ENERGY MACHINERIES ON A MANIFOLD;
APPLICATION TO THE CONSTRUCTION
OF NEW ENERGY REPRESENTATIONS
OF GAUGE GROUPS

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Abstract
The introduction of the concepts of energy machinery and energy structure on a manifold makes it possible a large class of energy representations of gauge groups including, as a very particular case, the ones known up to now. By using an adaptation of methods initiated by I.M. Gelfand, we provide a sufficient condition for the irreducibility of these representations.

Introduction

a) Let $X$ be a smooth Riemannian manifold, let $G$ be a compact semisimple Lie group with Lie algebra $g$, let $\mathcal{D}(X, G)$ be the gauge group consisting of all the $G$-valued and compactly supported smooth mappings on $X$, let $\mathcal{D}_1(X, g)$ be the nuclear space of all the $G$-valued and compactly supported smooth 1-forms on $X$, and let $b : \mathcal{D}(X, G) \rightarrow \mathcal{D}_1(X, g)$ be the Maurer-Cartan cocycle $g \rightarrow b(g) = dg.g^{-1}$.

An energy representation of $\mathcal{D}(X, G)$ is a unitary representation $\Pi$ of $\mathcal{D}(X, G)$ into the symmetric Fock space of some complex Hilbert space $(\mathcal{H}, ||||)$ such that:

(1) there exists a continuous morphism $\Phi$ from $\mathcal{D}_1(X, G)$ into $\mathcal{H}$;
(2) the spherical function $\varphi$ of $\Pi$ with respect to the vacuum vector $\text{EXPO}$ is given by

$$g \rightarrow \varphi(g) = \langle \Pi(g) \text{EXPO}, \text{EXPO} \rangle = \exp \left\{ -\frac{1}{2} ||\Phi(b(g))||^2 \right\}.$$
b) In [11] I have exposed why, from several points of view, such representations of gauge groups are interesting; at that time only two kinds of energy representations were known: those constructed by Ismagilov [7] and Gelfand et al. [16], [17] associated with Riemannian structures on $X$, and the ones constructed in [9] which are associated to non vanishing vector fields on $X$ (for more details, see the Historical Note in §7).

c) The present paper tries to bring a substantial answer (although incomplete) to the open questions exposed in [11]. More precisely, we introduce the concepts of energy machinery and of measurable field of such objects, which allow to endow some spaces related to $\mathcal{D}(X,g)$ with $\mathcal{D}(X,G)$-invariant Euclidean structures, namely the energy structures. Then, the Maurer-Cartan cocycle $b$ and the method exposed in [6], [7], [17] allow to associate to each energy structure an energy representation of $\mathcal{D}(X,G)$ (Theorem 1).

d) The last part of the paper (§§ 7, 8, 9) is devoted to extend the papers [16], [17], [18] and [2] in order to get a sufficient condition for the irreducibility of the energy representations with support $X$, the so-called $\Pi$-property, which turns the irreducibility into a result about a kind of strong disjointness of Gaussian measures closely connected with the restriction of the energy representations to $\mathcal{D}(X,T)$, where $T$ is some Cartan subgroup of $G$ (Theorem 3).

1. Preliminaries and Notations

a) In all this paper $X$ is a $n$-dimensional smooth Riemannian manifold; as usual, $TX$ specifies its tangent bundle, and $T^*X$ its contangent bundle; $\mathcal{D}(T^*X)$ is the space of real valued and compactly supported smooth 1-forms on $X$ endowed with the Schwartz topology of compactly supported smooth sections; it is well-known that $\mathcal{D}(T^*X)$ is a LF-space, that is to say a real nuclear space, inductive limit of Fréchet spaces (see e.g. [15]).

b) The set $\mathcal{R}(X)$ of Riemannian structures on $X$ is by definition the subset of $C^\infty(S^2T^*X)$ consisting of sections $\tau$ which induce a positive definite inner product $q_{\tau,x}$ on each tangent space $T_xX$, and therefore a volume measure $dv^\tau$ on $X$. Let $\tau$ be in $\mathcal{R}(X)$ and let $r$ be in the set $C^1_+(X)$ of all the strictly positive $C^1$-functions on $X$; an element $x$ of $X$ being given, one gets a positive definite inner product $q_{\tau,x}^*$ on the cotangent space $T^*_xX$ by

$$q_{\tau,x}^*(u_x,v_x) = tr(u_x^*,v_x), \quad (u_x,v_x) \in (T^*_xX)^2,$$

where $u_x^* : H \to T^*_xX$ denotes the adjoint of $u_x$ with respect to $q_{\tau,x}$. One gets also a positive definite inner product $<,>$ on the space $\mathcal{D}(T^*X)$ of all the compactly supported real smooth 1-forms on $X$, given by:

$$<\omega,\omega'>_{\tau,x} = \int_X q_{\tau,x}^*(\omega(x),\omega'(x))r(x)dv^\tau(x).$$
c) Let \( F = \bigcup_{x \in X} F_x \) be a smooth subbundle of \( TX \), and let \( Y \) be a connected smooth submanifold of \( X \). \( F^Y \) will be the restricted subbundle \( \bigcup_{x \in Y} F_x \), and \( M^+(Y) \) will be the set of strictly positive measures on \( Y \). An element \( \omega \) in \( \mathcal{D}(T^*X) \) being given, \( \omega^F \) will be the mapping from \( X \) into the dual bundle \( F^* = \bigcup_{x \in X} F_x^* \) of \( F \), such that for all \( x \) in \( X \), \( \omega^F(x) \) is the restriction of \( \omega(x) \) to \( F_x \), and \( \omega^F \) will be the restriction of \( \omega^F \) to the submanifold \( Y \). An euclidean structure \( q \) on \( F^Y \) is a family \( q = (q_x)_{x \in Y} \) such that for all \( x \) in \( Y \), \( q_x \) is a scalar product on \( F_x \); of course, one gets a scalar product \( q_x^* \) on \( F_x^* \), \( x \in Y \), by:

\[
q_x^*(u_x, v_x) = \text{tr}(u_x^* \cdot v_x), \quad (u_x, v_x) \in F_x^* \times F_x^*,
\]

where \( u_x^* \) denotes the adjoint of \( u_x \) with respect to the scalar product \( q_x \) on \( F_x \).

d) Let \( \tau \) be an element of \( \mathcal{R}(X) \), let \( F \) be a smooth subbundle of \( TX \), and let \( Y \) be a submanifold of \( X \). \( F^\perp = \bigcup_{x \in X} F_x^\perp \), where \( F_x^\perp \) is the orthogonal complement of \( F_x \) with respect to the scalar product \( q_x \). \( \tau \)-orthogonal complement of \( F \), and we have a canonical identification of the space \( \{ \omega^F / \omega \in \mathcal{D}(T^*X) \} \) with the subspace of \( \mathcal{D}(T^*X) \):

\[
(2) \quad \mathcal{D}(T^*X)^F = \{ \omega \in \mathcal{D}(T^*X) / \omega^{F^\perp} = 0 \}.
\]

From proposition 50.1 of [15] it follows that \( \mathcal{D}(T^*X)^F \), as subspace of \( \mathcal{D}(T^*X) \), is a real nuclear space, inductive limit of Fréchet spaces.

\( \mathcal{D}(T^*X)^F \) will be the space \( \{ \omega^F / \omega \in \mathcal{D}(T^*X)^F \} \); one has also:

\[
(3) \quad \mathcal{D}(T^*X)^F = \{ \omega^F / \omega \in \mathcal{D}(T^*X) \}.
\]

2. Energy Machineries on the Manifold \( X \)

Definition 1. An energy machinery on the manifold \( X \) is a quadruplet \( \varepsilon = (Y, d\nu, F, q) \) such that:

1. \( Y \) is a connected submanifold of \( X \) endowed with a strictly positive measure \( d\nu \);
2. \( F \) is a smooth subbundle of \( TX \), and \( q = (q_x)_{x \in Y} \) is an Euclidean structure on \( F^Y \);
3. for all \( \omega, \omega' \) in \( \mathcal{D}(T^*X) \) the mapping:

\[
x \mapsto q_x^*(\omega^F(x), \omega'^F(x)), \quad x \in Y
\]

is \( d\nu \)-integrable.

An obvious example of energy machinery is the following:

Let \( \tau \) be in \( \mathcal{R}(X) \), and let \( \rho \) be in \( C_+(X) \); it follows from (1) that \( (X, \rho d\nu^*, TX, (q_{\tau, x})_{x \in X}) \) is an energy machinery. We shall give later on numerous other examples.
Lemma 1. Let $\mathcal{E} = (Y, dv, F, q)$ be an energy machinery on $X$.

(i) The mapping: $(\omega^F, \omega'^F) \rightarrow (\omega^Y, \omega'^Y)_e = \int_Y q^*_e(\omega^F(x), \omega'^F(x))dv(x)$ is a positive definite inner product on $D(T^*X)^F$.

(ii) The mapping: $(\omega, \omega') \rightarrow (\omega, \omega')_e = (\omega^Y, \omega'^Y)_e$ is a positive inner product on $D(T^*X)^F$.

(iii) The mapping: $(\omega^F, \omega'^F) \rightarrow (\omega^Y, \omega'^Y)_e$ is a positive inner product on $D(T^*X)^F$.

Proof: The lemma is an obvious consequence of the above definition. 

3. Complete $\tau$–Energy Machineries on $X$

a) Definition 2. Let $\tau$ be in $\mathcal{R}(X)$; the energy machineries of the form $\mathcal{E} = (X, rdv^\tau, F, q)$ are called the complete $\tau$–energy machineries on $X$.

From Lemma 1, it follows easily:

Lemma 2. Let $\mathcal{E}$ be a complete $\tau$–energy machinery of the form $\mathcal{E} = (X, rdv^\tau, TX, q)$. The mapping:

$$(\omega, \omega') \rightarrow (\omega, \omega')_e = \int_X q^*_e(\omega(x), \omega'(x))r(x)dv^\tau(x)$$

is a positive definite inner product on $D(T^*X)$.

