WEIGHTED NORM INEQUALITIES
FOR AVERAGING OPERATORS
OF MONOTONE FUNCTIONS

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Abstract

We prove weighted norm inequalities for the averaging operator
\[ A_f(x) = \frac{1}{x} \int_0^x f \, dx \]
of monotone functions.

1. Introduction

This paper is concerned with weighted Hardy type inequalities of the form

\[ \int_0^\infty \left( \frac{1}{x} \int_0^x f \right)^p w(x) \, dx \leq c \int_0^\infty f(x)^p u(x) \, dx. \]

Muckenhoupt [6] has given necessary and sufficient conditions for (*) to hold for arbitrary \( f \).

In their paper [1] Ariño and Muckenhoupt studied the problem when the Hardy-Littlewood maximal operator is bounded on Lorentz spaces and observed that this leads to the study of (*) for non-increasing \( f \). There are more weights in this case than for general \( f \) [1]. They solved the problem for \( w = u \) by the condition \( B_p \), i.e., \( w \in B_p \) if and only if

\[ \int_r^\infty \left( \frac{1}{x} \int_0^x f \right)^p w(x) \, dx \leq c \int_0^r w(x) \, dx, \]

\( r > 0 \). The proof is rather lengthy and first establishes that \( B_p \) implies \( B_{p-\epsilon} \)
(Lemma 2.1 of [1]).

The purpose of this paper is

(i) to give a much shorter proof of a somewhat more general version of (*)
without \( B_p \) implies \( B_{p-\epsilon} \),
(ii) to prove then \( B_p \) implies \( B_{p-\epsilon} \), using an iterated version of (*),
(iii) to investigate the reverse inequalities
\[ \int_0^\infty f(x)^p u(x) \, dx \leq c \int_0^\infty \left( \frac{1}{x} \int_0^x f \right)^p u(x) \, dx, \]
(iv) to study the same questions for non-decreasing functions, and finally
(v) to present some properties of \( B_p \)-weights suggested by the analogous
properties of \( A_p \)-weights as, e.g. the \( A_1 \cdot A_1^{1-p} \) factorization of an \( A_p \)-weight [3].
We point out that the double weight inequality (*) has been characterized in a recent paper by E. Sawyer [7] for non-increasing functions with the $q$-norm of the averaging operator on the left and the $p$-norm on the right. It is also possible to prove some of our results by the methods developed in the paper by D.W. Boyd [2].

Throughout the paper we shall use the following notation. The symbol $f \uparrow (f \downarrow)$ means $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ non-decreasing (non-increasing). For $f \downarrow$ we define $f^{-1}(t) = \inf\{\tau : f(\tau) \leq t\}$ with an analogous statement for $f \uparrow$. In proving (*) for monotone functions we may restrict ourselves to homeomorphisms since a general monotone function can be approximated by homeomorphisms. For $0 < r < \infty$, let $x_r(x) = \chi_{[r, \infty)}(x)$ and $x^r(x) = \chi_{(0, r]}(x)$. By a weight $w$ we mean any measurable $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

2. Non-increasing functions

For the norm inequalities for the averaging operator $Af(x) = \frac{1}{x} \int_0^x f$ we need the following lemma.

Lemma 2.1. Let $\phi \downarrow$ and let $W$ be a weight. Then

(i) $\int_0^\infty \int_0^\infty x\phi(y)(x)W(x)dx dy = \int_0^\infty \phi^{-1}(x)W(x)dx$

(ii) $\int_0^\infty \int_0^\infty \chi_{\phi(y)}(x)\left(\frac{\phi(y)}{x}\right)^p W(x)dx dy

= \int_0^\infty \left\{ \frac{1}{x^p} \int_0^x \phi^{-1}(u)d(u^p) - \phi^{-1}(x) \right\} W(x)dx.$

Proof: (i) We interchange the order of integration and get

$\int_0^\infty \int_0^{\phi^{-1}(x)} W(x)dy dx = \int_0^\infty \phi^{-1}(x)W(x)dx.$

(ii) The left side is, after interchanging the order of integration,

$\int_0^\infty \int_0^{\phi^{-1}(x)} \frac{W(x)}{x^p}(\phi(y))^p dy dx$

and the inner integral in $y$ is

$\int_{\phi^{-1}(x)}^\infty (\phi(y))^p dy = \int_0^{x^p} \phi^{-1}(t^{1/p})dt - x^p\phi^{-1}(x)$

$= \int_0^x \phi^{-1}(u)d(u^p) - x^p\phi^{-1}(x).$
This can be seen by comparing areas of the regions under the curve $t = (\varphi(y))^p$ or $y = \varphi^{-1}(t^{1/p})$.

**Definition.** For $1 \leq p < \infty$ and $n$ a positive integer we write $(w, v) \in B(p, n)$ if and only if there is $0 < c < \infty$ such that for every choice $0 < r_1, r_2, \cdots, r_n < \infty$,

$$
\int_0^\infty \left\{ \prod_{j=1}^n \left( \chi_{r_j}(x) + \chi_{r_j}(x) \left( \frac{r_j}{x} \right)^p \right) \right\} w(x) dx \\
\leq c \int_0^\infty \left\{ \prod_{j=1}^n \chi_{r_j}(x) \right\} v(x) dx.
$$

**Remark.**

(i) In case $w = v$, we simply write $w \in B(p, n)$.

(ii) If $n = 1$, then $(w, v) \in B(p, 1)$ means

$$
\int_0^r w + \int_r^\infty \left( \frac{r}{x} \right)^p w(x) dx \leq c \int_0^r v,
$$

$r > 0$. Hence, if $v = w$, we get the equivalent condition

$$
\int_0^\infty \left( \frac{r}{x} \right)^p w(x) dx \leq c \int_0^r w
$$


(iii) The smallest $c$ in the above expressions will be referred to as the $B_p(w)$-constant of $w$ or the $B(p, n)$-constant of $(w, v)$.

(iv) If we let $r_n \to \infty$ we see that $B(p, n) \subset B(p, n-1)$.

