phi_\infty^q(y,t)\varphi(y) &= \left(\frac{p}{q}\right)^{\alpha/p} \sup_{y \geq t} \Phi_\infty^q(y,t) \left(\int_y^\infty \left(\int_s^\infty h \right)^{\frac{p}{q}-1} h(s) ds \right)^{q/p} \\
&\leq \left(\frac{p}{q}\right)^{q/p} \sup_{y \geq t} \left(\int_y^\infty \left(\int_s^\infty h \right)^{\frac{p}{q}-1} \Phi_\infty^q(s,t) h(s) ds \right)^{q/p}.
\end{aligned}$$

Now, applying Minkowskii's inequality, we get

$$\begin{aligned}
&\int_0^\infty \left[\sup_{y \geq t} \Phi_\infty^q(y,t)\varphi(y) \right] w(t)dt \\
&\leq \left(\frac{p}{q}\right)^{q/p} \int_0^\infty \left(\int_t^\infty \left(\int_s^\infty h \right)^{\frac{p}{q}-1} \Phi_\infty^p(s,t) h(s) ds \right)^{q/p} \\
&\leq \mathbb{A}^q \left(\frac{p}{q}\right)^{q/p} \int_0^\infty \left(\int_t^\infty \left(\int_s^\infty h \right)^{\frac{p}{q}-1} h(s) V(s) ds \right)^{q/p} \\
&= \mathbb{A}^q \left(\varphi^{p/q} v\right)^{q/p}.
\end{aligned}$$

The finish of the proof of the upper bound $J \leq \mathbb{A}$ follows by applying Lemma 2.1 and Monotone Convergence Theorem . □

Remark 3.3. In the case

$$\Phi(y,t) = \operatorname*{esssup}_{s \in [t,y]} u(s),$$

where $u(s) \geq 0$ is a measurable function, we obtain

$$R\varphi(t) = \operatorname*{esssup}_{s \in [t,\infty]} u(s)\varphi(s)$$

and Theorem 3.2 extends ([1], Theorem 3.2 (i)), where the function $u(s)$ was supposed to be continuous.

4. The case $q < p$

Definition 4.1. A measurable function $\Phi(x,y) \geq 0$ on $\{(x,y) : x \geq y \geq 0\}$, we name *Oinarov kernel*, $\Phi(x,y) \in \mathcal{O}$, if there exist a constant $D \geq 1$, independent of x, y and z such, that

$$D^{-1}\left(\Phi(x,z) + \Phi(z,y)\right) \leq \Phi(x,y) \leq D\left(\Phi(x,z) + \Phi(z,y)\right) \tag{4.1}$$

for all $x \geq z \geq y \geq 0$.

Proposition 4.2. *Let $\Phi(x,y) \in \mathcal{O}$. Then $\Phi_\infty(x,y) \in \mathcal{O}$.*

Proof. For all $x \geq y \geq 0$ from (3.1) and (3.2) we have

$$\Phi_\infty(x,y) = \operatorname*{esssup}_{s\in[y,x]} \Phi(s,y) = \sup_{s\in[y,x]} \operatorname*{esssup}_{t\in[y,s]} \Phi(t,y) = \sup_{s\in[y,x]} \Phi_\infty(s,y). \tag{4.2}$$

Then $\Phi_\infty(x,y)$ is non-decreasing with respect to x for $x \in [y,\infty)$. Similarly, using (4.1), we find

$$\Phi_\infty(x,y) = \operatorname*{esssup}_{s\in[y,x]} \Phi(s,y) = \sup_{s\in[y,x]} \operatorname*{esssup}_{t\in[s,x]} \Phi(t,y) \geq D^{-1} \sup_{s\in[y,x]} \Phi(s,y) \geq D^{-1}\Phi(x,y).$$

Therefore,

$$\Phi(x,y) \leq D\Phi_\infty(x,y) \tag{4.3}$$

for all $x \geq y \geq 0$. Let $x \geq z \geq y \geq 0$. hen it follows from (4.2), that

$$\Phi_\infty(x,y) \geq \Phi_\infty(z,y). \tag{4.4}$$

Moreover, again using (3.2) and (4.1), we find

$$\Phi_\infty(x,y) = \sup_{s\in[y,x]} \operatorname*{esssup}_{t\in[s,x]} \Phi(t,y) \geq \sup_{s\in[z,x]} \operatorname*{esssup}_{t\in[s,x]} \Phi(t,y)$$

$$\geq D^{-1} \sup_{s\in[z,x]} \operatorname*{esssup}_{t\in[s,x]} \Phi(t,s) = D^{-1} \sup_{s\in[z,x]} \Phi_\infty(x,s) \geq D^{-1}\Phi_\infty(x,z). \tag{4.5}$$