Definition 3. The complete $\tau$–energy machineries of the form $\mathcal{E} = (X, rdv^\tau, TX, q)$ are called regular complete $\tau$–energy machineries.

$(X, rdv^\tau, TX, (q_\tau, x)_x \in X)$ is the basic example of regular complete $\tau$–energy machineries and for a given subbundle $F \neq (0)$ of $TX$, $(X, rdv^\tau, F; (q_\tau, x)_x \in X)$ is a basic example of complete $\tau$–energy machineries.

We have to give now other examples.

b) Let us suppose that the Euler number $e(X)$ of $X$ is zero; as it is well-known (see e.g. [14], § 39.8) we can find an integer $k$, $1 \leq k \leq n$, and a $k$–frame:

$$\zeta = (\xi_1, \ldots, \xi_k)$$

of smooth vector fields on $X$ which generates a smooth subbundle:

$$(4) \quad F(\xi) = \bigcup_{x \in X} P_x(\xi) \text{ with } F_x(\xi) = \bigoplus_{i=1}^k \mathcal{R}\xi_i(x);$$

we endow $F(\xi)$ with the Euclidean structure $g^\xi = (g^\xi_2)_{x \in X}$ such that:

$$g^\xi_i(\sum_{i=1}^k a_i \xi_i(x), \sum_{i=1}^k b_i \xi_i(x)) = \sum_{i=1}^k a_i b_i;$$

it follows that, an element $\tau$ of $\mathcal{R}(X)$ and an element $r$ of $C^1_\tau(X)$ being given, for all $\omega, \omega'$ in $D(T^*X)$ the mapping:

$$\tau \rightarrow (g^\xi_2)^*(\omega^F(\xi)(x), \omega'^F(\xi)(x))$$

is $rdv^\tau$–integrable on $X$. So:
Lemma 3. Let \( r \in \mathcal{R}(X) \), let \( r \in C^1_+(X) \), and let us suppose that \( X \) has a \( k \)-frame of smooth vector fields \( \xi = (\xi_1, \ldots, \xi_k) \), \( k \geq 1 \). \((X, r dv^r, F(\xi), q^\xi)\) is a complete \( r \)-energy machinery.

c) There exists a universal way to get complete \( r \)-energy machineries: let \( r \in \mathcal{R}(X) \), let \( r \in C^1_+(X) \) and let \( F \neq (0) \) be a smooth subbundle of \( TX \); \( k \) will be the common dimension of the fibers \( F_x \) of \( F \). Let us consider the corresponding infinite Grassmann manifold \( G_k(\mathbb{R}^\infty) \), and let \( \gamma_k \) be universal bundle with base \( G_k(\mathbb{R}^\infty) \) (see e.g. [12], § 5.8). The set of continuous bundle morphisms from \( F \) into \( \gamma_k \) is not empty ([12], theorem 5.6). Let \( \theta \) be such a morphism, and for each \( x \) in \( X \), let \( \theta_x \) be the corresponding linear isomorphism from \( F_x \) onto some \( k \)-dimensional subspace of \( \mathbb{R}^\infty \), \( \mathbb{R}^\infty \) being endowed with its canonical scalar product \( < , > \); one gets a scalar product \( q^\theta_x \) on \( F_x \) by \( q^\theta_x(u, v) = < \theta_x(u), \theta_x(v) > \). Let \( q^\theta \) be the family \((q^\theta_x)_{x \in X}\). From the continuity of \( \theta \), it follows that for all \( \omega, \omega' \) in \( D(T^*X) \) the mapping \( x \to (q^\theta_x)^*(\omega^F(x), \omega'^F(x)) \) is \( rdv^r \)-integrable on \( X \), and then \((X, r dv^r, F, q^\theta)\) is a complete \( r \)-energy machinery.

d) Let \( \varepsilon = (X, r dv^r, F, q^\varepsilon) \) be a complete \( r \)-energy machinery, and let \( \Gamma^1_+(X) \) be the set of global sections \( \sigma \) of class \( C^1 \) of the bundle \( Hom(F, F) \), such that, for all \( x \) in \( X \), \( \sigma(x) \) is a positive definite operator on \( F_x \). Each element \( \sigma \) of \( \Gamma^1_+(X) \) gives rise to a new scalar product \( \sigma(q)^x \) such that \( \sigma(q)^x(u, v) = q_x(\sigma(x)u, \sigma(x)v) \). From the continuity of \( \sigma \) it follows that \((X, r dv^r, F, \sigma(q))\), with \( \sigma(q) = (\sigma(q)^x)_{x \in X} \), is a complete \( r \)-energy machinery.

e) Let us suppose that \( \dim(X) \geq 2 \), let \( r \) be in \( \mathcal{R}(X) \), and let \( F \) be a proper smooth subbundle of \( TX \), i.e. \( F \neq (0) \) and \( F \neq TX \). \( \varepsilon = (X, r dv^r, F, q^F_r) \) is a complete \( r \)-energy machinery, where \( q^F_r \) is the Euclidean structure coming from \( r \) and restricted to \( F \). Now let us consider the orthogonal subbundle \( F^\perp \), and let \( r' \) be another Riemannian structure on \( X \); \( q'^{F, r'}_r \) will denote the restriction to \( F^\perp \) of the Euclidean structure coming from \( r' \).

One gets a new Euclidean structure \( q^{r, F, r'}_x = (q^{r, F} x, q^{F, r'}_x)_{x \in X} \) on \( TX = F \oplus F^\perp \) such that, for all \( x \) in \( X \):

\[
q^{r, F, r'}_x(u_x + u^\perp_x, v_x + v^\perp_x) = q^F_r(u_x, v_x) + q^{F^\perp}_r(u^\perp_x, v^\perp_x).
\]

with \((u_x, v_x)\) in \( F_x x F_x \), and \((u^\perp_x, v^\perp_x)\) in \( F^\perp_x x F^\perp_x \).

From the fact that one has the orthogonal sum:

\[
D(T^*X) = D(T^*X)^F \oplus F(T^*X)^{F^\perp}
\]

with respect to \( q_r \), corresponding to the decomposition \( \omega \to \omega^F + \omega^{F^\perp} \), it follows that for all pair \((\omega, \omega')\) of elements in \( D(T^*X) \), and each \( x \) in \( X \) one has:

\[
(q^{r, F, r'}_x)^*(\omega(x), \omega'(x)) = \omega^F(x) + (q^{F^\perp}_x)^*(\omega^{F^\perp}(x), \omega^{F^\perp}(x)).
\]
As \((X, r\nu^r, F, q^F_r)\) is an energy machinery, the mapping \(x \rightarrow (q^F_{r, x})^*(\omega^F(x))\) is \(r\nu^r\)-integrable on \(X\); for the same reason, the mapping

\[ x \rightarrow (q^F_{r, x})^*(\omega^F(x), \omega^F_{r, x}(x)) \]

is \(d\nu^r\)-integrable on \(X\).

As \(d\nu^r\) and \(d\nu^{r'}\) are smooth strictly positive measures, there exists a strictly positive \(C^\infty\)-function \(\lambda\) on \(X\) such that \(d\nu^r = \lambda d\nu^{r'}\); it follows that:

\[ x \rightarrow (q^F_{r, x})^*(\omega^F(x), \omega^F_{r, x}(x)) \]

is \(d\nu^r\)-integrable on \(X\).

and then

\[ x \rightarrow (q^F_{r, x})^*(\omega(x), \omega'(x)) \]

is \(d\nu^r\)-integrable.

From the above study it follows:

**Lemma 4.** Let \(F\) be a proper smooth subbundle of \(TX\), with \(\dim(X) \geq 2\), and let \(\varepsilon = (X, r\nu^r, F, q^F_r)\) be a basic complete \(r\)-energy machinery according to \(F\). Another Riemannian structure \(r'\) on \(X\) being given, the quadruplet \(\varepsilon^{r'} = (X, r\nu^{r'}, TX, q^{r', F, r'})\) is a regular complete \(r\)-energy machinery. \(\varepsilon^{r'}\) will be called the \(r'\)-regularization of \(\varepsilon\).

**Remark:** If \(r' = r\), \(q^r, r, q^r\) is \(q_r\) and then \(\varepsilon^r\) is the basic regular complete \(r\)-energy machinery \((X, r\nu^r, TX, q_r)\).

4. \((\Lambda, d\ell)\)-Measurable Field of Energy Machineries of Type \(F\)

a) By standard Borel measure space \((\Lambda, d\ell)\), we mean here a standard Borel space \(\Lambda\) endowed with a positive Borel measure \(d\ell\) (of course \(\sigma\)-finite).

**Definition 4.** Let \((\Lambda, d\ell)\) be a standard Borel measure space, let \(F\) be a smooth subbundle of \(TX\), and for each \(\alpha\) in \(\Lambda\), let \(\varepsilon_\alpha = (Y_\alpha, d\nu_\alpha, F_q)\) be an energy machinery on \(X\). The assignment \(\alpha \rightarrow \varepsilon_\alpha\) will be a \((\Lambda, d\ell)\)-measurable field of energy machineries of type \(F\) if:

1. \(\alpha \neq \alpha'\) implies \(Y_\alpha \cap Y_{\alpha'} = \emptyset\);
2. \(\alpha \rightarrow d\nu_\alpha\) is a \(d\ell\)-integrable field of measures;
3. for all pair \((\omega, \omega')\) of elements in \(D(T^*X)\) the mapping \(\alpha \rightarrow (\omega_{Y_\alpha}, \omega'_{Y_\alpha})_{\kappa_\alpha}\) is \(d\ell\)-integrable on \(\Lambda\);
4. let \(X(\Lambda) = \bigcup_{\alpha \in \Lambda} Y_\alpha\); the mapping \((\omega_{X(\Lambda)}, \omega'_{X(\Lambda)}) \rightarrow \int_{\Lambda} (\omega_{Y_\alpha}, \omega'_{Y_\alpha})_{\kappa_\alpha} d\ell(\alpha)\)

is a positive definite inner product on \(D(T^*X)_{X(\Lambda)}\).

