

# TENSENESS OF RIEMANNIAN FLOWS

HIRAKU NOZAWA AND JOSÉ IGNACIO ROYO PRIETO

ABSTRACT. We show that any transversally complete Riemannian foliation  $\mathcal{F}$  of dimension one on any possibly non-compact manifold  $M$  is tense; namely,  $(M, \mathcal{F})$  admits a Riemannian metric such that the mean curvature form of  $\mathcal{F}$  is basic. This is a partial generalization of a result of Domínguez, which says that any Riemannian foliation on any compact manifold is tense. Our proof is based on some results of Molino and Sergiescu, and it is simpler than the original proof by Domínguez. As an application, we generalize some well known results including Masa's characterization of tautness.

## 1. INTRODUCTION

1.1. **Background.** A foliated manifold  $(M, \mathcal{F})$  is called *taut* if  $M$  admits a metric  $g$  such that every leaf of  $\mathcal{F}$  is a minimal submanifold of  $(M, g)$ ; in other words,  $M$  admits a metric such that the mean curvature form of  $\mathcal{F}$  is trivial. The tautness of foliated manifolds has been studied from the dynamical or geometric point of view after the characterization of tautness in terms of foliation cycles due to Sullivan [Su79]. For Riemannian foliations, tautness is remarkably of topological nature, and its relation to cohomology has been studied by many authors [KT83, Ca84, Gh84, Ha85, MS85, EH86, Ma92, AL92]. In particular, as conjectured by Carrière [Ca84] and finally proved by Masa [Ma92] using Sarkaria's smoothing operator [Sar78], an oriented and transversally oriented Riemannian foliation is taut if and only if the top degree component of the basic cohomology is nontrivial. Álvarez López [AL92] defined the so-called *Álvarez class* to characterize tautness of Riemannian foliations and removed the assumption of the orientability from Masa's characterization.

Based on these works on tautness of Riemannian foliations, Domínguez proved the following result.

---

The first author is supported by the EPDI/JSPS/IHÉS Fellowship and the Spanish MICINN grant MTM2011-25656. This paper was written during the stay of the first author at Centre de Recerca Matemàtica (Bellaterra, Spain), Institut Mittag-Leffler (Djursholm, Sweden) and Institut des Hautes Études Scientifiques (Bures-sur-Yvette, France); he is very grateful for their hospitality.

The second author is partially supported by the UPV/EHU grant EHU09/04 and by the Spanish MICINN grant MTM2010-15471. This paper was written during the stay of the second author at Centre de Recerca Matemàtica (Bellaterra, Spain); he is very grateful for their hospitality.

**Theorem 1.1** ([Do98, Tenseness Theorem in p. 1239]). *Any Riemannian foliation on a closed manifold is tense.*

Here, recall that a foliated manifold  $(M, \mathcal{F})$  is called *tense* if  $M$  admits a metric such that the mean curvature form of  $\mathcal{F}$  is basic. Theorem 1.1 can be regarded as a generalization of Masa's characterization of tautness, and has many applications in the study of geometrical and cohomological properties of Riemannian foliations (see, for example, [KT83b, KT84, To97, RP01, RPSAW08, RPSAW09]).

**1.2. Main result.** In this article we will generalize Theorem 1.1 to Riemannian foliations of dimension one on possibly non-compact manifolds.

There is one remarkable difference between the non-compact and the compact cases. By [KT83b, Eq. 4.4], the mean curvature form  $\kappa$  of a Riemannian foliation with a tense metric on a compact manifold is always closed. On the other hand, there exists a Riemannian foliation on a non-compact manifold with a tense metric whose  $\kappa$  is not closed [CE97, Example 2.4]. Based on this fact, we say that a foliated manifold  $(M, \mathcal{F})$  is *strongly tense* if  $M$  admits a Riemannian metric such that the mean curvature form of  $\mathcal{F}$  is basic and closed.

The main result of this paper is the following.

**Theorem 1.2.** *Any transversally complete Riemannian foliation of dimension one on a manifold is strongly tense.*

We refer to Definition 2.1 for the definition of transversally completeness of Riemannian foliations. Note that some authors use this terminology with a different meaning.

*Remark 1.3.* For Riemannian foliations of any dimension which can be suitably embedded into a singular Riemannian foliation on a compact manifold, strongly tenseness was proved in [RPSAW08, RPSAW09] by the application of Domínguez's theorem. For any Riemannian foliation such that the space of leaf closures is compact, the strongly tenseness was proved in [No12, Theorem 1.9].

Based on Theorem 1.2 we ask the following question.

**Question.** Is any complete Riemannian manifold strongly tense?

Here a Riemannian foliation is complete if the holonomy pseudogroup is complete as a pseudogroup. Due to Salem [Mo88, Appendix by Salem], it is the largest known class of Riemannian foliations for which Molino's structure theorems hold.

The first essential point in the proof of Theorem 1.2 is the following dichotomy, which is specific for dimension one (see Remark 3.2 for a preceding result of Molino).

**Theorem 1.4.** *Let  $M$  be a connected manifold with a transversally complete Riemannian foliation  $\mathcal{F}$  of dimension one. Then, one of the following holds:*

- (i)  $(M, \mathcal{F})$  is an  $\mathbb{R}$ -bundle or
- (ii) the closure of every leaf of  $\mathcal{F}$  is compact.

Since Theorem 1.2 is clearly true for  $\mathbb{R}$ -bundles, it is essential to prove Theorem 1.2 in the case where the closure of every leaf is compact. In turn, Molino's structure theorem remains true in this case even if  $M$  is non-compact (Theorem 4.2). Thus we can apply some results of Molino and Sergiescu involving reductions of the structure group of torus bundles (Section 4.3) to show Theorem 1.2. Even in the case where  $M$  is compact, our proof is new and simpler than the original proof of Theorem 1.1 due to Domínguez, as we make no use of Sarkaria's smoothing operator [Sar78].

**1.3. The Álvarez class.** For a Riemannian foliation  $\mathcal{F}$  on a closed manifold  $M$  with a bundle-like metric  $g$ , the orthogonal projection  $\kappa_b$  of the mean curvature form  $\kappa$  to the space of basic 1-forms with respect to the natural inner product is closed [AL92, Corollary 3.5]. The cohomology class  $[\kappa_b] \in H^1(M/\mathcal{F})$  is independent of  $g$  ([AL92, Theorem 5.2]) and called the *Álvarez class* of  $(M, \mathcal{F})$ . The triviality of the Álvarez class of  $(M, \mathcal{F})$  characterizes tautness [AL92, Theorem 6.4]. In turn, the example [CE97, Example 2.4] of a Riemannian foliation on a non-compact manifold with basic but non-closed  $\kappa$  shows that the Álvarez class is not defined in general for Riemannian foliations on non-compact manifolds. Nevertheless, Theorem 1.4 implies the following result for Riemannian foliations of dimension one (see Section 4.5 for the proof).

**Theorem 1.5.** *Let  $(M, \mathcal{F})$  be a manifold with a transversally complete Riemannian foliation of dimension one with a strongly tense metric  $g$ . If  $(M, \mathcal{F})$  is not an  $\mathbb{R}$ -bundle, then the cohomology class of the mean curvature form  $\kappa$  is given by the logarithm of the holonomy homomorphism  $\pi_1 M \rightarrow \mathbb{R}$  of the determinant line bundle of the Molino's commuting sheaf of  $(M, \mathcal{F})$ . In particular,  $[\kappa]$  is independent of  $g$ .*

We will use the following terminology below, which is well-defined by Theorem 1.5.

**Definition 1.6.** For a manifold  $M$  with a transversally complete Riemannian foliation  $\mathcal{F}$  of dimension one which is not an  $\mathbb{R}$ -bundle, the cohomology class  $[\kappa]$  of the mean curvature form of any strongly tense metric is called the *Álvarez class* of  $(M, \mathcal{F})$ .

*Remark 1.7.* It is easy to see that an  $\mathbb{R}$ -bundle is always taut. But there exists an  $\mathbb{R}$ -bundle with a strongly tense metric such that the cohomology class of the mean curvature form is nontrivial (see [No12, Proposition 9.3]). Below the Álvarez class of an  $\mathbb{R}$ -bundle is defined by the trivial class for a conventional reason.

