Abstract

A ring is said to be strongly right bounded if every nonzero right ideal contains a nonzero ideal. In this paper strongly right bounded rings are characterized, conditions are determined which ensure that the split-null (or trivial) extension of a ring is strongly right bounded, and we characterize strongly right bounded right quasi-continuous split-null extensions of a left faithful ideal over a semiprime ring. This last result partially generalizes a result of C. Faith concerning split-null extensions of commutative FPF rings.

Examples of strongly right bounded rings are: right duo rings (e.g., commutative rings and strongly regular rings) [8], [18] and [26]; right subdirectly irreducible rings [9] and [10]; right valuation rings which are not subdirectly irreducible [24, p. 216]; and bounded principal ideal domains [20, p. 41]. In [13, p. 364] an example of a strongly left bounded right primitive ring is given. In [16, p. 5.3] an example of a strongly right bounded right self-injective ring which is not left selfinjective is presented. Strongly right bounded rings play a fundamental role in the theory of FPF rings (e.g., a strongly right bounded right selfinjective ring is right FPF and the basic ring of a semiperfect right FPF ring is strongly right bounded [16]). In fact, according to [17, p. 310], C. Faith has conjectured that a right FPF ring is Morita equivalent to a strongly right bounded ring.

All rings are associative, $R$ denotes a ring with unity and $M$ will always be a unital $(R, R)$-bimodule. The split-null (or trivial extension) $S(R, M)$ of $M$ by $R$ is the ring formed from the Cartesian product $R \times M$ with component-wise addition and with multiplication given by $(a, m)(b, k) = (ab, ak + mb)$ (cf., [12], [15], and [22]). Annihilators will be symbolized as $t_A(X) = \{a \in A | aX = 0\}$ and $r_A(X) = \{a \in A | Xa = 0\}$. A (ring) direct summand of $R$ will mean a right ideal generated by a (central) idempotent. From [16], $R$ is right FPF if every finitely generated faithful right $R$-module generates the category $\text{mod-}R$. From [3], $R$ is right quasi-FPF if, whenever a faithful right $R$-module is a direct sum of finitely many cyclic modules, then it is a generator for $\text{mod-}R$. A ring $R$ is (quasi-) Baer (cf., [7] and [23]) if the right annihilator of every
(ideal) nonempty subset of $R$ is a direct summand of $R$. Semiprime right FPF rings are quasi-Baer [11, p. 168]. From [6] a ring is right CS if every right ideal is essential in a direct summand. From [21], $R$ is right quasi-continuous (also known as π-injective [19]) if it is right CS and if $P$ and $Q$ are direct summands of $R$ such that $P \cap Q = 0$, then $P \oplus Q$ is a direct summand of $R$. Note that if $R$ is right CS and every idempotent is central, then $R$ is right quasi-continuous. Thus in [14, p. 83] Faith has shown that every commutative FPF ring is quasi-continuous. $R$ satisfies the intersection left annihilator sum property, ILAS, if whenever $X$ and $Y$ are right ideals such that $X \cap Y = 0$, then $I_R(X)R + I_R(Y)R = R$ (e.g., right uniform rings, right selfinjective rings [25, p. 275], and right quasi-FPF rings [3, Lemma 1]).

**Proposition 1.** The following conditions are equivalent:

(i) $R$ is a strongly right bounded ring.

(ii) If $xR$ is a faithful cyclic module, then $rR(x) = 0$.

(iii) $R$ is directly finite and every faithful cyclic module is isomorphic to $R$.

**Proof:**

(i) $\Rightarrow$ (ii). If $rR(x) \neq 0$, then there exists a nonzero ideal $Y \subseteq rR(x)$. Hence $xRY = 0$. Contradiction!

(ii) $\Rightarrow$ (iii). Assume $R = X \oplus S$ where $X$ and $S$ are right ideals and $S$ is isomorphic to $R$. Hence $R/X$ is faithful. Therefore, $X = 0$. Consequently, $R$ is directly finite. Clearly every faithful cyclic module is isomorphic to $R$.

(iii) $\Rightarrow$ (i). Let $X$ be a right ideal containing no nonzero ideals. Then $R/X$ is isomorphic to $R$. Hence $R = X \oplus S$ where $S$ is a right ideal. Since $R$ is directly finite, $X = 0$. Consequently, $R$ is strongly right bounded.