**Remarks:** Let us suppose that \(\Lambda\) is reduced to a single point: \(\Lambda = \{\alpha_0\}\), endowed with its canonical counting measure \(dn_1\), and let \(\varepsilon = (Y, d\nu, F, q)\) be an energy machinery; the assignment \(\alpha \in \Lambda = \{\alpha_0\} \rightarrow \varepsilon_\alpha = \varepsilon\) is obviously a \((\Lambda, dn_1)\)-measurable field of energy machineries of type \(F\); it follows
that any energy machinery can be viewed as \((A, dl)\)-measurable field of energy machineries.

b) Let \( \gamma \) be a configuration in \( X \), i.e. a non empty locally finite subset of \( X \); we can find a subset \( N(\gamma) \) of the set of strictly positive integers \( N^* \), with \( \mathbb{N}(\gamma) = \{1, \ldots, p\} \) if \( \gamma \) is a finite configuration (this is always the case if \( X \) is a compact manifold), and with \( \mathbb{N}(\gamma) = N^* \) if \( \gamma \) is countable, such \( i \rightarrow x_i \) is a one-to-one mapping from \( \mathbb{N}(\gamma) \) onto \( \gamma \). \( N(\gamma) \) is endowed with its counting measure \( dn_\gamma \), and each subset \( \{x_i\} \) in \( \mathbb{N}(\gamma) \), with its counting measure \( dl \).

Lemma 5. Let \( \gamma \) be a configuration in \( X \), let \( \rho \) be a strictly positive function on \( \gamma \), and let \( q = (q_i)_{i \in \mathbb{N}(\gamma)} \) be a family of scalar products, such that for all \( i \) in \( \mathbb{N}(\gamma) \), \( q_i \) is a scalar product on \( T_{x_i}X \).

The assignment \( i \rightarrow \xi_i = ([x_i], \rho dx_i, TX, q_i) \) is a \( (\mathbb{N}(\gamma), dn_\gamma) \)-measurable field of energy machineries of type \( TX \).

Proof: Let \( i \) be in \( \mathbb{N}(\gamma) \); \( x \rightarrow q_\gamma^* (\omega(x), \omega'(x)) \) is obviously \( dl \)-integrable on \( \{x_i\} \), for all \( \omega, \omega' \) in \( \mathcal{D}(T^*X) \). Identifying \( \mathcal{D}(T^*X)_{\{x_i\}} \) with \( T_{x_i}X \), it follows that \( \xi_i \) is an energy machinery, the scalar product being given on \( \mathcal{D}(T^*X)_{\{x_i\}} \) by \( (\omega(x_i), \omega'(x_i))_{x_i} = \rho(x_i) q_\gamma^* (\omega(x_i), \omega'(x_i)) \), which, for a given pair \( (\omega, \omega') \) of compactly supported 1-forms, equals to zero outside a finite subset of \( \mathbb{N}(\gamma) \); it follows that \( i \rightarrow (\omega(x_i), \omega'(x_i))_{x_i} \) is \( dn_\gamma \)-integrable and:

\[
\int_{\mathbb{N}(\gamma)} (\omega(x_i), \omega'(x_i))_{x_i} dn_\gamma(i) = \sum_{x_i \in \gamma} \rho(x_i) q_\gamma^* (\omega(x_i), \omega'(x_i)).
\]

Proof completed.

c) In the following example we suppose that \( \dim(X) \geq 2 \). Let \( N^* \) be the set \( N^* \cup \{\infty\} \); if \( k \) is an element of \( N^* \), \([k]\) will be the set \( \{1, 2, \ldots, k\} \); of course, \( [\infty] = N^* \).

Definition 5. Let \( F \) be a smooth subbundle of \( TX \). A family \( \tilde{F} \) of connected submanifolds of \( X \) will be said subordinate to \( F \) if:

1. There exists \( k \) in \( N^* \) such that \( \tilde{F} = (Y_i)_{i \in [k]} \), and for all \( i \) in \( [k] \) the restricted bundle \( F_{Y_i} = TY_i \);
2. \( Y_i \cap Y_j = \emptyset \) for all \( i, j \) in \( [k] \) such that \( i \neq j \);
3. for any compact subset \( K \) of \( X \) there exists a finite subset \( \tilde{K} \) of \( [k] \) such that \( Y_i \cap K = \emptyset \) for all \( i \) in \( [k] - \tilde{K} \).

Remarks:

1) All the submanifolds of a family \( \tilde{F} \) subordinate to \( F \) have the same dimension \( \dim(F) \).

2) Let \( F \) be a smooth subbundle of \( TX \); any finite family of connected submanifolds of \( X \) satisfying properties (1) and (2) of the above definition is subordinate to \( F \).
3) Let $F$ be an integrable smooth subbundle of $TX$; any almost countable family of leaves of the foliation coming from $F$ (see e.g. [5], [13]) and satisfying (3) is subordinate to $F$.

Let $\tilde{F} = (Y_i)_{i \in [k]}$ be a family of connected submanifolds of $X$ subordinate to some subbundle $F$ of $TX$, and let $\tau$ be a Riemannian structure on $X$; $\tau$ induces on each submanifold $Y_i$, $i \in [k]$, a Riemannian structure $\tau^i$, together with a volume measure $dv^i$ and an Euclidean structure $g^{\tau^i}$ on $TY_i$; it follows that for all families $(\rho_i)_{i \in [k]}$ such that $\rho_i$ is in $C_1^r(Y_i)$, for all $i$ in $[k]$, the quadruplet $\varepsilon^i = (Y_i, \rho_i dv^i, F, g^{\tau^i})$ is an energy machinery.

Let $dn_{[k]}$ be the counting measure on $[k]$.

Let $\omega, \omega'$ be in $D(T^*X)$; as $\omega$ and $\omega'$ are compactly supported $\int_{Y_i} (\omega^F, \omega'^F)_{\varepsilon^i} = \int_{Y_i} (\omega(x), \omega'(x))_{\rho_i(x)} dv^i(x)$ is zero except for a finite subset of $[k]$; it follows that $i \mapsto (\omega^F, \omega'^F)_{\varepsilon^i}$ is $dn_{[k]}$-integrable.

From the above study it follows that Lemma 6 holds:

**Lemma 6.** Let $F$ be a smooth subbundle of $TX$, let $\tau$ be a Riemannian structure on $X$, let $\tilde{F} = (Y_i)_{i \in [k]}$ be a family of connected submanifolds of $X$ subordinated to $F$, where $k$ is some element of $N^r$, and let $(\rho_i)_{i \in [k]}$ be a family such that for all $i$ in $[k]$, $\rho_i$ is an element of $C_1^r(Y_i)$. The assignment $i \mapsto \varepsilon^i = (Y_i, \rho_i dv^i, F, g^{\tau^i})$ is a $([k], dn_{[k]})$-measurable field of energy machineries of type $F$.

**Definition 6.** The fields of energy machineries of the type given in Lemma 6 are called $(F, [k], \tau)$-fields of energy machineries. When the codimension of $F$ is 1, they are called maximal $(F, [k], \tau)$-fields.

**5. $(\tau, \mathcal{H}, F, \Lambda, d\ell)$-Energy Structures**

**Definition 7.** Let $X$ be endowed with the Riemannian structure $\tau$, let $\mathcal{H}$ be a non zero smooth subbundle of $TX$, let $F$ be a smooth subbundle of $\mathcal{H}$, and let $(\Lambda, d\ell)$ be a standard Borel measure space. A $(\tau, \mathcal{H}, F, \Lambda, d\ell)$-energy structure on $X$ is a family $E = (E_p, E_c)$ of energy machineries such that:

1. $E_p = (X, \tau dv^\Lambda, \mathcal{H}, q)$ is a complete $\tau$-energy machinery,
2. $E_c = (\varepsilon_\alpha)_{\alpha \in \Lambda}$, with $\varepsilon_\alpha = (Y_\alpha, dv_\alpha, F, q_\alpha)$, is a $(\Lambda, d\ell)$-measurable field of energy machineries of type $F$.

$E_p$ and $E_c$ are respectively called the principal part and the complementary part of $E$.

**Lemma 7.** Let $E$ be a $(\tau, \mathcal{H}, F, \Lambda, d\ell)$-energy structure on $X$: $E = (E_p, E_c = (Y_\alpha, dv_\alpha, F, q_\alpha)_{\alpha \in \Lambda})$. The mapping:

$$
(\omega^\mathcal{H}, \omega'^\mathcal{H}) \mapsto (\omega^\mathcal{H}, \omega'^\mathcal{H})_E = (\omega^\mathcal{H}, \omega'^\mathcal{H})_{E_p} + \int_{\Lambda} (\omega^F_{\varepsilon_\alpha}, \omega'^F_{\varepsilon_\alpha})_{\varepsilon_\alpha} d\ell(\alpha)
$$
is a positive definite inner product on $D(T^*X)^H$.

Proof: Let $E_p = (X, r\, dv^r, \mathcal{H}, q)$ be the principal part of $E$; from Lemma 1 it follows that:

$$
(\omega^H, \omega^H) \rightarrow (\omega^H, \omega^H)_{E_p} = \int_X q^*(\omega^H(x), \omega^H(x)) r(x) dv^r(x)
$$

is a positive definite inner product on $D(TX)^F$. Moreover, for all $w$ in $D(T^*X)$, since $F \subset \mathcal{H}$, the restriction of $\omega^H$ to $F$ is exactly $\omega^F$; it follows from Definition 4 that $(\omega^F_{X(\Lambda)}, \omega^F_{X(\Lambda)}) \rightarrow \int_{\mathcal{A}} (\omega^F_{X(\Lambda)} \omega^F_{X(\Lambda)}) d\lambda(\alpha)$ is a positive definite inner product on $(TX)_{X(\Lambda)}^F$, from which it follows that: $(\omega^H, \omega^H) \rightarrow \int_{\mathcal{A}} (\omega^H \omega^H)^{\epsilon^\alpha} d\lambda(\alpha)$ is a positive inner product on the space of 1-forms $D(T^*X)^H$. $(\ , \ )_E$ is then the sum of a positive definite inner product and of a positive inner product, hence is a positive definite inner product on $D(T^*X)^H$.

