ON Σ^1_1 EQUIVALENCE RELATIONS OVER THE NATURAL NUMBERS

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Abstract. We study the structure of Σ^1_1 equivalence relations on hyperarithmetical subsets of ω under reducibilities given by hyperarithmetical or computable functions, called h-reducibility and FF-reducibility, respectively. We show that the structure is rich even when one fixes the number of properly Σ^1_1 (i.e. Σ^1_1 but not Δ^1_1) equivalence classes. We also show the existence of incomparable Σ^1_1 equivalence relations that are complete as subsets of $\omega \times \omega$ with respect to the corresponding reducibility on sets. We study complete Σ^1_1 equivalence relations (under both reducibilities) and show that existence of infinitely many properly Σ^1_1 equivalence classes that are complete as Σ^1_1 sets (under the corresponding reducibility on sets) is necessary but not sufficient for a relation to be complete in the context of Σ^1_1 equivalence relations.

1. Introduction

In [8, 10] the notion of hyperarithmetical and computable reducibility of Σ^1_1 equivalence relations on hyperarithmetical subsets of ω was used to study the question of completeness of natural equivalence relations on hyperarithmetical classes of computable structures as a special class of Σ^1_1 equivalence relations on ω. In this paper we use this approach to study the structure of Σ^1_1 equivalence relations on ω as a whole.

In descriptive set theory, the study of definable equivalence relations under Borel reducibility has developed into a rich area. The notion of Borel reducibility allows one to compare the complexity of equivalence relations on Polish spaces, for details see e.g. [12, 15, 16]. As proved by Louveau and Velickovic in [20], the partial order of inclusion modulo finite sets on $\mathcal{P}(\omega)$ can be embedded into the partial order of Borel equivalence relations modulo Borel reducibility. Thus, the structure of Borel equivalence relations under $\leq_B$ is shown to be very rich.

In computable model theory equivalence relations have also been a subject of study, e.g. [2, 5, 17], etc. In these papers equivalence relations of rather low complexity were studied (computable, Σ^0_1, Π^0_1, having degree in the Ershov hierarchy). In [8] Σ^1_1 equivalence relations on computable structures were investigated. The authors used the notions of hyperarithmetical and computable reducibility

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of Σ^1_1 equivalence relations on ω to estimate the complexity of natural equivalence relations on hyperarithmetical classes of computable structures.

In this paper we take up the general theory of Σ^1_1 equivalence relations on hyperarithmetical subsets of ω. We show that the general structure of Σ^1_1 equivalence relations on hyperarithmetical subsets of ω under reducibilities given by hyperarithmetical or computable functions is very rich. Namely, the structure of Σ^1_1 sets under hyperarithmetical many-one-reducibility (*hm*-reducibility) is embeddable into the structure of Σ^1_1 equivalence relations under reducibility given by a hypearithmetical function. Moreover, this embedding can be taken to have range within the class of Σ^1_1 equivalence relations with a unique properly Σ^1_1 equivalence class. Furthermore, we show that there are properly Σ^1_1 equivalence relations with only finite equivalence classes, and there are Σ^1_1 relations with exactly n properly Σ^1_1 equivalence classes, for $n \leq \omega$. We also show that a Σ^1_1 equivalence relation with infinitely many properly (moreover, *hm*-complete) Σ^1_1 classes need not be complete with respect to the hyperarithmetical reducibility.

2. Background

Here we list some definitions and facts that we will use throughout the paper. We assume the familiarity with the main notions from recursion theory. The standard references are [23, 25].

2.1. **Linear orderings.**

Definition 1. *Let K be a class of structures closed under isomorphism and K^c be the set of its computable members.*

1. *An* enumeration *of $K^c/_{\cong}$ is a sequence $(\mathcal{A}_n)_{n\in\omega}$ of elements of K^c representing each isomorphism type in K^c at least once.*
2. *An enumeration $(\mathcal{A}_n)_{n\in\omega}$ of $K^c/_{\cong}$ is* computable (hyperarithmetical) *if there is a computable (hyperarithmetical) function f which, for every n, gives a computable index $f(n)$ for the computable structure $\mathcal{A}_n$.*

As proved in [14]:

Proposition 1. *There exists a computable enumeration of all isomorphism types for computable linear orderings.*

Thus, we can consider ω as a set of effective codes for computable linear orderings. We will denote by L_n the n-th computable linear order in this enumeration. We will abbreviate the set of codes for linear orderings as LO and the set of codes for well-orderings as WO.

Theorem 1 (e.g. [23], Chapter 16, Corollary XXa)**.** *The set* WO *is a* Π^1_1*-complete set, moreover there exists a computable function $f(z,x)$ such that for every z, the Π^1_1 set with the Π^1_1 index z is 1-reducible to* WO *by the function $\lambda x[f(z,x)]$.*

In view of Theorem 1 one can think about Π^1_1 sets in the following way. Let A be a Π^1_1 set and let m be its Π^1_1 index. Then for every $x \in A$, the ordinal isomorphic to $L_{f(m,x)}$ may be considered as "the level" at which the membership of x is determined.

Theorem 2 (Bounding). *For each computable ordinal α, let WO_α denote the set of codes for computable well-orderings isomorphic to an ordinal less than α. Then if F is a hyperarithmetical function from a hyperarithmetical subset of ω into* WO*, there exists a computable α such that the range of F is contained in WO_α.*

Theorem 3 (Uniformization). *Every Π^1_1 binary relation on $X \times Y$, where $X, Y \subseteq \omega$ are hyperarithmetical contains a Π^1_1 (hyperarithmetical) function with the same domain.*

2.2. **Reducibilities on Σ^1_1 equivalence relations.** The following definitions were introduced in [8][1]:

Definition 2. *Let E, E' be Σ^1_1 equivalences relations on hyperarithmetical subsets $X, Y \subseteq \omega$, respectively.*

(1) The relation E is h*-reducible to E', denoted by $E \leq_{\mathrm{h}} E'$, iff there exists a hyperarithmetical function f such that for all $x, y \in X$,*

$$xEy \iff f(x)E'f(y).$$

(2) The relation E is FF*-reducible to E', denoted by $E \leq_{\mathrm{FF}} E'$, iff there exists a partial computable function f with $X \subseteq dom(f), f[X] \subseteq Y$ such that for all $x, y \in X$,*

$$xEy \iff f(x)E'f(y).$$

Remark. A definition analogous to that of FF-reducibility was introduced in [1] for the case of c.e. equivalence relations.