**Theorem 2.2.** Let $1 \leq p < \infty$ and let $f_j \downarrow$, $j = 1, \cdots, n$. Then

$$
\int_0^\infty \left\{ \prod_{j=1}^n \left( \frac{1}{x} \int_0^x f_j \right)^p \right\} w(x) dx \leq c \int_0^\infty \left\{ \prod_{j=1}^n f_j \left( \frac{1}{x} \int_0^x f_j \right)^{p-1} \right\} v(x) dx
$$

if and only if $(w, v) \in B(p, n)$ with $c$ equal to the $B(p, n)$-constant of $(w, v)$.

**Proof.** If $f_j = \chi_{r_j}$, $j = 1, \cdots, n$, then the norm inequality easily gives $(w, v) \in B(p, n)$. We do the converse for $n = 2$; the general case is obtained by repeating the argument.

Let $\varphi_j \downarrow$, $j = 1, 2$, and let $r_j = \varphi_j(y_j)$, where $0 < y_1, y_2 < \infty$. We next integrate the condition $B(p, 2)$ over $\{(y_1, y_2) : y_1, y_2 > 0\}$ and obtain

$$
L \equiv \int_0^\infty \int_0^\infty \int_0^\infty \psi_1(x, y_1) \psi_2(x, y_2) w(x) dx dy_1 dy_2 \\
\leq c \int_0^\infty \int_0^\infty \int_0^\infty \chi_{\varphi_1(y_1)}(x) \chi_{\varphi_2(y_2)}(x) w(x) dx dy_1 dy_2 \equiv R,
$$
where \( \psi_j(x, y_j) = \chi_{\varphi_j(y_j)(x)} + \chi_{\varphi_j(y_j)}(x) \left( \frac{\varphi_j(y_j)}{x} \right)^p \). By Lemma 2.1,

\[
R = \int_0^\infty \int_0^\infty \varphi_1^{-1}(x) \chi_{\varphi_2(y_j)(x)} u(x) dx dy_j \\
= \int_0^\infty \varphi_1^{-1}(x) \varphi_2^{-1}(x) u(x) dx.
\]

The inner 2 integrals of \( L \) can be written as

\[
\int_0^\infty \int_0^{\psi_1(y_1)} \psi_2(x, y_2) w(x) dx dy_1 \\
+ \int_0^\infty \int_{\varphi_1(y_1)}^\infty \psi_2(x, y_2) \left( \frac{\varphi_1(y_1)}{x} \right)^p w(x) dx dy_1 = I_1 + I_2.
\]

By (i) of Lemma 2.1 with \( W = \psi_2 w \), \( I_1 = \int_0^\infty \varphi_1^{-1}(x) \psi_2(x, y_2) w(x) dx \). Similarly, by (ii) of Lemma 2.1,

\[
I_2 = \int_0^\infty \left\{ \frac{1}{x^p} \int_0^x \varphi_1^{-1}(u) d(u^p) - \varphi_1^{-1}(x) \right\} \psi_2(x, y_2) w(x) dx.
\]

Hence \( I_1 + I_2 = \int_0^\infty \left\{ \frac{1}{x^p} \int_0^x \varphi_1^{-1}(u) d(u^p) \right\} \psi_2(x, y_2) w(x) dx \). We integrate this expression in \( y_2 \) and repeat the argument to get

\[
L = \int_0^\infty \left\{ \frac{1}{x^p} \int_0^x \varphi_1^{-1}(u) d(u^p) \right\} \left\{ \frac{1}{x^p} \int_0^x \varphi_2^{-1}(u) d(u^p) \right\} w(x) dx.
\]

We thus obtain

\[
\int_0^\infty \left\{ \frac{1}{x^p} \int_0^x \varphi_1^{-1}(u) d(u^p) \right\} \left\{ \frac{1}{x^p} \int_0^x \varphi_2^{-1}(u) d(u^p) \right\} w(x) dx \\
\leq c \int_0^\infty \varphi_1^{-1}(x) \varphi_2^{-1}(x) v(x) dx.
\]

We remark here that the constant \( c \) is the same as the \( c \) in \( B(p, 2) \).

We now let \( \varphi_j^{-1}(u) = f_j(u) \left( \frac{1}{u} \int_0^u f_j \right)^{p-1} \), \( j = 1, 2 \), and observe that

\[
\frac{1}{x^p} \int_0^x \varphi_j^{-1}(u) d(u^p) = \frac{1}{x^p} \int_0^x f_j(u) \left( \int_0^u f_j \right)^{p-1} du \\
= \frac{1}{x^p} \left( \int_0^x f_j \right)^p.
\]
This completes the proof of Theorem 2.2. □

**Remark.** It may be of interest to point out that there is an easy condition for equality in Theorem 2.2. Let

(i) \[ \int_0^\infty A f^P w = \int_0^\infty f A f^{P-1} v, \]

(ii) \[ v(t) = pt^{p-1} \int_t^\infty \frac{w(x)}{x^p} \, dx. \]

If (i) holds for \( f \downarrow \), then (ii) follows. Simply let \( f = \chi_t \) and differentiate the resulting equation \[ \int_0^t v = \int_0^t w + \int_t^\infty \left( \frac{t}{x} \right)^p w(x) \, dx. \] Conversely, if (ii) holds, then (i) is valid for any \( f : \mathbb{R}_+ \to \mathbb{R}_+ \). This can be seen by replacing \( v \) in (i) by (ii) and then integrating by parts.

We state the special case \( p = 1 \) of Theorem 2.2 as

**Corollary 2.3.** If \( f_j \downarrow, j = 1, \ldots, n \), then

\[ \int_0^\infty \left\{ \prod_{j=1}^n \left( \frac{1}{x} \int_0^x f_j \right) \right\} w(x) \, dx \leq c \int_0^\infty \left\{ \prod_{j=1}^n f_j(x) \right\} v(x) \, dx \]

if and only if \( (w, v) \in B(1, n) \).

The case \( w = v \) of Theorem 2.2 yields as a special case the Ariño-Muckenhoupt weighted norm inequality for non-increasing functions [1].