*Remark 1.8.* For Riemannian foliations of any dimension which can be suitably embedded into a singular Riemannian foliation on a compact manifold, the

Álvarez class is well-defined as shown in [RPSAW08, RPSAW09] by the application of Domínguez's theorem.

#### 1.4. Applications.

1.4.1. *Characterization of tautness.* The basic cohomology is the de Rham cohomology of the leaf space in a sense (see, for example, [Mo88, Appendix B]), and its relation to tautness of Riemannian foliations was studied by many authors mentioned in the introduction.

First, we state the twisted Poincaré duality of the basic cohomology, which is a consequence of a theorem of Sergiescu and the dichotomy theorem (Theorem 1.4). In [KT84, Theorem 3.1], Kamber-Tondeur proved that any orientable and transversally oriented tense Riemannian foliation  $\mathcal{F}$  of codimension  $q$  on a compact manifold  $M$  satisfies

$$(1) \quad H_c^\bullet(M/\mathcal{F}) \cong H_\kappa^{q-\bullet}(M/\mathcal{F})^*,$$

where the  $\kappa$ -twisted basic cohomology  $H_\kappa^\bullet(M/\mathcal{F})$  stands for the cohomology of the basic de Rham complex with the twisted differential  $d_\kappa\omega = d\omega - \kappa \wedge \omega$  [KT83b, p. 121]. In [Se85, Section 1], Sergiescu defined the orientation sheaf  $\mathcal{P}$  of  $(M, \mathcal{F})$  and proved the Poincaré duality [Se85, Théorème I] on basic cohomology of  $(M, \mathcal{F})$ . His argument shows the isomorphism

$$(2) \quad H_c^\bullet(M/\mathcal{F}) \cong H^{q-\bullet}(M/\mathcal{F}; \mathcal{P}^*)^*$$

for any complete Riemannian foliation on a possibly non-compact manifold whose closures of leaves are compact (see [Ha85, Proposition 3.2.9.1]). Here a Riemannian foliation is called complete if the canonical transverse parallelism of its lift to the orthonormal frame bundle consists of complete vector fields [Mo88, Remark on p. 88]. In the case where  $M$  is compact,  $H_\kappa^\bullet(M/\mathcal{F}) \cong H^\bullet(M/\mathcal{F}; \mathcal{P}^*)$  by [Do98, Theorem 5.9 (iii)]. So (2) coincides with (1) if  $M$  is compact. The twisted duality (1) for Riemannian foliations of any dimension which can be suitably embedded into a singular Riemannian foliation on a compact manifold was proved in [RPSAW09]. Note that, it is not clear if (1) always follows from (2) in the case where  $M$  is non-compact.

A priori, a transversally complete Riemannian foliation may not be complete in the sense of [Mo88, Remark on page 88], but we get the following result by Theorems 1.4, 1.5 and [Se85, Théorème I].

**Corollary 1.9.** *For a transversally oriented and transversally complete Riemannian foliation  $\mathcal{F}$  of dimension one and codimension  $q$  on a possibly non-compact manifold  $M$ , we have the isomorphisms (2) and*

$$(3) \quad H_c^\bullet(M/\mathcal{F}) \cong H_\kappa^{q-\bullet}(M/\mathcal{F})^*,$$

where  $\kappa$  is a representative of the Álvarez class of  $(M, \mathcal{F})$ .

*Proof.* If  $(M, \mathcal{F})$  is an  $\mathbb{R}$ -bundle, then  $\kappa$  is trivial and (3) follows from the Poincaré duality of the leaf space  $M/\mathcal{F}$ . So, by Theorem 1.4, we can assume that the closure of each leaf of  $(M, \mathcal{F})$  is compact. In this case, Molino's structure theorems (Theorem 4.2) remain valid. Thus the proof of [Se85, Théorème I] for the compact case can be applied to show (2). Note that the holonomy homomorphisms of Sergiescu's orientation sheaf  $\mathcal{P}$  and the determinant line bundle of Molino's commuting sheaf are equal up to sign by definition of  $\mathcal{P}$ . Thus the latter part of Theorem 1.5 implies  $H_\kappa^\bullet(M/\mathcal{F}) \cong H^\bullet(M/\mathcal{F}; \mathcal{P}^*)$ . Hence we get (3).  $\square$

Theorem 1.2 and Corollary 1.9 give us the following characterization of tautness in terms of basic cohomology (see Section 4.5 for the proof).

**Corollary 1.10.** *Let  $\mathcal{F}$  be a transversally oriented and transversally complete Riemannian foliation of dimension one and codimension  $q$  on a possibly non-compact manifold  $M$ . Let  $\kappa$  be a representative of the Álvarez class of  $(M, \mathcal{F})$ . Then, the following are equivalent:*

- (i)  $\mathcal{F}$  is taut;
- (ii)  $H_c^q(M/\mathcal{F}) \cong \mathbb{R}$ ;
- (iii)  $H_\kappa^0(M/\mathcal{F}) \cong \mathbb{R}$ ;
- (iv) the image of the holonomy homomorphism  $\pi_1 M \rightarrow \text{Aut}(\mathbb{R}) \cong \mathbb{R}^\times$  of Sergiescu's orientation sheaf of  $(M, \mathcal{F})$  is contained in  $\{\pm 1\}$ .
- (v) the image of the holonomy homomorphism  $\pi_1 M \rightarrow \text{Aut}(\mathbb{R}) \cong \mathbb{R}^\times$  of Molino's commuting sheaf of  $(M, \mathcal{F})$  is contained in  $\{\pm 1\}$ .

Otherwise,  $H_c^q(M/\mathcal{F}) = 0$ .

Corollary 1.10 generalizes [MS85, Théorème A] and [Ma92, Minimality Theorem] to Riemannian foliations of dimension one on possibly non-compact manifolds. For Riemannian foliations of any dimension which can be suitably embedded into a singular Riemannian foliation on a compact manifold, the equivalence of the first three statements of Corollary 1.10 are shown in [RPSAW08, RPSAW09] by the application of Domínguez's theorem.

1.4.2. *The Euler class and the Gysin sequence.* In [RP01], the Euler class and the Gysin sequence of Riemannian flows on compact manifolds were obtained by using Domínguez's tenseness theorem. Theorems 1.2 and 1.4 allow us to obtain the Euler class and the Gysin sequence of transversally complete Riemannian flows on possibly non-compact manifolds.

**Corollary 1.11.** *Let  $\mathcal{F}$  be an oriented transversally complete Riemannian flow on a possibly non-compact manifold  $M$ . Then, we get the following long exact sequence:*

$$\dots \rightarrow H^i(M/\mathcal{F}) \longrightarrow H^i(M) \longrightarrow H_\kappa^{i-1}(M/\mathcal{F}) \xrightarrow{\wedge^e} H^{i+1}(M/\mathcal{F}) \rightarrow \dots ,$$

where  $\kappa$  is a representative of the Álvarez class of  $(M, \mathcal{F})$  and the connecting morphism is the multiplication by the Euler class  $e$  defined in the  $(-\kappa)$ -twisted basic cohomology  $H_{-\kappa}^2(M/\mathcal{F})$ .

*Outline of the proof.* If  $(M, \mathcal{F})$  is an  $\mathbb{R}$ -bundle, then the Euler class and the Gysin sequence of  $(M, \mathcal{F})$  are trivial. Thus, by Theorem 1.4, it is essential to construct them in the case where the closure of every leaf of  $\mathcal{F}$  is compact. In this case, with Theorem 1.2, the construction of [RP01] of the Euler class and the Gysin sequence can be carried out without any modification. Note that, since the closure of every leaf is compact, any leaf has a good saturated neighborhood described by Carrière in [Ca84, Proposition 3], which is called a Carrière neighborhood in [RP01].  $\square$

1.4.3. *Generalization of Tondeur's theorem.* By Theorem 1.2, we obtain the generalization of the main result of [To97].