**Lemma 2.** Let $R$ be a strongly right bounded ring.

(i) Every nonzero right ideal is an essential extension of an ideal of $R$.

(ii) $R$ is right nonsingular if and only if $R$ is semiprime if and only if $R$ is reduced (i.e., $R$ has no nonzero nilpotent elements).

**Proof:** Part (i) is in [16, Note 1.3D]. Part (ii) is in [4, Proposition 1].

**Proposition 3.** Let $R$ be a strongly right bounded ring. Then the following conditions are equivalent:

(i) $R$ is quasi-Baer.

(ii) $R$ is semiprime right quasi-continuous.

(iii) $R$ is semiprime right quasi-FPF.

**Proof:** This result follows from [2, Proposition 1.2], [3, Propositions 4 and 6], and Lemma 2.

The following notation will be used: if $V \subseteq S(R, M)$, then $V_1$ and $V_2$ are the sets of first and second components of $V$, respectively.
Lemma 4.

(i) If $V$ is a right ideal of $S(R, M)$, then $V_1$ is a right ideal of $R$, $V_2$ is a right $R$–submodule of $M$, and $\{0\} \times V_1 M$ is a right $S(R, M)$–submodule of $V$.

(ii) If $W$ is a right ideal of $R$ and $K$ is a right $R$–submodule of $M$ such that $WM \subseteq K$, then $W \times K$ is a right ideal of $S(R, M)$.

(iii) Let $V \subseteq S(R, M)$. Then $[I_R(V_1) \cap I_R(V_2)] \times I_M(V_1) \subseteq I_S(R,M)(V)$.

(iv) The right ideal $\{0\} \times M$ is right essential in $S(R, M)$ if and only if $M$ is left faithful (i.e., $I_R(M) = 0$).

(v) If $V$ and $W$ are right ideals of $S(R, M)$ such that $V \cap W = 0$, then $V_1 M \cap W_1 M = 0$.

(vi) Let $S(R, M)$ be strongly right bounded where $M$ is an ideal of $R$. Then $R$ is strongly right bounded and if $I_R(M) \neq 0$, then $I_R(M) \cap I_R(M) \neq 0$.

(vii) Let $M$ be a module such that whenever $A \cap B = 0$, then $AM \cap BM = 0$ where $A$ and $B$ are right ideals of $R$ (e.g., $M$ is an ideal). If $S(R, M)$ satisfies the ILAS condition, then $R$ satisfies the ILAS condition.

(viii) Let $M$ be an ideal of $R$. Then $S(R, M)$ is right uniform if and only if $R$ is right uniform and $M$ is left faithful.

Proof:

(i) Clearly $V_1$ is a right ideal of $R$ and $V_2$ is a right $R$–submodule of $M$. Let $w \in V_1$ and $m \in M$. There exists $k \in V_2$ such that $(w, k) \in V$. Then $(w, k)(0, m) = (0, wm) \in V$. Thus $\{0\} \times V_1 M$ is a right $S(R, M)$–submodule of $V$.

(ii) and (iii) are straightforward.

(iv) Suppose $\{0\} \times M$ is right essential in $S(R, M)$ and $0 \neq t \in I_R(M)$. There exists $(w, m) \in S(R, M)$ such that $0 \neq \langle t, 0 \rangle (w, m) \in \{0\} \times M$. Contradiction! Hence $M$ is left faithful. Conversely, let $(w, m) \in S(R, M)$. If $w = 0$, we are finished. So assume $w \neq 0$. There exists $k \in M$ such that $0 \neq (w, m)(0, k) = (0, wk) \in \{0\} \times M$. Hence $\{0\} \times M$ is right essential in $S(R, M)$.

(v) Assume $wm = wk \in V_1 M \cap W_1 M$ where $v \in V_1$, $w \in W_1$, and $m, k \in M$. There exists $x \in V_2$ and $y \in W_2$ such that $(v, x) \in V$ and $(w, y) \in W$. Consider $(v, x)(0, m) = (0, vm) = (0, wk) = (w, y)(0, k) \in V \cap W = 0$. Therefore, $V_1 M \cap W_1 M = 0$.

