# KURANISHI TYPE MODULI SPACES FOR PROPER CR SUBMERSIONS FIBERING OVER THE CIRCLE

LAURENT MEERSSEMAN

Abstract. Kuranishi's fundamental result (1962) associates to any compact complex manifold $X_0$ a finite-dimensional analytic space which has to be thought of as a local moduli space of complex structures close to $X_0$. In this paper, we give an analogous statement for Levi-flat CR manifolds fibering properly over the circle by describing explicitely an infinite-dimensional Kuranishi type local moduli space of Levi-flat CR structures. We interpret this result in terms of Kodaira-Spencer deformation theory making clear the likenesses as well as the differences with the classical case. The article ends with applications and examples.

## Introduction

In 1962, M. Kuranishi proved that any compact complex manifold $X_0$ has a versal (also called semi-universal) finite-dimensional analytic space of deformations $K$ (see [Ku1] for the original paper, [Ku2] and [Ku3] for more accurate versions and simpler proofs). Roughly speaking, this means that every deformation of the complex structure of $X_0$ is encoded in a family defined over this finite-dimensional analytic space $K$ (the family is said to be complete); and that this family is minimal amongst all complete families. This fundamental result is the crowning achievement of the famous deformation theory K. Kodaira developed in collaboration with D.C. Spencer (see [K-S1], [K-S2]). The Kuranishi space $K$ must be thought of as a substitute for a local moduli space of complex structures, which is known not to exist in general.

It is natural to ask for generalizations of this result to other geometric structures, and in particular to foliations. It is known that a Kuranishi's Theorem is valid for holomorphic foliations and even for transversely holomorphic foliations



2010 *Mathematics Subject Classification.* 32G07, 32V05, 57R30.

*Key words and phrases.* Levi-flat CR structures, foliations by complex manifolds, deformations of CR structures and foliations, local moduli space.

This work was partially supported by project COMPLEXE (ANR-08-JCJC-0130-01) from the Agence Nationale de la Recherche. It is part of Marie Curie project DEFFOL 271141 funded by the European Community. I enjoyed the warmful atmosphere of the CRM of Bellaterra during the elaboration of this work. I would like to thank the ANR, the CRM and the European Community. This work benefited from fruitful discussions with Marcel Nicolau to whom I am especially grateful.

(see [G-H-S]). In this last case, the foliation is smooth, but its normal bundle is endowed with a complex structure which is invariant by holonomy.

However, in the case of foliations by complex manifolds (or, equivalently, Levi-flat CR structures), the situation is much more complicated. Here the tangent bundle to the foliation is endowed with a complex structure, so that this is a case in a sense orthogonal to the previous one.

The main problem is that it is easy to find examples of foliations by complex manifolds with no finite-dimensional complete family. Indeed, this is the most common case and it seems very difficult to give a useful criterion to ensure finite-dimensional completeness. For example, in the case of an irrational linear foliation of the 3-torus by curves, finite-dimensionality is related to the arithmetic properties of the irrational slope (see [Sl] or [EK-S]).

So one is faced with the following dilemma: either studying the rare examples where finite-dimensionality occurs (cf. [M-V] for such an example), or dealing with infinite-dimensional families.

If one follows this last option and try to construct an infinite-dimensional versal space, many technical obstacles come out. The classical proof of versality in Kuranishi's Theorem breaks down completely since it uses heavily elliptic theory, and begins with showing that the candidate for versal space is tangent to the kernel of an auto-adjoint elliptic operator. But this implies that this space is finite-dimensional. Hence there is no natural way of attacking the problem.

The starting point of this paper is the following easy remark. If we consider the Levi-flat CR manifold $\mathbb{E}_\tau \times \mathbb{S}^1$ (for $\mathbb{E}_\tau$ the elliptic curve of modulus $\tau \in \mathbb{H}$), then a complete space for close Levi-flat CR structures [1] is given by the set of smooth maps from $\mathbb{S}^1$ to $\mathbb{H}$ close to the constant map $\tau$. It is even a local moduli space if $\tau$ is not a root of unity.

The main result of this paper (Theorem 2) shows that this picture is still valid for Levi-flat CR manifolds which are proper submersions over the circle. Starting with any such CR manifold $\mathcal{X}_0$, there exists a finite-dimensional analytic space $K_{c_0}$ with a marked loop $c_0$ such that a neighborhood of $c_0$ in the loop space of $K_{c_0}$ contains all small deformations of $\mathcal{X}_0$. This loop space is the base of an infinite-dimensional family which is complete for $\mathcal{X}_0$ (Theorem 4).

If all the fibers of the submersion $\mathcal{X}_0 \to \mathbb{S}^1$ are biholomorphic, the space $K_{c_0}$ is just the common Kuranishi space of all fibers. However, if the submersion has non-isomorphic fibers, one has to build this space from different Kuranishi spaces by gluing them together. This causes many technical problems. First these spaces

[1] Here, we consider only Levi-flat CR structures on the tangent bundle to the elliptic factor, that is we fix the differentiable type of the induced smooth foliation. Moreover, we identify two such structures if they are isomorphic through a CR isomorphism which is the identity on the $\mathbb{S}^1$-factor.

may not have the same dimension, hence we have to fat them. Then, there is no canonical choice for the gluing maps, hence we have to make all the choices coherent. Finally, we have to make sure that the resulting glued space is really an analytic space, especially that it is Hausdorff.

It is important to notice that this complete family is not always versal. But it is minimal in a reasonable sense as explained in Corollary 4. Indeed, we give examples in Sections V.3 and V.4 which suggest that there is no versal family. We also give a complete characterization of versality of our family in Theorem 4.

The paper is organized as follows. In Part I, we give some preliminary material. While some of it is very classical, the version of Kuranishi's Theorem as well as the definitions of CR deformations are not. In Part II, we study the neighborhood of a loop in the set of complex structures on a fixed differentiable manifold. It is the first step in the construction of the Kuranishi type moduli space and is completed in Theorem 1. Part III contains the construction of the Kuranishi type moduli space $C^\infty(\mathbb{S}^1, K_{c_0})$ and the proof of Theorem 2, our main Theorem. Part IV develops the necessary deformation machinery to interpret Theorem 2 in terms of deformation theory. Part V contains applications to connectedness and rigidity of the Kuranishi type moduli space as well as examples showing that it can be explicitely computed.

## I. Preliminaries

**1. Notations and Framework.** The notations we introduce in this section will be used throughout the article.

Let $X^{diff}$ be a smooth (i.e. $C^\infty$) connected compact manifold of dimension $2n$. We fix a smooth riemannian metric $g$ on its tangent bundle $TX^{diff}$.

Given any locally trivial smooth bundle $E$ over $X^{diff}$, we denote by $\Sigma(E)$ the space of sections of $E$. More generally, we denote by $\Sigma(E, B)$, or simply by $\Sigma(E)$ when the context is clear, the space of smooth sections of a locally trivial bundle $E$ with base $B$. We topologize such spaces of sections using a Sobolev norm and consider sections belonging to the Sobolev class $W_2^r$ (cf. [Ku3, **?**] for more details). They form a Hilbert space. We assume that $r$ is big enough to ensure that all these sections are at least of class $C^{2n+1}$. We drop any reference to this $r$ in the sequel, since it does not play any specific role.

We denote by $\mathcal{E}$ the set of almost-complex structures of class $W_2^r$ on $X^{diff}$, and by $\mathcal{I}$ the subset of $\mathcal{E}$ formed by the integrable ones. We assume that both $\mathcal{E}$ and $\mathcal{I}$ are non-empty. Observe that our choice of $r$ implies that an element of $\mathcal{I}$ gives rise to a structure of complex manifold on $X^{diff}$ by Newlander-Nirenberg Theorem [N-N].

Any almost-complex operator $J$ is diagonalizable over $\mathbb{C}$ with eigenvalues $i$ and $-i$ and conjugated eigenspaces. This induces a splitting of the complexified tangent bundle of $X^{diff}$

$$T_{\mathbb{C}}X^{diff} = T \oplus \bar{T} \tag{1}$$

Conversely, any such splitting define a unique almost-complex structure on $X^{diff}$, which is equal (as an operator) to the multiplication by $i$ on $T$ and by $-i$ on $\bar{T}$. Therefore, denoting by $\text{Gr}(T_{\mathbb{C}}X^{diff})$ the bundle over $X^{diff}$ whose fiber at $x$ is the grassmannian of complex $n$-planes of $(T_{\mathbb{C}}X^{diff})_x$, we may identify $\mathcal{E}$ to an open subset of $\Sigma(\text{Gr}(T_{\mathbb{C}}X^{diff}))$. We set, more precisely

$$\mathcal{E} = \{T \in \Sigma(\text{Gr}(T_{\mathbb{C}}X^{diff})) \quad | \quad \bar{T} \cap T = \{0\}\} \tag{2}$$

And we have

$$\mathcal{I} = \{T \in \mathcal{E} \quad | \quad [T, T] \subset T\} \tag{3}$$

making $\mathcal{I}$ a closed subset of $\mathcal{E}$.

We will make use of the the formalism of Banach manifolds and Banach analytic spaces as defined in [Do1] and [Do2]. Since $\text{Gr}(T_{\mathbb{C}}X^{diff})$ is a complex bundle, its space of sections is a Banach manifold over $\mathbb{C}$ with tangent space at some section $\sigma$ equal to the Banach space of sections of Sobolev class $W_2^r$ of the complex vector bundle $\sigma^*T\text{Gr}(T_{\mathbb{C}}X^{diff})$. Moreover, the open set $\mathcal{E}$ is a $\mathbb{C}$-Banach manifold and its subset $\mathcal{I}$ is a $\mathbb{C}$-analytic Banach space (cf. [Do1]).

Given $J \in \mathcal{E}$ (respectively $J \in \mathcal{I}$), we denote by $X_J$ the almost-complex (respectively complex) manifold $(X^{diff}, J)$.

Let $Diff(X^{diff})$ be the group of diffeomorphisms of class $W_2^{r+1}$ of $X^{diff}$. It acts on $\mathcal{E}$ as follows: for $J \in \mathcal{E}$ and $f \in Diff(X^{diff})$, we have

$$(f \cdot J)_x(v) = d_x f \circ J_{f^{-1}(x)} \circ (d_x f)^{-1}(v) \tag{4}$$

for $(x, v) \in TX^{diff}$, or, using the presentation (3),

$$(f \cdot T)_x = (d_x f)(T_{f^{-1}(x)}). \tag{5}$$

This group being an open subset of the Banach manifold of $W_2^{r+1}$ maps from $X^{diff}$ to the complex manifold $X$, is also a Banach manifold and the action (4) is $\mathbb{C}$-analytic (cf. [Do1]). Besides, by definition, it preserves the almost-complex structures, that is $f$ realizes an isomorphism between $X_J$ and $X_{f \cdot J}$.

Given two subsets $U$ and $V$ of $\mathcal{E}$ with $U$ open and an analytic map $F$ from $U$ to $V$, we say that $F$ is *almost-complex preserving* if, for each $J \in U$, the manifolds $X_J$ and $X_{F(J)}$ are CR isomorphic. We note

$$U \xrightarrow[a.c.]{F} V$$

Notice that an a.c. map is a special type of equivariant map. Indeed, the set of a.c. maps corresponds exactly to the set of equivariant maps which descends as the identity on the quotient space $\mathcal{E}/Diff(X^{diff})$. We extend this notion to the following case. Letting $U$ and $V$ as before and letting $W$ be an open subset of a topological $\mathbb{C}$-vector space, we say that an analytic map $F$ from $U$ to $V \times W$ (respectively from $U \times W$ to $V$) is *almost-complex preserving* if the composition

$$U \xrightarrow{\quad F \quad} V \times W \xrightarrow{\text{1st projection}} V$$

respectively

$$U \xrightarrow{\text{inclusion}} U \times \{w\} \subset U \times W \longrightarrow V$$

is almost-complex preserving (respectively almost-complex preserving for all $w \in W$).

## 2. Kuranishi's Theorem revisited.

In this Section, we state a version of Kuranishi's Theorem which is suited for our purposes. Although it is very close to the statements of [Ku2], this slight reformulation will be crucial in the proof of our main results.

Let $J_0 \in \mathcal{I}$. Set $X_0 := X_{J_0}$. Let $W$ be a neighborhood of the identity in $Diff(X^{diff})$. Observe that, if $W$ is small enough, every $\phi \in W$ can be constructed as follows. There exists a smooth vector field $\xi$ close to 0 such that the map

$$x \in X^{diff} \overset{e(\xi)}{\longmapsto} \gamma_{x,\xi(x)}(1) \tag{6}$$

is exactly $\phi$. Here

$$\gamma_{x,\xi(x)} \colon \mathbb{R}^+ \longrightarrow X^{diff}$$

is the geodesic starting at $x$ with initial velocity $\xi(x)$.

Conversely, there exists an open neighborhood V of 0 in $\Sigma(TX^{diff})$ such that, for every $\xi \in V$, the map $e(\xi)$ defined by (6) is a diffeomorphism of $X^{diff}$. Hence we constructed in that way an isomorphism $e$ between $V$ and $W$.

Now, let $H_0$ be the subspace of $\Sigma(TX^{diff})$ consisting of the real parts of the holomorphic vector fields of $X_0$. Choose a decomposition

$$\Sigma(TX^{diff}) = H_0 \oplus L_0 \tag{7}$$

for some *closed* subspace $L_0$. Observe that such a closed complementary subspace always exists, since $H_0$ is finite-dimensional hence closed and since we are in a Hilbert space. Nevertheless, we want to emphasize that *we do not ask* (7) *to be orthogonal with respect to any product.* Indeed, in the sequel, we will use the fact that $L_0$ is not unique and that we can choose it.

Kuranishi's Theorem may be rephrased as:

**Theorem.** There exists an open neighborhood $U_0$ of $J_0$ in $\mathcal{E}$, an open neighborhood $W_1$ of 0 in $L_0$ and an analytic map

$$U_0 \xrightarrow[a.c.]{\Xi_0} \Delta_0 \subset U_0 \tag{8}$$

such that

(i) The set

$$K_0 := \Xi_0(U_0 \cap \mathcal{I})$$

is a (finite-dimensional) analytic set of (embedding) dimension at 0 equal to

$$h^1(0) := \dim H^1(X_0, \Theta).$$

(ii) The map

$$(J, \xi) \in K_0 \times W_1 \xrightarrow{\Phi_0} e(\xi) \cdot J \in U_0 \cap \mathcal{I} \tag{9}$$

is an a.c. isomorphism whose inverse has component in $K_0$ equal to $\Xi_0$.

*Remark.* In the classical presentation of [Ku2], a slightly different splitting is used. From the decomposition (1), Kuranishi defines $A^p$ as the space of $(0,p)$-forms of $X_0$ with values in $T$. Hence $A^0$ is the space of $(1,0)$-vectors and we may write

$$A^0 = H^0 \oplus {}^\perp A^0 \tag{10}$$

for $H^0$ the subspace of holomorphic vector fields and ${}^\perp A^0$ its orthogonal with respect to the $L^2$-norm for some fixed hermitian metric of $X_0$. And he encodes the small diffeomorphisms of $X^{diff}$ through the map

$$\xi \in A^0 \longmapsto e(\xi + \bar{\xi}).$$

The version we present can easily be deduced from the classical one, the decomposition (7) playing the role of (10). However, the crucial point is that (7) gives a splitting of the space of smooth vector fields, which is obviously independent of the complex structure $J_0$, whereas (10) gives a splitting of $A^0$, whose definition depends on $J_0$. So using (7) instead of (10) will allow us to compare different splittings based at different points.