Definition 3. *We say that equivalence relations E, F are* h*-equivalent (*FF*-equivalent), denoted by $E \equiv_{\mathrm{h}} F$ ($E \equiv_{\mathrm{FF}} F$, respectively), if $E \leq_{\mathrm{h}} F$ and $F \leq_{\mathrm{h}} E$ ($E \leq_{\mathrm{FF}} F$ and $F \leq_{\mathrm{FF}} E$, respectively).*

Obviously, every Σ^1_1 equivalence relation on a hyperarithmetical subset of ω is h-equivalent to a Σ^1_1 equivalence relation on ω. For FF-reducibility the situation is different:

Fact 1. *There exists a Σ^1_1 equivalence relation E on a hyperarithmetical subset X of ω such that for no Σ^1_1 equivalence relation E' on ω, $E \equiv_{\mathrm{FF}} E'$.*

[1]In [8], we used the term "*tc*-reducible" for "FF-reducible", by analogy with the reducibility defined in [3] for classes of countable structures. Later J. Knight suggested the term "FF-reducibility" which was used in [10]. In the current work we follow J. Knight's suggestion.

Proof. Consider an arbitrary Σ^1_1 equivalence relation on a hyperarithmetical set X and suppose there exists a relation E' on ω such that $E \equiv_{\mathrm{FF}} E'$. Let f be a computable function which witnesses $E' \leq_{\mathrm{FF}} E$. Then $f(\omega)$ is a c.e. subset of X. Therefore if a Σ^1_1 equivalence relation is defined on a hyperarithmetical set without a c.e. subset, it cannot be FF-equivalent to an equivalence relation on ω. □

From [13], every *computable* equivalence relation on ω is FF-equivalent to one of the following:

(1) For some finite n, the equivalence relation $x \equiv y \ mod \ n$, which defines a computable equivalence relation with exactly n infinite equivalence classes and no finite classes.
(2) The equality relation, which defines a computable equivalence relation with infinitely many classes of size one, and no other classes.

Thus, the partial ordering of the computable equivalence structures, modulo the FF-reducibility, is isomorphic to $\omega + 1$.

In the current paper we are mostly interested in properly Σ^1_1 equivalence relations, i.e. equivalence relations that are Σ^1_1 but not Δ^1_1. The reason is the following:

Fact 2. *Let id_ω denote the equality on ω.*

(1) $id_\omega \leq_{\mathrm{h}} E$ for any Σ^1_1 equivalence relation E with infinitely many equivalence classes.
(2) Any Δ^1_1 equivalence relation on a hyperarithmetical subset of ω is h*-reducible to id_ω.*

Proof. Define a function $f\colon \omega \to X$, where $X = dom(E)$ is hyperarithmetical, in the following way:

$$f(x) = \mu y[y \in X \& \bigwedge_{z \leq x} \neg f(z) E y].$$

By its definition, f is a Π^1_1 function with $dom(f) = \omega$, thus f is a hyperarithmetical function. Obviously, $x = y \iff f(x) E f(y)$.

To prove the second statement, let E be a Δ^1_1 equivalence relation on a hyperarithmetical set X. Without loss of generality we assume $0 \notin X$. Consider a function $f(x)$ defined on X in the following way:

$$f(x) = \mu z[xEz].$$

For $x \notin X$ define $f(x) = 0$. Then the function f is hyperarithmetical and $xEy \Longleftrightarrow f(x) = f(y) \neq 0$. □

Therefore all the Δ^1_1 equivalence relations on ω with infinitely many equivalence classes are h-equivalent.

The question we study in the present paper is the following:

Question 1. *How complicated is the structure of all Σ^1_1 equivalence relations on ω under* h*-reducibility (or* FF*-reducibility)?*

2.3. **Hyperarithmetical many-one reducibility on Σ^1_1 sets.** In what follows we use the standard notions of m-reducibility and 1-reducibility [25]:

Definition 4.

(1) *A set $A \subseteq \omega$ is* many-one reducible *(m-reducible) to a set $B \subseteq \omega$, denoted by $A \leq_m B$, if there exists a computable function f such that for every $n \in \omega$,*
$$n \in A \iff f(n) \in B.$$
(2) *A set $A \subseteq \omega$ is* 1-reducible *to a set $B \subseteq \omega$, denoted by $A \leq_1 B$, if A is m-reducible to B via a $1-1$ computable function.*

These reducibilities will be useful for the study of the structure of Σ^1_1 equivalence relations with respect to FF-reducibility.

Consider a hyperarithmetical version of the m-reducibility on subsets of ω. It will play an important role in the investigation of complexity of the structure of Σ^1_1 equivalence relations relative to h-reducibility.

Definition 5. *Let A, B be subsets of ω. We say that A is hyperarithmetically m-reducible to B, denoted by $A \leq_{hm} B$, iff there exists a hyperarithmetical function f with $A \subseteq dom(f)$, such that for every $n \in \omega$,*
$$n \in A \iff f(n) \in B.$$

Every equivalence relation can also be considered as a set of pairs, thus, compared to other sets via m- or hm-reducibilities. The following is straightforward:

Fact 3. *Let E, F be Σ^1_1 equivalence relations on hyperarithmetical subsets of ω.*

(1) *If $E \leq_{\mathrm{FF}} F$ then $E \leq_m F$;*
(2) *if $E \leq_{\mathrm{h}} F$ then $E \leq_{hm} F$.*

We state that the structure of hm-degrees of Σ^1_1 subsets of ω is rather complicated.

Theorem 4. *The countable atomless Boolean algebra may be embedded into the hm-degrees of Π^1_1 subsets of ω.*

Proof. We start as in the proof of Theorem 2.1, Chapter IX in [25]. Let $(\alpha_i)_{i\in\omega}$ be a uniformly computable sequence of computable subsets of ω which form a dense Boolean algebra under $\cup, \cap$. For each $i \in \omega$, we are going to build a Π^1_1 set A_i such that the mapping
$$\alpha \mapsto A_\alpha = \{\langle i, x\rangle | i \in \alpha, x \in A_i\}$$
gives the desired embedding, i.e.,

(1) $\alpha \subseteq \beta$ iff $A_\alpha \leq_{hm} A_\beta$;
(2) $\deg(A_{\alpha\cap\beta}) \leq \deg(A_\alpha), deg(A_\beta)$;

(3) $\deg(A_{\alpha\cup\beta}) \geq deg(A_\alpha), \deg(A_\beta)$.