**Corollary 1.12.** *Let  $\mathcal{F}$  be a foliation of dimension one on a possibly non-compact manifold  $M$ . Then  $\mathcal{F}$  is transversally complete Riemannian if and only if there exists a complete metric  $g$  on  $M$  such that the tangent bundle of  $\mathcal{F}$  is locally generated by Killing vector fields on  $(M, g)$ .*

*Outline of the proof.* The if part is proved by an argument similar to [To97] with Theorem 1.2. To prove the only if part, note that, by Theorem 1.4, any transversally complete Riemannian foliation of dimension one admits a complete bundle-like metric.  $\square$

*Organization of the article.* Section 2 is devoted to recall the definition of fundamental notions. In Section 3, the dichotomy result (Theorem 1.4) is proved. In Section 4, we analyze the special case of linearly foliated torus bundles. In Section 5, the main result (Theorem 1.2) is proved based on Theorem 1.4 and the results in Section 4.

## 2. FUNDAMENTAL NOTIONS

2.1. **Foliations and metrics.** We recall some notions on foliated manifolds and metrics on them. A *Haefliger cocycle* (of codimension  $q$ ) on a manifold  $M$  is a triple  $(\{U_i\}, \{\pi_i\}, \{\gamma_{ij}\})$  consisting of

- (i) an open covering  $\{U_i\}$  of  $M$ ,
- (ii) submersions  $\pi_i: U_i \rightarrow \mathbb{R}^q$ ,
- (iii) local diffeomorphisms  $\gamma_{ij}: \pi_j(U_i \cap U_j) \rightarrow \pi_i(U_i \cap U_j)$  such that  $\pi_i = \gamma_{ij} \circ \pi_j$ .

Two Haefliger cocycles on  $M$  are said to be equivalent if their union is a Haefliger cocycle on  $M$ . Recall that a codimension  $q$  foliation of  $M$  is defined by an equivalence class of Haefliger cocycles of codimension  $q$ .

A foliation  $\mathcal{F}$  is called *Riemannian* if there exist Riemannian metrics  $h_i$  on  $\pi_i(U_i)$  such that  $\gamma_{ij}^* h_i = h_j$ . Let  $\nu\mathcal{F}$  denote the normal bundle  $TM/T\mathcal{F}$  of  $(M, \mathcal{F})$ . Here  $(\pi_i)_*: \nu_x\mathcal{F} \rightarrow T_{\pi_i(x)}\mathbb{R}^q$  is an isomorphism at each point  $x \in U_i$ . By pulling back the metric  $h_i$  by  $(\pi_i)_*$  to  $\nu_x\mathcal{F}$  at each point  $x \in U_i$ , we get a metric

on  $\nu\mathcal{F}|_{U_i}$ . This gives rise to a well-defined metric  $g$  on  $\nu\mathcal{F}$ . Such metric on  $\nu\mathcal{F}$  constructed from  $\{h_i\}$  is called *holonomy invariant*. We will say that a metric  $g$  on  $(M, \mathcal{F})$  is *bundle-like* if the metric induced on  $\nu\mathcal{F}$  via the identification  $\nu\mathcal{F} \cong (T\mathcal{F})^\perp$  is holonomy invariant.

It is easy to see that any manifold with a Riemannian foliation admits a bundle-like metric. In [Re59, Proposition 2] (see also [Mo88, Proposition 3.5]), Reinhart proved that a metric on  $(M, \mathcal{F})$  is bundle-like if and only if a geodesic whose initial vector is orthogonal to  $\mathcal{F}$  is orthogonal to  $\mathcal{F}$  everywhere. The following is the notion of completeness in the transverse direction of Riemannian foliations.

**Definition 2.1.** A Riemannian foliation  $\mathcal{F}$  on a connected manifold  $M$  is *transversally complete* if there exists a Riemannian metric  $g$  which is both bundle-like and transversally complete; namely, at every point, on any maximal geodesic that is orthogonal to the leaves, the natural parameter changes from  $-\infty$  to  $\infty$ .

*Remark 2.2.* The definition of transversally completeness we use here is different from the one adopted in [Mo88, Definition 4.1].

**2.2. Tautness and tenseness.** For a given Riemannian metric  $g$  on a foliated manifold  $(M, \mathcal{F})$ , the *mean curvature form*  $\kappa \in \Omega^1(M)$  of  $\mathcal{F}$  at  $x$  is the mean curvature form of the leaf which goes through  $x$  (see, for example, [CC00, Section 10.5] for formulas of  $\kappa$  in terms of  $g$ ).

Recall that, on a foliated manifold  $(M, \mathcal{F})$  represented by a Haefliger cocycle  $(\{U_i\}, \{\pi_i\}, \{\gamma_{ij}\})$ , a  $k$ -form  $\alpha$  is called *basic* if, for every  $i$ , there exists a  $k$ -form  $\alpha_i$  on  $\pi_i(U_i)$  such that  $\alpha|_{U_i} = \pi_i^* \alpha_i$ . Let us recall the following terminologies.

**Definition 2.3.** A Riemannian metric  $g$  on a foliated manifold  $(M, \mathcal{F})$  is said to be *tense* (resp., *taut*) if the mean curvature form  $\kappa$  is basic (resp., trivial).

We similarly define the following notion according to strongly tenseness of foliated manifolds.

**Definition 2.4.** A Riemannian metric  $g$  on a foliated manifold  $(M, \mathcal{F})$  is said to be *strongly tense* if the mean curvature form  $\kappa$  is basic and closed.

Strongly tenseness can be considered as a variant of tautness twisted with a real line bundle (see [No12, Proposition 7.5]).

*Remark 2.5.* By a result of Kamber-Tondeur [KT83b, Eq. 4.4], if  $M$  is compact, then any tense metric on  $(M, \mathcal{F})$  is strongly tense.

**2.3. Characteristic forms.** For a Riemannian manifold  $(M, g)$  with an oriented  $p$ -dimensional foliation  $\mathcal{F}$ , the characteristic form  $\chi \in \Omega^p(M)$  is defined by

$$(4) \quad \chi(X_1, \dots, X_p) = \det(g(X_i, E_j)_{ij}), \quad \forall X_1, \dots, X_p \in C^\infty(TM),$$

where  $\{E_1, \dots, E_p\}$  is a local oriented orthonormal frame of  $T\mathcal{F}$ .

The mean curvature form is determined by the characteristic form by Rumm-ler's formula [Ru79]:

$$(5) \quad \kappa(Y) = -d\chi(Y, E_1, \dots, E_p), \quad \forall Y \in C^\infty((T\mathcal{F})^{\perp g}),$$

where  $\{E_1, \dots, E_p\}$  is a local oriented orthonormal frame of  $T\mathcal{F}$ . Notice that  $\chi$  is determined by the orthogonal complement  $(T\mathcal{F})^{\perp g}$  and the metric along the leaves, hence so is  $\kappa$ .

**Definition 2.6.** The characteristic form  $\chi$  of an oriented foliation  $\mathcal{F}$  on a Riemannian manifold  $(M, g)$  is said to be *tense*, *strongly tense* or *taut* if  $g$  is tense, strongly tense or taut, respectively.

### 3. A DICHOTOMY ON LEAVES

Let  $M$  be a smooth manifold with a transversally complete Riemannian foliation  $\mathcal{F}$  of dimension one. We show the following dichotomy.

**Theorem 1.4 (bis.).** *Let  $M$  be a connected manifold with a transversally complete Riemannian foliation  $\mathcal{F}$  of dimension one. Then, one of the following holds:*

- (i)  *$(M, \mathcal{F})$  is an  $\mathbb{R}$ -bundle or*
- (ii) *the closure of every leaf of  $\mathcal{F}$  is compact.*

*Remark 3.2.* A result [Mo82, Lemme 3] of Molino implies that the leaves of any Lie foliation of dimension one which is transversally complete in the sense of [Mo88, Definition 4.1] are either all closed or all have compact closures. Theorem 1.4 is its generalization proved by a similar argument.

We will need the following lemma to prove Theorem 1.4.