**Corollary 2.4.** Let \( 1 \leq p < \infty \) and \( f_j \downarrow, j = 1, \ldots, n \). Then

\[ \int_0^\infty \left\{ \prod_{j=1}^n \left( \frac{1}{x} \int_0^x f_j \right)^p \right\} w(x) \, dx \leq c \int_0^\infty \left\{ \prod_{j=1}^n f_j(x)^p \right\} w(x) \, dx \]

if and only if \( w \in B(p, n) \).

**Proof:** The necessity follows from \( f_j = \chi_{r_j} \), and for the sufficiency we apply Hölder’s inequality to obtain

\[ \int_0^\infty \left\{ \prod_{j=1}^n f_j \right\} \cdot \prod_{j=1}^n \left( \frac{1}{x} \int_0^x f_j \right)^{p-1} w(x) \, dx \]

\[ \leq \left\{ \int_0^\infty \left\{ \prod_{j=1}^n f_j \right\}^p \, dx \right\}^{1/p} \left\{ \int_0^\infty \left\{ \prod_{j=1}^n \left( \frac{1}{x} \int_0^x f_j \right)^p \right\} w \, dx \right\}^{1/p}. \]

Divide by the last factor to obtain the norm inequality. □

**Remark.** (i) For a single weight the conditions \( B(p, n) \) and \( B_p \) are equivalent, i.e., \( w \in B(p, n) \) iff \( w \in B_p \). Since the implication \( B(p, n) \subset B_p \) was
already observed in (iv) of the previous remark, we only need to show that $B_p \subseteq B(p, n)$. It is clear that if $u \downarrow$ and $w \in B_p$, then $uw \in B_p$. Let now $f_j \downarrow$, $j = 1, 2$, and let $w \in B_p$. Then $A_j f_j(x)^p w(x) \in B_p$, and hence

$$\int_0^\infty A_j f_j^p A_j f_j^p w \leq c \int_0^\infty f_j^p A_j f_j^p w.$$ 

Since $f_j^p w \in B_p$, we can continue this inequality $\leq c \int_0^\infty f_j^p f_j^p w$, i.e., $w \in B(p, 2)$.

(ii) Results related to the above Corollaries can also be found in [2].

We will now show that an iterated version of Corollary 2.4 provides a short proof of $B_p$ implies $B_{p-\varepsilon}$, the basic Lemma in [1]. Similar ideas for the Hardy-Littlewood maximal operator and the “$A_p$ implies $A_{p-\varepsilon}$” case can be found in [4],[5].

**Theorem 2.5.** Let $1 \leq p < \infty$ and let $w \in B(p, 1)$. Then there is $\varepsilon > 0$ such that $w \in B(p-\varepsilon, 1)$.

**Proof:** Fix $r > 0$ and let $f = \chi_r$. If $A_n f(x)$ is the $n$-times iterated averaging operator, i.e., $A_0 f(x) = f(x)$, $A_1 f(x) = \frac{1}{x} \int_0^x f$, then for $n \geq 1$,

$$A_n f(x) = \begin{cases} 1, & 0 < x \leq r \\ \frac{r}{x} \sum_{j=0}^{n-1} \frac{1}{j!} \log^j \left( \frac{x}{r} \right), & x > r. \end{cases}$$

Since $w \in B(p, 1)$ we have from Corollary 2.4,

$$\int_0^\infty A_n f(x)^p w(x) dx \leq c^n \int_0^\infty f(x)^p w(x) dx$$

$$= c^n \int_0^r w(x) dx.$$ 

For $x > r$,

$$A_n f(x)^p = \left( \frac{r}{x} \right)^p \left( \sum_{j=0}^{n-1} \frac{1}{j!} \log^j \left( \frac{x}{r} \right) \right)^p$$

$$\geq \left( \frac{r}{x} \right)^p \left( \sum_{j=0}^{n-1} \frac{1}{j!} \log^j \left( \frac{x}{r} \right) \right) \geq \left( \frac{r}{x} \right)^p \frac{1}{(n-1)!} \log^{n-1} \left( \frac{x}{r} \right),$$

where the next to the last inequality follows since $\sum_{j=0}^{n-1} j \geq 1$. We substitute this in our norm inequality and get

$$\int_r^\infty \left( \frac{r}{x} \right)^p \frac{1}{(n-1)!} \log^{n-1} \left( \frac{x}{r} \right) w(x) dx \leq c^n \int_r^\infty w(x) dx.$$
Let $s > c$. Then
\[
\int_{r}^{\infty} \left( \frac{r}{x} \right)^{p} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left( \log \frac{x}{s} \right)^{n-1} w(x)dx \leq C \int_{0}^{r} w(x)dx
\]
or
\[
\int_{r}^{\infty} \left( \frac{r}{x} \right)^{p-1/s} w(x)dx \leq C \int_{0}^{r} w(x)dx,
\]
i.e. $w \in B\left(p - \frac{1}{s}, 1\right)$. 

3. The case $n = 1$ and reverse inequalities

We begin by asking for which averaging operator is $(w, v) \in B(p, 1)$ a necessary and sufficient condition for a weighted norm inequality. The case $p = 1$ is handled by Corollary 2.3 with $A f(x) = \frac{1}{x} \int_{0}^{x} f$. For $1 < p < \infty$ we define
\[
A_{p} f(x) = \left\{ \frac{1}{x^{p}} \int_{0}^{x} f(u)^{p} d(u^{p}) \right\}^{1/p}.
\]

**Theorem 3.1.** If $f \downarrow$ and $1 < p < \infty$, then
\[
\int_{0}^{\infty} A_{p} f(x)^{p} w(x)dx \leq c \int_{0}^{\infty} f(x)^{p} v(x)dx
\]
if and only if $(w, v) \in B(p, 1)$.

**Proof:** The necessity follows by taking $f = x_{r}$.

For the sufficiency simply let $\varphi^{-1}(u) = f(u)^{p}$ in the proof of Theorem 2.2.