Let us give the core of the proof and explain why it is possible to replace the splitting (10) with the splitting (7).

Let $\delta$ be the adjoint (from differential operator theory) with respect to a fixed hermitian metric on $X_0$ of the $\bar{\partial}$-operator extended to the forms with values in the holomorphic bundle $T$. This $\bar{\partial}$-operator acting on the spaces $A^p$ defines an elliptic complex, hence the associated Laplace-type operator $\Delta$ is elliptic.

Also, by ellipticity, we have a direct sum decomposition

$$A^1 = \text{Im } \bar{\partial} \oplus \text{Ker } \delta. \tag{11}$$

Using the splitting (10), consider the smooth map

$$(\xi, \omega) \in {}^\perp A^0 \times \text{Ker } \delta \longmapsto e(\Re\xi) \cdot \omega \in A^1. \tag{12}$$

*Remark.* In order to make (11) and (12) precise, one should add that $A^1$ means the set of 1-forms of class $W_2^r$, and $A^0$ means the -forms of class $W_2^{r+1}$.

A direct computation shows that its differential at 0 is given by

$$(\eta,\alpha)\in{}^{\perp}A^0\times\text{Ker }\delta\longmapsto\alpha+\bar\partial\eta\in A^1 \tag{13}$$

and that (13) is invertible with inverse given by

$$\omega=\bar\partial\eta+\alpha\in\text{Im }\bar\partial\oplus\text{Ker }\delta\longmapsto(G\delta\bar\partial\eta,\alpha)\in{}^{\perp}A^0\times\text{Ker }\delta. \tag{14}$$

Here $G$ denotes the Green operator associated to $\Delta$.

By application of the inverse function theorem on Banach spaces, the map (12) is a local diffeomorphism.

The Kuranishi space of $X_0$ is then defined as

$$K_0=\{\omega\in A^1\quad|\quad\bar\partial\omega-[\omega,\omega]=\delta\omega=0\}. \tag{15}$$

The restriction of (12) to ${}^{\perp}A^0\times K_0$ gives a map similar to (9), which appears in the classical statement of Kuranishi's Theorem.

If we use now the splitting (7), we just have to modify the previous formulas as follows.

We consider the map

$$(\xi,\omega)\in L_0\times\text{Ker }\delta\longmapsto e(\xi)\cdot\omega\in A^1 \tag{16}$$

instead of (12), whose differential at 0 is

$$(\eta,\alpha)\in L_0\times\text{Ker }\delta\longmapsto\alpha+\bar\partial\tau\eta\in A^1. \tag{17}$$

This is completely analogous to (13), the only difference being the use of the identification

$$\tau\colon\xi\in\Sigma(TX^{diff})\longmapsto\xi-iJ\xi\in A^0. \tag{18}$$

This identification (18) maps $\Sigma(TX^{diff})$ onto $A^0$, and $H_0$ onto $H^0$. But it does not map $L_0$ onto ${}^{\perp}A^0$. Nevertheless, since $\tau L_0$ and ${}^{\perp}A^0$ are complementary to the same finite-dimensional space, they are isomorphic, so that we can twist (18) into an identification

$$\tilde\tau\colon H_0\oplus L_0\longrightarrow H^0\oplus{}^{\perp}A^0 \tag{19}$$

which preserves the direct sum decompositions. More precisely, we define $\tilde\tau$ as the map

$$\tilde\tau(\xi_0\oplus\xi_{L_0})=\tau\xi_0\oplus(\tau\xi)^{\perp} \tag{20}$$

where

$$\xi=\xi_0\oplus\xi_{L_0}\in H^0\oplus L_0\qquad\text{and}\qquad\tau\xi=(\tau\xi)^0\oplus(\tau\xi)^{\perp}\in H^0\oplus{}^{\perp}A^0.$$

From (19) and (20), we infer that the formula for the inverse of (17), analogous to (14), is

$$(21)\qquad \omega=\bar\partial\tau\eta+\alpha\in \text{Im }\bar\partial\oplus\text{Ker }\delta\longmapsto(\tilde\tau^{-1}G\delta\bar\partial\tau\eta,\alpha)\in L_0\times\text{Ker }\delta.$$

This shows that the map (16) is a local diffeomorphism. Its restriction to $L_0\times K_0$ gives the map (9). The first component of its inverse gives the map (8).

We call a map $\Xi_0$ as defined in (8) a *Kuranishi map* based at $J_0$. Its domain of definition is called a *Kuranishi domain.* The analytic space $K_0$ is called the *Kuranishi space* of $X_0$. We note the following unicity property.

**Corollary.** The Kuranishi space $K_0$ of $X_0$ is unique in the following sense.

(i) If $L'$ is another closed complementary subspace to $H_0$, then the corresponding space $K_0'$ is a.c. isomorphic to $K_0$.

(ii) If $U'$ is another neighborhood of $J_0$ in $\mathcal{E}$, then the restrictions of $\Xi_0$ and of the corresponding map $\Xi_0'$ to $U\cap U'\cap\mathcal{I}$ have a.c. isomorphic images.

In particular, it is unique as a germ of analytic space at 0. In this paper, thinking of this corollary, we say that $K_0$ is *the* Kuranishi space of $X_0$, even if it depends on $U_0$ and on $L_0$.

In the sequel, when we will refer to Kuranishi's Theorem, we will always refer to this version of Kuranishi's Theorem.

## 3. Deformations.

Let $J_0\in\mathcal{I}$ and set $X:=X_{J_0}$. Recall the following classical definitions (cf. [Su] and [Ko] for additional details).

**Definitions.** An *analytic deformation* of $X$ is a flat morphism $\Pi\colon\mathcal{X}\to B$ onto a (possibly non-reduced) analytic space, together with a base-point $0\in B$ and a marking, that is a holomorphic identification $i\colon X\to\Pi^{-1}\{0\}$.

A *smooth deformation* of $X$ is a smooth submersion $\Pi\colon\mathcal{X}\to B$ onto a smooth manifold, together with a base-point and a marking. The total space $\mathcal{X}$ is a endowed with a Levi-flat CR structure whose associated leaves are the level sets of $\Pi$ and the marking is assumed to be holomorphic.

Now, deformations can also be defined as analytic (resp. smooth) families of complex operators. To be more precise,

**Definition.** Let $B$ be an analytic space (resp. a smooth manifold) with base-point 0. Consider a family $(J_t)_{t\in B}$ of elements of $\mathcal{I}$. We say that $(J_t)_{t\in B}$ is *analytic* (resp. *smooth*) if endowing each fiber $X^{diff}\times\{t\}$ of the projection $X^{diff}\times B\to B$ with the complex structure $J_t$ turns it into an analytic (resp. smooth) deformation of $X$.

The Kuranishi space $K$ of $X$ defines such a family of complex operators. It follows from [Ku3, **?**] that this family is analytic. Hence the Kuranishi space $K$ of $X$ naturally defines an analytic deformation $\Pi\colon \mathcal{K} \to K$ once chosen a marking. It is called the *Kuranishi family.* It has the following properties.

**Corollary.** Let $\Pi\colon \mathcal{K} \to K$ be the Kuranishi family of a compact complex manifold $X$. Then it is versal at 0, that is

(i) It is complete at 0: any holomorphic (resp. smooth) deformation $\mathcal{X} \to B$ of $X$ is locally isomorphic at 0 to the pull-back of $K$ by some analytic (resp. smooth) map $f$ from $(B, 0)$ to $(K, 0)$. Moreover this local isomorphism may be asked to preserve the markings.

(ii) The (embedding) dimension of $K$ at 0 is minimal amongst the bases of complete families for $X$.

Property (ii) is known to be equivalent to the following. Given a deformation $\mathcal{X} \to B$, the map $f$ given by completeness is in general not unique, but its differential at 0 is, provided only marking preserving isomorphisms are used. It can also be proven that there exists a unique germ of versal family up to isomorphism. This gives another unicity property of the Kuranishi space.

Moreover, from the existence of the Kuranishi's family, we obtain the following. Given an analytic (resp. smooth) map $f\colon B \to \mathcal{I}$, the family $(f(t))_{t\in B}$ is analytic (resp. smooth).

To finish with this section, we add the somehow less classical definitions.

**Definition.** A *Levi-flat CR space Z* is a second-countable Hausdorff space for which there exists a covering by open subsets $V_\alpha$ and homeomorphisms

$$F_\alpha\colon V_\alpha \longrightarrow \mathbb{R}^p \times W_\alpha$$

for some analytic sets $W_\alpha \subset \mathbb{C}^{n_\alpha}$, such that the changes of charts

$$F_{\alpha\beta} := F_\beta \circ F_\alpha^{-1}$$

are smooth, respects the foliation by copies of $W_\alpha$, and are analytic in the second variable; that is, setting

$$F_{\alpha\beta}\colon (x, z) \longmapsto (f_{\alpha\beta}, g_{\alpha\beta})(x, z)$$

we have that $f_{\alpha\beta}$ does not depend on $x$, and that, for all $x$, the map

$$z \longmapsto g_{\alpha\beta}(x, z)$$

is analytic.

A Levi-flat CR space is just a special case of ringed space, and of mFB space (see [F-K]). But we really want to consider it as a Levi-flat CR manifold with singularities. As in the smooth case, it is foliated, the leaves being obtained by gluing the $W_\alpha$ via $g_{\alpha\beta}$. The leaves are analytic, but, unlike the smooth case, they may have singularities. A trivial example is given by a product of an analytic space with a smooth manifold.

**Definition.** A *CR deformation* of $X$ is a Levi-flat CR space $\mathcal{Z}$ together with a proper and transflat CR morphism $\Pi\colon \mathcal{Z} \to B$, for $B$ a Levi-flat CR space, a base-point and a marking.

By transflat, we mean that there are submersion charts

$$\begin{array}{ccc} z \in U \subset \mathcal{Z} & \xrightarrow{\text{CR iso.}} & \Pi(U) \times \mathbb{R}^p \\ \Pi \big\downarrow & & \big\downarrow \text{1st projection} \\ \Pi(z) \in \Pi(U) \subset B & \xrightarrow[Id]{} & \Pi(z) \in \Pi(U) \subset B \end{array}$$

for all points $z \in \mathcal{Z}$ (see [Sc]).

In [F-K] and [Sc], such a deformation is called a "relativ-analytisch Deformazion". Observe that analytic and smooth deformations are particular cases of CR deformations. Observe also that if $B$ is a product $B_1 \times B_2$ with $B_1$ smooth and $B_2$ analytic, then for every $x \in B_1$, the induced deformation over $B_2$ obtained by restricting $\mathcal{Z}$ to $\Pi^{-1}(\{x\} \times B_2)$ is analytic; whereas for every $z \in B_2$, the induced deformation over $B_1$ obtained by restricting $\mathcal{Z}$ to $\Pi^{-1}(B_1 \times \{z\})$ is smooth.

Observe that, if $f\colon B \to \mathcal{I}$ is CR, then the family $(J_f(t))_{t \in B}$ defines a CR deformation of $X$. We call such a family a *CR family*.

Finally, note that all the previous definitions of deformations hold for infinite-dimensional analytic spaces as bases, using the formalism of [Do].This of course has no interest in the case of compact complex manifolds, since Kuranishi's Theorem implies the existence of finite-dimensional complete families, but it will be used in Section IV, when dealing with deformations of proper CR submersions over the circle. For example, the infinite-dimensional analytic set $\mathcal{I}$ defined in (3) is the base of such an infinite-dimensional analytic deformation of $X_0$, once a base-point corresponding to $X_0$ is fixed. This comes from the fact that, using the map (9) at each point of $\mathcal{I}$, one shows that this family is locally obtained by pull-back from the Kuranishi family. Since the last one is flat, so is the first one.

## II. Structure of $\mathcal{I}$ in the neighborhood of a compact set

**1. Foliation of the neighborhood of a compact set.** Recall that $Diff(X^{diff})$ acts on $\mathcal{E}$. Given a closed vector subspace $L$ of the space $\Sigma(TX^{diff})$, set

$$\Gamma_L = \langle e(W_L) \rangle \tag{22}$$

where $e$ was defined in (6), where $W_L$ is an open neighborhood of 0 in $L$ small enough to be in the domain of definition of $e$, and where $\langle\ \rangle$ means the group generated by. Notice that (22) does not depend on the choice of $W_L$.

**Proposition 1.** Let $C$ be a compact set of $\mathcal{E}$. Then, for any sufficiently small neighborhood $U$ of $C$ in $\mathcal{E}$, there exists a closed subspace $L$ of $\Sigma(TX^{diff})$ of finite codimension such that $\Gamma_L$ foliates $U \cap \mathcal{I}$. More precisely, there exists a finite open covering

$$U = U_1 \cup \ldots \cup U_k \tag{23}$$

such that

(i) For every $i$, there exists an analytic space $K_i$ such that $U_i \cap \mathcal{I}$ is a.c. isomorphic to the product of some neighborhood $W_i$ of 0 in $L$ with $K_i$.

(ii) When defined, the composition of two such isomorphisms preserves the plaques $W_i \times \{Cst.\}$.

(iii) The induced leaves are in one-to-one correspondance with the connected components of the action of $\Gamma_L$ restricted to $U \cap \mathcal{I}$.

*Remark.* Item (iii) means the following. The leaf through a point $x \in U \cap \mathcal{I}$ is the connected component at $x$ of the intersection of the $\Gamma_L$-orbit of $x$ with $U$.

*Proof.* Begin with choosing an open neighborhood $U$ of $C$ in $\mathcal{E}$ and a finite open covering of $U$ by Kuranishi domains $U_1, \ldots, U_k$ based at $J_1, \ldots, J_k$, points of $C$. Set

$$H_i := H_{J_i}$$

and denote by $\text{Kur}_i$ the Kuranishi space of $J_i$. Choose a splitting (7) for each $i$. It follows from Kuranishi's Theorem that for every $i$ and every $J \in U_i \cap \mathcal{I}$, we have

$$L_i \cap H_J = \{0\}. \tag{24}$$

We claim that the set

$$\mathcal{H} = \{v \in \Sigma(TX^{diff}) \quad | \quad \|v\| = 1 \text{ and } v \in H_J \text{ for some } J \in C\}$$

is compact in $\Sigma(TX^{diff})$. Indeed, if $(v_n)$ is a subsequence of $\mathcal{H}$, setting $v_n \in H_{J'_n}$, then $(J'_n)$ admits a converging subsequence to some $J \in C$ by compacity. Now such a bounded sequence of holomorphic vector fields converges to an element $v$ of $H_J$, proving compacity.

From this claim, we infer that we may assume, shrinking $U$ if necessary, that the covering $(U_i)$ is such that the sets

$$\mathcal{H}_i = \{v \in \Sigma(TX^{diff}) \quad | \quad \|v\| = 1 \text{ and } v \in H_J \text{ for some } J \in U_i\}$$

are contained in the ball centered in $J_i$ with radius equal to a fixed real number $\epsilon$.

Let $\mathcal{S}$ be the set of closed vector subspaces $S$ of $\Sigma(TX^{diff})$ having finite codimension and satisfying

$$S \cap H_J = \{0\} \qquad \text{for all } J \in U. \tag{25}$$

It follows from (24) and from the hypothesis on the radius of $\mathcal{H}_i$ that a well-chosen complementary subspace to

$$H := H_1 + \ldots + H_k \tag{26}$$

belongs to $\mathcal{S}$. Hence, we may choose $L \in \mathcal{S}$ having minimal codimension. We claim that this $L$ satisfies the requirements of Proposition 1.