Notice that the implication from left to right of the first property, as well as the second and the third properties follow from the definition of $\mathcal{A}_{\alpha_i}$. To ensure the implication from right to left of the first property, we will use the ideas of metarecursion [24]. We will build the Π^1_1 sets A_i's in ω_1^{CK} steps in such a way that no A_i is hm-reducible to the set $A_{\neq i} = \{\langle k, x\rangle | k \in \omega, k \neq i, x \in A_k\}$.

The whole construction will take now ω_1^{CK} steps, but as only the Π^1_1 subsets of ω are considered, there will be only ω-many requirements. Thus, each of them may be injured only finitely many times. This approach is used, for example, in [24], Chapter VI, Theorems 2.1, 2.4.

Let $(f_j)_{j\in\omega}$ be a universal Π^1_1 enumeration of all Π^1_1 functions on ω. Such an enumeration exists, e.g., by [23], Chapter 16.5. Recall that the hyperarithmetical functions are the total Π^1_1 functions. Then our requirements are:

$$R_{i,j} : A_i \neq f_j^{-1}[A_{\neq i}] \text{ and } A_i \text{ is co-infinite.}$$

We build our sets in stages $\sigma < \omega_1^{\text{CK}}$. We assign requirements to stages in such a way that each requirement is assigned to cofinally many stages. At stage 0 we do nothing.

At stage $0 < \sigma < \omega_1^{\text{CK}}$, let $R_{i,j}$ be the current requirement. The strategy to satisfy $R_{i,j}$ is the following. Look for an $n > 2^j$ such that $f_j^\sigma(n) \downarrow \notin A^\sigma_{\neq i}$. Put n into A_i and restrain $f_j^\sigma(n)$ from entering $A_{\neq i}$. This may injure requirements with lower priority.

Lemma 1. *For all i, j, the requirement $R_{i,j}$ acts only finitely many times.*

Proof. This is because the requirements are ordered in order type omega, and between any two stages at which the $(n + 1)$-st requirement acts, one of the first n requirements must have acted. It follows by induction on n that the n-th requirement only acts finitely many times. □

Lemma 2. *For all $i, j \in \omega$, $A_i \neq f_j^{-1}[A_{\neq i}]$.*

Proof. Assume the opposite, i.e. for some $i \in \omega$, $A_i \leq_{hm} A_{\neq i}$ via f_j. Choose a stage σ where requirement $R_{i,j}$ is considered and requirements of higher priority have ceased to act; also choose an $n > 2^j$ such that $f_j^\sigma(n) \downarrow$ and $f_j^\sigma(n) \notin A^\sigma_{\neq i}$. Such a n exists, as at most 2^k numbers less than 2^{k+1} are added to A_i for each k and therefore A_i is co-infinite. But then at stage σ a number was added to A_i to violate the reduction f_j, contradiction. □

The lemmas above prove the theorem. □

Corollary 1. *The countable atomless Boolean algebra may be embedded into the hm-degrees of Σ^1_1 subsets of ω.*

Note that there are, or course, much deeper statements about the structure of c.e. m-degrees (e.g., [6, 19, 22]) that one could try to lift to hm-degrees of Π^1_1 sets.

However, Corollary 1 provides enough evidence that the structure of *hm*-degrees of Σ^1_1 sets is rich.

3. A complete Σ^1_1 equivalence relation

We start the section by establishing some general properties of Σ^1_1 equivalence relations.

Definition 6. *An equivalence relation E is* complete *in a class $\mathcal{R}$ of equivalence relations (with a specified reducibility), if $E \in \mathcal{R}$ and every equivalence relation from $\mathcal{R}$ is reducible to E (with respect to the chosen reducibility).*

Theorem 5. *(1) There exists a universal Σ^1_1 enumeration of all Σ^1_1 equivalence relations on ω.*
(2) There exists a complete Σ^1_1 equivalence relation U (with respect to h- *or* FF*-reducibility).*

Proof. Let $\{A_e\}_{e\in\omega}$ be the standard Σ^1_1 enumeration of all Σ^1_1 subsets of $\omega \times \omega$ (for instance, as in [23]). Define the equivalence relation R_e as the reflexive transitive closure of A_e, i.e.

$$xR_ey \iff x = y \vee (\exists z_0, \dots, z_k)[z_0 = x\& \dots \&z_k = y\&(\forall i < k)(\langle z_i, z_{i+1}\rangle) \in A_e] \vee (\exists z_0, \dots, z_k)[z_0 = y\& \dots \&z_k = x\&(\forall i < k)(\langle z_i, z_{i+1}\rangle) \in A_e].$$

Then every Σ^1_1 equivalence relation appears in this enumeration, moreover from the properties of the enumeration $\{A_e\}_{e\in\omega}$, the enumeration $\{R_e\}_{e\in\omega}$ is universal.

Now define an equivalence relation R as follows:

$$\langle x, e\rangle R\langle y, e\rangle \iff xR_ey.$$

Then R is an h- and FF-complete Σ^1_1 equivalence relation. □

A useful and rather straightforward property of complete Σ^1_1 equivalence relations is the following:

Proposition 2. *An* h*-complete (or* FF*-complete) Σ^1_1 equivalence relation has infinitely many properly Σ^1_1 equivalence classes.*

Proof. Under h- or FF-reducibility properly Σ^1_1 equivalence classes are mapped to properly Σ^1_1 equivalence classes. In Theorem 10 below we show that there exist Σ^1_1 equivalence relations with infinitely many properly Σ^1_1 equivalence classes. Thus, a complete Σ^1_1 equivalence relation must also have this property. □

Recall the notion of *hm*-reducibility on subsets of ω introduced in Section 2.3. There exist Σ^1_1 equivalence relations with infinitely many *hm*-complete classes (e.g., as in Theorem 10 below). Therefore,

Corollary 2. *An* h*-complete (*FF*-complete) Σ^1_1 equivalence relation must have infinitely many properly Σ^1_1 equivalence classes that are hm-complete (m-complete, respectively).*

In a following section we will show that this condition is necessary but not sufficient for a relation to be h- or FF-complete among Σ^1_1 equivalence relations.