We will now characterize the weights $(w, v)$ for which the reverse inequality holds for $f \downarrow$. 

**Theorem 3.2.** Let $f \downarrow$ and $1 \leq p < \infty$. Then
\[
\int_{0}^{\infty} f(x)^{p} w(x)dx \leq c \int_{0}^{\infty} \left( \frac{1}{x} \int_{0}^{x} f \right)^{p} v(x)dx
\]
if and only if \[ \int_0^\infty \int_0^\infty w(x) dx dy = \int_0^\infty \phi^{-1}(x)w(x)dx \]

Proof: The necessity follows with \( f = \chi_r \). For the sufficiency, let \( \phi \downarrow \) and let \( \tau = \varphi(y) \). Then as in the proof of Theorem 2.2,

\[
\int_0^\infty \int_0^\infty w(x) dx dy = \int_0^\infty \phi^{-1}(x)w(x)dx
\]
and

\[
\int_0^\infty \int_0^\infty \phi(y) v(x) dx dy + \int_0^\infty \int_0^\infty w(x) \phi(y) dx dy
\]
\[
= \int_0^\infty \int_0^\infty \phi^{-1}(x)v(x) dx + \int_0^\infty \frac{1}{x^p} \int_0^x \phi^{-1}(u) d(u^p)v(x) dx
\]
\[
- \int_0^\infty \phi^{-1}(x)v(x) dx = \int_0^\infty \frac{1}{x^p} \int_0^x \phi^{-1}(u) d(u^p)v(x) dx.
\]

We let now \( \phi^{-1}(u) = f(u) \left( \frac{1}{u} \int_0^u f \right)^{p-1} \) and obtain

\[
\int_0^\infty f(x) \left( \frac{1}{x} \int_0^x f \right)^{p-1} w(x)dx \leq c \int_0^\infty \left( \frac{1}{x} \int_0^x f \right)^p v(x)dx.
\]

We complete the proof by noting that \( \frac{1}{x} \int_0^x f \geq f(x) \) since \( f \downarrow \). \( \blacksquare \)

We will now characterize the single weights, i.e., \( w = v \), for which the above reverse inequality holds for a given \( 0 < c < 1 \).

Theorem 3.3. The following statements are equivalent for \( f \downarrow \), \( 0 < c < 1 \), \( 1 < p < \infty \), and \( w \in L^1_{loc}(\mathbb{R}^+) \).

1. \( \int_0^\infty f^pw \leq c \int_0^\infty Af^pw \)
2. \( B_p(w(1-p')) \leq c \frac{c}{1-c}. \)

Proof: (1) \( \Rightarrow \) (2). If \( f = \chi_r \), we get

\[
\int_0^r w \leq c \left( \int_0^r w + \int_r^\infty \left( \frac{r}{x} \right)^p w(x)dx \right).
\]

We let \( x = y^{1-p'} \) and get

\[
\int_0^r w(x)dx = (p' - 1) \int_{r^{1-p'}}^\infty w(1-p') \frac{dy}{y^{p'}},
\]
\[
r^p \int_r^\infty \frac{w(x)}{x^p} dx = (p' - 1)r^p \int_0^{r^{1-p'}} w(1-p')dy.
\]
Hence
\[(1 - c)(p' - 1) \int_{r^{1-p}}^{\infty} w(y^{1-p'}) \frac{dy}{y^{p'}} \leq c(p' - 1)r^p \int_0^{r^{1-p}} w(y^{1-p'}) dy.\]

If we set \( \rho = r^{1-p} \), then \( r^p = \frac{1}{\rho^{p'}} \) and (2) follows.

(2) \( \rightarrow \) (1). We have
\[\int_r^{\infty} \left( \frac{r}{y} \right)^p w(y^{1-p'}) \frac{dy}{y^{p'}} \leq \frac{c}{1 - c} \int_0^r w(y^{1-p'}) dy.\]

Let \( y = x^{1-p} \). Then, again
\[\int_r^{\infty} \left( \frac{r}{y} \right)^p w(y^{1-p'}) \frac{dy}{y^{p'}} = r^p (p - 1) \int_0^{r^{1-p'}} w(x) dx \]
\[\int_0^r w(y^{1-p'}) \frac{dy}{y^{p'}} = (p - 1) \int_{r^{1-p'}}^{\infty} \frac{w(x)}{x^p} dx.\]

Thus, with \( \rho = r^{1-p'} \) we get
\[\int_0^\rho w(x) dx \leq \frac{c}{1 - c} \int_\rho^{\infty} \left( \frac{\rho}{x} \right)^p w(x) dx.\]

We add \( \frac{c}{1 - c} \int_0^\rho w \) to both sides and get
\[\int_0^\rho w \leq c \left( \int_0^\rho w + \int_\rho^{\infty} \left( \frac{\rho}{x} \right)^p w(x) dx \right).\]

Apply now Theorem 3.2. ■

Remark. (2) of Theorem 3.3 reminds one of the duality \( w \in A_p \) iff \( w^{1-p'} \in A_{p'} \).

4. Non-decreasing functions

We will not dwell on the straightforward results of \( f \uparrow \) that one gets from our previous results via the change of variables \( x \rightarrow \frac{1}{x} \). In particular we have

Theorem 4.1. If \( f \uparrow \) and \( 1 \leq p < \infty \), then
\[\int_0^\infty \left( \frac{x}{x^2} f(u) \frac{du}{u^2} \right) \frac{w(x) dx}{x^p} \leq c \int_0^\infty f(x)p w(x) dx \]
if and only if \( \int_0^r \left( \frac{r}{x} \right)^p w(x) dx \leq c \int_r^{\infty} w(x) dx, r > 0.\)

In order to see what type of results one has for the averaging operator \( \frac{1}{x} \int_0^x f \)
for \( f \uparrow \) we need a lemma similar to Lemma 2.1.
Lemma 4.2. Let \( \varphi \uparrow \) with \( \varphi(0) = 0 \), and let \( W \) be a weight. Then

\[
(i) \quad \int_0^\infty \int_0^\infty \chi^{\varphi(y)}(x)W(x)dx \, dy = \int_0^\infty \varphi^{-1}(x)W(x)dx
\]

\[
(ii) \quad \int_0^\infty \int_0^\infty \chi^{\varphi(y)}(x)\left(\frac{y - \varphi(y)}{x}\right)^p W(x)dx \, dy = \int_0^\infty \left\{ \frac{1}{x^p} \int_0^x \varphi^{-1}(x - u)d(u^p) \right\} W(x)dx.
\]

Proof: For (i) we simply interchange the order of integration. The left side of (ii) is

\[
\int_0^\infty \int_0^\infty \chi^{\varphi(y)}(x)\left(\frac{y - \varphi(y)}{x}\right)^p W(x)dx \, dy = \int_0^\infty \varphi^{-1}(x)W(x)dx
\]

as can be seen by interpreting the integral as area under \( t = (x - \varphi(y))^p \).