(vi) Let $Y$ be a nonzero right ideal of $R$. There exists an ideal $J$ of $S(R, M)$ such that $J$ is essential in $Y \times YM$. Since $J_1$ and $J_2$ cannot both be zero, $Y$ contains a nonzero ideal. Hence $R$ is strongly right bounded. If $I_R(M) \neq 0$, then there exists a nonzero ideal $H \subseteq I_R(M) \times \{0\}$. Hence $H_1$ is a nonzero ideal of $R$ and $(\{0\} \times M)H = \{0\} \times MH_1 \subseteq H$. Therefore, $0 \neq H_1 \subseteq I_R(M) \cap I_R(M)$.

(vii) Let $A$ and $B$ be right ideals of $R$ such that $A \cap B = 0$. Let $A^* = A \times AM$ and $B^* = B \times BM$. Hence $A^* \cap B^* = 0$. Now $I_{S(R, M)}(A^*) = I_R(A) \times I_M(A)$ and $I_{S(R, M)}(B^*) = I_R(B) \times I_M(B)$. Consequently, $I_R(A)R +
\[ l_R(B)R = R. \]

(viii) Assume \( S(R, M) \) is right uniform and let \( Y \) be a nonzero right ideal of \( R \). By part (iv) \( M \) is left faithful. Let \( 0 \neq w \in R \). There exists \((t, m) \in S(R, M)\) such that \( 0 \neq (w, 0)(t, m) = (wt, w\cdot m) \in Y \times YM \). Therefore, \( R \) is right uniform. Conversely, let \( V \) be a nonzero right ideal of \( S(R, M) \) and \( 0 \neq (t, m) \in S(R, M) \). By part (iv) \( 0 \neq V \cap (\{0\} \times M) = \{0\} \times V_2 \) is essential in \( V \). If \( t \neq 0 \), there exists \( y \in R \) such that \( 0 \neq ty \in V_2 \). Since \( M \) is left faithful, there exists \( k \in M \) such that \( 0 \neq tyk \in V_2 \). Thus \( 0 \neq (t, m)(0, yk) = (0, tyk) \in \{0\} \times V_2 \). If \( t = 0 \), then \( m \neq 0 \) and there exists \( q \in R \) such that \( 0 \neq mq \in V_2 \). Thus \( 0 \neq (t, m)(q, 0) = (0, mq) \in \{0\} \times V_2 \). Consequently, in all cases \( \{0\} \times V_2 \) is right essential in \( S(R, M) \). Therefore, \( S(R, M) \) is right uniform.

We note that if \( R \) is commutative and \( M \) is an ideal of \( R \), then \( S(R, M) \) is commutative. However, in Example 9 we shall provide a strongly right bounded ring \( T_1 \) and an ideal \((T, 0)\) such that \( S(T_1, (T, 0)) \) is not strongly right bounded. Also in [9, Example 2.2] the ring \( R \) is a strongly right bounded ring; however, from Lemma 4 (vi), \( S(R, R(z_1, 0)R) \) is not strongly right bounded. Thus it is natural to investigate conditions on \( R \) and \( M \) which insure that \( S(R, M) \) is strongly right bounded. We say \( M \) is a strongly right bounded module if every nonzero right \( R \)-submodule contains a nonzero \((R, R)\)-bisubmodule of \( M \).

**Theorem 5.** Let \( R \) be a strongly right bounded ring. If either of the following conditions is satisfied, then \( S(R, M) \) is a strongly right bounded ring.

(i) \( M \) is a strongly right bounded module such that \( l_R(M) \) contains no nonzero nilpotent ideals of \( R \) and \( l_R(M) \subseteq r_R(M) \).

(ii) \( M \) is an ideal of \( R \) such that \( l_R(M) \cap M = 0 \).