Remark. In [8] the authors showed that, in fact, the natural equivalence relation of bi-embeddability on the class of computable trees (here we mean the standard model-theoretic notion of embedding of structures) is FF-complete (thus, also h-complete) for the class of all Σ^1_1 equivalence relations on ω, where trees are considered in the signature with one unary function symbol interpreted as the predecessor function. Furthermore, [10] showes that the isomorphism relation on many natural classes of computable structures is FF-complete among Σ^1_1 equivalence relations.

By the above results, there exist the h-degrees formed by Δ^1_1 equivalence relations with exactly n equivalence classes, for $n \leq \omega$, and a greatest h-degree of Σ^1_1 equivalence relations, namely, that of a complete Σ^1_1 equivalence relation. The next step is to show that the structure of h-degrees of properly Σ^1_1 equivalence relations is not trivial:

Proposition 3. *There exists a Σ^1_1 equivalence relation on ω which is neither Δ^1_1 nor* h-*complete.*

Proof. Let $(L_m)_{m\in\omega}$ be the numbering of all computable linear orderings on ω. Consider the following equivalence relation $E_{\omega_1^{\mathrm{CK}}}$:

$$mE_{\omega_1^{\mathrm{CK}}}n \iff \text{either } L_m, L_n \text{ are not well-orders, (i.e. } m,n \notin \mathrm{WO}) \\ \text{or } L_m \cong L_n.$$

The relation $E_{\omega_1^{\mathrm{CK}}}$ is Σ^1_1 but not Δ^1_1 as otherwise the equivalence class consisting of non-well-orderings would be a Δ^1_1 set, a contradiction. Moreover, for every computable ordinal α, the equivalence class of $E_{\omega_1^{\mathrm{CK}}}$ containing α is hyperarithmetical. The only properly Σ^1_1 equivalence class is the class consisting of the computable non well-orderings. As the complete relation R constructed above has infinitely many properly Σ^1_1 equivalence classes, it cannot be reduced to $E_{\omega_1^{\mathrm{CK}}}$. Thus $E_{\omega_1^{\mathrm{CK}}}$ is not complete. □

We would like to mention another natural example of an incomplete properly Σ^1_1 equivalence relation: namely, the relation of bi-embeddability on the class of linear orders studied in [21]. Recall the notion of Scott rank: it is a measure of model theoretic complexity of countable structures. For a computable structure, the Scott rank is at most $\omega_1^{\mathrm{CK}}+1$ (see, for instance, [4] for a definition and an overview of results about the Scott rank of computable structures). In the class of computable linear orderings with the relation of bi-embeddability, the only equivalence class that contains structures of high (i.e. non-computable) Scott rank is the class of the dense linear order η. All other equivalence classes contain only structures of computable Scott rank (see [21] for details). If bi-embeddability on linear orderings were complete, it would necessarily have infinitely many equivalence classes with structures of high Scott rank. Therefore, bi-embeddability on linear orders cannot be complete.

4. Embedding Σ_1^1 sets into Σ_1^1 relations

For the reasons stated in Fact 2 we are interested in the structure of properly Σ_1^1 equivalence relations, i.e. relations that are Σ_1^1 but not Δ_1^1. In this section we will prove the following theorem:

Theorem 6. *The structure of properly Σ_1^1 sets with the relation of m-reducibility is order-preservingly (and effectively) embedded into the structure of properly Σ_1^1 equivalence relations with the relation of FF-reducibility, i.e. one can assign to every properly Σ_1^1 set A a properly Σ_1^1 equivalence relation E_A such that for any properly Σ_1^1 sets A, B,*

$$A \leq_m B \iff E_A \leq_{\mathrm{FF}} E_B.$$

Before we give the proof of this theorem we will show the following:

Theorem 7. *The structure of properly Σ_1^1 sets with the relation of 1-reducibility is order-preservingly (and effectively) embedded into the structure of properly Σ_1^1 equivalence relations with the relation of FF-reducibility where the reducing function is $1-1$.*

Proof. Let A be a properly Σ_1^1 set.

Define the relation E_A in the following way:

$$\begin{aligned} xE_Ay \iff & x, y \in A \\ & \text{or } x = y. \end{aligned}$$

The relation E_A is properly Σ_1^1.

Lemma 3. *For all properly Σ_1^1 sets A, B,*

$$A \leq_1 B \iff E_A \leq_{\mathrm{FF}} E_B,$$

where the FF-reducibility is witnessed by a computable $1-1$ function.

Proof. The direction from right to left is obvious. To prove the direction from left to right suppose $A \leq_1 B$ via a computable $1-1$ function f. Consider x, y such that xE_Ay. By definition of E_A, E_B and by properties of f,

$$xE_Ay \iff x, y \in A \text{ or } x = y \iff f(x), f(y) \in B \text{ or } f(x) = f(y) \iff f(x)E_Bf(y).$$

We use the fact that f is injective to prove the equivalence of the 3rd and the 2nd statement. □

The lemma proves the theorem. □

Remark. Relations of this kind for Σ_1^0 sets were considered in [13].

Proposition 4. *There exists an effective procedure which transforms a properly Σ^1_1 set A into a properly Σ^1_1 set A^* in such a way that*

$$A \leq_m B \Longrightarrow A^* \leq_1 B^*;$$
$$A^* \leq_m B^* \longrightarrow A \leq_m B.$$

Proof. For every set A, define $A^* = A \times \omega = \{\langle x, i\rangle | x \in A, i \in \omega\}$. For every i, denote by A_i the set $\{\langle x, i\rangle | x \in A\}$. Then $A^* = \cup_i A_i$. Note that by definition of A^*,

$$x \in A \iff \forall i \langle x, i\rangle \in A^* \iff \exists i \langle x, i\rangle \in A^*.$$

Suppose $A \leq_m B$ via a computable function f. We define a computable function h in the following way: for $x' = \langle x, i\rangle$ let $h(x') = \langle f(x), \langle x, i\rangle\rangle$, i.e. we send every $x' \in A_i$ to an element of $B_{x'}$. It guarantees that the function h is $1-1$. Thus we only need to show that h witnesses the 1-reduction of A^* to B^*:

$$x' \in A^* \iff x \in A \iff f(x) \in B \iff \langle f(x), \langle x, i\rangle\rangle \in B^*.$$