Remark: When $E = (E_p, E_c)$ is a $(\tau, \mathcal{H}, (0)\Lambda, d\ell)$-energy structure, then $(\ , \ )_E = (\ , \ )_{E_p}$. This fact allows to identify a $(\Lambda, d\ell)$-measurable field of energy machineries of type $(0)$ with an object that we shall call the vacuum energy machinery and that we shall denote $\phi$. It follows that energy structures of the form $E = (E_p, \phi)$ will be identified with the complete $\tau$-energy machineries: $E = E_p = (X, r\, dv^r, \mathcal{H}, q)$.

Definition 8.

(1) The energy structures of the form: $E = (X, r\, dv^r, \mathcal{H}, q)$ are called the simple $\tau$-energy structures.

(2) Among the simple $\tau$-energy structures the ones of the form $E = (X, r\, dv^r, TX, q)$ are called the principal $\tau$-energy structures.

(3) Among the principal $\tau'$-energy structures the ones of the form $E = (X, r\, dv^r, TX, q')$ are called the basic $\tau$-energy structures.

(4) The energy structures of the form $E = (E_p, E_c)$ with $E_p$ a basic $\tau$-energy structure, and with $E_c$ a $(F, [k], \tau')$-field of energy machineries are called $(\tau, F, [k], \tau')$-energy structures; if moreover the codimension of $F$ is 1, $E$ is called a maximal $(\tau, F, [k], \tau')$-energy structure.

As a corollary of Lemma 7 one gets:

Lemma 8. Let $E = (E_p, E_c)$ be an energy structure whose principal part $E_p$ is a principal $\tau$-energy structure; $(\ , \ )_E$ is a positive definite inner product on $D(T^*X)$.

6. The Orthogonal Representation $V^E$ of a Gauge Group

a) Let $X$ be a smooth connected Riemannian manifold and let $G$ be a compact semisimple Lie group with Lie algebra $\mathfrak{g}$ endowed with its canonical scalar product $\langle \ , \ \rangle$ given by the opposite of its Killing form (which is invariant
by the adjoint representation $Ad$ of $G$ into $g$). In accordance with the practice in quantum field and gauge field theories the nuclear Lie group $\mathcal{D}(X,G)$ of all the $G$-valued compactly supported smooth mappings on $X$ will be called a gauge group (see e.g. [2]); its Lie algebra is the space $\mathcal{D}(X,g)$ of $g$-valued compactly supported smooth mappings on $X$. For any subspace $V$ of $g$, $\mathcal{D}(X,V)$ will be the nuclear space of $V$-valued compactly supported smooth mappings on $X$, and $\mathcal{D}_1(X,V)$ will be the nuclear space of $V$-valued compactly supported smooth 1-forms on $X$. If $F$ is a smooth subbundle of $TX$, $\mathcal{D}_1(X,V)^F$ will be the space of restrictions $\omega^F$ to $F$ of the elements $\omega$ of $\mathcal{D}_1(X,V)$.

We have of course the following equalities:

\[ D_1(X,V) = D(T^*X) \otimes V \]
\[ D_1(X,V)^F = D(T^*X)^F \otimes V \]

b) Let $E$ be a $(\tau,\mathcal{H},F,\Lambda,d\Omega)$-energy structure on $X$ and let $V \neq \{0\}$ be a subspace of $g$. It follows from Lemma 7 that we can endow $\mathcal{D}_1(X,V)^\mathcal{H}$ with a positive definite inner product $\langle , \rangle_E$ such that for all elements $\omega^\mathcal{H} \otimes u, \omega'^\mathcal{H} \otimes u'$ in the space $D(TX)^\mathcal{H} \otimes V = D_1(X,V)^\mathcal{H}$:

\[ \langle \omega^\mathcal{H} \otimes u, \omega'^\mathcal{H} \otimes u' \rangle_E = (\omega^\mathcal{H}, \omega'^\mathcal{H})_E \cdot \langle u, u' \rangle. \]

For all $\bar{g}$ in $\mathcal{D}(X,G)$ let us consider the operator $V^E(\bar{g})$ on the real prehilbertian space:

\[ (\mathcal{D}_1(X,g)^\mathcal{H}, \langle , \rangle_E) \]

such that, for all $\omega^\mathcal{H}$ in $\mathcal{D}_1(X,g)^\mathcal{H}$, $V^E(\bar{g})\omega^\mathcal{H}$ is the 1-form:

\[ x \rightarrow (V^E(\bar{g})\omega^\mathcal{H})(x) = Ad(\bar{g})(x) \cdot \omega^\mathcal{H}(x). \]

As $G$ acts unitarily on $g$ (with respect to $\langle , \rangle$ by its adjoint representation $Ad$, it follows that:

**Lemma 9.** Let $E$ be a $(\tau,\mathcal{H},F,\Lambda,d\Omega)$-energy structure on $X$; the assignment $\bar{g} \rightarrow V^E(\bar{g})$ is an orthogonal representation of $\mathcal{D}(X,G)$ into $(\mathcal{D}_1(X,g)^\mathcal{H}, \langle , \rangle_E)$. Of course the continuity of $Ad$ on $G$ implies the continuity of $V^E$ on $\mathcal{D}(X,G)$; consequently we can extend $V^E$ in the following two ways:

i) Firstly, we extend $V^E$ into a continuous unitary representation of $\mathcal{D}(X,G)$ into the complex Hilbert space $h^E(g)$ generated by $\mathcal{D}_1(X,g)^\mathcal{H}$ with respect to $\langle , \rangle_E$;

ii) Secondly, we extend $V^E$ by transposition into a continuous representation of $\mathcal{D}(X,G)$ into the dual space $\mathcal{D}_1(X,g)^\mathcal{H}$ of the nuclear space $\mathcal{D}_1(X,g)^\mathcal{H}$:

\[ \langle V^E(\bar{g})\xi, \omega^\mathcal{H} \rangle = \langle \chi, V^E(\bar{g}^{-1})\omega^\mathcal{H} \rangle, \] for all $\bar{g}$ in $\mathcal{D}(X,G)$, all $\chi$ in $\mathcal{D}_1(X,g)^\mathcal{H}$, and all $\omega^\mathcal{H}$ in $\mathcal{D}_1(X,g)^\mathcal{H}$.  


These extensions remain denoted by $V^E$.

c) Let us consider now the so-called Maurer–Cartan cocycle $b : \mathcal{D}(X, G) \to \mathcal{D}_1(X, g)$, given for all $g$ in $\mathcal{D}(X, G)$ by:

\begin{equation}
(10) \quad b(g) = dg \cdot \bar{g}^{-1}
\end{equation}

It is well-known that for each $x$ in $X$ and all $g$, $g'$ in $\mathcal{D}(X, G)$ one has:

\begin{equation}
(11) \quad b(\bar{g}g')(x) = b(g)(x) + Adg(x) \cdot b(g')(x).
\end{equation}

Let $E$ be a $(\tau, \mathcal{H}, F, \Lambda, d\ell)$–energy structure on $X$, and let $b^\mathcal{H} : \mathcal{D}(X, G) \to \mathcal{D}_1(X, g)^\mathcal{H}$ the mapping $\bar{g} \to b^\mathcal{H}(\bar{g}) = (b(g))^\mathcal{H}$. From the definition of $V^E$ and from (11) it follows that for all $g, g'$ in $\mathcal{D}(X, G)$:

\begin{equation}
(12) \quad b^\mathcal{H}(\bar{g}g') = b^\mathcal{H}(g) + V^E(g)b^\mathcal{H}(g').
\end{equation}

Let $(\bar{g}_p)_p$ be a sequence in $\mathcal{D}(X, G)$ and let $\bar{g}$ be in $\mathcal{D}(X, G)$ such that $\lim_{p \to +\infty} \bar{g}_p = \bar{g}$ with respect to the Schwartz topology of the nuclear Lie group $\mathcal{D}(X, G)$; the sequence $(d\bar{g}_p)_p$ converges, with respect to the Schwartz topology, to the corresponding differential mapping $d\bar{g}$ of $\bar{g}$, and then, with respect to the Schwartz topology of $\mathcal{D}_1(X^*, g)^\mathcal{H} = \mathcal{D}(TX)^\mathcal{H} \otimes g$, $\lim_{p \to +\infty} b^\mathcal{H}(\bar{g}_p) = b^\mathcal{H}(\bar{g})$.