Since $L$ has finite codimension in $\Sigma(TX^{diff})$, we may choose some finite-dimensional vector subspaces $\tilde{H}_i$ such that, for all $i$, we have

$$\Sigma(TX^{diff}) = L \oplus \tilde{H}_i \oplus H_i. \tag{27}$$

Set

$$\tilde{L}_i := L \oplus \tilde{H}_i. \tag{28}$$

We may then replace $L_i$ with $\tilde{L}_i$ and obtain new Kuranishi maps and domains covering $U$. We still denote by the same symbols $U_i$ this refined covering. Remembering (28), Kuranishi's Theorem implies that these Kuranishi maps induce a.c. isomorphisms between $U_i \cap \mathcal{I}$ and the product of the finite-dimensional analytic space

$$K_i := \mathrm{Kur}_i \times B_i \tag{29}$$

(for $B_i$ an open neighborhood of 0 in $\tilde{H}_i$) with $W$, an open neighborhood of 0 in $L$. Hence they give foliated charts as wanted in (i).

*Remark.* If the Kuranishi space is not reduced, then we define $K_i^{red}$, the reduction of $K_i$, as the product of the reduction of the Kuranishi space with $\tilde{H}_i$, and we put on each irreducible component of $K_i^{red}$ the multiplicity of the corresponding component of the Kuranishi space.

Moreover, still by Kuranishi's Theorem, the plaques $W \times \{Cst\}$ correspond to the local orbits of the $\Gamma_L$-action. Since the compositions involved are equivariant, they must preserve the connected component of the action of $\Gamma_L$, hence preserve the plaques. This proves (ii) and then (iii). □

**Definition.** Let $U$ satisfying the hypotheses of Proposition 1. Then *an adapted covering* of $U$ is a finite covering (23) satisfying the conclusions of Proposition 1.

We note the following easy fact.

**Proposition 2.** The subspace $L$ is unique in the following sense. If $L$ and $L'$ are two vector subspaces of $\Sigma(TX^{diff})$ such that

(i) $\Gamma_L$ and $\Gamma'_L$ foliate $U \cap \mathcal{I}$ as in the statement of Proposition 1.

(ii) Both $L$ and $L'$ are elements of $\mathcal{S}$ of minimal codimension. Then $L$ and $L'$ are isomorphic.

Observe that, if $\Gamma_L$ foliates $U \cap \mathcal{I}$ as in Proposition 1, then $L$ must be an element of $\mathcal{S}$. And of course, if they do not have the same codimension, we cannot expect

them to be isomorphic. So condition (ii) in the statement of Proposition 1 is not a restriction but an obvious necessary condition to have unicity.
*Proof.* This is standard linear algebra. Set

$$I = L \cap H \qquad J = L' \cap H$$

-see (26)- and

$$\begin{aligned} \Sigma(TX^{diff}) &= H \oplus L_0 \quad \text{ with } L = L_0 \oplus I \\ &= H \oplus L'_0 \quad \text{ with } L' = L'_0 \oplus J. \end{aligned}$$

So $L_0$ and $L'_0$ are isomorphic as complementary subspaces of the same subspace $H$, and have thus the same codimension. So $I$ and $J$ are isomorphic and we may extend the isomorphism between $L_0$ and $L'_0$ to an isomorphism between $L$ and $L'$. □

Nevertheless, we do not know if the foliations by $\Gamma_L$ and $\Gamma'_L$ are equivalent under the hypotheses of Proposition 2.

**2. The case of a neighborhood of a path.**

Let $c$ be a continuous path into $\mathcal{I}$ and let $U$ be a connected neighborhood of $c$ in $\mathcal{E}$ for which Proposition 1 is valid. In this particular case, we can give a much more precise description of the $\Gamma_L$-foliation.

**Theorem 1.** Let $\Gamma_L$ foliating $U \cap \mathcal{I}$ as in Proposition 1. Then, if $U$ is small enough, the foliation is given by the level sets of a submersion. To be more precise, there exists an analytic space $K_U$ and an a.c. morphism

$$U \cap \mathcal{I} \xrightarrow[a.c.]{\Xi_U} K_U$$

such that

(i) $K_U$ is the leaf space of the $\Gamma_L$-foliation and the leaves are given by the level sets of $\Xi_U$.

(ii) The map $\Xi_U$ is locally a projection: in the neighborhood $V$ of any point $x \in U \cap \mathcal{I}$, we have a commutative diagram

$$\begin{array}{ccc} x \in V & \xrightarrow{\text{a.c. isomorphism}} & \Xi_U(V) \times W \\ {\scriptstyle \Xi_u}\big\downarrow & & \big\downarrow{\scriptstyle \text{1st projection}} \\ \Xi_U(V) \subset K_U & \xrightarrow[\text{Identity}]{} & \Xi_U(V) \subset K_U \end{array} \tag{30}$$

for some open neighborhood $W$ of 0 in $L$.

Moreover, the $\Gamma_L$-foliation is smoothly trivial, that is there exists a smooth injection

$$K_U \xrightarrow[a.c.]{i_U} U \cap \mathcal{I} \tag{31}$$

and, up to shrinking $U$ if necessary, a diffeomorphism

$$(J, f) \in K_U \times W_U \xrightarrow[a.c.]{\Phi_U} f \cdot i_U(J) \in U \cap \mathcal{I} \tag{32}$$

where $W_U$ is an open neighborhood of the identity in $\Gamma_L$.

*Remark.* We use the word "submersion" in an extended way, since $K_U$ is an analytic space. Indeed, we just mean that there exist submersion charts (30) at any point. Observe that we cannot say that $\Xi_U$ is a flat morphism since it is not proper. We prefer keeping the word flat for proper maps, and thus use the word submersion in this quite unusual way.

*Remark.* The open set $U$ may contain "large" open sets of $\Gamma_L$-orbits, so that we cannot ensure that they can be completely encoded via the map $e$. This is typically the case if the path $c$ is included in a single $\Gamma_L$-orbit. This explains why the set $W_U$ of (32) is included in $\Gamma_L$ and not in $L$.

*Proof.* If $c$ is constant or is contained in a single Kuranishi domain, then there is nothing to do: we just take $U$ as a Kuranishi domain based at $c(0)$ and the map (9) given by Kuranishi's Theorem as trivialization chart (30). This map is indeed global, so that (32) is satisfied with an analytic isomorphism.

So assume that $c$ is not contained in a single Kuranishi domain. We just make use of Proposition 1. We first assume that $c$ is not a loop. We cover it by an adapted covering. We assume without loss of generality that

$$\overline{U_i} \cap \overline{U_j} \neq \emptyset \iff |i - j| = 0 \text{ or } 1 \tag{33}$$

and that every non-empty intersection $U_i \cap U_j$ is connected.

We also assume that every intersection $U_i \cap c$ and $U_i \cap U_j \cap c$ is connected.

Furthermore, we assume that the $U_i$ are sufficiently small so that, for any pair $(x, y)$ of points of $U_i \cup U_{i+1}$ with $U_i \cap U_j \neq \emptyset$, if $x$ and $y$ belong to the same orbit in $U_i \cup U_{i+1}$, we have

$$y = e(\xi) \cdot x \qquad \text{for some } \xi \in W_i \cap W_{i+1} \tag{34}$$

where the local charts

$$\Phi_i \colon K_i \times W_i \xrightarrow{a.c.} \mathcal{I} \tag{35}$$

given by Proposition 1 are supposed to cover an open set containing $U_i \cap \mathcal{I}$.

We want to construct $K_U$ by gluing each $K_i$ to the next one using the changes of foliated charts. To be more precise, we proceed by induction.

Firstly, consider, for $i = 1, 2$, the two local charts (35) and the associated change of chart

$$(z, v) \xrightarrow[a.c.]{\Phi_{12} := \Phi_2^{-1} \circ \Phi_1} (\Psi(z), \chi(z, v)). \tag{36}$$

By abuse of notation, we will denote by the same symbol $K_i$ and the image $\Phi_i(K_i \times \{0\})$. We assume, composing $\Phi_1$ and $\Phi_2$ with translations on the $W$-factor if necessary, that $K_1$ and $K_2$ have disjoint images in $\mathcal{I}$.

Consider the space obtained by gluing $K_1$ to $K_2$ through the map $\Psi$. We claim that, up to shrinking $U_1$ and $U_2$, it is an analytic space. This can be proved as follows.

We define the open set $V_1 \subset K_1$ by the following relation

$$x \in V_1 \iff \exists v \in W_1 \quad \text{such that } \Phi_1(x,v) \in U_2. \tag{37}$$

By (35), observe that $V_1$ contains all the points whose leaf in $U_1 \cap U_2$ meets $U_2$. Denoting as usual by $\Xi_1$ the map contracting $U_1 \cap \mathcal{I}$ onto $K_1$, observe that $\Xi_1(c \cap U_1) \cap V_1$ is non-empty. Then, define

$$x \in V_1 \xrightarrow[a.c.]{g} \Xi_2 \circ \Phi_1(x,v) \in K_2 \tag{38}$$

where $v$ is any vector of $W_1$ such that (37) holds. Observe that the previous map does not depend on the particular choice of $v$ because of (35).

This is the gluing map we want to use. Set $V_2 = g(V_1) \subset K_2$. It follows from the proof of Proposition 1 that there exists an analytic map

$$\eta \colon V_1 \subset K_1 \longrightarrow L$$

such that

$$g \equiv e(\eta). \tag{39}$$

Now, choose some open set $V_1'$ included in $V_1$ such that $\Xi_1(c \cap U_1) \cap V_1'$ is non-empty. Let $\chi$ be a bump function defined on $U_1$, equal to 1 on $V_1'$ and to 0 on $K_1 \setminus V_1$. Let $V_2'$ be the image $g(V_1')$. Define

$$\eta_1 := \chi \cdot \eta \colon U_1 \longrightarrow L. \tag{40}$$

Then the map $g_1 := e(\eta_1)$ is an a.c. diffeomorphism from $K_1$ to its image. Moreover, we may assume that $g_1(K_1)$ intersects $K_2$ along

$$\tilde{V}_2' = \{x \in \overline{V_2'} \quad | \quad \Gamma_L^{U_1 \cup U_2}(y) \cap \overline{V_1'} \neq \emptyset\}. \tag{41}$$

where $\Gamma_L^{U_1 \cup U_2}(y)$ denotes the leaf of $y$ in $U_1 \cup U_2$, and where the closure is taken in $K_2$ (respectively $K_1$).

We would like that the set $\tilde{V}_2'$ is open and that the union $g_1(K_1) \cup K_2$ cuts every leaf of $U_1 \cup U_2$ into a single point. Taking into account the definition of $\tilde{V}_2'$, this would imply that the image through $g_1$ of $\overline{V_1}' \setminus V_1'$ does not belong to $K_2$. From this, it is easy to check that $K_1 \cup_{g_1} K_2$ would be Hausdorff, hence an analytic space since it is obtained by gluing two analytic spaces along an open set; also this would imply that it is the leaf space of $U_1 \cup U_2$.

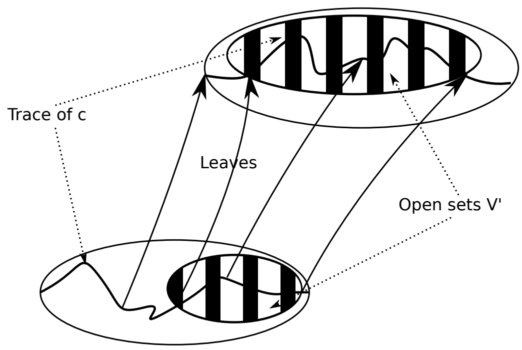


FIGURE 1. non-Hausdorff gluing.

*Remark.* To clarify the proof, let us emphasize the following obvious point: $K_1 \cup_{g_1} K_2$ will be constructed as an *abstract* analytic space *homeomorphic* to $g_1(K_1) \cup K_2$. But obviously, this last set is not an analytic subspace of $\mathcal{I}$.

Nevertheless, it is not true in general. Indeed, it fails every time that $\tilde{V}_2'$ contains a point of $\overline{V_2'} \setminus V_2'$.

We claim that it is enough to shrink $U_1$ and $U_2$ to ensure

$$\tilde{V}_2' = V_2' \tag{42}$$

and thus to solve our problem.

In figure 1 above, the big ellipses represent $K_1$ and $K_2$ and the small ones represent $V_1'$ and $V_2'$. The arrows on the leaves just suggest the identification. The glued space is obtained by gluing along the *open* shaded parts. Clearly, since $V_1'$ has boundary points in $K_1$ which correspond through the identification to boundary points of $V_2'$ in $K_2$, the resulting space is not Hausdorff.

The claim is that it is possible to shrink $U_1$ and $U_2$ so that the picture is like picture 2 (see next page). The big ellipses represent the domains of $K_1$ and $K_2$ onto which the new sets $U_1$ and $U_2$ retract. In this new case, the boundary points of $V_1'$ are not in correspondance with the boundary points of $V_2'$, hence the gluing is Hausdorff. Notice that, in the case of $U_2$ (the upper ellipse), a subset of the former trace of $c$ in $K_2$ is now out of the domain. This has no consequence since this part of $c$ has still a trace in $K_1$, but it helps reducing the boundary of $V_2'$.

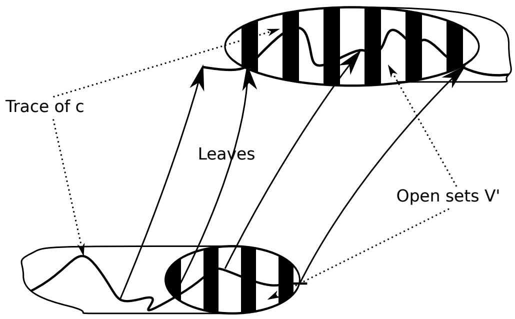


FIGURE 2. Hausdorff gluing.

First, observe that shrinking $U_i$ to, say, $U_i'$, we may assume that it transforms (35) into

(43) $$\Phi_i\colon K_i' \times W_i' \subset K_i \times W_i \xrightarrow{a.c.} U_i' \cap \mathcal{I}$$

However, $W_i' \subset W_i$ *may not contain* 0. Then $g_1(K_1')$ intersects $K_2'$ along $\tilde{V}_2' \cap K_2'$. The claim is that, after shrinking and after taking the intersection with $K_2'$, we may assume that (42) holds.

To go further, we distinguish cases.

1st case. We assume that $\Xi_i(c \cap U_i)$ does not intersect the boundary $\overline{V_i'} \setminus V_i'$. This is typically the case is $c$ is included in a single orbit of $\Gamma_L$, hence its trace in each $K_i$ is a single point belonging to $V_i'$. Then there is no problem. We set

$$U_i = \Xi_i^{-1}(V_i') \quad \text{ for } \quad i = 1, 2.$$

Geometrically, $K_1$ and $K_2$ are simply identified one to the other.

2nd case. We assume that the intersection of $\Xi_i(c \cap U_i)$ with the boundary $\overline{V_i'} \setminus V_i'$ is connected. Then it is enough to shrink $U_2$ setting

$$U_2 = \Xi_2^{-1}(V_2').$$

Geometrically, $K_2$ is identified to an open subset of $K_1$.

3rd case. We assume that the intersection of $\Xi_i(c \cap U_i)$ with the boundary $\overline{V_i'} \setminus V_i'$ is disconnected. This is the case treated in figures 1 and 2.

Set

(44) $$I = \{t \in [0, 1] \quad | \quad \Xi_2(c(t)) \cap \overline{V_2'} \setminus V_2' \neq \emptyset\}.$$

It has at least two connected components and we may find two disjoint open intervals $I_1$ and $I_2$ such that

$$I \subset I_1 \sqcup I_2. \tag{45}$$

Then, shrink $U_1$ and $U_2$ such that (33) and (34) and all the above hypotheses are still satisfied, as well as the additional hypothesis

$$U'_i \cap c(I) \subset c(I_i). \tag{46}$$

Properties (46) and (45) imply that (42) holds after taking the intersection with $K'_2$.