**Lemma 3.3** ([Mo88, Proposition 6.6]). *Let  $M$  be a connected manifold with a Riemannian foliation  $\mathcal{F}$  and a transversally complete bundle-like metric  $g$ . Let  $P$  be a plaque of a leaf of  $\mathcal{F}$ . The foliation naturally induced on the normal bundle  $\nu\mathcal{F}|_P$  restricted to  $P$  is denoted by  $\mathcal{G}_P$ . The exponential map  $\exp: (\nu\mathcal{F}|_P, \mathcal{G}_P) \rightarrow (M, \mathcal{F})$  is a well defined foliated map. If  $P$  is relatively compact, then there exists an open neighborhood  $U$  of the zero section of  $\nu\mathcal{F}|_P$  such that  $\exp|_U$  is a diffeomorphism.*

*Proof of Theorem 1.4.* Let  $L$  a leaf of  $\mathcal{F}$  and  $T$  a transversal of  $\mathcal{F}$  which intersects with  $L$  at  $x \in L$ . Let  $K$  be a relatively compact neighborhood of  $x$  in  $T$ . Assume that  $L$  is non-compact and non-proper. We will show that  $\bar{L}$  is compact. We identify  $L$  with  $\mathbb{R}$ . Since  $L$  is not proper, we can assume that there exists a strictly monotonically increasing sequence  $\{x_i\}_{i \in \mathbb{Z}_{\geq 0}}$  in  $L \cap K$  such that  $\lim_{i \rightarrow \infty} x_i = \infty$ .

First, by reductio ad absurdum, we will show that there exists a strictly monotonically increasing sequence  $\{x_i\}_{i \in \mathbb{Z}_{< 0}}$  in  $L \cap K$  such that  $\lim_{i \rightarrow -\infty} x_i = -\infty$ . Assume that such  $\{x_i\}_{i \in \mathbb{Z}_{< 0}}$  does not exist. Then there exists  $z$  in  $L \cap K$  such that  $K \cap \{x \in L \mid x < z\} = \emptyset$ . For  $r > 0$ , let  $D(r)$  be the disk of radius  $r$  centered at  $z$  in  $K$ . We consider the metric on  $K$  induced from the transverse metric of

$(M, \mathcal{F})$ . Take a small  $r$  so that each pair of points in  $D(r)$  is connected with a unique geodesic (see, for example, [DC92, Proposition 4.2]). Without losing generality we can assume that  $D(r)$  is orthogonal to  $\mathcal{F}$  and that  $x_0, x_1$  and  $x_2$  belong to  $D(r/2)$ . Here, the unique geodesic on  $M$  which connects two given points on  $D(r)$  is contained in  $D(r)$  by the property of bundle-like metrics. Hence, at each point  $y$  in  $D(r/2)$ , the image of the disk of radius  $r/2$  centered at 0 in  $\nu_y \mathcal{F}$  under the exponential map at  $y$  is contained in  $D(r)$ . Let  $J$  be the closed segment in  $L$  which connects  $x_0$  and  $x_2$ . Then, by Lemma 3.3, a leaf of  $\nu \mathcal{F}|_J$  is mapped to an open neighborhood  $S$  of  $z$  in  $L$  by the normal exponential map along  $J$ . By construction,  $S$  intersects  $D(r)$  at its endpoints. Thus it contradicts with the hypothesis  $K \cap \{x \in L \mid x < z\} = \emptyset$ .

Thus, there exists a strictly monotonically increasing sequence  $\{x_i\}_{i \in \mathbb{Z}}$  in  $L \cap K$  such that

$$\lim_{i \rightarrow -\infty} x_i = -\infty, \quad \lim_{i \rightarrow \infty} x_i = \infty.$$

Let  $J_i$  be the segment in  $L$  which connects  $x_i$  and  $x_{i+1}$ . We fix  $r > 0$  and a bundle-like metric  $g$ . By the transversally completeness of  $\mathcal{F}$  and Lemma 3.3, we get a foliated chart  $U_i$  such that  $J_i \subset U_i \subset J_i(r_i)$ , where  $r_i > 0$  and  $J_i(r_i)$  is the  $r_i$ -neighborhood of  $J_i$  with respect to the metric on  $M$  induced from  $g$ . Then  $\{U_i \cap K\}_{i \in \mathbb{Z}}$  covers  $K$ . We choose a finite subset  $I \subset \mathbb{Z}$  so that  $\{U_i \cap K\}_{i \in I}$  covers  $K$ . Then, by construction,  $\bar{L}$  is contained in  $\cup_{i \in I} U_i$ , which is a bounded set. So  $\bar{L}$  is compact.

Let  $L$  be a leaf of  $\mathcal{F}$  such that  $\bar{L}$  is compact. Denote the metric on  $M$  induced from  $g$  by  $d$ . If another leaf  $L'$  of  $\mathcal{F}$  has a point  $x'$  such that  $d(x', \bar{L}) = l$ , then, the transversally completeness implies that, for any point  $y \in L'$ , there exists a geodesic which connects  $y$  and  $\bar{L}$  of length  $l$ . Thus  $d(y, \bar{L}) \leq l$ , which implies the compactness of  $\bar{L}'$ . So the union of the leaves with compact closure is open in  $M$ .

By a theorem of Žukova [Žu87, Theorem 1], any non-compact proper leaf  $L$  of  $\mathcal{F}$  admits an open tubular neighborhood  $U$  such that  $(U, \mathcal{F}|_U)$  is an  $\mathbb{R}$ -bundle. So the union of non-proper leaves is open in  $M$ . Thus, by the connectivity argument, if the closure of a leaf of  $\mathcal{F}$  is compact, then so is any other leaf of  $\mathcal{F}$ . If  $\mathcal{F}$  admits a non-compact proper leaf, then  $\mathcal{F}$  is an  $\mathbb{R}$ -bundle.  $\square$

A trivial  $\mathbb{R}$ -bundle is taut with a product metric constructed with a trivialization. A non-trivial  $\mathbb{R}$ -bundle is taut with a metric which comes from a double cover. Thus, any  $\mathbb{R}$ -bundle is taut. We get the following corollary:

**Corollary 3.4.** *If  $\mathcal{F}$  admits a leaf whose closure is non-compact, then  $\mathcal{F}$  is taut.*

Below we will consider the case where the closure of every leaf is compact.

#### 4. LINEARLY FOLIATED TORUS BUNDLES

**4.1. Definition of  $(\mathbb{T}^k, \mathcal{F}_v)$ -bundles.** Let  $v$  be a nonzero vector in  $\mathbb{R}^k$ . Let  $\mathcal{F}_v$  be the linear flow of slope  $v$  on the torus  $\mathbb{T}^k$ .

**Definition 4.1.** A linearly foliated torus bundle, or a  $(\mathbb{T}^k, \mathcal{F}_v)$ -bundle is a torus bundle equipped with a defining 1-cocycle valued in  $\text{Diff}(\mathbb{T}^k, \mathcal{F}_v)$ .

We will always assume that the leaves of  $\mathcal{F}_v$  are dense in  $\mathbb{T}^k$ . Notice that the total space of a  $(\mathbb{T}^k, \mathcal{F}_v)$ -bundle has a one dimensional foliation, which is Riemannian. This foliation is called the *canonical foliation* of the  $(\mathbb{T}^k, \mathcal{F}_v)$ -bundle.

**4.2. Reduction to linearly foliated torus bundles.** The importance of the linearly foliated torus bundles comes from the following version of Molino structure theory for Riemannian foliations such that the closures of leaves are compact.