Definition. Let \( n \) be a positive integer and \( 1 < p < \infty \). We say that \((w, v) \in C(p, n)\) if and only if there is \( 0 < c < \infty \) such that for every choice \( 0 < r_1, r_2, \ldots, r_n < \infty \),

\[
\int_0^\infty \left\{ \prod_{j=1}^n \chi^{r_j}(x) \right\} w(x)dx \leq c \int_0^\infty \left\{ \prod_{j=1}^n \chi^{r_j}(x) \left(\frac{x - r_j}{x}\right)^p \right\} v(x)dx.
\]

Theorem 4.3. Let \( f_j, j = 1, \ldots, n \). Then

\[
\int_0^\infty \left\{ \prod_{j=1}^n f_j(x) \right\} w(x)dx \leq c \int_0^\infty \left\{ \prod_{j=1}^n \left(\frac{1}{x} \int_0^x f_j \right) \right\} v(x)dx
\]

if and only if \((w, v) \in C(1, n)\).

Proof: The necessity follows by taking \( f_j = \chi^{r_j} \). As in Theorem 2.2 we prove the converse for \( n = 2 \); the general case is obtained by repeating the argument. We let \( \varphi_j 1, \varphi_j(0) = 0, \) and \( r_j = \varphi_j(y_j), j = 1, 2, \) where \( 0 < y_1, y_2 < \infty \). We next integrate the \( C(1, n) \) condition over all such \((y_1, y_2)\) and obtain

\[
L \equiv \int_0^\infty \int_0^\infty \int_0^\infty \chi^{\varphi_1(y_1)}(x)\chi^{\varphi_2(y_2)}(x)w(x)dx \, dy_1 \, dy_2
\]

\[
\leq c \int_0^\infty \int_0^\infty \int_0^\infty \psi_1(x, y_1)\psi_2(x, y_2)v(x)dx \, dy_1 \, dy_2 \equiv R,
\]
where \( \psi_j(x,y_j) = \chi_{r_j}(y_j)(x) \left( \frac{x - \varphi_j(y_j)}{x} \right) \). By (i) of Lemma 4.2,

\[
L = \int_0^\infty \varphi_j^{-1}(x)\varphi_1^{-1}(x)w(x)dx,
\]

and by (ii) with \( p = 1 \),

\[
R = \int_0^\infty \left( \frac{1}{x} \int_0^x \varphi_j^{-1} \right) \left( \frac{1}{x} \int_0^x \varphi_1^{-1} \right) v(x)dx.
\]

From this we get the theorem by letting \( \varphi_j^{-1}(t) = f_j(t) \) if \( f_j(0) = 0 \). Otherwise, let \( \epsilon_n(x) = nx \), if \( 0 \leq x \leq \frac{1}{n} \), and \( \epsilon_n(x) = 1 \), \( x > \frac{1}{n} \). If \( \varphi_j^{-1}(t) = \epsilon_n(t)f_j(t) \), then we get the weighted norm inequality for \( \epsilon_n f_j \), and the final result by letting \( n \to \infty \).

**Corollary 4.4.** Let \( f \uparrow \) and \( n \) a positive integer. Then

\[
\int_0^\infty f(x)^nw(x)dx \leq c \int_0^\infty \left( \frac{1}{x} \int_0^x f \right)^n v(x)dx
\]

if and only if \((w,v) \in C(1,n)\).

**Proof:** If \((w,v) \in C(1,n)\), then the inequality follows from Theorem 4.3 by letting \( f_1 = f_2 = \cdots = f_n \). Conversely, let \( f = \prod_{1}^{n} \chi_{r_j} \). Then \( f = f^n \) and by Hölder's inequality

\[
\left( \frac{1}{x} \int_0^x f \right)^n \leq \prod_{1}^{n} \left( \frac{1}{x} \int_0^x \chi_{r_j} \right) = \prod_{1}^{n} \chi_{r_j}(x) \left( \frac{x - r_j}{x} \right). \]

**Remark.** We were unable to find a characterization of

\[
\int_0^\infty f(x)^pw(x)dx \leq c \int_0^\infty \left( \frac{1}{x} \int_0^x f \right)^p v(x)dx
\]

for \( f \uparrow \) and \( p \) not a positive integer. However, as we shall see, \((w,v) \in C(p,1)\) controls the averaging operator

\[
A_p f(x) = \frac{1}{x^p} \int_0^x f(x-u)du^p.
\]

We observe that, when \( p \) is a positive integer, then \( \int_0^x f(x-u)du^p \) is, apart from a multiplicative constant, the \( p \)-times iterated integral of \( f \).
Theorem 4.5. Let $f$ ↑ and $1 \leq p < \infty$. Then

(i) $\int_0^\infty A_p f(x) w(x) dx \leq c \int_0^\infty f(x) v(x) dx$ if and only if $\int_r^\infty \left( \frac{x - r}{x} \right)^p w(x) dx \leq c \int_r^\infty v(x) dx$, $r > 0$.

(ii) $\int_0^\infty f(x) w(x) dx \leq c \int_0^\infty A_p f(x) v(x) dx$ if and only if $\int_r^\infty \left( \frac{x - r}{x} \right)^p v(x) dx \leq c \int_r^\infty x w(x) dx$, $r > 0$, i.e., $(w, v) \in C(p, 1)$.