In other words, set

$$K_{12} := K'_1 \cup_{g_1} K'_2 \qquad \text{with } g_1 \colon K'_1 \cap V'_1 \to K'_2 \cap V'_2.$$

Then $K_{12}$ is an analytic space. It is by construction the leaf space of the $\Gamma_L$-action restricted to $U'_1 \cup U'_2$.

*Remark.* If $K_1$ (or equivalently $K_2$) is not reduced, then we perform the previous construction with their reduction and put on each component of the resulting space the common multiplicity of the corresponding components of $K_1$ and $K_2$.

Besides the two maps

$$U'_1 \cap \mathcal{I} \xrightarrow[a.c.]{\Phi_1^{-1}} K'_1 \times W'_1 \xrightarrow{\text{1st projection}} K'_1$$

and

$$U'_2 \cap \mathcal{I} \xrightarrow[a.c.]{\Phi_2^{-1}} K'_2 \times W'_2 \xrightarrow{\text{1st projection}} K'_2$$

glue into a single map

$$\Xi \colon (U'_1 \cup U'_2) \cap \mathcal{I} \xrightarrow{a.c.} K_{12} \tag{47}$$

with charts (30) by construction.

It follows now from (47) that the inclusions

$$i_1 \colon z \in K'_1 \longmapsto \Phi_1(z, \eta_1(z)) \in U'_1 \cap \mathcal{I}$$

and

$$i_2 \colon z \in K'_2 \longmapsto \Phi_2(z, 0) \in U'_2 \cap \mathcal{I}.$$

glue naturally into a smooth inclusion

$$i_{12} \colon K_{12} \xrightarrow{a.c.} (U'_1 \cup U'_2) \cap \mathcal{I} \tag{48}$$

yielding (31) and a smooth trivialization (32) (shrinking $U$ if necessary).

Repeating the process, we construct the map $\Xi_U$ as desired as well as the smooth trivialization.

Assume now that $c$ is a loop. We cover it by adapted domains and we assume without loss of generality that

$$k > 2 \quad \text{and} \quad \overline{U_i} \cap \overline{U_j} \neq \emptyset \iff |i-j| = 0 \text{ or } 1 \text{ or } k-1, \tag{49}$$

that $U_i \cap U_j$, $U_i \cap c$ and $U_i \cap U_j \cap c$ are connected when non-empty. Assume also that (34) is satisfied.

We proceed as before, but we have now to perform an ultimate gluing between $U_k$ and $U_1$ to obtain $K_U$, still using the changes of charts of Proposition 1. Let $K$ denote the analytic space obtained by making all the gluings except for the last one. We also have a smooth map

$$i\colon K \xrightarrow{a.c.} U \cap \mathcal{I}$$

analogous to (31). The last gluing to perform is defined through an analytic map

$$V_k \subset K_k \xrightarrow[a.c.]{g} K_1$$

which is equal as before to $e(\eta)$ for some analytic map $\eta$ from $K_k \cap U_1$ into $L$ because of (34). So we may proceed as before and extend smoothly $\eta$ to

$$\eta_1\colon K \longrightarrow L$$

equal to $\eta$ on $V'_k \subset K_k$ for some open set $V'_k$ included in $V_k$. We assume it meets $\Xi_k(c \cap U_k)$. Now, up to shrinking the $U_i$'s, we have that $e(\eta) \cdot K$ is a smooth global transverse section to the $\Gamma_L$-foliation on $U \cap \mathcal{I}$. This proves at the same time that the space $K_U$ obtained from $K$ after performing the last gluing is homeomorphic to $e(\eta) \cdot K$, hence Hausdorff and by construction an analytic space; and that the foliation is smoothly trivial.

The injection (31) and the trivialization (32) are then obtained as before. □



**Corollary 1.** The analytic space $K_U$ is unique up to a.c. isomorphism, that is does not depend on the choice of the adapted covering and of the Kuranishi maps.

*Proof.* This is a direct consequence of the fact that $K_U$ is the leaf space of the $\Gamma_L$-foliation restricted to $U$. Hence it is unique. □

For the same reason, we also have

**Corollary 2.** Let $U$ and $U'$ be two connected neighborhoods of $c$ for which a smooth trivialization (32) exists. Then the restrictions of $K_U$ and $K'_U$ to $U \cap U' \cap \mathcal{I}$ (via the trivializations (32)) are a.c. isomorphic.

However, it is worth to emphasize that $K_U$ *depends on the choice of* $L$.

## III. The Kuranishi type moduli space of a proper CR submersion over the circle

### 1. Proper CR submersions.

**Definition.** A proper smooth submersion $\pi\colon \mathcal{X} \to \mathbb{S}^1$ is called a *a proper CR submersion* if $\mathcal{X}$ is endowed with a Levi-flat integrable CR structure which is tangent to the fibers of $\pi$.

As a smooth manifold, a proper CR submersion is a locally trivial smooth fiber bundle over the circle, thanks to Ehresmann's Lemma. The fiber, that we denote by $X^{diff}$, is a smooth compact manifold. We assume that it is connected. In other words, $\mathcal{X}$ is diffeomorphic to

$$X_\phi := (X^{diff} \times [0,1])/\sim \qquad \text{where} \qquad (x,0) \sim (x',1) \iff x' = \phi(x). \tag{50}$$

Here $\phi$ is a fixed diffeomorphism of $X^{diff}$, classically called the *monodromy* of $X_\phi$. Recall also that $X_\phi$ and $X'_\phi$ are diffeomorphic if $\phi$ and $\phi'$ are isotopic.

As a CR manifold, each fiber of $\mathcal{X}$ is a copy of $X^{diff}$ equipped with a complex structure. The only difference between a proper CR submersion and a smooth deformation over the circle is that here there is no marked point.

By Fischer-Grauert's Theorem (see [Me2] for the version we use), if all the fibers of a proper CR submersion are biholomorphic to a fixed manifold $X_0$, then it is locally trivial, that is satisfies

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\text{CR isomorphism}} & U \times X_0 \\ \pi \big\downarrow & & \big\downarrow \text{1st projection} \\ U & \xrightarrow[\text{Identity}]{} & U \end{array}$$

in a neighborhood $U$ of any point of the circle. We call such a locally trivial CR submersion a *CR bundle.*

Choosing a smooth model (50) for $\mathcal{X}$, we may identify it as a CR manifold with a smooth path $c$ in $\mathcal{I}$ such that

$$c(1) = \phi \cdot c(0) \tag{51}$$

and

$$c_\phi\colon t \in (-\epsilon, \epsilon) \longmapsto \begin{cases} c(1+t) & \text{if } t \leq 0 \\ \phi \cdot c(t) & \text{if } t \geq 0 \end{cases} \qquad \text{is smooth.} \tag{52}$$

that is, $\mathcal{X}$ is CR isomorphic to the family

$$(\pi^{-1}(\exp 2i\pi t), c(t))_{\exp 2i\pi t \in \mathbb{S}^1},$$

the identification of the endpoints being realized thanks to (51) and (52).

**2. The set of CR submersions structures on a fixed smooth proper submersion over the circle.**

Let $\pi^{diff}\colon \mathcal{X}^{diff} \to \mathbb{S}^1$ be a smooth proper submersion. Let $X^{diff}$ be its fiber. Define $\mathcal{E}$ and $\mathcal{I}$ as in (3) and (4). Choose a monodromy $\phi$ so that $\mathcal{X}^{diff}$ is diffeomorphic to $X_\phi$ (see (50)).

As proved in the preceding section, turning $\pi^{diff}$ into a CR submersion $\pi$ means choosing a path $c$ in $\mathcal{I}$ whose endpoints satisfy (51) and (52). So the space of CR submersions compatible with $\pi^{diff}$ is just the set

$$\mathcal{I}(\mathcal{X}^{diff}) = \{c\colon [0,1] \to \mathcal{I} \quad | \quad c \text{ is smooth and satisfies (51) and (52)}\} \tag{53}$$

and we may define similarly the space of *almost CR submersions* as the set

$$\mathcal{E}(\mathcal{X}^{diff}) = \{c\colon [0,1] \to \mathcal{E} \quad | \quad c \text{ is smooth and satisfies (51) and (52)}\}. \tag{54}$$

We want to give a Theorem describing locally the set $\mathcal{I}(\mathcal{X}^{diff})$, analogous to that of Kuranishi describing locally $\mathcal{I}$. Before stating and proving this result in the next section, we first make some useful preliminary comments.

Start with a CR submersion $\pi\colon \mathcal{X}_0 \to \mathbb{S}^1$ compatible with $\pi^{diff}$, which we represent by a smooth map $c_0$ satisfying (51) and (52). Let $U$ be a connected neighborhood of $c_0$ in $\mathcal{E}$. As usual, we cover $U$ by an adapted covering.

If $\mathcal{X}^{diff} \to \mathbb{S}^1$ is trivial, we assume that

$$U_k = U_1 \qquad \phi \equiv \text{Identity} \qquad c_0 = \text{loop}$$

and finally we define $U_\phi$ to be $U$.

If $\mathcal{X}^{diff} \to \mathbb{S}^1$ is not trivial, then, shrinking $U$ if necessary, we may assume that the Kuranishi domains satisfy (33) with all non-empty intersections being connected. In particular, we assume that $U_k$ is disjoint from $U_1$ but equal to $\phi \cdot U_1$. This means that we choose $\phi$ so that it is not a biholomorphism of $X_{c_0(0)}$. Thus $c_0(1)$ is different from $c_0(0)$. This is always possible, since $\phi$ can be chosen arbitrarily in a fixed isotopy class of diffeomorphisms of $X^{diff}$. We define now $U_\phi$ as the set obtained from $U$ by identifying each point $J$ of $U_1$ with the corresponding point $\phi \cdot J$ of $U_k$. We put on $U_\phi$ the quotient topology.

Observe that we may assume it is Hausdorff, shrinking $U$ if necessary. As in the proof of Theorem 1, the only problem that could appear is the following. If we can find a sequence $(x_n)$ in $U_1$ such that

$$\lim x_n \in \overline{U \cap U_1} \setminus \overline{U} \tag{55}$$

and

$$y := \lim \phi(x_n) \in \overline{U \cap U_k} \setminus \overline{U} \tag{56}$$

then $U_\phi$ is not separated at $y$. This can be avoided as follows.

First, since $c_0$ is not constant, we may assume that $c_0'(0) \neq 0$, changing the base point 0 otherwise. As $c_0$ maps into a Banach manifold locally modeled on a Hilbert space, we may work locally in a chart based at $c_0(0)$ and define

$$H := (c_0'(0))^\perp \tag{57}$$

We then extend $c_0$ to $(-\epsilon, 0]$ for $\epsilon$ small enough so that

$$\begin{aligned} \langle \overrightarrow{c_0(0)c_0(t)}, c_0'(0)\rangle > 0 &\iff 0 < t < \epsilon \\ < 0 &\iff -\epsilon < t < 0 \end{aligned} \tag{58}$$

Shrinking $U$ if necessary, we may assume that $U_1$ is a ball and that $H$ cuts $U_1$ into two connected components, say $H^+$ and $H^-$. Moreover, we assume that $U_2 \subset H^+$ and, using $U_k = \phi \cdot U_1$, that $U_{k-1} \subset \phi \cdot H^-$.

Let now $x$ be a point of $\overline{U \cap U_1} \setminus \overline{U}$. Then $x$ belongs to $U_2$, hence to $H^+$. As a consequence, if $(x_n)$ converges to $x$, then $\phi(x_n)$ belongs to $\phi \cdot H^+$ for $n$ big enough. Since $\overline{U \cap U_k} \setminus \overline{U}$ is included in $\phi \cdot H^-$, we see that it is not possible to find a sequence $(x_n)$ satisfying (55) and (56).

Hence $U_\phi$ is Hausdorff. Since it is obtained from the Banach manifold $U$ by an open gluing, it is also a Banach manifold. We may thus speak of smooth maps into $U_\phi$.

If $\mathcal{X}$ is a CR submersion close to $\mathcal{X}_0$, it is represented by a smooth path $c$ in $U$ whose endpoints belong to $U_1$ and $U_k$ respectively, and which satisfies (52) and (51). Hence it is represented by a smooth loop in $U_\phi$. Reciprocally it is clear that any loop in $U_\phi$ formed by integrable structures lifts to a point of $\mathcal{I}(\mathcal{X}^{diff})$ close to $c_0$, thus defining a CR submersion close to $\mathcal{X}_0$.

In other words, denoting abusively $U_\phi \cap \mathcal{I}$ the subset of points of $U_\phi$ corresponding to integrable structures, we may take the space $C^\infty(\mathbb{S}^1, U_\phi)$ as a neighborhood of $c_0$ in $\mathcal{E}(\mathcal{X}^{diff})$; and the space $C^\infty(\mathbb{S}^1, U_\phi \cap \mathcal{I})$ as a neighborhood of $c_0$ in $\mathcal{I}(vX^{diff})$.

Observe that $U_\phi$ being obtained from $U$ by an a.c. gluing, we have

**Proposition 3.** Let $U$ and $U_\phi$ be as above. Then the $\Gamma_L$-foliation of $U$ descends as a foliation of $U_\phi$.

**Definition.** We call this foliation the $\Gamma_L$-foliation of $U_\phi$.

Let $Diff_\pi(\mathcal{X}^{diff})$ be the group of bundle isomorphisms of $\pi^{diff}$ which descend as the identity on the base $\mathbb{S}^1$. It acts on $C^\infty(\mathbb{S}^1, U_\phi)$ in the obvious way. That is, given $c \in C^\infty(\mathbb{S}^1, U_\phi)$ and $f \in Diff_\pi(\mathcal{X}^{diff})$, we have

$$\forall t \in \mathbb{S}^1 \qquad (f \cdot c)_t = f_t \cdot c_t$$

where the action on the right-hand side is defined in (5).

Consider the set $\Sigma_\pi(T\mathcal{X}^{diff})$ of smooth vector fields of $vX^{diff}$ tangent to the fibers of $\pi$. We have

$$\begin{aligned}\Sigma_\pi(T\mathcal{X}^{diff}) &\simeq \{\sigma\colon t\in\mathbb{S}^1\longmapsto \sigma_t\in\Sigma(T\pi^{-1}(t)\}) \\ &\simeq \{\sigma\colon t\in\mathbb{S}^1\longmapsto \sigma_t\in\Sigma(TX^{diff})\} \\ &\simeq C^\infty(\mathbb{S}^1,\Sigma(TX^{diff}))\end{aligned}$$

where we use the fact that $\ker d\pi\to\mathbb{S}^1$ is a smooth vector bundle over the circle with fiber $TX^{diff}$, hence is trivial. Using (6), we may easily define a map

$$e_\pi\colon C^\infty(\mathbb{S}^1,\Sigma(TX^{diff}))\longrightarrow Diff_\pi(\mathcal{X}^{diff}) \tag{59}$$

which satisfies, for all $t\in\mathbb{S}^1$,

$$(e_\pi)_t\colon \xi\in\Sigma(TX^{diff})\simeq\Sigma(T\pi^{-1}\{t\})\longmapsto e(\xi)\in Diff(X^{diff})\simeq Diff(\pi^{-1}\{t\})$$

As in the pointwise case, $e_\pi$ realizes a diffeomorphism between an open neighborhood of the zero-section in $\Sigma_\pi(T\mathcal{X}^{diff})$ and a neighborhood of the identity in $Diff_\pi(\mathcal{X}^{diff})$.

