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FACTORING ROSENTHAL OPERATORS

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Abstract

In this paper we show that a Rosenthal operator factors through a Banach space containing no isomorphs of l_1.

All spaces are to be Banach spaces. If X is a Banach space, X'' denotes the bidual of X, J_X is the canonical isometry embedding X in X'', B_X is the closed unit ball of X and I_X denotes the identity operator on X. Let $\mathrm{L}(X,Y)$ be the set of all continuous linear operators from X into Y.

Recall that an operator $T \in \mathrm{L}(X,Y)$ is said to be *completely continuous* if every weakly convergent sequence (x_n) is mapped into a norm convergent sequence (Tx_n). The class of all completely continuous operators from X into Y is denoted by $\mathrm{Cc}(X,Y)$ and $\mathrm{Co}(X,Y)$ will denote the space of all compact operators from X into Y.

Let $T \in \mathrm{L}(X,Y)$. Then T is called a *Rosenthal operator* if $ST \in \mathrm{Co}(X,Y_0)$ for all $S \in \mathrm{Cc}(Y,Y_0)$, where Y_0 is an arbitrary Banach space. Using a theorem which is due to Rosenthal [**8**] and Dor [**2**] in the real and complex case, respectively, one gets that T is a Rosenthal operator if and only if it maps bounded sequences into sequences possessing weak Cauchy subsequences (see for example [**6**]), and hence T is a Rosenthal operator if and only if $\mathrm{T}(B_X)$ is $\sigma(Y'',Y')$-relatively sequentially compact in Y''.

The Rosenthal operators are called weak Cauchy operators by [**5**] and conditionally weakly convergent operators by [**7**].

An operator $T \in \mathrm{L}(X,Y)$ is called a *ℓ_1-singular operator* if for each $S \in \mathrm{L}(\ell_1,X)$, the composition TS is not an isomorphic embedding [**5**]. So, $T \in \mathrm{L}(X,Y)$ is a Rosenthal operator if and only if is a ℓ_1-singular operator. $\mathrm{Ro}(X,Y)$ will denote the class of all Rosenthal operators from X into Y.

Let $N\ell_1$ be the space ideal of all Banach spaces containing no isomorphic copy of ℓ_1 and let $\mathrm{Op}(N\ell_1)$ be the operator ideal of all operators which factor through spaces in $N\ell_1$.

In this note we shall prove that $\mathrm{Op}(N\ell_1)$ coincides with the class of all Rosenthal operators. For this end we shall use the construction of Davies-Figiel-Johnson-Pelczynski [**1**]. Let W be a convex, symmetric and bounded subset of a Banach space X. For $n = 1,2,3,\dots$, the Minkowski functional $\|\ \|_n$ of the set $U_n = 2^n W + 2^{-n} B_X$ is a norm equivalent to $\|\ \|$. Define, for $x \in X$, $|||\, x\, ||| = (\sum_{n=1}^{\infty} \|\, x\, \|_n^2)^{\frac{1}{2}}$; let $Z = \{x \in X : |||\, x\, ||| < \infty\}$ and let j denote the identity embedding of Z into X; then $(Z, |||\ |||)$ is a Banach space and j is continuous [**1**, lemma 1, (ii)].

Theorem. *Let X, Y be Banach spaces and let $T \in \mathrm{L}(X,Y)$. Then the following properties are equivalent:*

(i) $T \in \mathrm{Ro}(X,Y)$

(ii) $T \in \mathrm{Op}(N\ell_1)(X,Y)$.

Proof: (i) $\Longrightarrow$ (ii). Suppose that $T \in \mathrm{Ro}(X,Y)$. With reference to [**1**], put $W = T(B_X)$, $M = \{y \in Y : |||\, y\, ||| < \infty\}$ and let j denote the identity embedding of M into Y. Let (y_n) be a sequence of elements of W, then (y_n) has a subsequence (y_{n_j}) such that $(J_Y\, y_{n_j})$ is (Y'', Y')-convergent in Y''. Hence $J_Y W$ is $\sigma(Y'', Y')$-sequentially compact in Y'' and so by virtue of [**1**, lemma 1, (xii)], $J_M B_M$ is $\sigma(M'', M')$-sequentially compact in M''. This implies that $M \in N\ell_1$.

The operators $j^{-1}T : X \longrightarrow M$ and $j : M \longrightarrow Y$ provide the required factorization.

(ii) $\Longrightarrow$ (i). It is trivial. ■

Remark 1. In a 1980 paper [**4**], S. Heinrich showed that if $T \in \mathrm{Ro}(X,Y)$ then $T \in \mathrm{Op}(N\ell_1)(X,Y)$ proving that the operator ideal of all Rosenthal operators is injective surjective and satisfy the Σ_p-condition for $1 < p < \infty$ .That is, for arbitrary Banach spaces $X_n, Y_n, n \in N$, the followings holds:

If $T \in \mathrm{T}((\Sigma X_n)_p, (\Sigma Y_n)_p)$, and $Q_n T P_m \in \mathrm{Ro}(X_m, Y_n)$, $n, m \in N$ then $T \in \mathrm{Ro}((\Sigma X_n), (\Sigma Y_n))$ where P_m and Q_n denote the projections of $(\Sigma X_n)_p, (\Sigma Y_n)_p$ onto the coordinates X_m and Y_n, respectively.

Remark 2. Other characterizations of the Rosenthal operators were obtained by A. Fakhoury [**3**].

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