ON THE JACOBIAN CRITERION OF FORMAL SMOOTHNESS

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Abstract

We give a short proof of the jacobian criterion of formal smoothness using the Lichtenbaum–Schlessinger cotangent complex.

The aim of this note is to get a proof of the following jacobian criterion of formal smoothness:

Theorem 1. Let $A$ be a ring, $B$ a noetherian $A$–algebra, $J$ an ideal of $B$, and $C = B/J$. Let us consider over $A$ and $B$ the discrete and $J$–adic topology, respectively. Then, the following statements are equivalent

1) The $A$–algebra $B$ is formally smooth

2) For every representation $B \simeq R/I$, where $R$ is a smooth $A$–algebra, the canonical homomorphism

$$I/I^2 \otimes_B C \rightarrow \Omega_{R|A} \otimes_R C$$

is left invertible.

This theorem has been obtained by M. André [1, prop. 16.17] using simplicial methods. We shall get a more elementary proof, based on the Lichtenbaum–Schlessinger (co–)homology theory [3], and a counter-example showing that this result is not true for arbitrary $B$.

First we recall the definition of the Lichtenbaum–Schlessinger (co–)homology functors. Let $A$ be a ring, $B$ an $A$–algebra and $M$ a $B$–module. Choose a polynomial algebra $R$ over $A$ such that $B \simeq R/I$, and a free $R$–module $F$ such that there exists an exact sequence of $R$–modules

$$0 \rightarrow U \rightarrow F \xrightarrow{j} I \rightarrow 0.$$

Let $U_0$ be the image of the homomorphism $\phi : F \otimes_R F \rightarrow F$, $\phi(x \otimes y) = j(x)y - j(y)x$ and consider the complex of $B$–modules

$$L_{B|A} : 0 \rightarrow U/U_0 \rightarrow F/IF \rightarrow \Omega_{R|A} \otimes_R B \rightarrow 0.$$
Then, $T_i(B|A,M) = H^i(L_{B|A} \otimes_B M)$, $T^1(B|A,M) = H^1(\text{Hom}_B(L_{B|A}, M))$, $i = 0, 1, 2$.

There exist isomorphisms: $T_0(B|A,M) \simeq \Omega_{B|A} \otimes_B M$, $T^0(B|A,M) \simeq \text{Der}_A(B,M) \simeq \text{Hom}_B(\Omega_{B|A}, M)$, $T^1(B|A,M) \simeq \text{Exalcom}_A(B,M)$ the set of equivalence classes of infinitesimal extensions of $B$ over $A$ by $M$, and $T^1(B|A,M) \simeq \text{Hom}_B(I/I^n, M)$ if $B \simeq A/I$.

**Proposition 1.** Let $A$ be a ring, $B$ an $A$-algebra, $J$ an ideal of $B$ and $C = B/J$. Assume that $B \simeq R/I$, where $R$ is a smooth $A$-algebra. Then, the following conditions are equivalent

1) The canonical homomorphism $\frac{I}{I^2} \otimes_B C \rightarrow \frac{\Omega_{R|A} \otimes_R C}{\Omega_{B|A} \otimes_B C}$ is left invertible
2) $T_1(B|A,C) = 0$ and $\Omega_{B|A} \otimes_B C$ is a projective $C$-module
3) $T_1(B|A,M) = 0$ for all $C$-module $M$.

**Proof:** Since $R$ is a smooth $A$-algebra, we have $T_1(R|A, -) = 0 = T^1(R|A, -)$ [3, prop. 3.1.3]. Hence there exist exact sequences [3, 2.3.5]

$0 \rightarrow T_1(B|A,C) \rightarrow \frac{I}{I^2} \otimes_B C \rightarrow \frac{\Omega_{R|A} \otimes_R C}{\Omega_{B|A} \otimes_B C} \rightarrow 0$

$0 \rightarrow \text{Hom}_C(\Omega_{B|A} \otimes_B C, M) \rightarrow \text{Hom}_C(\Omega_{R|A} \otimes_R C, M) \rightarrow \text{Hom}_C(\frac{I}{I^2} \otimes_B C, M) \rightarrow T^1(B|A,M) \rightarrow 0$,

where $M$ is a $C$-module.

The result follows from this sequences having into account that $\Omega_{R|A} \otimes_R C$ is a projective $C$-module [3, prop. 3.1.3].

For every $C$-module $M$, the homomorphisms $A \rightarrow B \rightarrow B_n = B/J^n$ induce an exact sequence

$T^1(B_n|B,M) \rightarrow T^1(B_n|A,M) \rightarrow T^1(B|A,M) \rightarrow T^2(B_n|B,M)$

and, therefore, an exact sequence

$\lim T^1(B_n|B,M) \rightarrow \lim T^1(B_n|A,M) \rightarrow T^1(B|A,M) \rightarrow \lim T^2(B_n|B,M)$.

The formal smoothness of $B$ over $A$ is equivalent; by [2, prop. 19.4.4], to the vanishing of $\lim T^1(B_n|A,M)$ for all $C$-module $M$. Then, Theorem 1 is a consequence of Proposition 1 and

**Proposition 2.** Let $A$ be a noetherian ring, $I$ an ideal of $A$, $A_n = A/I^n$, and $M$ an $A/I$-module. Then

i) $\lim T^1(A_n|A,M) = 0$.

ii) $\lim T^2(A_n|A,M) = 0$. 
Part i) is easy: \[ \lim T^1(A,|A, M) \simeq \lim \text{Hom}_{A/I^n}(I^n/I^{2n}, M) \simeq \lim \text{Hom}_{A/I}(I^n/I^{n+1}, M) = 0. \]
To prove part ii) we need the following lemmas.