It follows that $b^\mathcal{H}$ is a continuous 1–cocycle of $\mathcal{D}(X, G)$ with respect to $V^E$. Moreover, $b^\mathcal{H}$ cannot be a 1–coboundary; for all $g$ in $\mathcal{D}(X, G)$, $b^\mathcal{H}(\bar{g})$ depends on the first derivative of $\bar{g}$, while, for any element $\omega^\mathcal{H}$ of $h^E(g)$, the corresponding 1–coboundary $V^E(g)\omega^\mathcal{H}$ depends only on $\bar{g}$; this argument of order in the sense of [10] proves that $b^\mathcal{H}$ cannot be a 1–coboundary, and therefore we have:

\begin{lemma}
Let $E$ be a $(\tau, \mathcal{H}, F, \Lambda, d\ell)$–energy structure on $X$, and let $G$ be a compact semisimple Lie group with Lie algebra $g$. $b^\mathcal{H}$ is a continuous non trivial 1–cocycle of $\mathcal{D}(X, G)$ with respect to the continuous unitary representation $V^E$.
\end{lemma}

7. The $(\tau, \mathcal{H}, F, \Lambda, d\ell)$–Energy Representations of $\mathcal{D}(X, G)$

a) Let $\mathcal{D}(X, G)$ be a gauge group, and let $E$ be a $(\tau, \mathcal{H}, F, \Lambda, d\ell)$–energy structure on $X$; we shall denote by $Sh^E(g)$ the symmetric Hilbert space based on the complex Hilbert space $h^E(g)$ generated by $\mathcal{D}_1(X, g)^\mathcal{H}$ with respect to $<, >_E$. Taking into account the Lemma 10, the general procedure described in [6] yields a unitary representation $U^E$ of type $(S)$ of $\mathcal{D}(X, G)$ into $Sh^E(g)$ such that, on the total set $\text{EXP}(h^E(g))$, for all $g$ in $\mathcal{D}(X, G)$, and for all $\omega^\mathcal{H}$ in $h^E(g)$ (the notations being the ones used in [16]):

\begin{equation}
(13) \quad U^E(\bar{g})\text{EXP}(\omega^\mathcal{H}) = \exp \left\{-\frac{1}{2}\|b^\mathcal{H}(\bar{g})\|_E^2 - < V^E(\bar{g})\omega, b^\mathcal{H}(\bar{g}) >_E\right\} \text{EXP}(V^E(\bar{g}) + b^\mathcal{H}(\bar{g})).
\end{equation}

One easily sees that such a representation is of order 1 and its support is the whole manifold $X$. If follow that the following theorem holds:
Theorem 1. To each \((\tau, \mathcal{H}, F, \Lambda, d\ell)\)-energy structure \(E\) on the manifold \(X\) there is a continuous unitary representation \(\mathcal{U}^E\), given by (13), of the gauge group \(\mathcal{D}(X, G)\), with support \(X\) and order 1. The corresponding spherical function \(\varphi_E : \mathcal{D}(X, g) \rightarrow \mathbb{C}\) with respect to the vacuum vector \(\mathcal{E}^0\) is given by
\[
\varphi_E(g) = \exp\{-\frac{1}{2}||\mathcal{H}(\xi)||_F^2\},
\]

Definition 9: Let \(E\) be a \((\tau, \mathcal{H}, F, \Lambda, d\ell)\)-energy structure on \(X\); the corresponding representation \(\mathcal{U}^E\) of \(\mathcal{D}(X, G)\) is called a \((\tau, \mathcal{H}, F, \Lambda, d\ell)\)-energy representation.

b) Historical Note. The first energy representation \(\mathcal{U}^E\) was given by R. Ismagilov in [7], with \(X\) an open subset of \(\mathbb{R}^n\), \(G = SU(2)\), and \(E = (X, dx, X \times \mathbb{R}^n, g_0)\), where \(dx\) is the Lebesgue measure on \(X\), \(g_0\) being the canonical Euclidean structure on \(\mathbb{R}^n\). A series of papers of A.M. Vershik, I.M. Gelfand and M.I. Graev ([16], [17], [18]) followed; this first work gave, for any gauge group \(\mathcal{D}(X, G)\), the energy representations \(\mathcal{U}^E\) being a simple \(\tau\)-energy structure of the type \((X, dv^*, \mathcal{H}, q^*_T)\). In [1] S. Albeverio and R. Hoegh-Krohn gave another realization of the same \(\mathcal{U}^E\); it is in this paper that, for the first time, appeared the expression energy representation, which comes from the fact that the corresponding spherical function \(\varphi_E\) can be looked at as a kind of integral of energy. Then S. Albeverio, R. Hoegh-Krohn and D. Testard studied energy representations \(\mathcal{U}^E\) with \(E = (X, \rho dv^*, TX, q_\tau)\), \(\rho\) in \(C^1_c(X) \cap C^\infty(X)\), in [2]. In the case of a manifold \(X\) with Euler number \(e(X) = 0\) J. Marion, in [9], gave energy representations \(\mathcal{U}^E\) with \(E = (X, \rho dv^*, F(\xi), q_\tau)\) of the type described in Lemma 3, with \(\tau\) in \(C^1_c(X)\). A survey of these various \(\mathcal{U}^E\), \(E\) being always a simple \(\tau\)-energy machinery is given in [11]; in this paper was raised the question of the existence of other types of energy representations; a partial answer to this question is given in the present work.

c) Let us give another useful and convenient realization of the representation \(\mathcal{U}^E\). The spherical function \(\varphi_E\) of \(\mathcal{U}^E\) is a continuous function of definite positive type on \(\mathcal{D}(X, G)\); it induces a positive definite function \(\tilde{\varphi}_E\) on the real nuclear space \(\mathcal{D}_1(X, g)^H\), given by:
\[
\varphi_E(\omega^H) = e^{-\frac{1}{2}||\omega^H||_2^2}, \omega^F \in \mathcal{D}_1(X, g)^H.
\]

Now let us recall that \(\mathcal{D}_1(X, g)^H\) is a real nuclear space, inductive limit of Fréchet spaces; it follows then from the Bochner–Minlos theorem that there exists a unique gaussian measure \(\mu_E\) on the dual space \(\mathcal{D}_1'(X, g)^H\) whose Fourier transform \(\hat{\mu}_E\) is given by:
\[
\hat{\mu}_E(\omega^H) = \tilde{\varphi}_E(\omega^H) = e^{i - \frac{1}{2}||\omega^H||_2^2}, \omega^F \in \mathcal{D}_1(X, g)^H.
\]

The theorem 7.9 of [6] allows then the realization of \(\mathcal{U}^E\) in the Hilbert space \(L^2(\mathcal{D}_1'(X, g)^H; \mu_E)\); in this picture \(\mathcal{U}^E\) is given by:
\[
\mathcal{U}^E(\xi)\Phi(\chi) = \exp\{i < b^H(\xi), \chi > \} \cdot \Phi(V^E(\xi^{-1})\chi),
\]
\( V^E \) being here the representation of \( \mathcal{D}(X,G) \) extended by transposition into \( \mathcal{D}'((X,G)^N) \) (see § III.6,b), for all \( \tilde{g} \) in \( \mathcal{D}(X,G) \), \( \Phi \) in \( L^2(\mathcal{D}'((X,g)^N) ; \mu_E) \), \( \chi \) in \( \mathcal{D}'((X,g)^N) \).

d) The main question is now to recognize what are the energy representations which are irreducible (if so, their classe are \( G \)-distributions of order 1 and with support \( X \)), or, at least, what are the ones which are cyclic. The following lemma shows that if suffices to know the answer in the case of energy representations \( \mathcal{E}_E \), \( E \) being a \( (\tau, TX, F, \Lambda, \delta \ell) \)-energy structure, i.e. of the form \( E = (E_p, E_r) \) with \( E_p \) a principal \( \tau \)-energy structure \( (X, rd\nu^\tau, TX, q) \).

Let \( E = ((X, rd\nu^\tau, \mathcal{H}, q), E_r) \) be a \( (\tau, \mathcal{H}, F, \Lambda, \delta \ell) \)-energy structure with principal part \( E_p = (X, rd\nu^\tau, \mathcal{H}, q) \), \( H \neq TX \), let \( q^\tau \) be the restriction of the Euclidean structure \( q \), coming from \( \tau \) to the orthogonal subbundle \( \mathcal{H}^\perp \) of \( \mathcal{H} \) with respect to this Riemannian structure. \( E_r = (X, rd\nu^\tau, \mathcal{H}^\perp, q_r^\tau) \) is an energy machinery; moreover, let \( q \oplus q_r^\tau \) be the Euclidean structure on \( TX = \mathcal{H} \oplus \mathcal{H}^\perp \) such that \( q \oplus q_r^\tau \) restricted to \( \mathcal{H} \) equals \( q \), and such that \( q \oplus q_r^\tau \) restricted to \( \mathcal{H}^\perp \) equals \( q_r^\tau \). From Lemma 4 it follows that:

\[ E_p \ast E_r = (X, rd\nu^\tau, TX, q \oplus p_r^\tau) \]

is a regular complete \( \tau \)-energy machinery. As an obvious consequence it follows that \( E \ast E_r = (E_p \ast E_r, E_r) \) is a \( (\tau, TX, F, \Lambda, \delta \ell) \)-energy structure on \( X \). In accordance with the definition given in Lemma 4, \( E \ast E_r \) will be called the \( \tau \)-regularization of \( E \).

**Lemma 11.** Let \( E \) be a \( (\tau, \mathcal{H}, F, \Lambda, \delta \ell) \)-energy structure with \( \mathcal{H} \neq TX \), and let \( E \ast E_r \) be its \( \tau \)-regularization.

\[ \bigcup_{E \ast E_r} \text{ is unitarily equivalent to } \bigcup E \ast \bigcup E_r. \]

**Proof:** In the proof of Lemma 4 the orthogonal decomposition with respect to \( \tau \) was shown

\[ \mathcal{D}(T^*X) = \mathcal{D}(T^*X)^N \oplus \mathcal{D}(T^*X)^{N^\perp}. \]

This orthogonal decomposition remains true with \( \mathcal{D}(T^*X) \) endowed with the scalar product \( \langle \ , \rangle_{E \ast E_r} \) which is the sum of the scalar product \( \langle \ , \rangle_E \) on \( \mathcal{D}(T^*X)^N \) and of the scalar product \( \langle \ , \rangle_{E_r} \) on \( \mathcal{D}(T^*X)^{N^\perp} \).

It follows the orthogonal decomposition:

\[ \mathcal{D}_1(X,g) = \mathcal{D}_1(X,g)^N \oplus \mathcal{D}_1(X,g)^{N^\perp} \]

with respect to the scalar product \( \langle \ , \rangle_{E \ast E_r} \) which is the sum of \( \langle \ , \rangle_E \) on \( \mathcal{D}_1(X,g)^N \) and of \( \langle \ , \rangle_{E_r} \) on \( \mathcal{D}_1(X,g)^{N^\perp} \).