Now suppose $A^* \leq_m B^*$ via a computable function h. Define $f(x) = y \iff l(h(\langle x, 0\rangle)) = y$, i.e. $h(\langle x, 0\rangle) = \langle y, j\rangle$, for some $j \in \omega$. Then the function f m-reduces A to B:

$$x \in A \iff \langle x, 0\rangle \in A^* \iff h(\langle x, 0\rangle) = \langle y, j\rangle \in B^* \iff y \in B. \qquad \square$$

Proof of Theorem 6. The proof now follows directly from Proposition 4 and Theorem 7. □

Corollary 3. *For any $1 \leq n \leq \omega$, there exists an effective embedding of the structure of properly Σ^1_1 sets under m-reducibility into the structure of properly Σ^1_1 relations with exactly n properly Σ^1_1 equivalence classes under the* FF*-reducibility.*

In Section 2.3 we introduced the notion of hm-reducibility on sets which is a hyperarithmetical analogue of m-reducibility. We showed that the structure of hm-degrees of Σ^1_1 sets is complicated. Consider now a hyperarithmetical version of the 1-reducibility of subsets of ω:

Definition 7. *Let A, B be subsets of ω. We say that A is hyperarithmetically* 1*-reducible to B, denoted by $A \leq_{h1} B$, iff there exists a hyperarithmetical* $1-1$ *function f, such that for every $n \in \omega$,*

$$n \in A \iff f(n) \in B.$$

Using this definition and ideas from above one can show the following:

Theorem 8. *The structure of properly Σ^1_1 sets with the relation of hm-reducibility is order-preservingly (and effectively) embedded into the structure of properly Σ^1_1 equivalence relations with the relation of* h*-reducibility, i.e. one can assign to every properly Σ^1_1 set A a properly Σ^1_1 equivalence relation E_A such that for any properly Σ^1_1 sets A, B,*

$$A \leq_{hm} B \iff E_A \leq_{\mathrm{h}} E_B.$$

Moreover, for every $n \leq \omega$, there is such an embedding into the structure of properly Σ^1_1 equivalence relations with exactly n properly Σ^1_1 equivalence classes.

Thus, the structure of h-degrees of Σ^1_1 equivalence relations even with just one properly Σ^1_1 equivalence class is at least as rich as the structure of Σ^1_1 sets under hm-reducibility.

5. Properly Σ^1_1 Equivalence Relations with only hyperarithmetical equivalence classes

In this section we show that a properly Σ^1_1 equivalence relation need not contain properly Σ^1_1 equivalence classes. Moreover, the example we present contains only equivalence classes of size 1 or 2.

Let A be a Σ^1_1 subset of ω which is not Δ^1_1. Define the corresponding equivalence relation F_A on $\omega \times 2$ in the following way:

$$\begin{aligned}(m_0, n_0)F_A(m_1, n_1) \iff & m_0 = m_1 \in A \\ & \text{or } (m_0, n_0) = (m_1, n_1).\end{aligned}$$

The relation F_A is Σ^1_1. The equivalence classes of F_A are of the form $\{(m, n)|1 \leq n \leq 2\}$, if $m \in A$, and $\{(m, n)\}$, if $m \notin A$. In particular, every equivalence class has size 1 or 2. Again, similar relations constructed from Σ^0_1 sets were considered in [13].

Claim 1. *The equivalence relation F_A is properly Σ^1_1.*

Proof. If F_A were Δ^1_1, so would be the set A, as $A = \{m|(m, 0)F_A(m, 1)\}$, a contradiction. □

One can easily modify the example to get an equivalence relation with classes of size at most (and including) k, for $2 \leq k < \omega$.

Definition 8. *Following* [13]*, we call an equivalence relation k-bounded if all its equivalence classes have size at most k.*

Theorem 9. *There exists a properly Σ^1_1 equivalence relation S^{k+1} with all its equivalence classes containing at most $k+1$ element such that for no Σ^1_1 equivalence relation R with its equivalence classes containing at most k elements do we have $R^{k+1} \leq_{\mathrm{h}} S$ (hence, for no such R do we have $R^{k+1} \leq_{\mathrm{FF}} S$).*

Proof. As shown in [13], the analogous result is true for the case of c.e. relation. Simple transformation of this argument proves the theorem for Σ^1_1 equivalence relations. □

6. Equivalence Relations with finitely many properly Σ^1_1 classes

One can modify the example from the proof of Proposition 3 to get, for every finite $k \geq 2$, a Σ^1_1 equivalence relation which has exactly k properly Σ^1_1 equivalence classes:

Proposition 5. *For every finite $k \geq 1$ there exists a Σ^1_1 equivalence relation on ω with infinitely many equivalence classes, such that exactly k of them are properly Σ^1_1.*

Proof. Let $A_1, \ldots, A_k$ be disjoint properly Σ^1_1 sets. Consider the relation $E_{A_1,\ldots,A_k}$:

$$xE_{A_1\ldots,A_k}y \iff x = y \vee x, y \in A_1 \vee \ldots \vee x, y \in A_k.$$

Then $E_{A_1\ldots,A_k}$ has the desired properties. □

We give another example of equivalence relations with exactly k properly Σ^1_1 classes, for $k \geq 1$. The reason is that in the next section we will use a generalization of this example.