From this and (4.4) the left hand side of (4.1) follows for $\Phi_\infty(x,y)$. Let $x \geq y \geq 0$ and $z \in [y,x]$, $s \in [y,x]$. Then it follows from (4.1) that

$$\begin{aligned}
\Phi(s,y) &= \chi_{[y,z]}(s)\Phi(s,y) + \chi_{[z,x]}(s)\Phi(s,y) \\
&\leq D\left(\chi_{[y,z]}(s)\Phi(z,y) + \chi_{[z,x]}(s)\left(\Phi(s,z) + \Phi(z,y)\right)\right) \\
&= D\left(\Phi(z,y) + \chi_{[z,x]}(s)\Phi(s,z)\right).
\end{aligned}$$

From this and (4.3)

$$\Phi_\infty(x,y) \leq D\left(\Phi(z,y) + \operatorname*{esssup}_{s\in[z,x]} \Phi(s,z)\right) \leq D^2\left(\Phi_\infty(z,y) + \Phi_\infty(x,z)\right).$$

Hence, $\Phi_\infty(x,y) \in \mathcal{O}$ with a constant D^2 in (4.1). □

Remark 4.3. If $0 < q < \infty$ and $\Phi(x,y) \in \mathcal{O}$, that is (4.1) holds, then $\Phi^q(x,y) \in \mathcal{O}$, so that (4.1) holds with some constant D_q, dependant only on D and q.

Theorem 4.4. *Let* $0 < q < p < \infty$, $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$ *and* $\Phi(x,y) \in \mathcal{O}$. *Suppose* $\Phi(x,y)$ *be continuous with respect to* x *for* $x \in [y,\infty)$ *for all* $y \geq 0$ *and assume that the weight functions* v *and* w *such, that* $0 < V(t) := \int_0^t v < \infty$, $0 < W(t) := \int_0^t w < \infty$ *and* $V(\infty) = W(\infty) = \infty$. *If the supremum operator* R *and the functional* J *are defined by* (3.3) *and* (3.5), *respectively, and*

$$B := \left(\sup_{\{x_k\}} \sum_k \left(\int_{x_k}^{x_{k+1}} \Phi_\infty^q(x_{k+1},t) w(t)dt \right)^{\frac{r}{q}} [V(x_{k+1})]^{-\frac{r}{p}} \right)^{\frac{1}{r}}, \tag{4.6}$$

where the sup *is taken over all increasing sequences* $\{x_k\} \subset \mathbb{R}_+$, *then*

$$J \approx B. \tag{4.7}$$

Proof. We start with the proof of the upper bound $J \ll B$. To this end we note, that because of (3.4) and (4.5) we have $R\varphi(s) \leq DR\varphi(t)$ for all $s \geq t \geq 0$, if $\varphi \in \mathfrak{M}^{\downarrow}$. Let $a > 1$ be a number, which we choose later and let $\{x_k\}, \{y'_k\} \subset \mathbb{R}_+$ be such increasing sequences, that

$$W(x_k) = V(y'_k) = a^k, \quad k \in \mathbb{Z}. \tag{4.8}$$

We have

$$\int_0^\infty (R\varphi)^q w \leq D^q \sum_k [R\varphi(x_k)]^q \int_{x_k}^{x_{k+1}} w = D^q(a-1) \sum_k a^k \sup_{s \geq x_k} [\Phi_\infty(s,x_k)\varphi(s)]^q$$
$$=: D^q(a-1) \sum_k a^k I_k.$$

Since $\Phi(x,y) \in \mathcal{O}$, then $\Phi_\infty(x,y) \in \mathcal{O}$ by Proposition 4.2 and by Remark 4.3 we have $\Phi_\infty^q(x,y) \in \mathcal{O}$. Let $D_q \geq 1$ be a constant, for which (4.1) holds for $\Phi_\infty^q(x,y)$. Then, applying (4.1) with $\Phi_\infty^q(x,y)$ and D_q, we obtain