Let \( \mu_E, \mu_{E \ast E_r}, \mu_{E \ast E_\tau} \) be the Gaussian measures on respectively \( \mathcal{D}'(X,g)^N \), \( \mathcal{D}'(X,g)^{N^\perp} \) and \( \mathcal{D}'(X,g)^{N^\perp} \) with corresponding Fourier transforms given by:

\[ \hat{\mu}_E(w^N) = \exp \left\{ -\frac{1}{4} \| w^N \|_{E_r}^2 \right\}, \quad w^N \text{ in } \mathcal{D}(X,g)^N, \]
\[ \hat{\mu}_E(w^{N^\perp}) = \exp \left\{ -\frac{1}{4} \| w^{N^\perp} \|_{E_\tau}^2 \right\}, \quad w^{N^\perp} \text{ in } \mathcal{D}(X,g)^{N^\perp}, \]
\[ \hat{\mu}_{E \ast E_r}(\omega) = \exp \left\{ -\frac{1}{2} \| \omega \|_{E \ast E_r}^2 \right\}, \quad \omega \text{ in } \mathcal{D}(X,g). \]
From \( ||\omega||_{E^*E'}^2 = ||\omega^H||_E^2 + ||\omega^{H^*}||_{E'}^2 \), it follows that:
\[
\mu_{E^*E'} = \mu_E \oplus \mu_{E'},
\]
and then, taking account (17), one gets:
\[
L^2(D_1(X,g);\mu_{E^*E'}) \cong L^2(D_1(X,g)^H;\mu_{E}) \otimes L^2(1(X,g)^{H^*};\mu_{E'}) .
\]
It follows then from (16) that \( \sqcup E^*E' = \sqcup E \otimes \sqcup E' \). \( \blacksquare \)

Corollary. If \( \sqcup E^*E' \) is irreducible, \( \sqcup E \) is irreducible, too.

8. The Unitary Representation \( \Pi^E_A \) and its Spectral Measure

Let \( E \) be a \( (\tau, TX, F, \Lambda, d\ell) \)-energy structure on \( X \), let \( \sqcup E \) be the corresponding energy representation of the gauge group \( D(X, G) \), and let \( A \) be a Cartan subalgebra of \( g \); the orthogonal complement of \( A \) in \( g \) with respect to the canonical scalar product \( < , > \) on \( g \) will be denoted by \( A^\perp \), and the maximal torus \( \exp(A) \) in \( G \) will be denoted by \( T \).

We introduce here the spaces \( D_1(X, A) = D(T^*X) \otimes A, D_1(X, A^\perp) = D(T^*X) \otimes A^\perp \); from the orthogonal sum:
\[
g = A \oplus A^\perp \text{ (with respect to } < , > \)
\]

it follows the orthogonal decomposition with respect to \( < , >_E \):
\[
D_1(X, g) = D_1(X, A) \oplus D_1(X, A^\perp).
\]

Of course the Lie algebra of the abelian nuclear Lie group \( D(X, T) \) is the abelian Lie algebra \( D(X, A) \). The energy representation \( \sqcup E \) defines a unitary representation \( \Pi^E_A \) of the abelian nuclear group \( D(X, A) \) into the Hilbert space \( L^2(D_1(X, g);\mu_E) \) given by:
\[
\Pi^E_A(u) = \sqcup E(\exp u), \quad u \in D(X, A).
\]

The present section is devoted to the study of \( \Pi^E_A \).

a) Let \( d : D(X, A) \to D_1(X, A) \) be the exterior derivative, given by \( u \to du; d \) is continuous with respect to the Schwartz topologies of \( D(X, A) \) and \( D_1(X, A) \) and \( \ker d \), the space of constant functions in \( D(X, A) \), is a closed subspace which equals \( 0 \) if \( X \) is a non compact manifold; we shall denote by \( \tilde{D}(X, A) \) the space \( D(X, A)/\ker d \), and for any element \( u \) in \( D(X, A) \) by \( \tilde{u} \) its class in \( \tilde{D}(X, A) \), \( \tilde{D}(X, A) \) is a nuclear space and \( \tilde{d} : \tilde{D}(X, A) \to D_1(X, A) \) such that \( \tilde{d}(\tilde{u}) = du \) is a one-to-one continuous linear mapping which allows to endow \( \tilde{D}(X, A) \) with the positive definite inner product \( \tilde{E} \) given by:
\[
\tilde{E}(\tilde{u}, \tilde{v}) = < \tilde{d}\tilde{u}, \tilde{d}\tilde{v} >_E = < du, dv >_E.
\]

It follows that the following lemma holds:
Lemma 12. \( \hat{d} \) is an isometry with close range of the real prehilbertian space \( (\mathcal{D}(X, \mathcal{A}), \langle \cdot, \cdot \rangle_E) \) into the prehilbertian space \( (\mathcal{D}(X, \mathcal{A}), \langle \cdot, \cdot \rangle_E) \).

Let \( \mu_E \) be the Gaussian measure on the dual space \( \mathcal{D}'(X, \mathcal{A}) \) with Fourier transformation \( \hat{\mu}_E : \mu \rightarrow \hat{\mu}_E(\mu) = \text{exp} \left\{ -\frac{1}{2} \mathcal{E}(\mu, \mu) \right\} \), let \( \mu_{E,A} \) and \( \mu_{E,A_1} \) be the Gaussian measures on the dual spaces \( \mathcal{D}'(X, \mathcal{A}) \) and \( \mathcal{D}'(X, \mathcal{A}_1) \) given by
\[
\hat{\mu}_{E,A} : \omega \rightarrow \hat{\mu}_{E,A}(\omega) = \exp \left\{ -\frac{1}{2} \langle \omega, \omega \rangle_E \right\}, \omega \in \mathcal{D}(X, \mathcal{A}),
\]
\[
\hat{\mu}_{E,A_1} : \omega \rightarrow \hat{\mu}_{E,A_1}(\omega) = \exp \left\{ -\frac{1}{2} \langle \omega, \omega \rangle_E \right\}, \omega \in \mathcal{D}(X, \mathcal{A}_1).
\]

One gets:

Lemma 13. For all \( \overset{\sim}{u} \) in \( \mathcal{D}(X, \mathcal{A}) \) let \( \overset{\sim}{w}(\overset{\sim}{u}) \) be the operator on \( L^2(\mathcal{D}'(X, \mathcal{A}); \mu_{E,A}) \) given by:
\[
\overset{\sim}{w}(\overset{\sim}{u})\Phi(\chi) = \text{exp} \{ i < \chi, d\mu > \} \Phi(\chi).
\]

(i) \( \overset{\sim}{w} : \overset{\sim}{u} \rightarrow \overset{\sim}{w}(\overset{\sim}{u}) \) is a continuous unitary representation of \( \mathcal{D}(X, \mathcal{A}) \) into \( L^2(\mathcal{D}'(X, \mathcal{A}); \mu_{E,A}) \).

(ii) The spectral measure of \( \overset{\sim}{w} \) is equivalent to \( \mu_E \).

Proof:

(i) follows from an easy verification.

(ii): Let \( \overset{\sim}{d}^* \) be the transposed mapping of \( \overset{\sim}{d} \); from Lemma 12 \( \overset{\sim}{d}^* \) maps \( \mathcal{D}'(X, \mathcal{A}) \) onto \( (\mathcal{D}'(X, \mathcal{A}), \mathcal{D}(X, \mathcal{A})) \). As these two spaces are standard Borel spaces there exists a Borel section \( s \) of \( \overset{\sim}{d}^* \) such that the mapping:
\[
\overset{\sim}{s} : \chi \rightarrow (\overset{\sim}{d}^* \chi, s\overset{\sim}{d}^* \chi - \chi)
\]
is an isomorphism of Borel spaces from \( \mathcal{D}'(X, \mathcal{A}) \) onto \( \mathcal{D}'(X, \mathcal{A}) \times \ker \overset{\sim}{d}^* \), from which it follows that \( \overset{\sim}{s}(\mu_{E,A}) = \overset{\sim}{d}^* \mu_{E,A} \times \lambda \), \( \lambda \) being some Borel measure on \( \ker \overset{\sim}{d}^* \). One gets then an isomorphism of Hilbert spaces:
\[
L^2(\mathcal{D}'(X, \mathcal{A}); \mu_{E,A}) \simeq L^2(\mathcal{D}'(X, \mathcal{A}); \mu_E) \otimes L^2(\ker \overset{\sim}{d}^*; \lambda),
\]
such that for all \( \overset{\sim}{u} \) in \( \mathcal{D}(X, \mathcal{A}) \), \( \overset{\sim}{w}(\overset{\sim}{u}) \) is transformed into the operator \( \overset{\sim}{w}(\overset{\sim}{u}) \otimes \overset{\sim}{I} \), \( \overset{\sim}{w}' \) being the unitary representation of \( \mathcal{D}(X, \mathcal{A}) \) into \( L^2(\mathcal{D}'(X, \mathcal{A}); \overset{\sim}{d}^* \mu_{E,A}) \) given by:
\[
\overset{\sim}{w}'(\overset{\sim}{u})\psi(\chi) = \text{exp} \{ i < \chi, \overset{\sim}{d}\overset{\sim}{u} > \} \psi(\chi),
\]
\( \psi \) in \( L^2(\mathcal{D}'(X, \mathcal{A}); \overset{\sim}{d}^* \mu_{E,A}) \), \( \chi \in \mathcal{D}'(X, \mathcal{A}) \).

It follows that the spectral measure of \( \overset{\sim}{w} \) is equivalent to \( \overset{\sim}{d}^* \mu_{E,A} \); as all the spaces interfering here are nuclear spaces, owing to the uniqueness of gaussian measures given by their Fourier transforms, it follows that \( \overset{\sim}{d}^* \mu_{E,A} \) equals \( \mu_E \).

b) Let \( \Theta^E \) be the unitary representation of \( \mathcal{D}(X, \mathcal{A}) \) into \( L^2(\mathcal{D}'(X, \mathcal{A}_1); \mu_{E,A_1}) \) given by:
\[
\Theta^E(u)\Phi(\chi) = \Phi(V^E(\exp u)\chi),
\]
with \( \Phi \) in \( L^2(\mathcal{D}'(X, \mathcal{A}_1); \mu_{E,A_1}) \), \( \chi \) in \( \mathcal{D}'(X, \mathcal{A}_1) \).
Lemma 14. The spectral measure $\nu^E$ of $\Theta^E$ is equivalent to the infinite direct sum:

$$\bigoplus_{k \geq 0} (rdv^r \otimes N)^{\otimes k},$$

$N$ being the counting measure on the set $\Delta$ of roots of the Cartan pair $(g, A)$.