Again consider ω as a set of codes for linear orders. We will define relations F_k, for $k \geq 1$, on pairs of linear orders. First of all, we define additional hyperarithmetical equivalence relations E_k (here we identify natural numbers k, k' with ordinals):

$$\begin{aligned} n_1 E_k n_2 \iff &\text{either } L_{n_1} \cong L_{n_2} \cong k' < k-1 \\ &\text{or both } n_1, n_2 \text{ are not codes for well-orders of type } k' < k-1. \end{aligned}$$

By definition, E_k is hyperarithmetical and has exactly k equivalence classes. We now define F_k as follows: for $(m_i, n_i) \in \omega^2, i = 1, 2,$

$$\begin{aligned} (m_1, n_1)F_k(m_2, n_2) \iff &\text{either } (L_{m_1}, L_{m_2} \text{ are not well-orders and } n_1 E_k n_2) \\ &\text{or } (L_{m_1} \cong L_{m_2}). \end{aligned}$$

The idea is that we "cut" the properly Σ^1_1 class of $E_{\omega_1^{\mathrm{CK}}}$ (the relation defined in Proposition 3) into k properly Σ^1_1 pieces. The relations F_k, $k \geq 1$, have the necessary properties. Moreover,

Proposition 6. *For all $1 \leq k_1 < k_2 < \omega$, $F_{k_1} <_{\mathrm{h}} F_{k_2}$.*

Proof. Let f be a hyperarithmetical function which witnesses $E_{k_1} <_{\mathrm{h}} E_{k_2}$. Consider the function $g(m, n) = (m, f(n))$. It is hyperarithmetical and reduces F_{k_1} to F_{k_2}. The reduction is strict, as F_{k_1} has fewer properly Σ^1_1 equivalence classes than F_{k_2}. □

Remark. No F_k, for $k \geq 1$, is complete as no Σ^1_1 equivalence relation with only finitely many properly Σ^1_1 equivalence classes can be complete for the class of Σ^1_1 equivalence relations.

7. Equivalence Relations with infinitely many properly Σ^1_1 classes

In this section we show that an infinite number of properly Σ^1_1 equivalence classes does not guarantee the h- or FF-completeness of a Σ^1_1 equivalence relation.

Indeed, it is easy to construct a non-complete Σ^1_1 equivalence relations with infinitely many properly Σ^1_1 equivalence classes. Take a computable sequence

$(A_n)_{n\in\omega}$ of disjoint Σ^1_1 sets, such that none of them is complete and consider the relation R_∞ defined as follows:

$$xR_\infty y \iff x = y \vee \exists n(x, y \in A_n).$$

As the sequence $(A_i)_{i\in\omega}$ is computable, the relation R_∞ is Σ^1_1. Moreover, it is not complete as, for example, the relation R_B for a complete Σ^1_1 set B constructed as in Section 4 is not reducible to R_∞.

By Corollary 2, an h-complete (a FF-complete) Σ^1_1 equivalence relation must have infinitely many equivalence classes that are *hm*-complete (*m*-complete) as Σ^1_1 sets. Below we will show that this condition is not sufficient:

Theorem 10. *There exists a non-*h*-complete (non-*FF*-complete) Σ^1_1 equivalence relation with infinitely many classes that are hm-complete (m-complete) among Σ^1_1 sets.*

The proof of the theorem will follow from Proposition 8 below.

For every computable infinite ordinal α, we define equivalence relations E_α and F_α in the following way:

$$\begin{aligned} n_1 E_\alpha n_2 \iff & \text{ either } L_{n_1} \cong L_{n_2} \cong \alpha' < \alpha \\ & \text{ or [neither } n_1 \text{ nor } n_2 \text{ code well-orders of type } < \alpha]. \end{aligned}$$

In other words, for each $\alpha' < \alpha$, there is an equivalence class consisting of linear orders isomorphic to α'. All the linear orders that are not isomorphic to any $\alpha' < \alpha$ form a single equivalence class. By definition, if α is computable, then E_α is hyperarithmetical with infinitely many equivalence classes, provided α is infinite. Indeed, for a fixed $\alpha < \omega_1^{\mathrm{CK}}$ it is hyperarithmetical to check whether or not some $n \in \omega$ is a code for a well-order of type α or of type $< \alpha$. Then both the first and the second line of the definition give hyperarithmetical conditions. Hence, for infinite $\alpha < \omega_1^{\mathrm{CK}}$, all E_α are hyperarithmetical and h-equivalent to each other. Notice that there is some non-uniformity in the definition of E_α for finite (defined in the previous section) and infinite α.

Now define:

$$\begin{aligned} (m_1, n_1) F_\alpha (m_2, n_2) \iff & \text{ either } L_{m_1}, L_{m_2} \text{ are not well-orders and } n_1 E_\alpha n_2 \\ & \text{ or } L_{m_1} \cong L_{m_2}. \end{aligned}$$

Proposition 7. *For all computable infinite α_1, α_2, $F_{\alpha_1} \equiv_{\mathrm{h}} F_{\alpha_2}$.*

Proof. Consider the function h that witnesses the h-equivalence of the corresponding $E_{\alpha_1}, E_{\alpha_2}$. The function h' which sends a pair (m, n) into the pair $(m, h(n))$ gives the equivalence of $F_{\alpha_1}, F_{\alpha_2}$. □

Recall the definition of the relation $E_{\omega_1^{\mathrm{CK}}}$ from Section 3:

$$\begin{aligned} mE_{\omega_1^{\mathrm{CK}}} n \iff & \text{ either } L_m, L_n \text{ are not well-orders, (i.e. } m, n \notin \mathrm{WO}) \\ & \text{ or } L_m \cong L_n. \end{aligned}$$

Finally, we define an equivalence relation $F_{\omega_1^{\mathrm{CK}}}$ as follows:

$$(m_1, n_1)F_{\omega_1^{\mathrm{CK}}}(m_2, n_2) \iff \text{either } L_{m_1}, L_{m_2} \text{ are not well-orders and } n_1 E_{\omega_1^{\mathrm{CK}}} n_2 \\ \text{or } L_{m_1} \cong L_{m_2}.$$

Note that all $F_\alpha, \alpha < \omega_1^{\mathrm{CK}}$ and $F_{\omega_1^{\mathrm{CK}}}$ have infinitely many equivalence classes that are m-complete (thus, also hm-complete) among Σ^1_1 sets.

Proposition 8. *For every computable α,*

$$F_\alpha <_{\mathrm{h}} F_{\omega_1^{\mathrm{CK}}}.$$

Proof. Obviously, $F_\alpha \leq_{\mathrm{h}} F_{\omega_1^{\mathrm{CK}}}$: let f reduce E_α to $E_{\omega_1^{\mathrm{CK}}}$, then $g(m, n) = (m, f(n))$ reduces F_α to $F_{\omega_1^{\mathrm{CK}}}$. We only need to prove that $F_{\omega_1^{\mathrm{CK}}}$ is not reducible to F_α, for any computable α. Suppose that for some computable α there were such a hyperarithmetical reduction h:

$$(m_1, n_1)F_{\omega_1^{\mathrm{CK}}}(m_2, n_2) \iff h((m_1, n_1))F_\alpha h((m_2, n_2)).$$

Consider $n_1, n_2 \in \omega$. For every $m \notin \mathrm{WO}$ we have:

$$n_1 E_{\omega_1^{\mathrm{CK}}} n_2 \iff (m, n_1)F_{\omega_1^{\mathrm{CK}}}(m, n_2) \iff h((m, n_1))F_\alpha h((m, n_2)) \iff \\ \iff L_{m_1} \cong L_{m_2} \cong \gamma, \text{ where } \gamma \text{ is an ordinal, or } [m_1, m_2 \notin \mathrm{WO} \text{ and } l_1 E_\alpha l_2],$$

where $h(m, n_i) = (m_i, l_i), i = 1, 2$. Fix this notation for the rest of the proof.