Given two subsets $U$ and $V$ of $\mathcal{E}(\mathcal{X}^{diff})$ with $U$ open, $W$ an open subset of a topological $\mathbb{C}$ vector space, and an analytic map $F$ from $U$ to $V\times C^\infty(\mathbb{S}^1,W)$, we say that $F$ is *almost-complex preserving* if the composition

$$c\in U\xrightarrow{\ F\ } V\times C^\infty(\mathbb{S}^1,W)\xrightarrow{\text{1st projection}} G(c)\in V$$

is almost-complex preserving for each $t\in\mathbb{S}^1$, that is the complex manifolds $X_{c(t)}$ and $X_{G(c(t))}$ are isomorphic for each $t$.

Thus we extend the notion of a.c. maps to $\mathcal{E}(\mathcal{X}^{diff})$. Observe that an a.c. map is equivariant with respect to the action of $Diff_\pi(\mathcal{X}^{diff})$.

## 3. The Kuranishi space of a proper CR submersion.

We are now in position to prove the main result of this paper: a statement analogous to Kuranishi's Theorem for $\mathcal{I}(\mathcal{X}^{diff})$.

Let $\mathcal{X}_0$ be a CR submersion compatible with $\pi^{diff}$, represented by an element $c_0$ of $vI(vX^{diff})$. Identify a neighborhood of $c_0$ in $\mathcal{I}(\mathcal{X}^{diff})$ with $C^\infty(\mathbb{S}^1,U_\phi)$ as explained in the previous Subsection. Choose a closed vector subspace $L$ of $\Sigma(TX^{diff})$ satisfying (25) for all $J$ in the image of $c_0$ and having minimal codimension for this property. We have

**Theorem 2.** Shrinking $U_\phi$ if necessary, we can find an analytic space $K_{c_0}$ and an analytic map

$$U_\phi\cap\mathcal{I}\xrightarrow[a.c.]{\ \Xi_\phi\ } K_{c_0} \tag{60}$$

such that

(i) The set $K_{c_0}$ is a (finite-dimensional) analytic space of (embedding) dimension at $c_0(t)$ equal to

$$\begin{array}{ll} h_1(t) + \text{codim } L - h_0(t) + 1 & \text{if } \mathcal{X}_0 \text{ is a non-trivial CR bundle} \\ h_1(t) + \text{codim } L - h_0(t) & \text{otherwise} \end{array}$$

where codim $L$ is the codimension of $L$ in $\Sigma(TX^{diff})$.
(ii) If we are not in the special case of (i), the analytic set $K_{c_0}$ is the leaf space of the $\Gamma_L$-foliation of $U_\phi$. Otherwise, there exists a closed subspace $L'$ of $\Sigma(TX^{diff})$ containing $L$ as a codimension-one subspace and such that $K_{c_0}$ is the leaf space of the $\Gamma_{L'}$-foliation of $U_\phi$.

As a consequence, the (infinite-dimensional) analytic space $C^\infty(\mathbb{S}^1, K_{c_0})$ plays the role of the Kuranishi space $K_0$ in the classical case. Hence we define

**Definition.** The loop space $C^\infty(\mathbb{S}^1, K_{c_0})$ is called a *Kuranishi type moduli space* of $\mathcal{X}_0$. We denote it by $K^g$.

*Proof.* We keep the same notations as in the previous subsections and recall that $U_\phi$ is defined as the quotient of some open neighborhood $U$ of $c_0$ by $\phi$. By Theorem 1, shrinking $U$ and thus $U_\phi$ if necessary, attached to $U$ is an analytic space $K_U$ together with an analytic a.c. map of $U$ onto $K_U$ such that $K_U$ is the leaf space of the $\Gamma_L$-foliation of $U$. We want now to define an analytic space $K_{c_0}$ attached to $U_\phi$.

If $\phi$ is the identity, there is nothing to do. We have $U_\phi = U$ and we take $K_{c_0} = K_U$, that is we take exactly the analytic space given by Theorem 1.

To do the general case, it would be natural to define $K_{c_0}$ as the quotient of $K_U$ by the action of $\phi$. However, the resulting quotient space is not always an analytic space and we have to consider two different cases.

Indeed, when dealing in the previous subsection with the construction of $U_\phi$, we impose the condition of $\phi$ not being a biholomorphism. This forces $c_0$ to have distinct endpoints. It follows that, when gluing $U_k$ to $U_1$ through $\phi$, the resulting quotient space is Hausdorff (at least after shrinking). The fact that $U_k$ and $U_1$ may be supposed to be disjoint is fundamental in this process. In the same way, when performing the same gluing onto $K_U$, we must ensure that the glued pieces corresponding to $K_1$ and $K_k$ in $K_U$ are disjoint to obtain an analytic space. This is possible (shrinking $U$ if necessary) if and only if the image $c$ of $c_0$ in $K_U$ is not a loop.

Indeed, we may assume that

$$K_k = \phi \cdot K_1$$

and, after identification between open sets of $K_k$ and $K_1$ and open sets of $K_U$, this induces a well-defined analytic a.c. isomorphism between two open sets of $K_U$. As we just told, if we may assume that, after shrinking, these two pieces are disjoint, then we may proceed as in Section 2 (construction of $U_\phi$) and ensure that the gluing occurs exclusively on these open sets and that the resulting analytic space is the desired leaf space. Hence we are done.

Assume that $\mathcal{X}_0$ is not a CR bundle. Then, by Fischer-Grauert Theorem, we can find $t \neq t'$ such that $X_{c_0(t)}$ is not biholomorphic to $X_{c_0(t')}$. This implies that $c$ is not a constant path. It may of course be a loop, which is exactly the situation we would like to avoid, but since it is not a constant loop, we claim that, shrinking $U$, the image of $c_0$ in $K_U$ is not a loop but a path. Indeed, assume that $c$ is a loop. Then this means not only that $c_0(0)$ and $c_0(1)$ are in the same leaf of $\mathcal{I}$, but also that they are in *the same leaf of* $U$. Either $c_0(t)$ or $c_0(t')$ must be in a different $\Gamma_L$-orbit than that of the two endpoints. Say it is $c_0(t)$. By shrinking $U$, we may assume that $c_0(t)$ belongs to some domain $U_p$ with $p$ different from 1 and from $k$, and that this $U_p$ does not intersect the $\Gamma_L$-orbit of $c_0(0)$. Hence the intersection of this orbit with $U$ is disconnected and $c_0(0)$ from the one hand, and $c_0(1)$ from the other hand, belong to two different connected components. In other words, the common leaf of $c_0(0)$ and $c_0(1)$ in $\mathcal{I}$ disconnects into (at least) two leaves in $U$, one passing through $c_0(0)$, and the other passing through $c_0(1)$. Because of the trivialization (35), this prevents their images $c(0)$ and $c(1)$ to be the same point of $K_U$. So in this case, we may define $K_{c_0}$ as the quotient of $K_U$ by the action of $\phi$.

Assume now that $\mathcal{X}_0$ is a CR bundle. Then, we cannot exclude that $c$ is the constant loop even after shrinking $U$ (cf. Example V.3). The quotient of $K_U$ by the action of $\phi$ occurs in the neighborhood of the *point* $c$, which is fixed by $\phi$. As a consequence, it may not be Hausdorff, depending on the properties of $\phi$. We avoid this problem as follows. Choose $c_0$ so that $c$ is a point. Instead of using in Proposition 1 a subspace $L$ of minimal codimension as we did, we take $L'$ such that

$$L = L' \oplus L_1$$

where $L_1$ is one-dimensional. Then using Proposition 1 with $L'$ this time, we obtain in place of $K_U$ the space $K_U \times W_1$ for some open set $W_1 \subset L_1$. Now, we may assume that the image of $c_0$ in $K_U \times W_1$ is not constant (for example that the projection onto $L_1$ is not constant). Taking this image as the new path $c$, we may now finish the argument with this $c$, defining $K_{c_0}$ as the gluing of $K_U \times W_1$ through $\phi$ as before. Observe that the gluing is given by the associated map

$$(J, v) \in K_U \times W_1 \longmapsto (\phi \cdot J, d\phi \cdot v) \in K_U \times W_1$$

the action on the first coordinate being defined in (4), and the action on the second coordinate being that of the differential of $\phi$ on $\Sigma(TX^{diff})$.

Notice that, when dealing with $L$ of minimal codimension, that is excluding the case where $\mathcal{X}_0$ is a non-trivial CR bundle, it follows from Proposition 1 and Theorem 1 that $K_U$, hence also $K_{c_0}$, is complete at $c(t)$ but not always versal, being the product of the Kuranishi space with a $\mathbb{C}$-vector space of dimension

$$\text{codim } L - \text{codim } L_0 = \text{codim } L - h^0(t).$$

Hence it has dimension

$$h^1(t) + \text{codim } L - h^0(t)$$

as stated. In the case of a non-trivial CR bundle, we have to increase the dimension by one.

To define the map $\Xi_\phi$, we proceed as follows. Assume that we are not in the special case. We already have an a.c. projection

$$U \cap \mathcal{I} \xrightarrow{a.c.} K_U \tag{61}$$

by Theorem 1. Since both the construction of $U_\phi$ and that of $K_{c_0}$ consist in taking the quotient by $\phi$, it follows that the projection (61) descends as a map

$$U_\phi \cap \mathcal{I} \xrightarrow{\Xi_\phi} K_{c_0} \tag{62}$$

as desired and that $K_{c_0}$ is the leaf space of the $\Gamma_L$-foliation of $U_\phi$. The special case is handled in the same way, just noting that, running the proof of Theorem 1 with $L'$ instead of $L$ yields an a.c. projection

$$U \cap \mathcal{I} \xrightarrow{a.c.} K_U \times W_1. \tag{63}$$

Using (63) instead of (61), we immediately see that it descends also as a map (62). □

*Remark.* If $K_U$ is not reduced, then we perform the previous construction with its reduction and put on each component of the resulting space the common multiplicity of the corresponding components of $K_U$.

**Corollary 3.** The map (62) is, after shrinking of $U_\phi$, a.c. diffeomorphic to a trivial bundle with base $K_{c_0}$ and fibre an open neighborhood of the identity in $\Gamma_L$ (respectively in $\Gamma_{L'}$ in the special case).

*Proof.* Use trivialization (32) and observe that the $\phi$-gluing respects the fibers of this trivialization. This shows that $U_\phi$ is a.c. diffeomorphic to a locally trivial bundle over $K_{c_0}$. Now there are two cases. Either the gluing occurs on the fibers (cf. case 1 and 2 in the proof of Theorem 2) hence the bundle is trivial; or the gluing occurs on the base (cf. case 3 of the proof of Theorem 2), but then its monodromy is isotopic to the identity (because it is given by some map (39)), and it is trivial.

Observe that, in the first case, the fiber is homotopic to a circle, whereas in the second case it is contractible but this time the base has a non-trivial fundamental group. □

**Corollary.** The analytic space $K_{c_0}$ is unique in the following sense.

(i) Up to a.c. isomorphism, it does not depend on the choice of the adapted covering and of the Kuranishi maps.
(ii) If $U'$ is another neighborhood of $c_0$, then the restrictions of $\Xi_\phi$ and of the corresponding map $\Xi'_\phi$ to $(U \cap U')_\phi \cap \mathcal{I}$ have a.c. isomorphic images.

*Proof.* This follows immediately from the fact that it is a leaf space. □

In other words, the germ of $K_{c_0}$ at $c_0$ is unique. However, once again, we want to emphasize that it depends on the choice of $L$, and of $L'$ in the special case.

## IV. Deformation Theory of proper CR submersions

### 1. Basic notions.

Let $\pi\colon \mathcal{X}_0 \to \mathbb{S}^1$ be a proper CR submersion. Recall the definitions given in Section I.3. The following definitions are inspired in [Bu].

**Definition.** Definitions A *holomorphic deformation* (resp. *smooth deformation*) of $\mathcal{X}_0$ is a Levi-flat CR space $\mathcal{Z}$ together with a proper and transflat CR morphism $\Pi\colon \mathcal{Z} \to B$ onto an analytic space (resp. a smooth manifold) $B$, a smooth transflat map $s\colon \mathcal{Z} \to \mathbb{S}^1$ and a marking $i\colon \mathcal{X}_0 \to \Pi^{-1}\{0\}$ such that, for all $t \in B$,
(i) The $\Pi$-fiber $\mathcal{Z}_t$ over $t$ is a Levi-flat CR submanifold of $\mathcal{Z}$.
(ii) The restriction $s_t$ of $s$ to $\mathcal{Z}_t$ is a proper CR submersion compatible with $\pi^{diff}$.
(iii) The composition $s_0 \circ i$ is equal to $\pi$.

This is a quite technical definition so let us highlight some of its principal features. Firstly, $\mathcal{Z}$ is locally diffeomorphic at $z \in \mathcal{Z}$ to $vX^{diff} \times U$, for $U$ a neighborhood of $\Pi(z)$ in $B$. Moreover, we have a commutative diagram of *diffeomorphisms*

$$
\begin{array}{ccc}
\mathbb{S}^1 & \xleftarrow{\ \pi^{diff}\ } & \mathcal{X}^{diff} \\
\uparrow s & & \uparrow \text{2nd projection} \\
\Pi^{-1}(U) \subset \mathcal{Z} & \xrightarrow{\ \simeq\ } & \mathcal{X}^{diff} \times U \\
\downarrow \Pi & & \downarrow \text{1st projection} \\
U & \xrightarrow[\text{Identity}]{} & U
\end{array}
$$

Secondly, choose any $t_0 \in \mathbb{S}^1$ and let $X_0$ be the fiber of $\mathcal{X}_0$ over $t_0$. Then, the map

$$P = (s, \Pi)\colon \mathcal{Z} \to \mathbb{S}^1 \times B \tag{64}$$

is a CR deformation of $X_0$ once chosen a marking.

### 2. Complete Families.

We are in position to prove the following *completeness* result.

**Theorem 3.** Let $\pi\colon \mathcal{X}_0 \to \mathbb{S}^1$ be a proper CR submersion. Choose a path $c_0$ representing $\mathcal{X}_0$ and let $K^g$ be a Kuranishi type moduli space of $vX_0$. Then there exists a (infinite-dimensional) holomorphic deformation $\Pi^g\colon vK^g \to K^g$ of $vX_0$ with base $K^g$. This family is complete at 0, that is any holomorphic (resp. smooth) deformation $\Pi\colon \mathcal{Z} \to B$ of $\mathcal{X}_0$ is locally isomorphic to the pull-back of $\mathcal{K}^g$ by some analytic (resp. smooth) map $f$ from $(B, 0)$ to $(K^g, 0)$.

Moreover, we may ask the local isomorphisms to preserve the markings.

*Proof.* Choose a path $c_0$ representing $\mathcal{X}_0$, define $K_{c_0}$ and $K^g$ as usual. Observe that $K_{c_0}$ being obtained from a finite number of Kuranishi spaces by gluing them through a.c. isomorphisms, it defines in a natural way an analytic family $\mathcal{K}_{c_0}$.