**Theorem 4.2.** *Let  $(M, \mathcal{F})$  be a manifold with a Riemannian foliation of dimension one and codimension  $q$  such that the closure of each leaf is compact. Let  $p: M^1 \rightarrow M$  be the orthonormal frame bundle of  $\nu\mathcal{F}$ . We have:*

- (i) *There exists an  $O(q)$ -invariant one dimensional Riemannian foliation  $\mathcal{F}^1$  on  $M^1$  such that the restriction of  $p$  to each leaf of  $\mathcal{F}^1$  is a covering map to a leaf of  $\mathcal{F}$ .*
- (ii) *There exists a smooth  $O(q)$ -equivariant  $(\mathbb{T}^k, \mathcal{F}_v)$ -bundle  $\pi_b: M^1 \rightarrow W$  whose fibers are the closures of the leaves of  $\mathcal{F}^1$ .*

*Outline of the proof.* We refer to [Mo88, Chapters 4 and 5] or [MM03, Chapter 4] for the terminologies on the Molino theory. The part (i) is a general construction valid for any Riemannian foliation. We give an outline of the proof of the part (ii).  $(M^1, \mathcal{F}^1)$  has a transverse parallelism  $\{X_1, \dots, X_m\}$  given by the basic connection on  $\nu\mathcal{F}$  and the canonical 1-form. Since the closure of each leaf of  $\mathcal{F}$  is compact, each  $X_i$  is complete on an open neighborhood of the closure of each leaf of  $\mathcal{F}^1$ . This implies that  $(M^1, \mathcal{F}^1)$  is homogeneous. In particular, the closures of leaves of  $\mathcal{F}^1$  define a foliation  $\overline{\mathcal{F}^1}$  of  $M^1$  whose leaf space  $W = M^1/\overline{\mathcal{F}^1}$  is a smooth manifold. The restriction of  $\mathcal{F}^1$  to a fiber  $F$  of  $M^1 \rightarrow W$  is a Lie foliation of dimension one. Thus, a theorem of Caron-Carrière [CC80] implies that  $(F, \mathcal{F}^1)$  is diffeomorphic to  $(\mathbb{T}^k, \mathcal{F}_v)$ .  $\square$

*Remark 4.3.* Theorem 4.2 and the following lemma reduces the proof of Theorem 1.2 to the case of linearly foliated torus bundles with a compact Lie group action.

Let  $(M^\sharp, \mathcal{F}^\sharp)$  be  $(M^1, \mathcal{F}^1)$  if  $\mathcal{F}^1$  is oriented, and otherwise a double cover of  $(M^1, \mathcal{F}^1)$  such that  $\mathcal{F}^\sharp$  is oriented. Let  $G$  be  $O(q)$  if  $\mathcal{F}^1$  is oriented, and otherwise  $O(q) \times \mathbb{Z}/2\mathbb{Z}$ .

**Lemma 4.4.** *If  $(M^\sharp, \mathcal{F}^\sharp)$  admits a  $G$ -invariant strongly tense metric  $g^\sharp$  such that the  $G$ -orbits are orthogonal to  $\mathcal{F}^\sharp$ , then  $(M, \mathcal{F})$  admits a strongly tense metric.*

*Proof.* We have a  $g^\sharp$ -orthogonal decomposition  $TM^\sharp = \ker \pi_* \oplus T\mathcal{F}^\sharp \oplus D$ , where  $D = (T\mathcal{F}^\sharp)^\perp \cap (\ker \pi_*)^\perp$ . As  $g^\sharp$  is  $G$ -invariant and  $\pi$  is a principal  $G$ -bundle,

there exists a Riemannian metric  $g$  on  $M$  such that  $g^\sharp(v, w) = g(\pi_*v, \pi_*w)$  for every  $v, w \in T_x\mathcal{F}^\sharp \oplus D_x$ . Notice that

$$(6) \quad \pi_*(T\mathcal{F}^\sharp) = T\mathcal{F} \quad \text{and} \quad \pi_*D = (T\mathcal{F})^{\perp g}.$$

We now show that  $g$  is strongly tense. Let  $U^\sharp$  be an open set of  $M^\sharp$  and  $U = \pi(U^\sharp)$ . We assume that  $U^\sharp$  and  $U$  are simply connected, and hence  $\mathcal{F}|_{U^\sharp}$  and  $\mathcal{F}|_U$  are orientable. We fix compatible orientations of  $\mathcal{F}|_{U^\sharp}$  and  $\mathcal{F}|_U$ . Let  $\chi_{U^\sharp}$  and  $\chi_U$  be characteristic forms. Then (4) and (6) imply that  $\pi^*\chi_U = \chi_{U^\sharp}$ . By Rummmler's formula (5), we get  $\pi^*(\kappa|_U) = \kappa^\sharp|_{U^\sharp}$ , and thus the proof is concluded.  $\square$

**4.3. Retracting the structure groups of  $(\mathbb{T}^k, \mathcal{F}_v)$ -bundles.** We consider the following group:

$$\mathrm{GL}_v(k; \mathbb{Z}) = \{A \in \mathrm{GL}(k; \mathbb{Z}) \mid Av = \lambda v, \exists \lambda \in \mathbb{R}\}.$$

We have a standard injection

$$\iota: \mathrm{GL}_v(k; \mathbb{Z}) \times \mathbb{T}^k \longrightarrow \mathrm{Diff}(\mathbb{T}^k, \mathcal{F}_v)$$

where  $\iota(A, y)(x) = Ax + y$  where  $+$  is the sum with respect to the standard coordinate on  $\mathbb{T}^k$ . We denote the linear part  $\mathrm{GL}_v(k; \mathbb{Z}) \rightarrow \mathrm{Diff}(\mathbb{T}^k, \mathcal{F}_v)$  of  $\iota$  by  $\iota_1$ .

**Lemma 4.5.** *The map  $\pi_0(\iota_1): \mathrm{GL}_v(k; \mathbb{Z}) \rightarrow \pi_0(\mathrm{Diff}(\mathbb{T}^k, \mathcal{F}_v))$  induced by  $\iota_1$  is bijective.*

*Proof.* Consider the natural map  $\rho_0: \pi_0(\mathrm{Diff}(\mathbb{T}^k, \mathcal{F}_v)) \rightarrow \mathrm{Aut}(H^1(\mathbb{T}^k; \mathbb{Z})) \cong \mathrm{GL}(k; \mathbb{Z})$ . Since  $\rho_0 \circ \pi_0(\iota_1)$  is injective as shown in the proof of [MS85, Lemma III.2],  $\pi_0(\iota_1)$  is injective. We will show that  $\pi_0(\iota_1)$  is surjective. It suffices to show that the image of  $\rho_0$  is contained in  $\mathrm{GL}_v(k; \mathbb{Z})$ . For every  $f \in \pi_0(\mathrm{Diff}(\mathbb{T}^k, \mathcal{F}_v))$ , we consider the following diagram

$$\begin{array}{ccc} H_1(\mathbb{T}^k; \mathbb{R}) & \xrightarrow{\psi} & H^1(\mathbb{T}^k/\mathcal{F}_v)^* \\ \rho_0(f) \downarrow & & \downarrow \rho_0(f)^{**} \\ H_1(\mathbb{T}^k; \mathbb{R}) & \xrightarrow{\psi} & H^1(\mathbb{T}^k/\mathcal{F}_v)^*, \end{array}$$

where  $\psi$  is the map induced from the canonical pairing  $H_1(\mathbb{T}^k; \mathbb{R}) \times H^1(\mathbb{T}^k/\mathcal{F}_v) \rightarrow \mathbb{R}$ . Since the kernel of  $\psi$  is generated by  $v$  and both vertical arrows are isomorphisms, it follows that  $v$  is an eigenvector of  $\rho_0(f)$ .  $\square$

**Proposition 4.6.**  *$\mathrm{Diff}(\mathbb{T}^k, \mathcal{F}_v)$  retracts to  $\mathrm{GL}_v(k; \mathbb{Z}) \times \mathbb{T}^k$ .*

*Proof.* By Lemma 4.5, the injection  $\mathrm{GL}_v(k; \mathbb{Z}) \times \mathbb{T}^k \rightarrow \mathrm{Diff}(\mathbb{T}^k, \mathcal{F}_v)$  induces the bijection on the groups of connected components. Then it suffices to show that the identity component  $\mathrm{Diff}_0(\mathbb{T}^k, \mathcal{F}_v)$  retracts to the identity component  $\mathbb{T}^k$  of  $\mathrm{GL}_v(k; \mathbb{Z}) \times \mathbb{T}^k$ . Since  $\mathrm{Diff}_0(\mathbb{T}^k, \mathcal{F}_v)$  acts trivially on the homology of  $\mathbb{T}^k$ , this is a direct consequence of [MS85, Lemmas II.2 and III.2].  $\square$

**4.4. Tenseness of linearly foliated torus bundles.** Let  $\pi: M \rightarrow W$  be an oriented  $(\mathbb{T}^k, \mathcal{F}_v)$ -bundle with canonical foliation  $\mathcal{F}$ . Assume that the structure group of  $\pi$  is a subgroup of  $\mathrm{GL}_v(k; \mathbb{Z}) \times \mathbb{T}^k$ . Then, for a simply connected open set  $U$  of  $W$ ,  $\pi^{-1}(U)$  has a  $\mathbb{T}^k$ -action up to conjugacy by  $\mathrm{GL}_v(k; \mathbb{Z})$ .