**Lemma 1.** Let \( A \) be a ring, \( I \) an ideal of \( A \), \( B = A/I \), and \( M \) a \( B \)-module. Then, there exists a natural monomorphism of \( B \)-modules \( T^2(B|A, M) \rightarrow \text{Ext}^1_A(I, M) \).

**Proof.** Let \( F \) be a free \( A \)-module such that there exists an exact sequence of \( A \)-modules \( 0 \rightarrow U \rightarrow F \rightarrow I \rightarrow 0 \). Let \( U_0 \) be the image of the homomorphism \( \phi : F \otimes_A F \rightarrow F, \phi(x \otimes y) = j(x)y - j(y)x \). Then, \( T^2(B|A, M) = \text{Coker}(\text{Hom}_A(F, M) \rightarrow \text{Hom}_A(U/U_0, M)) \). The result follows from the diagram of exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Hom}_A(F, M) & \rightarrow & \text{Hom}_A(U/U_0, M) & \rightarrow & T^2(B|A, M) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \text{Hom}_A(F, M) & \rightarrow & \text{Hom}_A(U, M) & \rightarrow & \text{Ext}^1_A(I, M) & \rightarrow & 0
\end{array}
\]

**Lemma 2.** Let \( A \) be a noetherian ring, \( I \) an ideal of \( A \) and \( n > 0 \) an integer number. Then, there exists \( s \geq n \) such that the canonical homomorphism

\[ \text{Tor}^1_A(I^n, A/I) \rightarrow \text{Tor}^1_A(I^n, A/I) \]

is trivial.

**Proof.** We have \( \text{Tor}^1_A(I^n, A/I) \simeq \text{Tor}^2_A(A/I^n, A/I) \simeq \text{Tor}^1_A(A/I^n, I) \). Let \( 0 \rightarrow U \rightarrow F \rightarrow I \rightarrow 0 \) be an exact sequence of \( A \)-modules where \( F \) is free and of finite type. Then

\[ \text{Tor}^1_A(A/I^n, I) \simeq (U \cap I^nF)/I^nU. \]

By Artin-Rees lemma [4, th. 15] there exists \( r > 0 \) such that \( I^tF \cap U = I^{t-r}(I^rF \cap U) \) for \( t > r \). Taking \( s = n + r \) we obtain \( I^sF \cap U \subseteq I^nU \) and therefore

\[ \text{Tor}^1_A(I^s, A/I) \rightarrow \text{Tor}^1_A(I^n, A/I) \]

is trivial.

We now prove part ii) of Proposition 2. By lemma 1 there exists a monomorphism

\[ \lim \lim T^2(A,|A, M) \rightarrow \lim \text{Ext}^1_A(I^n, M). \]
On the other hand, for each $n$ we have an exact sequence

$$0 \to \text{Ext}^1_{A/I}(I^n/I^{n+1}, M) \to \text{Ext}^1_A(I^n, M) \to \text{Hom}_{A/I}(\text{Tor}^A_1(I^n, A/I), M)$$

which is deduced, for instance, from the change-rings spectral sequence

$$E_2^{pq} = \text{Ext}^p_{A/I}(\text{Tor}^A_q(I^n, A/I), M) \Rightarrow \text{Ext}^{p+q}(I^n, M).$$

Since $\lim \text{Ext}^1_{A/I}(I^n/I^{n+1}, M) = 0$ and $\lim \text{Hom}_{A/I}(\text{Tor}^A_1(I^n, A/I), M) = 0$, by lemma 2, we obtain $\lim \text{Ext}^1_A(I^n, M) = 0$. Therefore, $\lim T^2(A_n|A, M) = 0$.

Theorem 1 is not true for arbitrary $B$. To exhibit a counter-example, we need the following result.

**Lemma 3.** Let $A$ be a ring and $I \subseteq T$ two ideals of $A$ such that $T^2 = T$ and $IT \neq I$. Let $B = A/I$, $J = T/I$ and $C = B/J$. Then, the $A$-algebra $B$ is formally smooth for the $J$-adic topology, but there exists a $C$-module $M$ such that $T^1(B|A, M) \neq 0$.

**Proof.** We have $J^2 = (T^2 + I)/I = (T + I)/I = T/I = J$. Hence, for each $C = A/T$-module $M$

$$\lim T^1(B_n|A, M) \simeq T^1(C|A, M) \simeq \text{Hom}_{A/T}(T/T^2, M) = 0,$$

where $B_n = B/J^n$. Therefore, $B$ is formally smooth.

On the other hand $T^1(B|A, M) \simeq \text{Hom}_B(I/I^2, M) \simeq \text{Hom}_{A/T}(I/IT, M)$. Since $I/IT \neq 0$, we obtain $T^1(B|A, I/IT) \neq 0$.

Counter-example. $A = C(R, R)$ the ring of all real-valued continuous functions on $R$, $I$ – principal ideal of $A$ generated by the identity function, and $T =$ maximal ideal of $A$ containing $I$ (see [6], Ch. 2, § 2, Ex. 15).

This counter-example solves negatively a question of A. Brezuleanu [6, Remark 1.3 (i)].

**References**


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