Indeed, going back to the proof of Theorem 1 and using the notations introduced there, let $K_1$ and $K_2$ be two Kuranishi spaces and let

$$K_{12} = K_1 \cup_g K_2 \tag{65}$$

be an analytic space obtained by gluing, the gluing $g$ being an a.c. isomorphism between an open set $V_1$ of $K_1$ and an open set $V_2$ of $K_2$. More precisely, $g$ is given as a map

$$g\colon J \in V_1 \subset K_1 \longmapsto G(J) \cdot J \in K_2 \tag{66}$$

where

$$G\colon V_1 \longrightarrow Diff(X^{diff}) \tag{67}$$

and where the $\cdot$ denotes the action (4). Consider now the families $\mathcal{K}_i$ induced above $K_i$. They are constructed as $K_i \times X^{diff}$, every fiber $\{J\} \times X^{diff}$ being endowed with the complex structure $J$. It follows that the map

$$(J, x) \in V_1 \times X^{diff} \longmapsto (g(J), G(J)(x)) \in V_2 \times X^{diff} \tag{68}$$

realizes an analytic isomorphism between open sets of $\mathcal{K}_i$ preserving the projections onto $K_i$. Gluing $\mathcal{K}_1$ and $\mathcal{K}_2$ through (68), we obtain a holomorphic family $\mathcal{K}_{12}$ over $K_{12}$. Repeating the process yields the family $\mathcal{K}_{c_0}$.

Set

$$\mathcal{K}^g := \{f^*\mathcal{K}_{c_0} \quad | \quad f \in C^\infty(\mathbb{S}^1, K_{c_0})\} \tag{69}$$

and let $\Pi^g$ be the induced projection from $\mathcal{K}^g$ to $K^g$. Call 0 the point of $K^g$ corresponding to $c_0$. Finally choose a marking

$$i^g\colon \mathcal{X}_0 \longrightarrow (\Pi^g)^{-1}(0).$$

As a consequence of property (64), a holomorphic (resp. smooth) CR deformation $\Pi\colon \mathcal{Z} \to B$ of $\mathcal{X}_0$ can be locally encoded as follows. Construct the neighborhood $U_\phi$ of $c_0$. Then $P$ induces an analytic (resp. smooth) map $f$ from a neighborhood of 0 in $B$ to $C^\infty(\mathbb{S}^1, U_\phi \cap \mathcal{I})$ such that $\mathcal{Z}$ is locally isomorphic to $\mathcal{X}^{diff} \times B$ endowed with the family of CR submersions structures $(f(t))_{t\in B}$.

It is enough to compose $f$ with the a.c. projection from $C^\infty(\mathbb{S}^1, U_\phi \cap \mathcal{I})$ to $K^g$ induced by the map $\Xi_\phi$ of (60) to prove completeness.

Finally, observe that if this composition, let us call it $g$, does not preserve the markings, then there exists a CR isomorphism $\Phi$ of $\mathcal{X}_0$ covering the identity of $\mathbb{S}^1$ such that

$$\begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{\Phi} & \mathcal{X}_0 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i^g} \\ (\mathcal{Z},\Pi^{-1}(0)) & \xrightarrow{G} & \mathcal{K}^g \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi^g} \\ (B,0) & \xrightarrow[g]{} & K^g \end{array}$$

But now, $\Phi$ acts on $\mathcal{K}^g$, so that we may replace the map $G$ by $\Phi^{-1} \cdot G$ so that we have now

$$\begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{i^g} & \mathcal{K}^g \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \text{Identity}} \\ (\mathcal{Z},\Pi^{-1}(0)) & \xrightarrow[\Phi^{-1}\cdot G]{} & \mathcal{K}^g \end{array}$$

and this time the map $g$ preserves the markings. □

**3. (Uni)versality.**

As recalled in Section I.3, the Kuranishi family is not only complete but also versal at 0 in the classical case. We will see that things are a bit different in the case of proper CR submersions.

First, recall that we gave two equivalent definitions of versality in the classical context. The first one deals with the Kuranishi space having minimal dimension at 0. Since our family is infinite-dimensional, we cannot use this definition. The other characterization deals with the unicity of the differential at 0 of the map $f$. This can be easily transposed to our context.

Moreover, recall that the Kuranishi space is *universal* if the germ of $f$ is unique. This definition can also be transposed to our context. It is known that, in the reduced case, the Kuranishi space is universal if and only if the function $h^0$ is constant along it (cf. [Wa1], [Wa2], [Me1]).

Let us fix some notations. We start from a deformation $\Pi\colon \mathcal{Z} \to B$ of $\mathcal{X}_0$ and we consider a map $f$ from some open set of $B$ to $K^g$ given by completeness. Its differential at the marked point 0 goes from the tangent space $T_0B$ to the tangent space $T_0K^g$. We only consider local isomorphisms from $\mathcal{Z}$ to $\mathcal{K}^g$ covering $f$ which preserve the markings. This prevents from composing with automorphisms of $\mathcal{Z}$ and $\mathcal{K}^g$ which descends as non-trivial automorphisms of the bases of the families.

CRM Preprint Series number 1121

Notice that this is necessary to hope the unicity of $d_0 f$. Even in the classical case, allowing these compositions, one loses the versality.

Nevertheless, we do not have versality in general. We will give some counterexamples in the next part. Indeed,

**Theorem 4.**
(i) The family $\Pi^g \colon \mathcal{K}^g \to K^g$ is versal at 0 if and only if

$$\forall t \in \mathbb{S}^1, \qquad h^0(t) = \text{codim } L. \tag{70}$$

(ii) The family $\Pi^g \colon \mathcal{K}^g \to K^g$ is universal at 0 for families over a reduced base if and only if

$$\forall J \in K_{c_0}, \qquad h^0(X_J) = \text{codim } L.$$

*Remark.* Observe that, if $\mathcal{X}_0$ is a trivial CR bundle, then (70) is automatically satisfied, hence $\mathcal{K}^g$ is versal. And it is universal if and only if the Kuranishi space of the fiber is universal. On the contrary, if $\mathcal{X}_0$ is a *non-trivial* CR bundle, the family $\mathcal{K}^g$ is never versal.

*Proof.* We first need to compute the tangent space of $K^g$. From its definition, one has that

$$T_c K^g = \{H \in C^\infty(\mathbb{S}^1, TK_{c_0}) \quad | \quad p \circ H \equiv c\} \tag{71}$$

where $p \colon TK_{c_0} \to K_{c_0}$ is the tangent sheaf of $K_{c_0}$. In particular, for $t \in \mathbb{S}^1$, we have a commutative diagram

$$\begin{array}{ccc} H \in T_c K^g & \xrightarrow{\text{evaluation at } t} & H(t) \in T_{c(t)} K_{c_0} \\ {\scriptstyle p\circ}\downarrow & & \downarrow{\scriptstyle p} \\ c \in K^g & \xrightarrow[\text{evaluation at } t]{} & p \circ H(t) = c(t) \in K_{c_0} \end{array} \tag{72}$$

Let $ev_t$ denote the evaluation map of the bottom arrow and $Ev_t$ that of the top arrow. Then analyzing (72) yields that the differential $d_c ev_t$ is equal to $Ev_t$.

Indeed, let $v$ be a vector of $T_c K^g$ and let

$$u \colon (-\epsilon, \epsilon) \longrightarrow K^g$$

be a smooth path whose derivative at 0 is $v$. Define

$$(s, \theta) \in (-\epsilon, \epsilon) \times \mathbb{S}^1 \longmapsto U(s, \theta) := ev_\theta \circ u(s).$$

Compute

$$\frac{d}{ds}|_{s=0}(ev_t \circ u(s)) = \frac{d}{ds}|_{s=0}(U(s,t)) = Ev_t \circ \frac{d}{ds}|_{s=0}(u(s))$$

that is

$$d_c(ev_t \circ u) = d_c ev_t(v) = Ev_t \circ v.$$

Let $\Pi\colon \mathcal{Z} \to B$ be a deformation of $\mathcal{X}_0$ and let $(f, F)$ be given by completeness. Consider now the submission

$$\Pi_t\colon \mathcal{Y}_t := s^{-1}(t) \xrightarrow{\Pi_{|\mathcal{Y}_t}} B. \tag{73}$$

This is a deformation of $X_t := \Pi^{-1}(t)$. But we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Y}_t \subset \mathcal{Z} & \xrightarrow{(F)_t} & \mathcal{K}_{c_0} \\ \Big\downarrow{\scriptstyle \Pi} & & \Big\downarrow{\scriptstyle \Pi^g} \\ (B, 0) & \xrightarrow[(f)_t]{} & (K_{c_0}, c_0(t)) \end{array} \tag{74}$$

where $(F)_t$ and $(f)_t$ are given by the evaluation at $t$ of the maps $F$ and $f$.

Observe that, if

$$i\colon \mathcal{X}_0 \longrightarrow \Pi^{-1}(0) \tag{75}$$

denotes the marking of our family $\Pi$, then

$$i_t\colon X_t \subset \mathcal{X}_0 \longrightarrow \Pi_t^{-1}(0) \tag{76}$$

is a marking for (73).

From (72) and (74), we obtain that

$$d_0(f)_t \equiv (d_0 f)_t\colon T_0 B \longrightarrow T_{c_0(t)} K_{c(0)}. \tag{77}$$

and we see that the versality at $c_0(t)$ of $K_{c_0}$ for any $t$ implies the versality at 0 of $K^g$. The markings used here are (76) and (75).

Conversely, assume that $K_{c_0}$ is not versal at some point $c_0(t_0)$ with $t_0 \in \mathbb{S}^1$. Then we can find a deformation

$$\mathcal{Y} \to (\mathbb{D}, 0)$$

of $X_{t_0}$ with marking $i_0$ and two holomorphic maps

$$(\mathbb{D}, 0) \xrightarrow{f_{t_0},\ g_{t_0}} (K_{c_0}, c_0(t_0))$$

with

$$\mathcal{Y} = f_{t_0}^* \mathcal{K}_{c_0} = g_{t_0}^* \mathcal{K}_{c_0},$$

respecting the markings $i_0$ and $i^g(c_0(t_0))$ and finally such that

$$d_0 f_{t_0} \not\equiv d_0 g_{t_0}. \tag{78}$$

Then we extend the map $f_{t_0}$ into a CR map

$$f\colon \mathbb{D} \times \mathbb{S}^1 \longrightarrow K_{c_0}$$

such that

$$\begin{cases} f_{|\mathbb{D}\times\{t_0\}} \equiv f_{t_0} \\ f_{|\{0\}\times\mathbb{S}^1} \equiv c_0 \end{cases}$$

To prove this, first observe that we may assume $K_{c_0}$ smooth. If not, just desingularize and lift both $f_{t_0}$ and $c_0$. Also observe that we may perform the extension step by step along the circle, from $t_0$ to a close $t_1$ and so on. Hence we reduce the problem to a local extension problem in $\mathbb{C}^n$.

*Remark.* There is no particular problem if the space $K_{c_0}$ is not reduced. Indeed, since we use maps from a reduced base (here a circle or an annulus) into $K_{c_0}$, they map into the reduction of $K_{c_0}$. Hence, we only have to desingularize the reduction of $K_{c_0}$.

Taking into account that we have a monodromy problem when coming back to $t_0$ after a complete turn, we finally see that the proof is completed with the following lemma.

**Lemma 1.** Let $u\colon [0,1] \to \mathbb{C}^n$ be a smooth path and let $f_i\colon \mathbb{D} \to \mathbb{C}^n$ be two $(i = 0, 1)$ holomorphic maps such that $f_i(0) = u(i)$.

Then, there exists a CR map

$$F\colon \mathbb{D} \times [0,1] \longrightarrow \mathbb{C}^n$$

such that

$$\begin{cases} F(-, i) \equiv f_i & \text{for } i = 0, 1 \\ F(0, t) = u(t) & \text{for } t \in [0,1] \end{cases}$$

*Proof of Lemma 1.* Consider the segment $[u(0), u(1)]$ and define the map

$$G\colon (z,t) \in \mathbb{D} \times [0,1] \longmapsto \begin{cases} f_0(z) + t(u(1) - u(0)) & \text{if } t \leq 1/2 \\ f_1(z) + (1-t)(u(1) - u(0)) & \text{if } t > 1/2 \end{cases}$$

This is not a CR map because it is not continuous at 1/2. To smooth it at this point, choose a smooth function

$$l\colon [0,1] \longrightarrow \mathbb{R}^{\geq 0}$$

satisfying

$$l(0) = l(1) = 1 \qquad \text{and} \qquad l(1/2) = 0.$$

Set

$$H(z,t) = l(t)G(z,t).$$

We are almost done, except for the fact that it does not satisfy $H(0,t) = u(t)$. To finish with the proof, just add a smooth translation factor as follows

$$F(z,t) = H(z,t) + u(t) - H(0,t). \ \square$$

Coming back to the proof of Theorem 4, define

$$\mathcal{Z} := f^*(\mathcal{K}_{c_0}) \longrightarrow \mathbb{D} \times \mathbb{S}^1 \xrightarrow{\text{1st projection}} \mathbb{D}. \tag{79}$$

This is a deformation of $\mathcal{X}_0$ with marking

$$i\colon \mathcal{X}_0 \longrightarrow c_0^*\mathcal{K}_{c_0}.$$

Now, taking into account that $K_{c_0}$ encodes complex operators, we may write

$$\mathcal{Y} = (X^{diff} \times \mathbb{D}, J_0) = (X^{diff} \times \mathbb{D}, J_1)$$

with

$$(F_{t_0})_* J_0 \equiv (G_{t_0})_* J_1$$

where $F_{t_0}$ (respectively $G_{t_0}$) is a map from $\mathcal{Y}$ to $\mathcal{K}_{c_0}$ covering $f_{t_0}$ (respectively $g_{t_0}$).

In other words, we may find some smooth family of diffeomorphisms $k_s$ of $X^{diff}$ parametrized over the disk such that, for every $s \in \mathbb{D}$, the map $k_s$ is a biholomorphism from $(X^{diff} \times \{s\}, J_0)$ to $(X^{diff} \times \{s\}, J_1)$.

We extend $J_0$ into a family of complex operators over $\mathbb{D} \times \mathbb{S}^1$ so that

$$\mathcal{Z} = (X^{diff} \times \mathbb{D} \times \mathbb{S}^1, J_0)$$

and we extend $J_1$ over the same base by defining

$$J_1 := (k_s)_* J_0$$

on the fiber over any point $(s, t) \in \mathbb{D} \times \mathbb{S}^1$.

Obviously, we also have

$$\mathcal{Z} = (X^{diff} \times \mathbb{D} \times \mathbb{S}^1, J_1).$$

Shrinking the base $\mathbb{D}$ if necessary, we may assume that the map

$$J_1\colon \mathbb{D} \times \mathbb{S}^1 \longrightarrow \mathcal{I}$$

has image in the open set $U_\phi \cap \mathcal{I}$ admitting a retraction (60) onto $K_{c_0}$. This allows us to extend the map $g_{t_0}$ into a map $g$ defined over $\mathbb{D} \times \mathbb{S}^1$ by stating

$$g(z, t) := \Xi_\phi((J_1)_{(z,t)})$$

where $(J_1)_{(z,t)}$ denotes the restriction of $J_1$ to the fiber over $(z, t)$. By construction, we have

$$\mathcal{Z} = g^*\mathcal{K}^g.$$

Because of (79), (78) and (77), the family $\mathcal{K}^g$ is not versal at $c_0$.

To finish with the proof of (i), we just have to show that $K_{c_0}$ is versal at each point $c_0(t)$ if and only if equality (70) holds.

The construction of $K_{c_0}$ given in the proof of Theorem 2 shows that it is complete at $c_0(t)$ with dimension

$$h^1(t) + \text{codim } L - h^0(t) \quad \text{or} \quad h^1(t) + \text{codim } L + 1 - h^0(t). \tag{80}$$

To be versal, it must have minimal dimension, that is dimension $h^1(t)$. Since we have

$$\text{codim } L \geq h^0(t) \qquad \text{for all } t$$

this yields the condition

$$h^0(t) = \text{codim } L. \tag{81}$$

Conversely, if this condition is fulfilled, then, from (80), the space $K_{c_0}$ is versal at each point of $c_0$, hence the differential $d_0(f)_t$ is uniquely determined for each $t \in \mathbb{S}^1$. By (77), which means that $d_0 f$ is uniquely determined, so the family $\Pi^g \colon \mathcal{K}^g \to K^g$ is versal at 0. This proves (i).