**Lemma 4.7.** *A characteristic form  $\chi$  of  $(M, \mathcal{F})$  is strongly tense if and only if, for any simply connected open set  $U$ , the characteristic form  $\chi|_{\pi^{-1}(U)}$  is invariant under any  $\mathbb{T}^k$ -action in the  $\mathrm{GL}_v(k; \mathbb{Z})$ -conjugacy class. If  $\chi$  is invariant under a  $\mathbb{T}^k$ -action in the  $\mathrm{GL}_v(k; \mathbb{Z})$ -conjugacy class globally defined on  $M$ , then  $\chi$  is taut after a multiplication of a positive function.*

*Proof.* Let  $x$  be a point of  $W$  and take a simply connected neighborhood  $U$  of  $x$  in  $W$ . Let  $X$  be a vector field on  $\pi^{-1}(U)$  tangent to  $\mathcal{F}|_{\pi^{-1}(U)}$  which generates an  $\mathbb{R}$ -subaction of a  $\mathbb{T}^k$ -action in the  $\mathrm{GL}_v(k; \mathbb{Z})$ -conjugacy class. We show the if part. By Rummmler's formula (5), we get

$$(7) \quad \kappa = \iota_{\frac{1}{\chi(X)}X}d\chi = \frac{1}{\chi(X)}\iota_X d\chi = -\frac{1}{\chi(X)}d(\chi(X)) = -d \log |\chi(X)|.$$

Since  $\log |\chi(X)|$  is a basic function and each  $\pi^{-1}(U)$  is saturated,  $\kappa$  is basic and closed in  $M$ .

We show the only if part. Assume that  $\chi$  is tense. Since  $\kappa = \iota_{\frac{1}{\chi(X)}X}d\chi$  is basic and closed, there exists a basic function  $h$  on  $(\pi^{-1}(U), \mathcal{F}|_{\pi^{-1}(U)})$  such that  $dh = \kappa$ . The Rummmler's formula implies that the mean curvature form of a characteristic form  $e^{-h}\chi$  is zero. Thus the closure of the flow generated by the vector field  $X$  tangent to  $\mathcal{F}$  which satisfies  $e^{-h}\chi(X) = 1$  is a  $\mathbb{T}^k$ -action which preserves  $e^{-h}\chi$ . Since  $e^{-h}$  is basic, the  $\mathbb{T}^k$ -action preserves  $\chi$ .

The latter part of the statement follows, because (7) implies that the mean curvature form of  $e^{-\chi(X)}\chi$  is zero.  $\square$

**4.5. Molino's commuting sheaf of linearly foliated torus bundles.** Let  $\pi: M \rightarrow W$  be a  $(\mathbb{T}^k, \mathcal{F}_v)$ -bundle with canonical foliation  $\mathcal{F}$  whose the structure group is a subgroup of  $\mathrm{GL}_v(k; \mathbb{Z}) \times \mathbb{T}^k$ . The Molino's commuting sheaf  $\mathcal{C}$  of  $(M, \mathcal{F})$  is determined by the structure group as follows (We refer to [Mo88, Section 5.3] for the definition of the Molino's commuting sheaf).

**Proposition 4.8** ([No10, Proposition 6]). *The holonomy homomorphism  $\mathrm{hol}(\mathcal{C})$  of  $\mathcal{C}$  is determined by*

$$\pi_1 M \longrightarrow \mathrm{GL}_v(k; \mathbb{Z}) \times \mathbb{T}^k \longrightarrow \mathrm{GL}_v(k; \mathbb{Z}) \xrightarrow{r} \mathrm{GL}(k-1; \mathbb{R}),$$

where the first arrow is the holonomy homomorphism of  $\pi$ , the second arrow is the first projection and the third arrow  $r$  is defined by sending  $A \in \mathrm{GL}_v(k; \mathbb{Z})$  to the map  $\mathbb{R}^k/\mathbb{R}v \rightarrow \mathbb{R}^k/\mathbb{R}v$  induced from  $A$ .

We will prove Theorem 1.5 by using Proposition 4.8.

*Proof of Theorem 1.5.* By Theorem 1.4, it suffices to prove the theorem in the case where the closure of each leaf is compact. Let  $M^1 \rightarrow W$  be the oriented linearly foliated torus bundle in Theorem 4.2. If we have a tense metric on  $(M, \mathcal{F})$  with mean curvature form  $\kappa$ , then we can construct a tense characteristic form on  $(M^1, \mathcal{F}^1)$  with mean curvature form  $\kappa^1$  such that  $\kappa^1 = \pi^*\kappa$ . Thus it suffices to prove the theorem in the case where  $(M, \mathcal{F})$  is an oriented linearly foliated torus bundle. Let  $\gamma: \mathbb{S}^1 \rightarrow W$  be a loop on  $W$ . The pull back of the oriented linearly foliated torus bundle to  $\mathbb{S}^1$  is a mapping torus  $N = \mathbb{T}^k \times [0, 1]/(A(x), 0) \sim (x, 1)$  for some  $A \in \mathrm{GL}_v(k; \mathbb{Z})$ . By Lemma 4.7, the restriction of  $\chi$  to  $\mathbb{T}^k \times \{t\}$  is linear with respect to the standard coordinate on  $\mathbb{T}^k$ . By [No10, Example 7.3], the class  $[\kappa]$  is determined by  $[\kappa]|_{\mathbb{T}^k \times \{t\}} = 0$  and  $\int_{\mathbb{S}^1} \kappa = \log \lambda$ , where  $\lambda$  is the eigenvalue of  $A$  with respect to  $v$ . Thus  $[\kappa]$  is determined by  $(M, \mathcal{F})$ . The latter part follows from Proposition 4.8.  $\square$

Let  $\det \mathcal{C}$  be the determinant line bundle of  $\mathcal{C}$ . Since it is easy to see that the composite of

$$\mathrm{GL}_v(k; \mathbb{Z}) \xrightarrow{r} \mathrm{GL}(k-1; \mathbb{R}) \xrightarrow{\det} \mathrm{GL}(1; \mathbb{R}),$$

is injective, we get the following consequence of Proposition 4.8, which is necessary in the proof of Corollary 1.10.

**Corollary 4.9.** *The image of the holonomy homomorphism of  $\det \mathcal{C}$  is contained in  $\{\pm 1\}$  if and only if so is the image of the holonomy homomorphism of  $\mathcal{C}$ .*

*Proof of Corollary 1.10.* The equivalence of the first three assertions is a formal consequence of Theorem 1.10 and Corollary 1.9 (see the proof of [RPSAW09, Theorems 3.3 and 3.5]). We will show the equivalence of (iii) and (iv). If  $(M, \mathcal{F})$  is an  $\mathbb{R}$ -bundle, then both (iii) and (iv) are true. By Theorem 1.4, we can assume that the closure of each leaf of  $(M, \mathcal{F})$  is compact. The equivalence of (iii) and (iv) follows from (2). Finally, since the holonomy homomorphisms of Sergiescu's orientation sheaf  $\mathcal{P}$  and the determinant line bundle of Molino's commuting sheaf are equal up to sign by definition of  $\mathcal{P}$ , the equivalence of (iv) and (v) follows from Corollary 4.9.  $\square$

The Molino's commuting sheaf and the Álvarez class are illustrated with Carrière's example [Ca84].