Proof: (i) For the necessity let $f = x^r$. To prove the sufficiency, let $\varphi \uparrow$, $\varphi(0) = 0$, and $r = \varphi(y)$, $0 < y < \infty$. Then

$$L \equiv \int_0^\infty \int_0^\infty \frac{w(x)}{x^p} (x - \varphi(y))^p dx \, dy \leq c \int_0^\infty \int_0^\infty v(x) dx \, dy \equiv R.$$ 

By Lemma 4.2, $R = \int_0^\infty \varphi^{-1}(x) v(x) dx$ and

$$L = \int_0^\infty \left\{ \frac{1}{x^p} \int_0^x \varphi^{-1}(x-u) d(u^p) \right\} w(x) dx.$$ 

The proof can be completed by letting $\varphi^{-1}(x) = f(x)$ if $f(0) = 0$; otherwise let $\varphi^{-1}(x) = \varepsilon_n(x) f(x)$ as in the proof of Theorem 4.3.

The proof of (ii) is the same as the one for (i). ■

5. More properties of weights

We begin with a "change of variables" result for $B_p$-weights.

Theorem 5.1. If $1 < q < p < \infty$ and $w \in B_q$, then $w \left( x^{\frac{p}{q}} \right) \in B_p$.

Proof: We set $I_r = \int_r^\infty \left( \frac{r}{x} \right)^p w \left( x^{\frac{p}{q}} \right) dx$ and let $u = x^\alpha$, $\alpha = \frac{p - 1}{q - 1}$. Then

$$I_r = c \int_r^\infty \left( \frac{r}{u^{1/\alpha}} \right)^p w(u) u^{\frac{1-\alpha}{\alpha}} du$$

$$= c \int_r^\infty \frac{r^p}{u^{(p+\alpha-1)/\alpha}} w(u) du.$$ 

We observe that $(p + \alpha - 1)/\alpha = q$ and so

$$I_r = \int_r^\infty \left( \frac{u^{\alpha}}{u} \right)^q w(u) du \cdot r^{p-\alpha q}.$$
Since $w \in B_q$ and $p - \alpha q = \frac{q - p}{q - 1} < 0$, we see that

$$I_r \leq cr^\frac{q-p}{q-1} \int_0^r w(u)du = cr^{1-\alpha} \int_0^r w(x^\alpha)x^{\alpha-1}dx$$

$$\leq c \int_0^r w(x^\alpha)dx.$$

The case $q = 1$ yields a slightly stronger result which we state as

**Theorem 5.2.** If $w \in B_1$ and $\alpha \geq 1$, then $w(x^\alpha) \in B_1$ with $B_1(w) = B_1(w(x^\alpha))$.

**Proof.** If $I_r = \int_r^\infty \left( \frac{r}{x} \right) w(x^\alpha)dx$ and $u = x^\alpha$, then

$$I_r = \frac{1}{\alpha} \int_{r^\alpha}^\infty \left( \frac{r}{u^{1/\alpha}} \right) w(u)u^{1/\alpha-1}du = \frac{r^{1-\alpha}}{\alpha} \int_{r^\alpha}^\infty \left( \frac{r^\alpha}{u} \right) w(u)du$$

$$\leq cr^{1-\alpha} \int_0^r w(x^\alpha)x^{\alpha-1}dx \leq c \int_0^r w(x^\alpha)dx,$$

since $\alpha \geq 1$. ■

The next result reminds one of the important $A_p$-property, i.e., $w \in A_p \rightarrow w^r \in A_p$ for some $r > 1$.

**Theorem 5.3.** If $w \in B_p$, then there is $\epsilon > 0$ such that $x^\epsilon w(x^{1+\epsilon}) \in B_p$.

**Proof.** Choose $\epsilon > 0$ so that $w \in B_{p/1+\epsilon}$ (Theorem 2.5), and note that

$$\int_r^\infty \left( \frac{r}{x} \right)^\epsilon x^\epsilon w(x^{1+\epsilon})dx = \frac{1}{1+\epsilon} \int_{r^{1+\epsilon}}^\infty \frac{r^p}{u^{p/1+\epsilon}} w(u)du$$

$$= \frac{1}{1+\epsilon} \int_{r^{1+\epsilon}}^\infty \left( \frac{r^{1+\epsilon}}{u} \right)^{p/1+\epsilon} w(u)du \leq \frac{c}{1+\epsilon} \int_0^r w(u)du$$

$$= c \int_0^r x^\epsilon w(x^{1+\epsilon})dx.$$

**Corollary 5.4.** If $w \in B_p$, then there is $\epsilon > 0$ such that $w(x^{1+\epsilon}) \in B_p$.

We are now ready to present a factorization theorem for $B_p$-weights similar to the factorization of $w \in A_p$ as $w = uv^{1-p}$, $u, v \in A_1$. 
Theorem 5.5. The following statements are equivalent for $1 < p < \infty$.

1. $w \in B_p$
2. $w(x) = u(x) \cdot x^{p-1}$ with $u(x^{1/p}) \in B_1$.

Proof: (1) $\Rightarrow$ (2). All we need to show is that $\frac{w(x^{1/p})}{x^{1/p'}} = u(x^{1/p})$ is in $B_1$, and this follows from

\[ \int_r^\infty \left( \frac{r}{x} \right)^p \frac{w(x^{1/p})}{x^{1/p'}} \, dx = c \int_r^\infty \left( \frac{r}{t^{1/p}} \right)^p \frac{w(t)}{t^{p/p'}} \, dt \]

\[ = c \int_r^\infty \left( \frac{r}{t} \right)^p w(t) \, dt \leq c \int_0^r \left( \frac{t}{t^{1/p}} \right)^p w(t) \, dt = c \int_0^r w(t^{1/p}) \, dt. \]

(2) $\Rightarrow$ (1). This is simply

\[ \int_r^\infty \left( \frac{t}{x} \right)^p u(x) \, dx = \frac{1}{p} \int_r^\infty \left( \frac{r}{t^{1/p}} \right)^p u(t^{1/p}) \, dt \]

\[ = \frac{1}{p} \int_r^\infty \left( \frac{t}{x} \right)^p u(t^{1/p}) \, dt \leq \frac{c}{p} \int_0^r u(t^{1/p}) \, dt = c \int_0^r u(x) \, dx. \]

Remark. By Theorem 5.2, if $u(x^{1/p}) \in B_1$, then $u(x) \in B_1$. Thus (2) can be written as $w = u \cdot \left( \frac{1}{x} \right)^{1-p}$, with $u \in B_1$. It is also clear that $\frac{1}{x} \in B_1$.