By definition $f$ is unique, yielding universality if $f_t$ is unique for all $t$. In other words, using (77), $K^g$ is universal if $K_{c_0}$ is universal at each point $c_0(t)$. Conversely, if $K_{c_0}$ is not universal at some point $c_0(t_0)$, then the same argument as above (just replacing (78) with $f_{t_0} \not\equiv g_{t_0}$) shows that $K^g$ is not universal at $c_0$.

Universality of $K_{c_0}$ implies also that it is versal at each point $c_0(t)$, that is that the function $h^0$ is equal to codim $L$ at every point $c_0(t)$. Now, by a Theorem of Wavrik (see [Me1, **?**] for the version we use), $K_{c_0}$ is universal at $c_0(t)$ for families over a reduced base if and only if $h^0$ is constant in a whole neighborhood of $c_0(t)$. So we finally obtain that the condition

$$h^0(X_J) = \text{codim } L \quad \text{ for all } J \in K_{c_0}$$

is sufficient to have universality for families over a reduced base. □

**4. Kodaira-Spencer map.**

We finish this Section with the construction of the Kodaira-Spencer map of a deformation of a proper CR submersion over the circle. In the classical case, the Kodaira-Spencer map of a deformation of $X_0$ takes value in the first cohomology group $H^1(X_0, \Theta_0)$, which can be identified with the tangent space at 0 of the Kuranishi space of $X_0$ in such a way that it corresponds to the differential $d_0 f$ of the map $f$ obtained by completeness. In our case, however, the Kodaira-Spencer map will take value in a first cohomology group which is different from $T_0 K^g$, except if (81) is satisfied for all $t \in \mathbb{S}^1$.

To do that, we start as usual with a proper CR submersion $\pi \colon \mathcal{X}_0 \to \mathbb{S}^1$ and we define $\Theta_\pi$ to be the sheaf of germs of CR vector fields of $\mathcal{X}_0$ tangent to the fibers of $\pi$. The first cohomology group $H^1(\mathcal{X}_0, \Theta_\pi)$ has a natural structure of the set of smooth sections of a sheaf of $\mathbb{C}$-vector spaces over the circle. The stalk at some point $t$ is the vector space $H^1(X_t, \Theta_t)$. Moreover, it is a vector bundle over the circle as soon as the function $h^1$ is constant along the circle (cf. [K-S1]).

Let $\Pi \colon \mathcal{Z} \to B$ be a deformation of $\mathcal{X}_0$. Let $\Theta_s$ be the sheaf of germs of CR vector fields of $\mathcal{Z}$ tangent to the fibers of $s$. Let also $\Theta_P$ be the sheaf of germs of CR vector fields of $\mathcal{Z}$ tangent to the fibers of $P$ (cf. (64)). Consider the following exact sequence of sheaves

$$0 \longrightarrow \Theta_P \longrightarrow \Theta_s \longrightarrow \Theta_s/\Theta_P \longrightarrow 0 \tag{82}$$

and observe that the quotient sheaf $\Theta_s/\Theta_P$ can be identified with the sheaf $\Theta_B$ of germs of CR vector fields on the base $B$. The long exact sequence associated to (82) runs as follows

$$\cdots \longrightarrow H^0(\mathcal{Z}, \Theta_s) \longrightarrow H^0(B, \Theta_B) \xrightarrow{\rho} H^1(\mathcal{Z}, \Theta_P) \longrightarrow \cdots \tag{83}$$

Observe now that the restriction of $\rho$ to the tangent space $T_0B$ gives a map

$$T_0B \xrightarrow{\rho_0} H^1(\mathcal{X}_0, \Theta_\pi). \tag{84}$$

**Definition.** Definition The map $\rho_0$ of (84) is called the *Kodaira-Spencer map* at 0 of the deformation $\Pi\colon \mathcal{Z} \to B$.

Roughly speaking, the map $\rho$ of (83) represents the complete obstruction to lift CR vector fields of $B$ to CR vector fields of $\mathcal{Z}$ respecting the fibers of $s$, thus trivializing the family. The Kodaira-Spencer map being the evaluation of this obtruction to the central fiber is the first obtruction to such a trivialization. It has the advantage to be defined on $\mathcal{X}_0$ and not on the whole deformation $\mathcal{Z}$ and thus can be computed explicitely in many cases.

**Theorem 5.** Let $\pi\colon \mathcal{X}_0 \to \mathbb{S}^1$ be a proper CR submersion. Assume that for all $t \in \mathbb{S}^1$, the identity (81) is fulfilled. Then there exists a fixed isomorphism

$$H^1(\mathcal{X}_0, \Theta_\pi) \xrightarrow{\varphi} T_0K^g \tag{85}$$

such that the following property holds.

Let $\Pi\colon \mathcal{Z} \to B$ be any deformation of $\mathcal{X}_0$. Let $\rho_0$ be its Kodaira-Spencer map at 0 and let $f\colon (B, 0) \to (K^g, 0)$ be given by completeness of $K^g$. Then we have

$$\varphi \circ \rho_0 \equiv d_0f \tag{86}$$

*Proof.* We have already seen that the natural projection

$$\mathcal{H}^1 = \bigcup_{t\in\mathbb{S}^1} H^1(X_t, \Theta_t) \longrightarrow \mathbb{S}^1$$

can be endowed with a structure of a sheaf over the circle with stalk $H^1(X_t, \Theta_t)$ at $t$. With this structure, $\mathcal{H}^1$ identifies with the cohomology group $H^1(\mathcal{X}_0, \Theta_\pi)$.

If $\Pi\colon \mathcal{Z} \to B$ is a deformation of $\mathcal{X}_0$, recall that (cf. (73))

$$\mathcal{Y}_t := s^{-1}(t) \xrightarrow{\Pi_t} B \tag{87}$$

is a deformation of $X_t := \pi^{-1}(t) \subset \mathcal{X}_0$. Associated to (87), when $B$ is finite-dimensional, we thus have a (classical) Kodaira-Spencer map

$$\rho_t\colon T_0B \longrightarrow H^1(X_t, \Theta_t) \tag{88}$$

and the family of these maps, when $t$ varies in $\mathbb{S}^1$, is exactly the Kodaira-Spencer map defined in (84).

We want to apply these considerations to the Kuranishi family of $\mathcal{X}_0$. In this case, because of (74), the deformation $\mathcal{Y}_t$ given by (87) reduces to a deformation over $(K_{c_0}, c_0(t))$. Indeed $\mathcal{Y}_t$ is equal to the pull-back of $\mathcal{K}_{c_0} \to (K_{c_0}, c_0(t))$ by the evaluation map at $t$. Hence (88) gives a decomposition of the Kodaira-Spencer map $\rho$ of the family $\Pi^g \colon \mathcal{K}^g \to K^g$ into a family

$$\rho_t \colon T_{c0(t)}K_{c_0} \longrightarrow H^1(X_t, \Theta_t) \tag{89}$$

and we have a commutative diagram

$$\begin{array}{ccc} T_0K^g & \xrightarrow{\ \rho\ } & H^1(\mathcal{X}_0, \Theta_\pi) \\ {\scriptstyle ev_t}\downarrow & & \downarrow{\scriptstyle ev_t} \\ T_{c0(t)}K_{c0} & \xrightarrow[\ \rho_t\ ]{} & H^1(X_t, \Theta_t) \end{array}$$

Since we assume that (81) is fulfilled for all $t$, the space $K_{c_0}$ is versal at $c_0(t)$ and all the $\rho_t$ are isomorphisms. So is the map $\rho$.

We define the map $\varphi$ of (85) to be $\rho^{-1}$.

Now, because of (72), (74) and (76), the property (86) is satisfied if and only if it is satisfied for each $t \in \mathbb{S}^1$, that is if

$$\varphi_t \circ \rho_t \equiv d_0 f_t$$

which is true by the chain-property of the (classical) Kodaira-Spencer map. □

## V. Applications and examples

**1. Connectedness and extension of deformations.** In the classical case of compact complex manifolds, Kuranishi's Theorem has as a consequence that every complex structure $J$ on $X^{diff}$ close enough to a fixed structure $J_0$ is connected to it. That is, there exists a 1-dimensional holomorphic (resp. smooth) deformation of $X_0$ that contains $X_J$. The proof just consists in choosing a disk (resp. a path) in the Kuranishi space of $X_0$ joining the base point to the point encoding $J$.

In our case, the same result is true.

**Theorem 6.** Let $\pi \colon \mathcal{X}_0 \to \mathbb{S}^1$ be a proper CR submersion, represented by an element $c_0$ of $\mathcal{I}(\mathcal{X}^{diff})$ as usual.

Then, if $\mathcal{X}$ is a proper CR submersion close enough to $\mathcal{X}_0$ (that is, if $\mathcal{X}$ can be represented by a path $c$ close enough to $c_0$ in the topological space $\mathcal{I}$), there exists a holomorphic (resp. smooth) 1-dimensional deformation joining $\mathcal{X}_0$ and $\mathcal{X}$.

*Proof.* Since $\mathcal{X}$ is close to $\mathcal{X}_0$, it is represented by a point in the Kuranishi space $K^g$ of $\mathcal{X}_0$. That means that there exists a loop $c$ in $K_{c_0}$ encoding $\mathcal{X}$.

Therefore, to construct a smooth deformation as desired, it is enough to construct an isotopy

$$H\colon \mathbb{S}^1\times[0,1]\longrightarrow K_{c_0}$$

such that

$$H_0 := H(-,0)\equiv c_0 \qquad \text{and} \qquad H_1 := H(-,1)\equiv c.$$

Now, it is a classical fact that two smooth loops in an analytic space are isotopic *as soon as they are close enough one from the other.* Indeed, this is clear for complex manifolds. For analytic spaces, we may first desingularize and extend the loops we want to isotope. Observe that the exceptional divisors being simply connected, there is no additional obstruction. Since we are in this case, the existence of $H$ follows.

*Remark.* There is no particular problem if the space $K_{c_0}$ is not reduced. Indeed, since we use maps from a reduced base (here a circle or an annulus) into $K_{c_0}$, they map into the reduction of $K_{c_0}$. Hence, we only have to desingularize the reduction of $K_{c_0}$. The same remark applies below and to the next results (Corollaries 5 and 6).

We treat the case of a 1-dimensional *holomorphic* family joining $\mathcal{X}_0$ to $\mathcal{X}$. That amounts to finding some CR morphism of $\mathbb{S}^1\times\mathbb{D}$ in $K_{c_0}$ whose image contains $c_0$ and $c$. Once again, desingularizing and extending $c_0$ and $c$ if necessary, we may assume that $K_{c_0}$ is smooth. For each $\exp 2i\pi\theta\in\mathbb{S}^1$, the corresponding holomorphic disk of $K_{c_0}$ must pass through $c_0(\exp 2i\pi\theta)$ and $c(\exp 2i\pi\theta)$. We can always construct such a disk $\mathbb{D}_\theta$ for $\theta$ fixed. And this can be done in a locally smooth way. The only problem that could appear is that, starting with $\mathbb{D}_0$ and constructing the family by extension, we finish with $\mathbb{D}_1$ different from $\mathbb{D}_0$. Lemma 1 allows us to solve this problem. □

Indeed, we have even a stronger connectedness result.

**Corollary 5.** Let $\pi\colon\mathcal{X}_0\to\mathbb{S}^1$ be a proper CR submersion.

If $X$ is a compact complex manifold close enough to some fiber $X_t$ of $\mathcal{X}_0$, then there exists a holomorphic (resp. smooth) 1-dimensional deformation $\mathcal{Z}$ of $\mathcal{X}_0$ such that, for some $z$ in the base, the $t$-fiber of $\mathcal{Z}_z$ is biholomorphic to $X$.

*Proof.* Choose $c$ in $\mathcal{I}(\mathcal{X}^{diff})$ close to $c_0$ and satisfying that $X_{c(t)}$ is biholomorphic to $X$ and apply Theorem 7. □

Finally, we prove that a 1-dimensional deformation of a fiber of $\mathcal{X}_0$ can be extended as a 1-dimensional deformation of $\mathcal{X}_0$.

**Corollary 6.** Let $\pi\colon\mathcal{X}_0\to\mathbb{S}^1$ be a proper CR submersion.

If $\mathcal{Y}\to B$ is a holomorphic (resp. smooth) deformation of some fiber $X_t$ of $\mathcal{X}_0$ over a 1-dimensional base, then there exists a holomorphic (resp. smooth) deformation $\mathcal{Z}$ of $\mathcal{X}_0$ over the same base inducing locally $\mathcal{Y}$, i.e. such that we have

$$(\mathcal{Y}_t)_{|U} \equiv \mathcal{Y}$$

for $U$ a neighborhood of 0 in $B$.

Recall the definition (73) of $\mathcal{Y}_t$.

*Proof.* We just do the holomorphic case. Since $K_{c_0}$ is complete at $c_0(t)$, we may assume that $\mathcal{Y}$ is obtained by pull-back of $K_t$ (the Kuranishi space of $X_t$) over some disk of the base. As before, we may assume that $K_{c_0}$ is smooth. Let

$$F\colon (\mathbb{D},0) \subset (B,0) \longrightarrow (K_{c_0}, c_0(t))$$

be the associated map. We just have to extend it into a CR map

$$H\colon \mathbb{S}^1 \times \mathbb{D} \longrightarrow K_{c_0}$$

such that

$$H_0 \equiv c_0 \qquad \text{and} \qquad H(1,-) \equiv F$$

which is not really different from what we did in the proofs of Theorems 4 and 6. □

**2. Rigidity.**

As in the classical case, we say that a proper CR submersion $\pi\colon \mathcal{X}_0 \to \mathbb{S}^1$ is *rigid* if any deformation $\Pi\colon \mathcal{Z} \to B$ of $\mathcal{X}_0$ is (locally) isomorphic to a product $\mathcal{X}_0 \times B$.

Here is a trivial example of a rigid CR submersion.

**Example 0.1.** Example Let $X_0$ be a rigid compact complex manifold (for example, $X$ is $\mathbb{P}^n$ for some $n > 0$). Let $\mathcal{X}_0$ be a CR bundle with fiber $X_0$.

If this bundle is trivial, then the space $K_{c_0}$ is nothing else that a point. So is $K^g$. By Theorem 2, $\mathcal{X}_0$ is trivial.

If it is not trivial, then $K_{c_0}$ can be taken as the unit disk (cf. Theorem 2), but the family $\mathcal{K}_{c_0} \to K_{c_0}$ is trivial by construction and every $\mathcal{X}$ close to $\mathcal{X}_0$ is isomorphic to

$$X_0 \times [0,1]/\sim \qquad \text{where} \qquad (z,0) \sim (\phi(z),1)$$

for $\phi$ a biholomorphism representing the monodromy of $\mathcal{X}_0$. Hence it is isomorphic to $\mathcal{X}_0$. The same argument shows that any deformation of $\mathcal{X}_0$ is trivial, hence $\mathcal{X}_0$ is trivial.

This is indeed the unique rigid example.

**Theorem 7.** Let $\pi\colon \mathcal{X}_0 \to \mathbb{S}^1$ be a proper CR submersion. Then $\mathcal{X}_0$ is rigid if and only if it is a CR bundle with rigid fiber.