If there exists an $m \notin \mathrm{WO}$ such that for all n_1, n_2 the corresponding $m_1, m_2 \notin \mathrm{WO}$, then the proposition is proved. Indeed, fix such an m. Then for all $n_1, n_2 \in \omega$,

$$n_1 E_{\omega_1^{\mathrm{CK}}} n_2 \iff (m, n_1)F_{\omega_1^{\mathrm{CK}}}(m, n_2) \iff (m_1, l_1)F_\alpha(m_2, l_2) \iff l_1 E_\alpha l_2,$$

which gives a hyperarithmetical reduction of $E_{\omega_1^{\mathrm{CK}}}$ to E_α, a contradiction.

Suppose now that for every $m \notin \mathrm{WO}$ there exist $n_1, n_2 \in \omega$ such that $L_{m_1} \cong L_{m_2} \cong \gamma$, for some $\gamma < \omega_1^{\mathrm{CK}}$. Define a Π^1_1 relation $R(m, n)$ as follows:

$$R(m, n) \iff (m \in \mathrm{WO} \wedge m = n) \\ \text{or } (n \in \mathrm{WO} \wedge L_n \cong L_{m_1} \cong L_{m_2} \text{ associated to some } h(m, n_1), h(m, n_2)).$$

By Uniformization, R can be uniformized by a Π^1_1 function f. The function f is total, thus hyperarithmetical from ω to WO. By Bounding, the range of f is bounded by a computable ordinal γ_0.

Consider now all $m \in \mathrm{WO}$, for which there exist n_1, n_2 such that $L_{m_1} \cong L_{m_2} \cong \gamma_0$. Then there is a computable bound α_0 on ordinals coded by such elements m.

Now we have

$$\mathrm{WO} = \{m | m \neq \mathit{code}(\beta) \text{ for } \beta \leq \alpha_0 \text{ and } \exists n_1, n_2 L_{m_1} \cong L_{m_2} \cong \gamma \leq \gamma_0\},$$

which gives a hyperarithmetical definition of WO, a contradiction. □

Remark. The process above of constructing of Σ^1_1 equivalence relations may be iterated further. In particular, the relation $F_{\omega_1^{\mathrm{CK}}}$ is not complete among Σ^1_1 equivalence relations.

8. More Results

The following result from [13] shows the difference between the theory of Σ^0_1 equivalence relations and that of Σ^1_1 equivalence relations:

Theorem 11. *Let $A_1, \ldots, A_n$ be disjoint c.e. sets the complement of whose union is infinite. Then*

$$id_\omega \leq_{\mathrm{FF}} R_{A_1,\ldots,A_n} \iff A_1 \cup \ldots \cup A_n \textit{ is not simple.}$$

Here

$$xR_{A_1,\ldots,A_n}y \iff x = y \vee \exists i \leq n(x, y \in A_i).$$

In the case of h-reducibility and disjoint Σ^1_1 sets $A_1, \ldots, A_n$,

$$\mathrm{id}_\omega \leq_{\mathrm{h}} R_{A_1,\ldots,A_n}$$

always holds. Indeed, the complement C of $\bigcup_{i\leq n} A_i$ is a Π^1_1 set, thus it contains a hyperarithmetical subset B. Then a $1-1$ hyperarithmetical function from ω onto B witnesses the reduction.

The analogy with c.e. equivalence relations might be more complete if we considered Π^1_1 equivalence relations.

Using ideas from [13] one can show the following:

Theorem 12. *There exist properly Σ^1_1 equivalence relations that are m-complete (hm-complete) as Σ^1_1 sets but* FF*-incomparable (respectively,* h*-incomparable) as Σ^1_1 equivalence relations.*

Proof. Let A be an m-complete, hence, also hm-complete Σ^1_1 set. Let E_A be a Σ^1_1 equivalence relation built from A as in Section 4. Let F_A be a Σ^1_1 equivalence relation with all its equivalence classes finite built from A as in Section 5. Then E_A and F_A are neither FF-comparable nor h-comparable.

Suppose E_A is reducible to F_A via a computable (or hyperarithmetical) function f. Fix an arbitrary $x_0 \in A$ and let $y_0 = f(x_0)$. Then $A = \{x | f(x)F_A y_0\}$, therefore $A \leq [y_0]_{F_A}$, where $[y_0]_{F_A}$ is finite. Thus A is computable (hyperarithmetical), a contradiction.

Suppose now that F_A is reducible to E_A via g. Consider the set $B = \{g(x)|x \in \omega\}$. Then $B \cap A \neq \emptyset$, otherwise F_A would be reducible to id_ω, thus hyperarithmetical. Now let $C = \{x|g(x) \in A\}$, then C is an equivalence class of F_A. Pick an arbitrary $y \in A$ and define $h(x)$ in the following way:

$$h(x) = \begin{cases} y, & \text{if } x \in C \\ g(x), & \text{otherwise} \end{cases}$$

All equivalence classes of F_A are finite, thus h is a computable (hyperarithmetical) function which reduces F_A to the equality on ω. □

9. Questions

If an equivalence relation E is reducible to an equivalence relation E' (under any of the two reducibilities considered here) then E is reducible to E' as sets (under the corresponding reducibility). On the other hand, if a Σ^1_1 equivalence relation is m-complete (hm-complete) as a Σ^1_1 set, it does not guarantee that it is FF-complete (h-complete) as a Σ^1_1 equivalence relation. Indeed, let A be an m-(hm-)complete Σ^1_1 set. Let E_A be a Σ^1_1 equivalence relation built as in Sections 4 or 5. Then E_A is not complete among Σ^1_1 relations but it is obviously complete as a Σ^1_1 set. One can also build such equivalence relations with any number of properly Σ^1_1 equivalence classes.

