INDUCTIVE LIMITS OF VECTOR-VALUED SEQUENCE SPACES

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Abstract

Let $L$ be a normal Banach sequence space such that every element in $L$ is the limit of its sections and let $E = \text{ind } E_n$ be a separated inductive limit of locally convex spaces. Then $\text{ind } L(E_n)$ is a topological subspace of $L(E)$.

The aim of this note is to prove the following result on the interchangeability of inductive limits and spaces of vector valued sequences: if $L$ is a normal Banach sequence space with the property that every element of $L$ is the limit of its sections and $E = \text{ind } E_n$ is a separated locally convex inductive limit, then the inductive limit $\text{ind } L(E_n)$ is a topological subspace of $L(E)$. The situation is completely different for the sequence space $L = 1^{\infty}$. In fact the first two authors showed in [2] that there are even strict inductive limits of Fréchet spaces $E = \text{ind } E_n$ such that the canonical injection $\text{ind } 1^{\infty}(E_n) \subset 1^{\infty}(E)$ is not open.

In what follows $(L, \|\|)$ denotes a normal Banach sequence space, i.e., a Banach space that satisfies

$(\alpha) \varphi \subset L \subset \omega$ algebraically and the inclusion $(L, \|\|) \subset \omega$ is continuous.

$(\beta) \forall a = (a_k)_{k \in \mathbb{N}} \in L, \forall b = (b_k)_{k \in \mathbb{N}} \in \omega$ such that $|b_k| \leq |a_k| \forall k \in \mathbb{N}$, we have that $b \in L$ and $\|b\| \leq \|a\|$. We will also assume the following property (cf [1])

$(\varepsilon) \lim_{n \to \infty} \|(0)_{k < n}, (a_k)_{k \geq n}\| = 0, \forall a = (a_k)_{k \in \mathbb{N}} \in L$.

This property is sometimes called AK-property. Clearly $(L, \|\|) = 1^{\infty}$ does not satisfy $(\varepsilon)$, whereas $(L, \|\|) = 1^p, 1 \leq p < \infty$ or $c_0$ has property $(\varepsilon)$.

We observe that there is $(\mu_k)_{k \in \mathbb{N}} \in L$ with $\mu_k > 0 (k \in \mathbb{N})$ and $\|\mu_k\| = 1$.

Given a locally convex space $E$, we denote by $cs(E)$ the family of all continuous seminorms on $E$. Given $E$ the vector valued sequence space $L(E)$ is defined by

$L(E) = \{z = (x_k)_{k \in \mathbb{N}} \in E^{\mathbb{N}}, (r(x_k))_{k \in \mathbb{N}} \in L \text{ for all } r \in cs(E)\}$
endowed with the locally convex topology defined by the seminorms

\[ x \rightarrow \| (r(x_k))_{k \in \mathbb{N}} \| \]

as \( r \) varies in \( cs(E) \). Clearly if \( (L, \| \|) \) satisfies property \((e)\), then the countable direct sum \( \oplus \{ E : n \in \mathbb{N} \} = E^{(\mathbb{N})} \) is dense in \( L(E) \).

Given a separated locally convex inductive limit \( E = \text{ind} E_n \) we are interested in the following question: is \( \text{ind} L(E_n) \) a topological subspace of \( L(E) \)? If \( (L, \| \|) = 1 \), a positive answer follows from a classical result of Grothendieck on projective tensor products (see e.g. [4]). If \( (L, \| \|) = c_0 \) the positive answer is a particular case of a result of Mujica [5,1,7]. We prove now that the answer is positive for arbitrary \( (L, \| \|) \) satisfying \((e)\).

1. Proposition. Let \( E \) be a locally convex space, \( F \) a closed subspace of \( E \) and \( q : E \to E/F \) the canonical surjection. The mapping \( Q : L(E) \to L(E/F) \) defined by \( Q((x_k)_{k \in \mathbb{N}}) : = (q(x_k))_{k \in \mathbb{N}} \) is open onto its image. If \( E \) is a Fréchet space then \( Q \) is also surjective.

Proof. Since \( E^{(\mathbb{N})} \) is a dense subspace of \( L(E) \) and \( Q(E^{(\mathbb{N})}) = (E/F)^{(\mathbb{N})} \), according to [4, 32, 5(3)] it is enough to show that \( Q : E^{(\mathbb{N})} \to (E/F)^{(\mathbb{N})} \) is open. To do this we fix \( r \in cs(E) \) and we show

\[ Q(\{ x \in L(E) ; x \in E^{(\mathbb{N})} \| (r(x_k))_{k \in \mathbb{N}} \| \leq 1 \}) \supset \{ \tilde{x} \in L(E/F) ; \tilde{x} \in (E/F)^{(\mathbb{N})} \| (\tilde{r}(\tilde{x_k}))_{k \in \mathbb{N}} \| \leq 2^{-1} \} \]