**Proof:** Let \( V \) be a nonzero right ideal of \( S(R, M) \). If \( V_1 = 0 \) or \( V \cap (\{0\} \times M) \neq 0 \), then there exists a nonzero \((R, R)\)-bisubmodule \( K \subseteq V_2 \) such that \( \{0\} \times K \subseteq V \) is an ideal of \( S(R, M) \). So assume \( V_1 \neq 0 \) and \( V \cap (\{0\} \times M) \neq 0 \). Let \( D \) be a nonzero ideal of \( R \) such that \( D \subseteq V_1 \). Note that with either condition (i) or (ii), \( V_1 M = 0 = MV_1 \). If condition (i) is satisfied, then \( V^2 = V^2_1 \times \{0\} \neq 0 \). Hence \( D \times \{0\} \subseteq V \) is a nonzero ideal of \( S(R, M) \). Now assume condition (ii) is satisfied. If \( V_2 = 0 \), then \( D \times \{0\} \subseteq V \) is a nonzero ideal of \( S(R, M) \). If \( V_2 \neq 0 \), then \( V_2 \times \{0\} \neq 0 \). But \( V(M \times \{0\}) = \{0\} \times V_2 M \subseteq V \cap (\{0\} \times M) = 0 \). Contradiction! Therefore, in all cases \( V \) contains a nonzero ideal of \( S(R, M) \). Consequently, \( S(R, M) \) is strongly right bounded.

We note that when \( M \) is an ideal of \( R \), then \( S(R, M) \) is isomorphic to a subring of \( T_2(R) \) (i.e., the \( 2 \times 2 \) lower triangular matrix ring over \( R \)). However, from [4, Proposition 10], \( T_n(R) \) is never strongly right bounded for \( n > 1 \).

**Corollary 6.** Let \( M \) be an ideal of \( R \). Then \( S(R, M) \) is strongly right bounded right uniform if and only if \( R \) is strongly right bounded right uniform and \( M \) is left faithful.
Proof: This result follows from Theorem 5 and Lemma 4 (viii).

Thus, if $R$ is a strongly right bounded domain and $M$ is any ideal of $R$, then $S(R, M)$ is a strongly right bounded right uniform ring. The ring $H[x]$ where $H$ denotes the real quaternions provides an example of a strongly bounded domain which is neither left nor right duo.

Proposition 7. Let $M$ be a left faithful ideal of $R$. Then the following equivalences are true:

(i) Every ideal of $R$ is right essential in a (ring) direct summand of $R$ if and only if every ideal of $S(R, M)$ is right essential in a (ring) direct summand of $S(R, M)$.

(ii) Every right ideal is right essential in a ring direct summand of $R$ if and only if the same is true for $S(R, M)$.

Proof:

(i) Let $S$ denote $S(R, M)$ and assume every ideal of $R$ is right essential in a direct summand of $R$. Let $Y$ be an ideal of $S$ and $V = Y \cap \{0\} \times M$. By Lemma 4 (iv), $V$ is right essential in $Y$, $V = \{0\} \times V_2$, and $V_2$ is an ideal of $R$. Hence there exists a (central) idempotent $e \in R$ such that $V_2$ is right essential in $eR$. Consider $(e, 0)S$. Let $(x, m) \in S$, then $(e, 0)(x, m) = (ex, em)$. Suppose $0 \neq (ex, em)$. If $ex \neq 0$, then there exists $t \in R$ such that $0 \neq ext \in V_2$. Hence $0 \neq (ex, em)(0, t) = (0, ext) \in V$. If $ex = 0$, then there exists $w \in R$ such that $0 \neq emw \in V_2$. Hence $0 \neq (ex, em)(w, 0) = (0, emw) \in V$. Therefore, in all cases, $V$ is right essential in $(e, 0)S$. Hence $Y$ is right essential in $(e, 0)S$. Consequently, every ideal of $S(R, M)$ is right essential in a (ring) direct summand of $S(R, M)$.

Conversely, suppose every ideal of $S$ is right essential in a (ring) direct summand of $S$. Let $K$ be an ideal of $R$. Then there exists a (central) idempotent $(e, m) \in S$ such that $\{0\} \times KM$ is right essential in $(e, m)S$. Note that $eme = 0$. Hence $(e, m)$ is central in $S$ if and only if $e$ is central in $R$ and $M = 0$. Now $\{0\} \times KM \subseteq (e, m)(\{0\} \times M) \subseteq (e, m)S$. Hence $KM$ is right essential in $eM$ and $eM$ is right essential in $eR$ because $M$ is left faithful in $R$. Since $K$ is an ideal and $KM$ is right essential in $K$, then $K$ is right essential in $eR$.

(ii) This part is proved in a manner similar to that of part (i).

In [15] Faith characterizes when $S(R, M)$ is $\mathcal{F}PF$ where $R$ is commutative and $M$ is faithful. He poses this characterization as an open problem when $R$ is noncommutative. The following result partially generalizes Faith's result.