Proof: We shall use the Fock realization of $L^2(D'_1(X, A^\perp); \mu_{E, A^\perp})$: let $h^E(A^\perp)$ be the complex Hilbert space spanned by $D_1(X, A^\perp)$ with respect to $\langle \cdot, \cdot \rangle_E$; from [6], theorem 7.2, $L^2(D'_1(X, A^\perp); \mu_{E, A^\perp})$ is isomorphic to the Fock space

$$S_h^E(A^\perp) = \sum_{p \geq 0} \otimes S^p h^E(A^\perp),$$

$S^p h^E(A^\perp)$ being the $p$-th symmetric tensor power of $h^E(A^\perp)$. Let $\Delta$ be the set of roots of $g$ with respect to the Cartan subalgebra $A$, and for $\alpha$ in $\Delta$, let $g^\alpha$ be the subspace of $g$ with weight $\alpha$; one gets:

$$A^\perp = \bigoplus_{\alpha \in \Delta} g^\alpha,$$

from which it follows that $h^E(A^\perp) = \bigoplus_{\alpha \in \Delta} h^E(g^\alpha), h^E(g^\alpha)$ being the complex Hilbert space spanned by $D_1(X, g^\alpha)$ with respect to $\langle \cdot, \cdot \rangle_E$. It follows that $S_h^E(A^\perp) = \bigotimes_{\alpha \in \Delta} S_h^E(g^\alpha)$. For all $\omega$ in $h^E(g^\alpha)$, for all $u$ in $D(X, A)$ one has:

$$\nu^E(\exp u)\omega : x \rightarrow e^{i\omega(u(x))}\omega(x), \quad x \in X.$$

Then the restriction of $\Theta^E$ on $S^p h^E(A^\perp)$ (which equals $\bigotimes_{\alpha \in \Delta} S^p h^E(g^\alpha)$) acts by multiplication by elements of the form $e^{\sum_{k=1}^p c_k u(x_k)}$, with $c_1, \ldots, c_p$ in $\Delta$ and $x_1, \ldots, x_p$ in $X$. It follows that the spectral measure $\nu^E$ of $\Theta^E$ is supported by the subset $\Gamma_A$ of $D'(X, A)$ consisting of functionals of the form:

$$\chi : u \rightarrow \chi(u) = \sum_{k=1}^p c_k u(x_k),$$

for all non-negative integer $p$, the $c_k$ being in $\Delta$ and the $x_k$ being in $X$.

We can then identify $\Gamma_A$ with $\bigcup_{p \geq 0} (X \times \Delta)^p_{sym}$, and $\nu^E$ with the Poisson measure whose restriction to $(X \times \Delta)^p_{sym}$ is given by $(rdv^r \otimes n)^{\otimes p}$, $N$ being the counting measure on $\Delta$, $rdv^r$ the measure on $X$ given by the principal part of $E$ (see e.g. [6], § 3). ($\blacksquare$

c) We come back now to the representation $\Pi_A^E$ of $D(X, A)$ into $L^2(D'_1(X, A); \mu_E)$ such that, for all $u$ in $D(X, A)$:

$$\Pi_A^E(u) = \Pi^E(\exp u).$$
From (18) it follows that we have the isomorphism:

\[ L^2(D^1(X, g); \mu_E) \simeq L^2(D^1(X, A); \mu_{E,A}) \otimes L^2(D^1(X, A^1); \mu_{E,A^1}) \]

so, if \( u \) is in \( D(X, A) \), and \( \Phi = \Phi_1 \otimes \Phi_2 \) in the above space, from the definitions of \( U^E \), \( \tilde{\omega} \) and \( \Theta^E \), and the fact that for all \( u \) in \( D(X, A) \), \( b(\exp u) = du \), it follows that:

\[ \Pi_A^E(u)(\Phi_1 \otimes \Phi_2) = (\tilde{\omega}(\tilde{u})\Phi_1) \otimes (\Theta^E(u)\Phi_2), \quad \text{i.e.:} \]

\[ \Pi_A^E(u) = \tilde{\omega}(\tilde{u}) \otimes \Theta^E(u). \quad (23) \]

From Lemma 13 and Lemma 14, one gets then:

**Theorem 2.** Let \( E \) be a \((t, TX, F, A, d\ell)\)-energy structure on the manifold \( X \), let \( U^E \) the corresponding energy representation of the gauge group \( D(X, G) \), let \( A \) be a Cartan subalgebra of \( g \), and let \( \Pi_A^E \) be the unitary representation of the abelian Lie group \( D(X, A) \) such that for all \( u \) in \( D(X, A) \):

\[ \Pi_A^E(u) = U^E(\exp u). \]

The spectral measure of \( \Pi_A^E \) is equivalent to the convolution \( \hat{\mu}_E \ast \nu^E \), \( \hat{\mu}_E \) being the Gaussian measure on \( D'(X, A) \) with Fourier transform \( \tilde{\mu}_E(\tilde{u}) = \exp \left\{-\frac{1}{2} ||\tilde{u}||^2_E \right\} \) and with \( \nu^E \simeq \bigoplus_{p \geq 0} (t^p \otimes N)^{\otimes p} \).

\[ \text{d) We want to give now a direct integral decomposition of } \Pi_A^E. \]

Let us recall that \( \Gamma_A \) is the subset of \( D'(X, A) \) of functionals of the form \( \chi = \sum_{k \in I \text{ finite}} \delta_{x_k}^\ast \), the \( \alpha_k \) being in \( \Delta \), the \( x_k \) in \( X \), and the functional \( \delta_{x_k}^\ast \) being given by \( \delta_{x_k}^\ast (u) = \alpha_k(u(x_k)) \), \( u \) in \( D(X, A) \) (see part b) above).

For each \( \chi \) in \( \Gamma_A \), let \( \gamma_\chi \) be the character of \( D(X, A) \) such that \( \gamma_\chi(u) = \exp(i \langle \chi, u \rangle) \), \( u \in D(X, A) \), and let \( \tilde{\omega}_\chi = \tilde{\omega} \otimes \gamma_\chi \) the unitary representation of \( D(X, A) \), \( \tilde{\omega} \) being the representation defined in Lemma 13. \( \tilde{\omega}_\chi \) can be written also \( \tilde{\omega}_\chi = \omega \otimes \gamma_\chi \), \( \omega \) being the unitary representation of \( D(X, A) \) defined by:

\[ w(u) = \tilde{\omega}(\tilde{u}), \quad u \in D(X, A). \quad (24) \]

It follows that \( \omega \) and \( \tilde{\omega} \) have the same spectral measure, which is equivalent to \( \hat{\mu}_E \) by Lemma 13, and then the spectral measure \( \hat{\mu}_E \) of \( \tilde{\omega}_\chi \) is the convolution of the spectral measure \( \hat{\mu}_E \) of \( \omega \) by the spectral measure of the character \( \gamma_\chi \). More precisely:

**Lemma 15.**

(i) One has the direct integral decomposition:

\[ \Pi_A^E = \int_{\Gamma_A} \tilde{W}_\chi d\nu^E(\chi). \]
(ii) $\bar{\mu}_E^\chi$ is equivalent to the translated of $\bar{\mu}_E$ by $-\chi$, i.e.:

$$\bar{\mu}_E^\chi \simeq \bar{\mu}_E(\cdot, -\chi).$$

**Proof:**

(i) From Lemma 14 one gets: $\Theta^E = \int_{\Gamma_A} \gamma_X d\nu^E(\chi)$; it follows then from (23) that:

$$\Pi_A^E = \tilde{\omega} \otimes \Theta^E = \tilde{\omega} \otimes \int_{\Gamma_A} d\nu^E(\chi) = \int_{\Gamma_A} (\tilde{\omega} \otimes \gamma_X) d\nu^E(\chi)$$

$$= \int_{\Gamma_A} \tilde{\omega}_X d\nu^E(\chi).$$

(ii) follows from the fact that the spectral measure of $\tilde{\omega}_X$ is the convolution of the spectral measure $\mu_E$ of $\tilde{\omega}$ by the spectral measure of the character $\gamma_X$ which is given by $\chi$, and then: $\bar{\mu}_E^\chi = \bar{\mu}_E(\cdot, -\chi)$.

Remarks:

1) Let $\mathcal{U}$ be the algebra of measurable subsets of $\mathcal{D}'(X, \mathcal{A})$; the mapping $(\chi, B) \in \Gamma \times \mathcal{U} \rightarrow \bar{\mu}_E^\chi(\chi, B)$ is measurable because $\bar{\mu}_E^\chi(\chi, B) = \mu_E(B - \chi)$.

2) The results of Lemmas 13, 14, 15 and of Theorem 2 are proved in [17] and [2] in the case of a non compact $X$ and of $E$ of the form $(X, r d\nu^r, TX, g_r)$. The proofs given here in the case of a general $(\tau, TX, F, \Lambda, d\ell)$–energy structure are of the same type; moreover the use of the space $\mathcal{D}(X, \mathcal{A})$ and for $\tilde{\omega}$, instead of $\mathcal{D}(X, \mathcal{A})$ and $d$, allows to include the case $X$ is compact.

9. A Sufficient Condition for the Irreducibility of $\sqcup^E$

The goal of this section is to prove that under some conditions about uniform disjointness of the Gaussian measures $\bar{\mu}_E^\chi$, $\sqcup^E$ is irreducible.