6. Weak type weights

We say that $w \in R_p$ iff $w(Ax, > y) \leq \frac{c}{y^p} \int_0^r w \, dx$, $r > 0$, and we say that $w \in W_p$ iff for $f \downarrow$, $w(Af, > y) \leq \frac{c}{y^p} \int_0^\infty f^p \, w$. The "R" in $R_p$ stands for "restricted".

We will study relationships among $R_p$, $W_p$, and $B_p$, and give a characterization of $B_p$.

Theorem 6.1. $w \in R_p$ iff there is $0 < c < \infty$ so that for $0 < r < s < \infty$,

\[ \frac{1}{sp} \int_0^s w \leq \frac{1}{rp} \int_0^r w. \]
Proof: First assume that \( w \in R_p \). The set \( \{ A \chi_r > y \} = (0, x_0) \), where \( \frac{r}{x_0} = y, 0 < y < 1 \). Hence \( \int_0^{x_0} w \leq \frac{c}{y^p} \int_0^r w \) from which

\[
\frac{1}{s^p} \int_0^s w \leq \frac{c}{r^p} \int_0^r w, \quad s = \frac{r}{y} > r.
\]

Conversely, for \( 0 < y < 1 \), with the same notation as above,

\[
w \{ A \chi_r > y \} = \int_0^{x_0} w = \frac{1}{y^p} \left( \frac{r}{x_0} \right)^p \int_0^{x_0} w \\
\leq \frac{c}{y^p} \int_0^r w.
\]

The next result shows how \( R_p \) and \( B_q \) are related.

**Theorem 6.2.** If \( w \in R_p \), then \( w \in B_q \) for \( q > p \).

**Proof:** From Theorem 6.1, for \( s > r \),

\[
\left( \frac{r}{s} \right)^p \int_0^s w \leq c \int_0^r w.
\]

Let \( t = \frac{r}{s} \). Then \( t^p \int_0^{r/t} w \leq c \int_0^r w, \) or, if \( 0 < \epsilon < 1 \),

\[
t^{p-\epsilon} \int_0^{r/t} w \leq ct^{-\epsilon} \int_0^r w, \quad 0 < t \leq 1.
\]

Hence

\[
L = \int_0^1 t^{p-\epsilon} \int_0^{r/t} w(x) dx dt \leq c \int_0^r w.
\]

We interchange the order of integration and see that

\[
L \geq \int_r^\infty \int_0^{r/x} w(x) t^{p-\epsilon} dt dx = c \int_r^\infty w(x) \left( \frac{r}{x} \right)^{p+1-\epsilon} dx.
\]

Thus \( w \in B_q, q = p + 1 - \epsilon. \)

**Example.** Let \( w(x) = x \). Then \( w \in R_2 \) but not in \( W_2 \) and thus not in \( B_2 \). For let \( f(x) = \frac{1}{x \log^{1/2} x} \cdot \chi_1(x) \). Then \( w \{ A f > y \} = \infty \), but \( \int f^2 w = \int_0^{1/e} \frac{dx}{x \log^{1/2} x} < \infty. \)

We will now show that the condition of Theorem 6.1 which characterizes \( R_p \) will, if slightly modified, characterize \( B_p \). We begin with
Lemma 6.3. Assume there exists $1 < a < \infty$ and $0 < c = c_a < 1$ such that
\[
\frac{1}{(ax)^p} \int_0^{ax} w dx \leq c \frac{1}{x^p} \int_0^x w, \ x > 0. \quad \text{Then} \ w \in B_p.
\]

Proof: For $0 < N < \infty$, let $w_N = w_{xN}$. Then $w_N$ satisfies the same hypothesis as $w$ with a constant $c = \max(c_a, 1/a^p) < 1$.

We have then
\[
L = \frac{1}{a^p} \int_0^\infty \frac{1}{x^{p+1}} \int_0^{ax} w_N(t) dt dx \leq c \int_0^\infty \frac{1}{x^{p+1}} \int_0^x w_N(t) dt dx = R.
\]

We interchange the order of integration and see that
\[
L \geq \frac{1}{a^p} \int_0^\infty \int_0^{ax} w_N(t) \frac{dx}{x^{p+1}} dt = \frac{1}{p} \int_0^\infty \frac{w_N(t)}{t^p} dt,
\]
\[
R = c \left\{ \int_0^x w_N(t) \frac{dt}{x^{p+1}} \right\} + \int_0^\infty \int_0^x w_N(t) \frac{dx}{x^{p+1}} dt
\]
\[
= c \left\{ \frac{1}{p} \int_0^x w_N(t) dt + \frac{1}{p} \int_0^\infty \frac{w_N(t)}{t^p} dt \right\}.
\]

The last integral
\[
\int_0^\infty \frac{w_N(t)}{t^p} dt = \left( \int_0^\infty + \int_0^x \right) \frac{w_N(t)}{t^p} dt \leq \frac{1}{r^p} \int_0^r w_N(t) dt + r \int_0^\infty \frac{w_N(t)}{t^p} dt.
\]

Hence
\[
R \leq c \left\{ \frac{1}{p} \int_0^x w_N(t) dt + \frac{1}{p} \int_0^\infty \frac{w_N(t)}{t^p} dt \right\}.
\]

From this we obtain,
\[
\frac{1}{p} (1 - c) \int_0^\infty \frac{w_N(t)}{t^p} dt \leq \frac{c}{pr^p} \int_0^r w_N(t) dt
\]
or
\[
\int_0^\infty \left( \frac{at}{t} \right)^p w_N(t) dt \leq \frac{ca^p}{1 - c} \int_0^r w_N(t) dt.
\]

We complete the proof by letting $N \to \infty$.