Corollary 8. Let $R$ be a semiprime or a right nonsingular ring and $M$ be a left faithful ideal of $R$. Then the following conditions are equivalent:

(i) $R$ is strongly right bounded and right quasi-continuous.
(ii) $S(R, M)$ is strongly right bounded and right quasi-continuous.
(iii) $S(R, M)$ is strongly right bounded and right quasi-FPF.

Proof:

(i) $\rightarrow$ (ii) By Lemma 2, $R$ is reduced. Hence every idempotent of $R$ is central. Thus every idempotent of $S(R, M)$ is central. By Theorem 5 and Proposition 7, $S(R, M)$ is strongly right bounded and right quasi-continuous.

(ii) $\rightarrow$ (iii) By Lemma 4 (vi) and Lemma 2, $R$ is reduced. Hence every idempotent of $S(R, M)$ is central. By [3, Proposition 6], $S(R, M)$ is right quasi-FPF.

(iii) $\rightarrow$ (i) By Lemma 4 (vi) and Lemma 2, $R$ is reduced strongly right bounded ring. By Lemma 4 (vi¡), $R$ satisfies the ILAS condition. From [1, Lemma 2.2] and Proposition 3, $R$ is right quasi-continuous.

When $R$ is quasi-Baer strongly right bounded and $M$ is a left faithful ideal of $R$, the sequence of embeddings

$$R \rightarrow S(R, M) \rightarrow T_2(R)$$

is interesting in that $S(R, M)$ is strongly right bounded (and right quasi-continuous) but not quasi-Baer (cf., Proposition 3) and $T_2(R)$ is quasi-Baer [23] but not strongly right bounded. ■

The following example is a special case of a general procedure indicated in [5].

Example 9. Let $I$ denote the ring of integers and $T$ the semigroup ring of $A$ over $I_2$ (i.e., integers modulo 2) where $A$ is the semigroup on the set \{a, b\} satisfying the relation $xy = y$ for $x, y \in A$. Thus $T = \{0, a, b, a + b\}$. Let $T_1$ denote the Dorroh extension of $T$ (i.e., the ring with unity formed from $T \times I$ with componentwise addition and with multiplication given by $(x, k)(y, n) = (xy + nx + ky, kn)$). $T_1$ has the following properties:

(i) The set of nilpotent elements of $T_1$, $N(T_1) = \{(0,0), (a + b,0)\}$, is the Jacobson radical and equals the right socle of $T_1$.

(ii) Every nonzero right ideal of $T_1$ contains either $N(T_1)$ or a nonzero ideal of the form $(0, 2ki) = \{(0, 2ki) \in T_1 | k$ is a fixed integer and $i \in I\}$. Therefore, $T_1$ is strongly right bounded.

(iii) $T_1$ is not right duo since $(a, 1)T_1$ is not an ideal.

(iv) $T_1$ is not strongly left bounded.

(v) $T_1$ does not satisfy the ILAS condition since $l_{T_1}(N(T_1)) + l_{T_1}((a + b, 2)T_1)T_1 \neq T_1$. However if $\{X_i\}$ is a nonempty set of ideals of $T_1$ such that $\cap X_i = 0$ then $R = \sum l_{T_1}(X_i)$, Thus $T_1$ satisfies the ILAS condition defined in [1].

(vi) $T_1$ is not right CS, since $(a + b, 2)T_1$ is not essential in a direct summand. However, every ideal is right essential in a direct summand of $T_1$. 

(vii) $S(I, N(T_1))$ (i.e., split-null extension) is ring isomorphic to the subring $(0, I) + N(T_1)$ of $T_1$. $S(I, N(T_1))$ provides an example for Theorem 5 (i).

(viii) $S(T_1, (0, k2I))$ provides an example for Theorem 5 (ii).

(ix) $S(T_1, (T, 0))$ is an example of a split-null extension of a strongly right bounded ring which is not strongly right bounded (cf. Theorem 5). To see this observe $((a, 1), (0, 0))S(T_1, (T, 0)) = \{(ka, k), (0, 0)\} | k \in I$ contains no nonzero ideals since $((b, 0), (0, 0))((ka, k), (0, 0)) = ((k(a + b), 0), (0, 0))$.

References

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1980 Mathematics Subject Classifications: 16A15

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Rebut el 12 d'Octubre de 1988