**Example 4.10.** Take  $A \in \mathrm{SL}(2; \mathbb{Z})$  such that  $\mathrm{tr} A > 2$ , let  $\lambda$  be one of its eigenvalues and denote by  $v = (a, b) \in \mathbb{R}^2$  the corresponding eigenvector. Notice that  $A$  induces a diffeomorphism  $\bar{A}$  on  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ . Consider the manifold  $\mathbb{T}_A^3 = (\mathbb{T}^2 \times [0, 1])/(\bar{A}x, 0) \sim (x, 1)$ , which is a  $\mathbb{T}^2$ -bundle over  $\mathbb{S}^1$ . Here  $\mathbb{T}_A^3$  admits the structure of a  $(\mathbb{T}^2, \mathcal{F}_v)$ -bundle whose structure group is the infinite cyclic subgroup of  $\mathrm{SL}_v(2; \mathbb{Z})$  generated by  $A$ . Let  $\mathcal{F}$  be the canonical foliation. Taking the standard coordinates  $(x, y, t)$  on  $\mathbb{T}_A^3$ , we have the parallelization

$$X = \lambda^t(a\partial/\partial x + b\partial/\partial y), \quad Y = \lambda^{-t}(-b\partial/\partial x + a\partial/\partial y), \quad T = \partial/\partial t.$$

Let  $g$  be the Riemannian metric on  $\mathbb{T}_A^3$  such that  $\{X, Y, T\}$  is an orthonormal parallelism. It is straightforward to check that  $g$  is bundle-like and that its mean curvature form  $\kappa$  is given by  $(\log \lambda)dt$  in the standard coordinates. So,  $g$  is strongly tense, while the Álvarez class  $[\kappa]$  of  $\mathcal{F}$  is not trivial. Thus  $\mathcal{F}$  is not taut.

## 5. INVARIANT TENSE METRICS ON LINEARLY FOLIATED TORUS BUNDLES

Let  $\pi: M \rightarrow W$  be a  $(\mathbb{T}^k, \mathcal{F}_v)$ -bundle and  $G$  a compact Lie group acting on  $M$  preserving the canonical foliation  $\mathcal{F}$ . We assume that  $\mathcal{F}$  is oriented. In this section, we will prove the following result, which completes the proof of Theorem 1.2 (see Remark 4.3):

**Theorem 5.1.**  *$(M, \mathcal{F})$  admits a  $G$ -invariant strongly tense metric. Moreover, if the  $G$ -action is locally free and  $\mathcal{F}$  is not tangent to any  $G$ -orbit, then we can take a  $G$ -invariant strongly tense metric so that the  $G$ -orbits are orthogonal to  $\mathcal{F}$ .*

By Proposition 4.6, we can assume that the structure group of  $\pi$  is  $\mathrm{GL}_v(k; \mathbb{Z}) \rtimes \mathbb{T}^k$ . Let  $\phi_M$  be the composite of

$$(8) \quad \pi_1 M \longrightarrow \mathrm{GL}_v(k; \mathbb{Z}) \rtimes \mathbb{T}^k \longrightarrow \mathrm{GL}_v(k; \mathbb{Z}) \longrightarrow \mathbb{R},$$

where the first arrow is the holonomy homomorphism of  $\pi$ , the second arrow is the first projection and the third arrow sends  $A \in \mathrm{GL}_v(k; \mathbb{Z})$  to the logarithm of the eigenvalue of  $A$  with respect to  $v$ .

**Lemma 5.2.** *Let  $K$  be a  $G$ -orbit in  $M$  and  $i_K: \pi_1 K \rightarrow \pi_1 M$  be the map induced from the inclusion. Then,  $\phi_M \circ i_K$  is trivial.*

*Proof.* Let  $E$  be the vector subbundle of  $\nu\mathcal{F}$  defined by the kernel of  $\pi_*: \nu\mathcal{F} \rightarrow TW$ . Here  $E$  is invariant under the  $G$ -action, because  $G$  preserves  $\mathcal{F}$  and the fibers of  $\pi$  are closures of the leaves of  $\mathcal{F}$ . Then  $E$  has a  $G$ -invariant metric by the compactness of  $G$ . Thus, for any loop  $\gamma$  in  $K$ , the holonomy map associated to  $\pi_*\gamma$  preserves a metric on  $E$ . This implies the triviality of  $\phi_M \circ i_K$ .  $\square$

The key of our proof of Theorem 5.1 is the following lemma, which was already used in [AL92, Lemma 6.3].

**Lemma 5.3.** *Let  $\chi$  be a characteristic form of  $(M, \mathcal{F})$ . Let  $\chi_1$  be defined by*

$$\chi_1 = \int_{g \in G} \epsilon(g)(g^*\chi)dg,$$

where  $dg$  is a Haar measure of  $G$  and  $\epsilon(g) = 1$  if  $g$  preserves the orientation of  $\mathcal{F}$  and  $\epsilon(g) = -1$  otherwise. Then  $\chi_1$  is a characteristic form of  $\mathcal{F}$ . Moreover, if  $\chi$  is taut, then so is  $\chi_1$ .

*Proof.* The first part follows from the fact that a 1-form  $\chi'$  on  $M$  is a characteristic form of  $\mathcal{F}$  if and only if  $\chi'$  is transverse to  $\mathcal{F}$ . The latter part follows from the Rummmler's formula (5) and  $d\chi_1 = \int_{g \in G} \epsilon(g)(g^*d\chi)dg$ .  $\square$

*Remark 5.4.* As we saw in the last lemma, the sum of two taut characteristic forms is taut, while the sum of two tense characteristic forms may not be tense. We can use this phenomenon to show Theorem 5.1 by taking a covering of  $(M, \mathcal{F})$  which is taut. Strongly tenseness of characteristic forms is not linear in a direct way. But there is a certain way to make the sum of two strongly tense characteristic forms to obtain a strongly tense one by using the interpretation of strongly tenseness as a twisted version of tautness (see [No12, Proposition 7.8]).

*Proof of Theorem 5.1.* By Proposition 4.6, the structure group of  $\pi$  can be reduced to  $\Gamma \ltimes \mathbb{T}^k$ , where  $\Gamma$  is a subgroup of  $\mathrm{GL}_v(k; \mathbb{Z})$ . Let  $p: (M', \mathcal{F}') \rightarrow (M, \mathcal{F})$  be the covering of  $(M, \mathcal{F})$  such that  $\pi_1 M' \cong \ker \phi_M$ , whose covering group is identified with  $\Gamma$ . For any  $G$ -orbit  $K$  in  $M$ , we have  $\pi_1 K \subset \pi_1 M'$  by Lemma 5.2. Thus the  $G$ -action on  $M$  lifts to a  $G$ -action  $\psi_G$  on  $M'$ . The  $\Gamma$ -action on  $M'$  via deck transformations commutes with  $\psi_G$ .

By construction,  $M'$  has a structure of a principal  $\mathbb{T}^k$ -bundle such that  $\mathcal{F}'$  is the orbit foliation of a dense  $\mathbb{R}$ -subaction of the principal  $\mathbb{T}^k$ -action  $\rho_0$ . Let  $X$  be a vector field which generates the dense  $\mathbb{R}$ -subaction of  $\rho_0$ . Let  $\chi$  be a  $\rho_0$ -invariant characteristic form of  $(M', \mathcal{F}')$  such that  $\chi(X) = 1$ . By the latter part of Lemma 4.7, we can assume that  $\chi$  is taut. Let  $\chi_1$  be the characteristic form of  $(M', \mathcal{F}')$  obtained from  $\chi$  like in Lemma 5.3. Let  $X_1$  be the vector field tangent to  $\mathcal{F}'$  such that  $\chi_1(X_1) = 1$ . Since  $\chi$  is taut, by Lemma 5.3, so is  $\chi_1$ . Thus  $X_1$  is a Killing vector field with respect to a Riemannian metric on  $M'$ . Then the closure of the flow generated by  $X_1$  is a principal  $\mathbb{T}^k$ -action  $\rho_1$  on  $M'$ . Since  $X_1$  is  $G$ -invariant up to sign determined by  $\epsilon$  in Lemma 5.3,  $\psi_G$  and  $\rho_1$  yield a  $(G \ltimes \mathbb{T}^k)$ -action on  $M'$ , where the semidirect product is defined by  $\epsilon: G \rightarrow \{\pm 1\}$ . Note that the  $\Gamma$ -action commutes with  $\psi_G$ . Here  $\rho_1$  and the  $\Gamma$ -action on  $M'$  yield a  $(\Gamma \ltimes \mathbb{T}^k)$ -action on  $M'$ , because, for  $h \in \Gamma$ , we have

$$h_* X_1 = \int_{g \in G} (h_* g_* X) dg = \int_{g \in G} (g_* h_* X) dg = \phi_M(h) \int_{g \in G} (g_* X) dg = \phi_M(h) X_1 .$$

In total, we get a  $((G \times \Gamma) \ltimes \mathbb{T}^k)$ -action on  $M'$  such that  $\mathcal{F}'$  is the orbit foliation of a dense  $\mathbb{R}$ -subaction of the principal  $\mathbb{T}^k$ -action  $\rho_1$ .