where \( \tilde{r}(z + F) := \inf \{ r(z + y) ; y \in F \} \) \( (z \in E) \) is the quotient seminorm. We fix \( (\mu_k)_{k \in \mathbb{N}} \in L, \mu_k > 0 (k \in \mathbb{N}), \| (\mu_k)_{k \in \mathbb{N}} \| = 1 \). Given \( \tilde{x} \in (E/F)^{(\mathbb{N})} \) with \( \| (\tilde{r}(\tilde{x_k}))_{k \in \mathbb{N}} \| \leq 2^{-1} \) we find \( 1 \in \mathbb{N} \) such that \( \tilde{x_k} = 0 \) for \( 1 \leq k \). For each \( k < 1 \) we select \( y \in F \) such that \( r(x_k + y_k) < \tilde{r}(x_k + F) + 2^{-1} \mu_k \). Then \( x = ((x_k + y_k)_{k \leq 1}, (0)_{1 < k}) \) belongs to \( E^{(\mathbb{N})} \subset L(E), Q(x) = \tilde{x} \) and \( \| (r(x_k + y_k)_{k \leq 1}, (0)_{1 < k}) \| \leq 1 \).

If \( E \) is also a Fréchet space, then \( Q(L(E)) \) is a Fréchet space dense in \( L(E/F) \). Consequently \( Q \) is surjective.

2. Proposition. Let \( (E_n)_{n \in \mathbb{N}} \) be a sequence of locally convex spaces. Then the map \( \psi : L(\oplus \{ E_n : n \in \mathbb{N} \}) \to \oplus \{ L(E_n) : n \in \mathbb{N} \} \) defined by

\[ \psi((x_n^k)_{n \in \mathbb{N}})_{k \in \mathbb{N}} : = ((x_n^k)_{k \in \mathbb{N}})_{n \in \mathbb{N}} \]

is a topological isomorphism.

Proof. Given \( x = ((x_n^k)_{n \in \mathbb{N}})_{k \in \mathbb{N}} \) in \( L(\oplus \{ E_n : n \in \mathbb{N} \}) \), to show that \( \psi(x) \in \oplus \{ L(E_n) : n \in \mathbb{N} \} \) it is enough to see that there is \( m \in \mathbb{N} \) such that \( x_n^k = 0 \) for all \( n \geq m, k \in \mathbb{N} \). If we assume the contrary we can find two strictly increasing
sequences \((k(j))_{j \in \mathbb{N}}\) and \((n(j))_{j \in \mathbb{N}}\) such that \(x_n^{k(j)} \neq 0\) for all \(j \in \mathbb{N}\) (recall that each \(x_n^k\) belongs to \(\oplus \{E_n : n \in \mathbb{N}\}\)). We select \((\lambda_k)_{k \in \mathbb{N}} \in \omega \setminus \mathbb{L}\) with \(\lambda_{k(j)} > 0\) for all \(j \in \mathbb{N}\) and \(\lambda_k = 0\) if \(k \notin \{k(j) ; j \in \mathbb{N}\}\). For all \(j \in \mathbb{N}\) we find \(r_j \in \text{cs}(E_n(j))\) with \(r_j(x_n^{k(j)})\) greater than \(\lambda_{k(j)}\). It is clear that \(r((z_n)_{n \in \mathbb{N}}) = \sum_{j=1}^{\infty} r_j(z_n(j))\) defines a continuous seminorm on \(\oplus \{E_n : n \in \mathbb{N}\}\). Therefore for \(z^k := (x_n^k)_{n \in \mathbb{N}} (k \in \mathbb{N})\), we have \((r(z^k)) \in L\). But \(r(z_n^{k(j)}) \geq r_j(x_n^{k(j)}) > \lambda_{k(j)}\), for all \(j \in \mathbb{N}\) and \(0 = \lambda_k \leq r(z^k)\) if \(k \notin \{k(j) ; j \in \mathbb{N}\}\). Consequently \((\lambda_k)_{k \in \mathbb{N}} \in L\), a contradiction. Therefore \(\psi\) is well defined. Clearly \(\psi\) is linear and injective.

To show that \(\psi\) is surjective, we take \(z = ((x_n^k)_{n \in \mathbb{N}})_{k \in \mathbb{N}}\) in \(\oplus \{L(E_n) : n \in \mathbb{N}\}\). Clearly \((x_n^k)_{n \in \mathbb{N}} \in \oplus \{E_n ; n \in \mathbb{N}\}\) for all \(k \in \mathbb{N}\), since \(x_n^k = 0\) for all \(n \geq m\) and \(k \in \mathbb{N}\). Given \(r \in \text{cs}(\oplus \{E_n ; n \in \mathbb{N}\})\) we can find \(r_n \in \text{cs}(E_n) (n \in \mathbb{N})\) with \(r(z) \leq \max (r_n(z_n) ; n \in \mathbb{N})\) for all \(z = (z_n) \in \oplus \{E_n ; n \in \mathbb{N}\}\). Therefore for all \(k \in \mathbb{N}\)

\[
\sum_{n=1}^{m} r_n(z_n^k) = \max (r_n(z_n^k) ; 1 \leq n \leq m) \leq \max (r_n(z_n^k) ; n \leq m)_k
\]

Since \((r_n(z_n^k)_{k \in \mathbb{N}}) \in L\) for \(1 \leq n \leq m\), we conclude \(y = ((x_n^k)_{n \in \mathbb{N}})_{k \in \mathbb{N}} \in L(\oplus \{E_n ; n \in \mathbb{N}\})\) and \(\psi(y) = z\).