*Proof.* We have already seen in the previous example that a CR bundle with rigid fiber is rigid. Conversely let $\mathcal{X}_0$ be rigid. If one of the fiber $X_t$ of $\mathcal{X}_0$ is not rigid, then by Corollary 6, there exists a non-trivial deformation of $\mathcal{X}_0$, so every fiber is rigid. By connectedness of the circle, this implies that all the fibers are biholomorphic, hence, by Fischer-Grauert's Theorem, it is a CR bundle. □

**3. Examples of CR bundles.**

**Example: the trivial case.** Assume that $\pi\colon \mathcal{X}_0 \to \mathbb{S}^1$ is a trivial CR bundle. Then $c_0$ is just a point of $\mathcal{I}$ and $K_{c_0}$ is the same as the Kuranishi space $K_0$ of the fiber $X_0$. So finally the Kuranishi space $K^g$ of $\mathcal{X}_0$ is $C^\infty(\mathbb{S}^1, K_0)$.

For example, if $\mathcal{X}_0$ is $\mathbb{E}_\tau \times \mathbb{S}^1$ (where $\mathbb{E}_\tau$ is the elliptic curve of modulus $\tau \in \mathbb{H}$), then $K_0$ is a neighborhood of $\tau$ in $\mathbb{H}$ and $K^g$ is the space of smooth maps from the circle to this neighborhood.

**Example: the non-trivial case.** Let $\omega$ be $\exp(2i\pi/3)$ and let $\mathbb{E}$ be the elliptic curve of modulus $\omega$. Let $\mathcal{X}_0$ be the CR bundle with fiber $\mathbb{E}$ and monodromy $\omega$. Here $c_0$ is also a point, but we are in the special case of Theorem 2 and we cannot take $K_{c_0}$ as a neighborhood of $\omega$ in $\mathbb{H}$. Let $\mathbb{D}_\omega$ be a disk centered at $\omega$ in $\mathbb{H}$. Let

$$V = \{z \in \mathbb{C} \quad | \quad \sup_{t\in[0,1]} |z - (t\omega + (1-t))| < \epsilon\}.$$

This is a neighborhood of the segment joining 1 to $\omega$ in $\mathbb{C}$. Let $\mathbb{D}_\epsilon$ be the open disk of radius $\epsilon$ centered at 1. Observe that $\mathbb{D}_\epsilon$ is a neighborhood of 1 included in $V$. Assume that $\epsilon$ is small enough to ensure that $\mathbb{D}_\epsilon$ and $\omega\mathbb{D}_\epsilon$ are disjoint.

Following the proof of Theorem 2, we define $K_{c_0}$ to be

$$K_{c_0} = \mathbb{D}_\omega \times V/\sim \qquad \text{with } (\tau, w) \in \mathbb{D}_\omega \times \mathbb{D}_\epsilon \sim (\omega\cdot\tau, \omega w)$$

where $\omega\cdot$ describes the action of the automorphism $\omega$ onto $\mathbb{H}$, that is

$$\omega\cdot\tau = \frac{-1-\tau}{\tau}.$$

So $K_{c_0}$ is biholomorphic to the product of a disk with an annulus. And $\mathcal{X}_0$ has non-trivial deformations, even if the situation may at first sight be rigid, due to the fact that no other elliptic curve than $\mathbb{E}$ admits $\omega$ as an automorphism. A CR submersion close to but different from $\mathcal{X}_0$ is encoded by a path $c$ in $\mathbb{H}$ with

$$c(1) = \omega\cdot c(0).$$

Such a structure is of course non-constant (in the sense that the fibers of the CR submersion are not all the same) and this explains how it is possible that the monodromy is not a biholomorphism of any fiber (cf. [M-V]). This is indeed an example of a non-versal Kuranishi family.

Observe that we are in the special case where the dimension of $K_{c_0}$ is one more than the dimension of the Kuranishi space it is constructed with. This extra-dimension comes from the fact that we need $\mathcal{X}_0$ to be represented by a path and not a loop in $K_{c_0}$. Without this trick, one should take as $K_{c_0}$ the quotient of $\mathbb{D}_\omega$ by the action generated by $\omega$; but we should then consider $K_{c_0}$ as *an orbifold.*

More generally, if we take as $\mathcal{X}_0$ a CR bundle with fiber $X_0$ a compact complex 2-torus and monodromy an automorphism of $X_0$ non-isotopic to the identity and non-periodic (such pairs exist, see [G-V]), the same construction yields as $K_{c_0}$ the product of the Kuranishi space $K_0$ of $X_0$, an open set of $\mathbb{C}^4$, with an annulus. Forgetting the extra-dimension, one should take the quotient of $K_0$ by the action generated by this automorphism; but we should then consider a non-Hausdorff space. Once again, our Kuranishi space is not versal.

On the contrary, when the monodromy of the fiber $X_0$ is not isotopic to the identity but extends as an automorphism of any manifold in the Kuranishi space $K_0$ of $X_0$, we may take $K_0$ as $K_{c_0}$ and gain one dimension with respect to our construction. However, with this "reduced" $K_{c_0}$, there is no equivalent to Theorem 3, because the space $\mathcal{K}^g$ of (69) is not complete. There is no hope to obtain a non-trivial CR bundle as pull-back by a *constant map*.

Hence, we see that in all these examples, our space $K_{c_0}$ is not versal but it is minimal with respect to properties of Theorems 2 and 3.

## 4. Hopf surfaces.

Consider the quotient of

$$\mathbb{C}^2 \setminus \{(0,0)\} \times \mathbb{S}^1$$

by the action generated by the map

$$(z, w, t) \longmapsto (2z + a(t)w^2, (2 + b(t))w, t) \tag{90}$$

where $a$ and $b$ are two smooth functions from $\mathbb{S}^1 \subset \mathbb{C}$ into $\mathbb{R}_{\geq 0}$ satisfying

$$a(-1) = a(1) = 0 \qquad \text{and} \qquad a(t) > 0 \text{ for } t \neq -1, 1 \tag{91}$$

and

$$b(1) = 0,\ b(-1) = 2 \qquad \text{and} \qquad 0 < b(t) < 2 \text{ for } t \neq -1, 1. \tag{92}$$

This defines a smoothly trivial CR submersion $\mathcal{X}_0$ over the circle with fibers (primary) Hopf surfaces. As usual, we denote by $X_t$ the fiber over $t$.

The following facts are well-known (cf. [Da] or [We]).

(i) The Hopf surface $X_1$ is the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by the linear action generated by the matrix $2Id$. Its Kuranishi space is smooth of dimension four and a Kuranishi domain can be identified with an open neighborhood $V$ of $2Id$ in $GL_2(\mathbb{C})$ under the correspondence

$$A \in V \longmapsto X_A := \left(\mathbb{C}^2 \setminus \{(0,0)\}\right) / \langle A \rangle. \tag{93}$$

(ii) The Hopf surface $X_{-1}$ is the quotient of the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by the linear action generated by the diagonal matrix with eigenvalues 2 and 4. Because 4 is the square of 2, we are in a resonant case and its Kuranishi space is a bit different from the previous one. It is smooth of dimension three and a Kuranishi domain can be identified with an open neighborhood $W$ of $(2,4,0)$ in $\mathbb{C}^3$ under the correspondence

$$(\alpha,\beta,s)\in W \longmapsto X_{(\alpha,\beta,s)} := \left(\mathbb{C}^2\setminus\{(0,0)\}\right)/\langle(z,w)\mapsto(\alpha z+sw^2,\beta w)\rangle. \tag{94}$$

(iii) For $t\neq -1,1$, the Hopf surface $X_t$ is biholomorphic to the linear Hopf surface defined by the matrix

$$A_\lambda(t)=\begin{pmatrix}2 & \lambda\\ 0 & 2+b(t)\end{pmatrix} \tag{95}$$

for any choice of $\lambda\in\mathbb{C}$. Its Kuranishi space is smooth of dimension two and a Kuranishi domain can be identified with an open neighborhood of $A_0(t)$ in the space of *diagonal* matrices.

(iv) The Kuranishi domain $V$ can be assumed to contain all matrices $A_{a(t)}(t)$ for $t$ different from $-1$. In the same way, the Kuranishi domain $W$ can be assumed to contain all triple $(2,2+b(t),a(t))$ for $t$ different from 1.

Taking these facts into account, we see that it is enough to choose adequately $V$ and $W$ and to glue $V$ to $W\times\mathbb{C}$ along a well-chosen open set to construct $K_{c_0}$.

To be more precise, consider the closed path

$$t\in\mathbb{S}^1\cap\{\Re t\geq 0\}\longmapsto c^+(t):=A_{a(t)}(t)\in GL_2(\mathbb{C}). \tag{96}$$

Let $\|-\|$ be the standard euclidian norm on $\mathbb{C}^4$. It induces a norm on $V\subset GL_2(\mathbb{C})$ by identifying $GL_2(\mathbb{C})$ to an open set of $\mathbb{C}^4$. Let $\epsilon$ be a positive real number and define

$$V=\{A\in GL_2(\mathbb{C})\quad|\quad \|A-c^+(t)\|<\epsilon \text{ for some } t\in\mathbb{S}^1\cap\{\Re t\geq 0\}. \tag{97}$$

Symmetrically, consider the closed path

$$t\in\mathbb{S}^1\cap\{\Re t\leq 0\}\longmapsto c^-(t):=(2,2+b(t),a(t),0)\in\mathbb{C}^4. \tag{98}$$

Define

$$W=\{A\in\mathbb{C}^4\quad|\quad \|A-c^-(t)\|<\epsilon \text{ for some } t\in\mathbb{S}^1\cap\{\Re t\leq 0\}. \tag{99}$$

Assume that $\epsilon$ is small enough so that

(i) The open sets $V$ and $W$ are Kuranishi domains.

(ii) The subset

$$V_0=\{A\in GL_2(\mathbb{C})\quad|\quad \|A-c^+(t)\|<\epsilon \text{ for } t=\pm i\}\subset V$$

has two connected components and any point $A=(A_{ij})_{i,j=1,2}$ of $V_0$ satisfies

$$0<|\lambda_1(A)|<|\lambda_2(A)|<4 \qquad\text{and}\qquad |A_{21}|>0 \tag{100}$$

where $\lambda_1(A)$ (respectively $\lambda_2(A)$) is the smallest (respectively biggest) eigenvalue of $A$.

(iii) The subset

$$W_0 = \{A \in \mathbb{C}^4 \quad | \quad \|A - c^-(t)\| < \epsilon \text{ for } t = \pm i\} \subset W$$

has two connected component and any point $A = (A_i)_{i=1,\ldots,4}$ of $W_0$ satisfies

$$0 < |A_1| < |A_2| < 4 \qquad \text{and} \qquad |A_3| > 0. \tag{101}$$

Then, because of (101) and (100), the map

$$A \in V_0 \longmapsto (\lambda_1(A), \lambda_2(A), A_{21}, A_{12}) \in W_0 \tag{102}$$

is a biholomorphism. Because of the facts recalled above, (102) is an a.c. isomorphism. Gluing $V$ to $W$ through (102) gives the analytic space $K_{c_0}$ we are looking for. Here it is smooth of dimension four and it has the homotopy type of a circle. The paths (98) and (96) glue together to give the path $c_0$.

According to Theorem 4, the associated space $K^g$ is not versal at $c_0$. This is easy to see in this case. Modifying the path (98) by replacing the zero fourth coordinate with any bump function (small enough in modulus) and gluing it to (96) gives a path encoding $\mathcal{X}_0$ although it is different from $c_0$.

Moreover, taking a bump function depending on a smooth parameter and performing the same construction, one obtains a trivial deformation of $\mathcal{X}_0$ over the interval with injective image in $K^g$. This contradicts versality.

## References


[Bu] T. Burel. *Déformations de feuilletages à feuilles complexes*, Thèse de doctorat, Institut de Mathématiques de Bourgogne, Dijon 2011.

[Da] K. Dabrowski. *Moduli spaces for Hopf surfaces*, Math. Ann. **259** (1982), 201–225.

[Do1] A. Douady. *Le problème des modules pour les variétés analytiques complexes*, Sém. Bourbaki **77** (1964/65).

[Do2] A. Douady. *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier **16** (1966), 1–95.

[EK-S] A. El Kacimi, J. Slimène. *Cohomologie de Dolbeault le long des feuilles de certains feuilletages complexes*, Ann. Inst. Fourier **60** (2010), 727–757.

[F-K] O. Forster, K. Knorr. *Relativ-analytische Räume und die Kohärenz von Bildgarben*, Invent. Math. **16** (1972), 113–160.

[G-V] E. Ghys, A. Verjovsky. *Locally free holomorphic actions of the complex affine group.* Geometric study of foliations (Tokyo, 1993), 201–217, World Sci. Publ., River Edge, NJ 1994.

[G-H-S] J. Girbau, A. Haefliger, D. Sundararaman. *On deformations of transversely holomorphic foliations*, J. Reine Angew. Math. **345** (1983), 122–147.

[Ko] K. Kodaira. *Complex manifolds and deformation of complex structures.* Classics in Mathematics, Springer, Berlin 2005.

[K-S1] K. Kodaira, D.C. Spencer. *On deformations of complex analytic structures I*, Ann. of Math. **67** (1958), 328–402.

[K-S2] K. Kodaira, D.C. Spencer. *On deformations of complex analytic structures II*, Ann. of Math. **67** (1958), 403–466.

[Ku1] M. Kuranishi. *On locally complete families of complex analytic structures*, Ann. of Math. **75** (1962), 536–577.

[Ku2] M. Kuranishi. *New Proof for the Existence of Locally Complete Families of Complex Structures.* Proc. Conf. Complex Analysis (Minneapolis, 1964), 142–154, Springer, Berlin 1965

[Ku3] M. Kuranishi. *Deformations of Compact Complex Manifolds.* Les presses de l'université de Montréal, Montréal 1971.

[Me1] L. Meersseman. *Foliated structure of the Kuranishi space and isomorphisms of deformation families of compact complex manifolds*, Ann. Sci. Éc. Norm. Supér. **44** (2011), 495–525.

[Me2] L. Meersseman. *Feuilletages par variétés complexes et problèmes d'uniformisation*, Panoramas & Synthèses. To appear.

[M-V] L. Meersseman, A. Verjovsky. *On the moduli space of certain smooth codimension-one foliations of the* 5*-sphere*, J. Reine Angew. Math. **632** (2009), 143–202.

[N-N] A. Newlander, L. Nirenberg. *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. **65** (1957), 391–404.

[Sl] J. Slimène. *Deux exemples de calcul explicite de cohomologie de Dolbeault feuilletée*, Proyecciones **27** (2008), 63–80.

[Sc] M. Schneider. *Halbstetigkeitssätze für relativ analytische Räume*, Invent. Math. **16** (1972), 161–176.

[Su] D. Sundararaman. *Moduli, deformations and classifications of compact complex manifolds.* Research Notes in Mathematics **45**, Pitman, Boston, Mass.-London 1980.

[Wa1] J.J. Wavrik. *Obstructions to the existence of a space of moduli.* Global Analysis. Papers in honor of K. Kodaira, 403–414. Princeton University Press, Princeton 1969.

[Wa2] J.J. Wavrik. *Deforming cohomology classes*, Trans. Amer. Math. Soc. **181** (1973), 341–350.

[We] J. Wehler. *Versal deformation of Hopf surfaces*, J. Reine Angew. Math. **328** (1981), 22–32.



Laurent Meersseman
Institut de Mathématiques de Bourgogne
B.P. 47870, 21078 Dijon Cedex, France
and
Centre de Recerca Matemàtica
08193 Bellaterra, Spain
*E-mail address*: `laurent.meerssemanu-bourgogne.fr, Lmeerssemancrm.cat`