As it follows from Theorem 12, two Σ^1_1 equivalence relations may be incomparable while both being m-complete among Σ^1_1 sets. However in the above example one of the relations had only finite classes while the other relation had an infinite class and all the other classes of size 1. Thus the following set of questions arises naturally:

Question 2. *Let E, E' be Σ^1_1 equivalence relations with only finite (or hyperarithmetical) equivalence classes. Suppose E, E' are both complete as sets (under m- or hm-reducibility). As follows from Theorem 9, it may be the case that $E < E'$. Is it possible that E and E' are incomparable?*

Question 3. *The same for relations with a fixed number of properly Σ^1_1 (Σ^0_1) equivalence classes.*

We studied properly Σ^1_1 equivalence relations according to the number of their properly Σ^1_1 equivalence classes. We saw examples of equivalence relations with only hyperarithmetical classes, with exactly n properly Σ^1_1 equivalence classes, for $n \in \omega$ and with infinitely many properly Σ^1_1 equivalence classes.

Question 4. *Does there exists a properly Σ^1_1 equivalence relation on (a hyperarithmetical subset of) ω with infinitely many equivalence classes such that all its classes are properly Σ^1_1?*

References

[1] C. Bernardi, and A. Sorbi, *Classifying positive equivalence relations*, J. Symb. Log. **48** (1983), 529–538.

[2] W. Calvert, D. Cenzer, V. Harizanov, and A. Morozov, *Effective categoricity of equivalence structures*, Ann. Pure Appl. Log. **141**(2006), 61–78.

[3] W. Calvert, D. Cummins, J. F. Knight, and S. Miller, *Comparing classes of finite structures*, Alg. Log. **43** (2004), 374–392.

[4] W. Calvert, E. Fokina, S. Goncharov, J. Knight, O. Kudinov, A. Morozov, and V. Puzarenko, *Index sets for classes of high rank structures*, J. Symb. Log. **72** (2007), 1418–1432.

[5] D. Cenzer, V. Harizanov, and J. Remmel, *Σ^0_1 and Π^0_1 equivalence structures*, in: Mathematical Theory and Computational Practice, Proceedings of the 5th Conference on Computability in Europe (CiE 2009) held in Heidelberg, July 19–24, 2009, edited by Klaus

Ambos-Spies, Benedikt Lowe and Wolfgang Merkle, Lecture Notes in Comput. Sci. **5635** (Springer, Berlin, 2009) 99–108 .

[6] S. D. Denisov, *Structure of the upper semilattice of recursively enumerable m-degrees and related questions*, Alg. Log., **17** (1978), 643–683.

[7] Yu. L. Ershov, Definability and Computability. Siberian School of Algebra and Logic (Consultants Bureau / Plenum, New York - London - Moskwa, 1996), pp. XIV+264.

[8] E. Fokina, and S. Friedman, *Equivalence relations on classes of computable structures*, in: Mathematical Theory and Computational Practice, Proceedings of the 5th Conference on Computability in Europe (CiE 2009) held in Heidelberg, July 19–24, 2009, edited by Klaus Ambos-Spies, Benedikt Lowe and Wolfgang Merkle, Lecture Notes in Comput. Sci. **5635** (Springer, Berlin, 2009) 198–207.

[9] E. Fokina, S. Friedman, and A. Törnquist, *The effective theory of Borel equivalence relations*, Ann. Pure Appl. Log. **161** (2010), 837–850.

[10] E. Fokina, S. Friedman, V. Harizanov, J. Knight, C. McCoy, and A. Montalbán, *Isomorphism and bi-embeddability relations on computable structures*, J. Symb. Log., to appear.

[11] Friedman, H., and L. Stanley, *A Borel reducibility theory for classes of countable structures*, J. Symb. Log. **54** (1989), 894–914.

[12] S. Gao, Invariant Descriptive Set Theory, Monographs and Textbooks in Pure and Applied Mathematics (Chapman & Hall/CRC, Boca Raton, FL - London - New York, 2008), pp. XIV+383.

[13] S. Gao, and P. Gerdes, *Computably enumerable equivalence relations*, Stud. Logica **67** (2001), 27–59.

[14] S. S. Goncharov, and J. F. Knight, *Computable structure and non-structure theorems*, Alg. Log. **41** (2002), 351–373.

[15] V. Kanovei, Borel Equivalence Relations. Structure And Classification, University Lecture Series, **44** (Am. Math. Soc., Providence, RI, 2008), pp. X+240.

[16] A. Kechris, *New directions in descriptive set theory*, Bull. Symb. Log. **5** (1999), 161–174.

[17] B. Khoussainov, F. Stephan, Y. Yang, *Computable categoricity and the Ershov hierarchy*, Ann. Pure Appl. Log. **156** (2008), 86–95.

[18] A. Lachlan, *A note on positive equivalence relations*, Z. Mat. Log. Grundl. Math. **33** (1987), 43–46.

[19] A. Lachlan, *Recursively enumerable many-one degrees*, Alg. Log. **11** (1972), 326–358.

[20] A. Louveau, and B. Velickovic, *A note on Borel equivalence relations*, Proc. Am. Math. Soc. **120** (1994), 255–259.

[21] A. Montalbán, *On the equimorphism types of linear orderings*, Bull. Symb. Log. **13** (2007), 71–99.

[22] S. Yu. Podzorov, *The universal Lachlan semilattice without the greatest element*, Alg. Log. **46(3)** (2007), 163–187.

[23] H. Rogers, Theory Of Recursive Functions And Effective Computability (McGraw-Hill, New York - St. Louis - San Francisco - Toronto - London - Sydney, 1967), pp. XXI+482.

[24] G. Sacks, Higher Recursion Theory. Perspectives in Mathematical Logic (Springer-Verlag, Berlin - Heidelberg - New York, 1989), pp. XIII+344.

[25] R. Soare, Recursively Enumerable Sets And Degrees. A Study Of Computable Functions And Computably Generated Sets. Perspectives in Mathematical Logic (Springer-Verlag, Berlin - Heidelberg - New York, 1999(1987)), pp. XVIII+437.

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