$$\begin{aligned}
I_k &\leq \sup_{x_k \leq s \leq x_{k+1}} \Phi_\infty^q(s,x_k)\varphi(s) + \sup_{s \geq x_{k+1}} \Phi_\infty^q(s,x_k)\varphi(s) \\
&\leq \sup_{x_k \leq s \leq x_{k+1}} \Phi_\infty^q(s,x_k)\varphi(s) \\
&\quad + D_q \left[\Phi_\infty^q(x_{k+1},x_k)\varphi(x_{k+1}) + \sup_{s \geq x_{k+1}} \Phi_\infty^q(s,x_{k+1})\varphi(s) \right] \\
&\leq (1+D_q) \sup_{x_k \leq s \leq x_{k+1}} \Phi_\infty^q(s,x_k)\varphi(s) + D_q \sup_{s \geq x_{k+1}} \Phi_\infty^q(s,x_{k+1})\varphi(s) \\
&=: (1+D_q)L_k + D_q I_{k+1}.
\end{aligned}$$

We find from this

$$\begin{aligned}
I &:= \sum_k a^k I_k \leq (1+D_q) \sum_k a^k L_k + D_q \sum_k a^k I_{k+1} \\
&= (1+D_q) \sum_k a^k L_k + \frac{D_q}{a} \sum_k a^k I_k = (1+D_q) \sum_k a^k L_k + \frac{D_q}{a} I.
\end{aligned}$$

Now we choose $a > 1$ such, that $a > 2D_q$. Then $I \le 2(1 + D_q) \sum_k a^k L_k$. Consequently,

$$\int_0^\infty (R\varphi)^q w \ll \sum_k a^k \left[\sup_{x_k \le s \le x_{k+1}} \Phi_\infty(s, x_k)\varphi(s) \right]^q.$$

It follows from the continuity of $\Phi(x, y)$ that $\Phi_\infty(x, y)$ is continuous. Moreover, taking into account Proposition 3.1 without a loss of generality we may and shall assume $\Phi_\infty^q(s, x_k)\varphi(s)$ to be continuous on $[x_k, x_{k+1}]$. while the upper bound is proving. Then there exist a point $z_k \in [x_k, x_{k+1}]$ such, that

$$\sup_{x_k \le s \le x_{k+1}} \Phi_\infty(s, x_k)\varphi(s) \le a^{1/q} \Phi_\infty(z_k, x_k)\varphi(z_k).$$

Now using (4.1) and $z_{k-2} \le x_{k-1} < x_k \le z_k$, it follows, that

$$\begin{aligned}
\int_0^\infty (R\varphi)^q w &\ll \sum_k \left(\int_{x_{k-1}}^{x_k} w(t)dt \right) \Phi_\infty^q(z_k, x_k)\varphi^q(z_k) \\
&\ll \sum_k \left(\int_{x_{k-1}}^{x_k} w(t)\Phi_\infty^q(z_k, t)dt \right) \varphi^q(z_k) \\
&\le \sum_k \left(\int_{z_{k-2}}^{z_k} w(t)\Phi_\infty^q(z_k, t)dt \right) \varphi^q(z_k) \\
&= \sum_k \left(\int_{z_{2k-2}}^{z_{2k}} w(t)\Phi_\infty^q(z_{2k}, t)dt \right) \varphi^q(z_{2k}) \\
&\quad + \sum_k \left(\int_{z_{2k-1}}^{z_{2k+1}} w(t)\Phi_\infty^q(z_{2k+1}, t)dt \right) \varphi^q(z_{2k+1}) \\
&=: S_{even} + S_{odd}.
\end{aligned}$$

Further we estimate only S_{even}, the arguments for S_{odd} are similar. Put

$$Y_k := \{l \in \mathbb{Z} : y_l' \in [z_{2k-2}, z_{2k}]\}, \quad Y := \{k \in \mathbb{Z} : Y_k \ne \emptyset\},$$

where y_l' are taken from the definition (4.8). Denote

$$\theta_k := \min\{y_l' : l \in Y_k\}, k \in Y; \;\; \Theta := \{\theta_k\}_{k \in Y} \subset \{y_l'\}_{l \in \mathbb{Z}}$$

and renumerate Θ so, that $\Theta =: \{y_n\}_{n \in \mathbb{Z}}$ and $y_n < y_{n+1}$. It is shown in the proof of ([1], Theorem 3.2 (ii)) that if $n, k \in \mathbb{Z}$ such, that $y_n < z_{2k} \le y_{n+1}$, then

$$[\varphi(z_{2k})]^q \ll \left(\int_0^{y_{n+1}} v \right)^{-q/p} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p v \right)^{q/p}. \tag{4.9}$$