Now the continuity of \(\psi^{-1} : \oplus \{L(E_n) ; n \in \mathbb{N}\} \rightarrow L(\oplus \{E_n ; n \in \mathbb{N}\})\) follows from the fact that its restriction to each \(L(E_n)\) is clearly continuous. Finally we show that \(\psi\) is continuous. To do this we consider \(r_n \in \text{cs}(E_n) (n \in \mathbb{N})\) and we observe that

\[
\sup_{n \in \mathbb{N}} \|((r_n(x_n^k))_{k \in \mathbb{N}})\| \leq \|(\sup_{n \in \mathbb{N}} (r_n(x_n^k))_{k \in \mathbb{N}})\|
\]

holds for every \(((x_n^k)_{n \in \mathbb{N}})_{k \in \mathbb{N}} \in L(\oplus \{E_n ; n \in \mathbb{N}\})\). $\blacksquare$

3. Theorem. Let \((L, \|\cdot\|)\) be a normal Banach sequence space with property (\(\varepsilon\)). Let \(E = \text{ind } E_n\) be a separated locally convex inductive limit. Then \(\text{ind } L(E_n)\) is a topological subspace of \(L(\text{ind } E_n)\).

Proof. We consider the following diagram

\[
\begin{array}{ccc}
L(\oplus \{E_n ; n \in \mathbb{N}\}) & \overset{Q_1}{\longrightarrow} & L(E) \\
\psi \uparrow & & \uparrow \phi \\
\oplus \{L(E_n) ; n \in \mathbb{N}\} & \overset{Q_2}{\longrightarrow} & \text{ind } L(E_n)
\end{array}
\]

where, for \(q_1 : \oplus \{E_n ; n \in \mathbb{N}\} \rightarrow E\) the canonical quotient map \(q_1((z_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} z_n\) we define \(Q_1((x_k)_{k \in \mathbb{N}}) = (q_1(x_k))_{k \in \mathbb{N}}\) for all \((x_k)_{k \in \mathbb{N}}\) in
$L(\oplus\{E_n; n \in \mathbb{N}\})$. $Q_2$ is the canonical quotient map and $\varphi$ is the canonical injection which is continuous. According to proposition 1, $Q_1$ is open onto its image. Certainly $Q_2$ is open and $\psi^{-1}$ is a topological isomorphism, according to proposition 2. Since the diagram is commutative, it follows that $\varphi$ is also open onto its image. Thus $\text{ind } L(E_n)$ is a topological subspace of $L(E)$.

4. Corollary. Let $(L, \|\|)$ be a normal Banach sequence space with property $(\varepsilon)$. Let $E = \text{ind } E_n$ be a strict inductive limit of locally convex spaces with $E_n$ closed in $E_{n+1}$ for all $n \in \mathbb{N}$. Then $L(E) = \text{ind } L(E_n)$ holds algebraically and topologically.

Proof: Only the algebraic identity needs a proof. It is clearly enough to show that for any $x = (x_k)_{k \in \mathbb{N}} \in L(E)$ there is $n \in \mathbb{N}$ with $x_k \in E_n$. If this is not satisfied we can find an increasing sequence $(k(n))_{n \in \mathbb{N}}$ in $\mathbb{N}$ such that $x_{k(n)} \notin E_n$, for all $n$ in $\mathbb{N}$. We select $(\gamma_k)_{k \in \mathbb{N}} \in \omega \setminus L$ with $\gamma_{k(n)} > 0$ $(n \in \mathbb{N})$ and $\gamma_k = 0$ if $k \notin \{k(n); n \in \mathbb{N}\}$. Now since $E_n$ is closed, there is $u_n \in E'$ with $u_n(x_{k(n)}) = \gamma_{k(n)}$ and $u_n|E_n = 0$. The equicontinuous sequence $(u_n)_{n \in \mathbb{N}}$ defines a continuous seminorm as follows:

$$
p(x) = \sup \{|u_n(x)|; n \in \mathbb{N}\}
$$

Thus $(p(x_k))_{k \in \mathbb{N}} \in L$, a contradiction, since $\gamma_k \leq p(x_k)$ for all $k \in \mathbb{N}$.

5. Remark: For an inductive limit $E = \text{ind } E_n$ and a normal Banach sequence space $(L, \|\|)$, the algebraic coincidence $L(E) = \text{ind } L(E_n)$ is a clearly equivalent to $\forall x \in L(E) \exists n \in \mathbb{N}$ with $x \in L(E_n)$. For instance if $(L, \|\|) = c_0$, then $L(E) = \text{ind } L(E_n)$ if and only if $E$ is a sequentially retractive (cf [3]).
References


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