**Definition 10:** Let $E$ be a $(\tau, TX, F, \Lambda, d\ell)$–energy structure on $X$, and let $\mathcal{D}(X, G)$ be a gauge group. We shall say that $\sqcup^E$ has the $\Gamma$–property if there exists a Cartan subalgebra $\mathcal{A}$ of $g$ such that for any pair $(\Gamma_1, \Gamma_2)$ of subsets of $\Gamma_A$ satisfying $\nu^E(\Gamma_1) > 0, \nu^E(\Gamma_2) > 0, \nu^E(\Gamma_1 \cap \Gamma_2) = 0$, the measures $\int_{\Gamma_1} \bar{\mu}_E^\chi d\nu^E(\chi)$ and $\int_{\Gamma_2} \bar{\mu}_E^\chi d\nu^E(\chi)$ are singular.

Remarks:

1) Two Cartan subalgebras $\mathcal{A}$ and $\mathcal{A}'$ of $g$ give isomorphic Borel measure spaces $(\Gamma_A, \nu^E)$ and $(\Gamma_{A'}, \nu^{E'})$; it follows that $\sqcup^E$ has the $\Gamma$–property with respect to some Cartan subalgebra if and only if $\sqcup^E$ has the $\Gamma$–property with respect to all Cartan subalgebras.

2) As $\bar{\mu}_E^\chi$ is equivalent to the spectral measure of $\tilde{\omega}_X$, the $\Gamma$–property is equivalent to the fact that, for $\Gamma_1, \Gamma_2$ as above, $\int_{\Gamma_1} \tilde{\omega}_X d\nu^E(\chi)$ and $\int_{\Gamma_2} \tilde{\omega}_X d\nu^E(\chi)$ contain no equivalent subrepresentations, i.e. are disjoints.

3) For all $\chi$ in $\Gamma_A$ let $\alpha_X$ be the spectral measure of the character $\gamma_X$, and let $\Gamma_1$, $\Gamma_2$ be as in Definition 10, let $\lambda_i$ be the measure $\int_{\Gamma_i} \alpha_X d\nu^E(\chi)$, $i = 1, 2$; the property $\Gamma$–property says that $\bar{\mu}_E \ast \lambda_1$ and $\bar{\mu}_E \ast \lambda_2$ are disjoints.
Lemma 16. Let $E$ be a $(\tau, TX, F, \Lambda, d\ell)$-energy structure, let $\mathcal{D}(X, G)$ be a gauge group, and let us suppose that $U^E$ has the $\Gamma$-property. The von Neumann algebra generated by $\Pi^E_A$ contains all operators of the form $1 \otimes \Theta^E(u)$ and $W(u) \otimes 1$, $u \in \mathcal{D}(X, A)$.

Proof: Let us recall that $W(u) = \bar{w}(u)$, $u \in \hat{\mathcal{D}}(X, A)$, and let us consider the family $(\bar{w}_\chi)_{\chi \in \mathcal{D}(X, A)}$ of unitary representations of $\mathcal{D}(X, A)$ given in Lemma 15. One has $\Pi^E_A = \int_{\Gamma} \bar{w}_\chi d\nu^E(\chi)$, and then $\Pi^E_A$ is equivalent to a representation of $\mathcal{D}(X, A)$ into a direct integral of Hilbert spaces $\int_{\Gamma} \mathcal{H}_\chi d\nu^E(\chi)$. Let $S$ be an operator commuting with the representation $\Pi^E_A$, and let $\Gamma'$ be a measurable subset of $\Gamma$ with $\nu^E(\Gamma') > 0, \nu^E(\Gamma - \Gamma') > 0$; from the $\Gamma$-property it follows that $\int_{\Gamma'} \mathcal{H}_\chi d\nu^E(\chi)$ and its orthogonal complement $\int_{\Gamma - \Gamma'} \mathcal{H}_\chi d\nu^E(\chi)$ are such that the restrictions $\pi^E_A$ to these two spaces contain no equivalent subrepresentations, and $\int_{\Gamma'} \mathcal{H}_\chi d\nu^E(\chi)$ is invariant by $S$. It follows that $S$ is decomposable into a direct integral with respect to $\nu^E$.

Let $N^E$ be the von Neumann algebra generated by $\Pi^E_A$. The operators $(1 \otimes \Theta^E(u))$ and $(W \otimes 1)(u)$ act on each Hilbert space $\mathcal{H}_\chi$ by multiplication by bounded $\nu^E$-measurable functions; it follows that they commute with all decomposable operators, in particular with operators $S$ which commute with $\Pi^E_A$; these operators are then in the bicommutant of $\Pi^E_A$, and then in $N^E$, by the von Neumann's theorem.

As a corollary one gets:

Corollary. $N^E$ contains all operators of multiplication by $e^{i \langle - , d\ell \rangle} E$, $u \in \mathcal{D}(X, A)$.

We come now to the main result about irreducibility.

Theorem 3. Let $\mathcal{D}(X, G)$ be a gauge group, and let $E$ be a $(\tau, TX, F, \Lambda, d\ell)$-energy structure. If $U^E$ has the $\Gamma$-property, $U^E$ is irreducible.

Proof: We shall use the realization of $U^E$ in the space $L^2(\mathcal{D}_1^1(X, g); \mu_E)$ given in (16), Section 7(c).

a) Let us prove, at first, that the vacuum vector $1 : \chi \rightarrow 1(\chi) = 1 \forall \chi \in \mathcal{D}_1^1(X, g)$, is cyclic for $U^E$. Let $L^E$ be the von Neumann algebra generated by $U^E$; as $\Pi^E_A(u) = U^E(\exp u)$, $u \in \mathcal{D}(X, A)$, the bicommutant of $\Pi^E_A$ is contained in the bicommutant of $U^E$; from the corollary of Lemma 16, it follows that if $U^E$ has the $\Gamma$-property, $L^E$ contains the operators of multiplication by $\exp(\langle - , d\ell \rangle)$, $u \in \mathcal{D}(X, A)$, for any Cartan subalgebra $A$ of $g$. As $g$ is the union of its Cartan subalgebras, it follows that $L^E$ contains all the operators of multiplication by $\exp(\langle - , d\ell \rangle)$ for all $u$ in $\mathcal{D}(X, g)$, and then, all the operators of the form:

$$n^E_u(\bar{g}) = U^E(\bar{g}) e^{i \langle - , d\ell \rangle} \cdot U^E(\bar{g}^{-1})$$
with \( g \) in \( D(X, G) \) and \( u \) in \( D(X, g) \), i.e. all the operators of multiplication by:

\[
e^{i<, V^g du> g} \in D(X, G), \ u \in D(X, g).
\]

It follows that \( L^E \) contains all the operators of multiplication by functions of the form:

\[
e^{i<, \sum_{k=1}^{k=n} e^{k}(g_{k}) du_{k}>}
\]

with \( g_1, \ldots, g_p \) in \( D(X, G) \) and \( u_1, \ldots, u_p \) in \( D(X, g) \).

By the lemma 3.5 of [17] one knows that the set \( \{ V^g du/g \in D(X, G), u \in D(X, g) \} \) is total in \( D_1(X, g) \); it follows that the functions:

\[
\chi \rightarrow n_u^E(g)(\chi), \ (gu) \in D(X, G) \times D(X, g)
\]

constitute a total set in \( L^2(D_1(X, g); \mu_E) \).

As \( \Pi^E (g) 1 = e^{i<, \xi(g)>} \), it follows that the smallest closed subspace of \( L^2(D_1(X, g); \mu_E) \) containing the functions \( \Pi^E (g) 1, g \in D(X, G) \), contains the space spanned by the functions \( n_u^E(g) \), and then, this space is exactly \( L^2(D_1(X, g); \mu_E) \); the cyclicity of \( 1 \) is then proved.

b) It remains now to prove the irreducibility of the cyclic component of \( U_1 \), \( \sqcup^E \) having the \( \Gamma \)-property.

Let \( Q \) be an operator belonging to the commutant of \( U_1(D(X, G)) \); \( Q \) commutes with \( \Pi^E_\mathfrak{A} \) for any Cartan subalgebra \( \mathfrak{A} \) of \( g \), and then, as it has been seen in the proof of Lemma 16, \( Q \) is decomposable with respect to the integral decomposition of \( \Pi^E_\mathfrak{A} \) given in Lemma 15, for any Cartan subalgebra \( \mathfrak{A} \) of \( g \). The projection of \( L^2(D_1(X, g); \mu_E) \) onto \( L^2(D_1(X, \mathfrak{A}); \mu_{E, \mathfrak{A}}) \) being diagonalizable, \( L^2(D_1(X, \mathfrak{A}); \mu_{E, \mathfrak{A}}) \) is invariant by \( Q \); it follows that \( Q1 \) belongs to all the spaces \( L^2(D_1(X, \mathfrak{A}); \mu_{E, \mathfrak{A}}) \) for all Cartan subalgebras \( \mathfrak{A} \). Owing to the semisimplicity of \( g \), the intersection of all the spaces \( L^2(D_1(X, \mathfrak{A}); \mu_{E, \mathfrak{A}}) \), \( \mathfrak{A} \) running in the set of Cartan subalgebras of \( g \), equals \( C1 \); it follows that \( Q \) is a scalar operator; as \( 1 \) is a cyclic vector for \( \sqcup^E \), it follows that \( \sqcup^E \) is irreducible. \( \blacksquare \)

Note: The Theorem 3 is, up to now, the only way known in order to prove the possible irreducibility of the energy representations, and is substantially the method given in [17], [2] in order to prove the irreducibility of \( \sqcup^E \) with \( E = (X, rdv', TX, g_1(\xi)) \) when \( \dim(X) \geq 3 \); in the case \( E = (X, rdv', F_\xi, g_\xi) \) the irreducibility of \( \sqcup^E \) when \( \dim(X) \geq 3 \) was also proved in [9] in the same way. We do not know whether the \( \Gamma \)-property of \( \sqcup^E \) is equivalent to its irreducibility.

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