Theorem 6.4. Assume that $w \in L^1_{loc}(\mathbb{R}_+)$, then $w \in B_p$ if $0 < \epsilon < 1$ implies the existence of $a_\epsilon > 1$ such that for $x > 0$,
\[
\frac{1}{a_\epsilon^p x^p} \int_0^{a_\epsilon x} w dx \leq c \frac{1}{x^p} \int_0^x w, \quad \text{a} \geq a_\epsilon.
\]

Proof. By Lemma 6.3 we only need to prove the necessity. By Theorem 2.5, there is $\eta > 0$ such that $w \in B_{p-\eta}$. Thus
\[
\frac{1}{a^p x^p} \int_0^{a^p x} w dx = \frac{1}{(ax)^{p-\eta}} \frac{1}{x^{p-\eta}} \int_0^x w \left( \frac{1}{a} \right)^\eta.
\]
Since $w \in B_{p-\eta} \subset R_{p-\eta}$, by Theorem 6.1 the first factor $\leq c$ and the proof is complete.

As an application of Theorem 6.4 we will prove
Theorem 6.5. Let \( w \in B_p \) and \( W(x) = \int_0^x w \). Then for \( 0 < \alpha < \infty \), \( W^\alpha \in B_{\alpha p+1} \).

Proof: We do \( \alpha = 1 \) first. Let \( 0 < \epsilon < \frac{1}{p+1} \). Then for \( a > a_\epsilon > 1 \) we have

\[
\frac{x^p}{(ar)^p} \int_0^{ar} w \leq \epsilon \int_0^x w = \epsilon W(x), \quad 0 < x \leq r.
\]

Thus

\[
L = \int_0^r \frac{x^p}{(ar)^p} \int_0^{ar} w \leq \epsilon \int_0^r W(x)dx,
\]

and

\[
L = \frac{1}{(p+1)} \frac{r^{p+1}}{(ar)^p} W(ar) = \frac{1}{(p+1)} \frac{1}{\alpha^{p+1}} (ar) W(ar)
\]

\[
\geq \frac{1}{p+1} \frac{1}{\alpha^{p+1}} \int_0^{ar} W,
\]

and so \( W \in B_{p+1} \).

For the general case, since

\[
W^\alpha(x) = \alpha \int_0^x W^{\alpha-1}w,
\]

we only need to verify that \( W^{\alpha-1}w \in B_{\alpha p} \). For some \( 0 < c < 1 \) and \( a > 1 \) we have

\[
\frac{1}{\alpha^{p+1}} \int_0^{ar} W^{\alpha-1}w = \frac{1}{\alpha^{p+1}} W^\alpha(ax) \leq \frac{1}{\alpha} \epsilon W^\alpha(x)
\]

\[
= c \int_0^r W^{\alpha-1}w. \, \Box
\]

7. The equality \( W_p = B_p \)

In this final section we will prove that \( W_p = B_p \) for \( 1 < p < \infty \), a situation quite analogous to the \( A_p \)-case. I am indebted to Richard Bagby for the original proof of this property. We will present a somewhat simplified version based on some of our previous results. For the definitions of \( R_p, W_p \) see the beginning of section 6.

Lemma 7.1. Let \( w \in R_p, 0 < a < \infty, \) and \( 1 < s < \infty \). Then

\[
\int_a^{as} \left( \frac{a}{u} \right)^p w(u)du \leq c(1 + \log s) \int_0^a w.
\]
Proof: We know that by Theorem 6.1,

\[ \frac{1}{t^p} \int_0^{t^a} w \leq c \int_0^a w, \quad t \geq 1. \]

Hence \( L = \int_1^{t^{p+1}} \int_0^{t^a} w \leq c \log s \int_0^a w \). We interchange the order of integration and get

\[ L \geq \int_a^{as} \int_{u/a}^s w(u) \frac{dt}{t^{p+1}} du = \frac{1}{p} \int_a^{as} w(u) \left[ \left( \frac{u}{a} \right)^p - \frac{1}{s^p} \right] du. \]

Hence

\[ \frac{1}{p} \int_a^{as} w(u) \left( \frac{u}{a} \right)^p du \leq c \log s \int_0^a w + \frac{1}{p s^p} \int_a^{as} w \]

\[ \leq c \log s \int_0^a w + c \int_0^a w, \]

since \( w \in \mathcal{R}_p \). \( \blacksquare \)

Theorem 7.2. \( \mathcal{W}_p = \mathcal{B}_p \) for \( 1 < p < \infty \).

Proof: The inclusion \( \mathcal{B}_p \subset \mathcal{W}_p \) is obvious, and for the reverse inclusion we consider for \( s > 1 \) the function \( f(x) = 1, 0 \leq x \leq a; = a/x, a \leq x \leq sa; \) and \( = 0, x > sa \). Then \( Af(as) = \frac{1 + \log s}{s} \). Since \( w \in \mathcal{W}_p \) we have that

\[ w\{Af(x) > y\} \leq \frac{c}{y^p} \int_0^\infty f^p w. \]

If \( y = \frac{1 + \log s}{s} \), we get

\[ \left( \frac{1 + \log s}{s} \right)^p \int_0^{as} w \leq c \left( \int_0^a w + \int_a^{as} \left( \frac{u}{a} \right)^p w(u) du \right) \leq c(1 + \log s) \int_0^a w \]

by Lemma 7.1. Thus

\[ \frac{1}{s^p} \int_0^{as} w \leq c(1 + \log s)^{1-p} \int_0^a w. \]

We choose \( s \) so large that \( c(1 + \log s)^{1-p} < 1 \) and apply Theorem 6.4. \( \blacksquare \)
Weighted norm inequalities

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