Let  $\chi_2$  be a  $\Gamma$ -invariant characteristic form of  $(M', \mathcal{F}')$ . Using the  $((G \times \Gamma) \ltimes \mathbb{T}^k)$ -action on  $M'$  obtained in the last paragraph, let

$$\chi_3 = \int_{u \in G \ltimes \mathbb{T}^k} \epsilon'(u) (u^* \chi_2) du ,$$

where  $du$  is a Haar measure of  $\mathbb{T}^k \times G$  and  $\epsilon'(u) = 1$  if  $u$  preserves the orientation of  $\mathcal{F}'$  and  $\epsilon'(u) = -1$  otherwise. Then  $\chi_3$  is a  $(\Gamma \ltimes \mathbb{T}^k)$ -invariant characteristic form, which is strongly tense by Lemma 4.7. Thus,  $(M', \mathcal{F}')$  admits a  $(G \times \Gamma)$ -invariant strongly tense metric, which induces a  $G$ -invariant strongly tense metric on  $(M, \mathcal{F})$ .

To show the latter part, it is sufficient to apply the same proof with  $\chi_2$  so that  $G$ -orbits are tangent to  $\ker \chi_2$ .  $\square$

## REFERENCES

- [AL89] J.A. Álvarez López, *A finiteness theorem for the spectral sequence of a Riemannian foliation*, Illinois J. Math. **33** (1989), 79–92.
- [AL92] ———, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. **10** (1992), 179–194.
- [AL96] ———, *On the first secondary invariant of Molino’s central sheaf*, Ann. Polon. Math. **64** (1996), 253–265.
- [CE97] G. Cairns, R. Escobales, *Further geometry of the mean curvature one-form and the normal plane field one-form on a foliated Riemannian manifold*, J. Austral. Math. Soc. Ser. A **62** (1997), 46–63.
- [CC00] A. Candel, L. Conlon, *Foliations I*, Grad. Stud. Math. 23, American Mathematical Society, Providence, RI, 2000.
- [CC80] P. Caron, Y. Carrière, *Flots transversalement de Lie  $\mathbb{R}^n$ , flots transversalement de Lie minimaux*, C. R. Acad. Sci. Paris Sér. A-B **291** (1980), 477–478.
- [Ca84] Y. Carrière, *Flots riemanniens*, Astérisque **116** (1984), 31–52.
- [DC92] M.P. do Carmo, *Riemannian Geometry*, Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.
- [Do98] D. Domínguez, *Finiteness and tensesness theorems for Riemannian foliations*. Amer. J. Math. **120** (1998), 1237–1276.
- [EH86] A. El Kacimi-Alaoui, G. Hector, *Décomposition de Hodge basique pour un feuilletage riemannien*, Ann. Inst. Fourier (Grenoble) **36** (1986), 207–227.
- [Gh83] É. Ghys, *Classification des feuilletages riemanniens totalement géodésiques de codimension un*, Comment. Math. Helv. **58** (1983), 543–572.
- [Gh84] ———, *Feuilletages riemanniens sur les variétés simplement connexes*, Ann. Inst. Fourier (Grenoble) **34** (1984), 203–223.
- [Ha80] A. Haefliger, *Some remarks on foliations with minimal leaves*, J. Differential Geom. **15** (1980), 269–284.
- [Ha85] ———, *Pseudogroups of local isometries*, Differential geometry (Santiago de Compostela, 1984), pp. 174–197, Res. Notes in Math., 131, Pitman, Boston, MA, 1985.
- [Ha88] ———, *Leaf closures in Riemannian foliations*, A fête of topology, Pap. Dedic. Iitiro Tamura, pp. 3–32 (1988).
- [KT75] F. Kamber, P. Tondeur, *Foliated bundles and characteristic classes*, Lecture Notes in Math. 493, Springer, New York, 1975.
- [KT83] ———, *Duality for Riemannian foliations*, Singularities, Part 1 (Arcata, Calif. 1981), pp. 609–618, Proc. Sympos. Pure Math. 40, Amer. Math. Soc., Providence, RI, 1983.
- [KT83b] ———, *Foliations and metrics*, Differential geometry (College Park, Md., 1981/1982), pp. 103–152, Progr. Math. 32, Birkhäuser Boston, Boston, MA, 1983.
- [KT84] ———, *Duality theorems for foliations*, Astérisque **116** (1984), 108–116.
- [Ma92] X. Masa, *Duality and minimality in Riemannian foliations*, Comment. Math. Helv. **67** (1992), 17–27.
- [MM03] I. Moerdijk, J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Stud. Adv. Math. 91, Cambridge University Press, Cambridge, 2003.
- [Mo82] P. Molino, *Feuilletages de Lie à feuilles denses*, Séminaire de Géométrie Différentielle 1982–83, Montpellier.

- [Mo88] ———, *Riemannian Foliations*, Progr. Math. 73, Birkhäuser Boston, Inc., Boston, MA, 1988.
- [MS85] P. Molino, V. Sergiescu, *Deux remarques sur les flots riemanniens*, Manuscripta Math. **51** (1985), 145–161.
- [No10] H. Nozawa, *Rigidity of the Álvarez class*, Manuscripta Math. **132** (2010), 257–271.
- [No12] ———, *Haefliger cohomology of Riemannian foliations*, Preprint (2012), [arXiv:1209.3817](https://arxiv.org/abs/1209.3817).
- [Re59] B. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. (2) **69** (1959), 119–132.
- [RP01] J.I. Royo Prieto, *The Euler class of Riemannian flows*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), 45–50.
- [RPSAW08] J.I. Royo Prieto, M. Saralegi-Aranguren, R. Wolak, *Tautness for Riemannian foliations on non-compact manifolds*, Manuscripta Math. **126** (2008), 177–200.
- [RPSAW09] ———, *Cohomological tautness for Riemannian foliations*, Russian J. Math. Phys. **16** (2009), 450–466.
- [Ru79] H. Rummier, *Quelques notions simples en géométrie riemannienne et leurs applications aux feuilletages compacts*, Comment. Math. Helv. **54** (1979), 224–239.
- [Sar78] K.S. Sarkaria, *A finiteness theorem for foliated manifolds*, J. Math. Soc. Japan **30** (1978), 687–696.
- [Se85] V. Sergiescu, *Cohomologie basique et dualité des feuilletages riemanniens*, Ann. Inst. Fourier (Grenoble) **35** (1985), 137–158.
- [Su79] D. Sullivan, *A homological characterization of foliations consisting of minimal surfaces*, Comment. Math. Helv. **54** (1979), 218–223.
- [To97] P. Tondeur, *A characterization of Riemannian flows*, Proc. Amer. Math. Soc. **125** (1997), 3403–3405.
- [Žu87] N. Žukova, *On the stability of leaves of Riemannian foliations*, Ann. Global Anal. Geom. **5** (1987), 261–271.

HIRAKU NOZAWA  
 INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES  
 LE BOIS-MARIE 35, ROUTE DE CHARTRES  
 91440 BURES-SUR-YVETTE, FRANCE  
*E-mail address:* nozawahiraku@06.alumni.u-tokyo.ac.jp

JOSÉ IGNACIO ROYO PRIETO  
 DEPARTAMENTO DE MATEMÁTICA APLICADA  
 UNIVERSIDAD DEL PAÍS VASCO UPV/EHU  
 ALAMEDA DE URQUIJO S/N  
 48013 BILBAO, SPAIN  
*E-mail address:* joseignacio.royo@ehu.es