Denote

$$A_n := \{k \in \mathbb{Z} : y_n < z_{2k} \le y_{n+1}\}, \;\; n \in \mathbb{Z}.$$

Then

$$S_{even} = \sum_n \sum_{A_n} \left(\int_{z_{2k-2}}^{z_{2k}} \Phi_\infty^q(z_{2k}, t) w(t) dt \right) \varphi^q(z_{2k}).$$

It follows from the properties of $\Phi_\infty(x, y) \in \mathcal{O}$ that

$$\Phi_\infty(z_{2k}, t) \ll \sup_{t \le s \le y_{n+1}} \Phi_\infty(s, t) \ll \Phi_\infty(y_{n+1}, t). \tag{4.10}$$

Applying (4.9), (4.10) and Hölder's inequality, we find

$$S_{even} \ll \sum_n \left(\int_0^{y_{n+1}} v \right)^{-q/p} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p v \right)^{q/p} \sum_{A_n} \int_{z_{2k-2}}^{z_{2k}} \Phi_\infty^q(y_{n+1}, t) w(t) dt$$

$$\le \sum_n \int_{y_{n-1}}^{y_{n+1}} \Phi_\infty^q(y_{n+1}, t) w(t) dt \left(\int_0^{y_{n+1}} v \right)^{-q/p} \left(\int_{y_{n-1}}^{y_{n+1}} \varphi^p v \right)^{q/p}$$

$$\le \left(\sum_n \left(\int_{y_{n-1}}^{y_{n+1}} \Phi_\infty^q(y_{n+1}, t) w(t) dt \right)^{r/q} \left(\int_0^{y_{n+1}} v \right)^{-r/p} \right)^{q/r} \left(\sum_n \int_{y_{n-1}}^{y_{n+1}} \varphi^p v \right)^{q/p}$$

$$\ll B^q \left(\int_0^\infty \varphi^p v \right)^{q/p}.$$

From this and analogous bound for S_{odd} the upper bound $J \ll B$ follows.

Let $\{x_k\} \subset \mathbb{R}_+$ be an arbitrary increasing sequence, N - any positive integer. The proof of the lower bound $J \gg B$ is proceeding with the help of the test function

$$\varphi_N(t) := \chi_{(0, x_{-N})}(t) \left(\sum_{i=-N}^N \alpha_i \right)^{1/p} + \sum_{k=-N}^N \chi_{[x_k, x_{k+1}]}(t) \left(\sum_{i=k}^N \alpha_i \right)^{1/p},$$

where

$$\alpha_i := \left(\int_{x_i}^{x_{i+1}} \Phi_\infty^q(x_{i+1}, t) w(t) dt \right)^{r/q} V^{-r/q}(x_{i+1}).$$

We have

$$\int_0^\infty [R\varphi_N]^q w \ge \sum_{k=-N}^N \int_{x_k}^{x_{k+1}} \sup_{t \le s \le x_{k+1}} \Phi_\infty^q(s, t) w(t) dt \left(\sum_{i=k}^N \alpha_i \right)^{q/p}$$

$$\ge \sum_{k=-N}^N \int_{x_k}^{x_{k+1}} \sup_{t \le s \le x_{k+1}} \Phi_\infty^q(x_{k+1}, t) w(t) dt \alpha_k^{q/p}$$

$$= \sum_{k=-N}^N \left(\int_{x_k}^{x_{k+1}} \Phi_\infty^q(x_{k+1}, t) w(t) dt \right)^{r/q} V^{-r/p}(x_{k+1}) =: B_N^r.$$

On the other hand

$$\begin{aligned}\int_0^\infty \varphi_N^p v &= \sum_{k=-N}^{N} \alpha_k \left(\int_0^{x_{-N}} v\right) + \sum_{k=-N}^{N}\sum_{i=k}^{N} \alpha_i \left(\int_{x_k}^{x_{k+1}} v\right) \\ &= \sum_{i=-N}^{N} \alpha_i \left(\int_0^{x_{-N}} v\right) + \sum_{i=-N}^{N} \alpha_i \left(\sum_{k=-N}^{i} \int_{x_k}^{x_{k+1}} v\right) \\ &= \sum_{i=-N}^{N} \alpha_i \left(\int_0^{x_{i+1}} v\right) = B_N^r.\end{aligned}$$

Consequently, $J \gg B_N$ and the lower bound $J \gg B$ follows. □

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