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HOW TO SOLVE AN OPERATOR EQUATION

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Abstract

This article summarizes a series of lectures delivered at the Mathematics Department of the University of Leipzig, Germany in April 1991, which were to overview techniques for solving operator equations on C^*-algebras connected with methods developed in a Spanish-German research project on "Structure and Applications of C^*-Algebras of Quotients" (SACQ). One of the researchers in this project was Professor Pere Menal until his unexpected death this April. To his memory this paper shall be dedicated.

1. Introduction

Solving equations belongs to the fundamental tasks of mathematics. Many problems in the sciences lead to equations involving numbers, mappings, and other quantities. In fact, it frequently occurs that eventually a question can be phrased as an "equation", although, at first, it appeared to be of a rather different nature. To find a solution of an equation generally implies both the existence as well as the uniqueness problem. There is no universal procedure for solving, but the devices invented seem to be as manifold as the possible questions, and only allow a rather rough classification such as numerical, approximative, algebraic methods etc. However, it is always an important step to determine the common features in solving a certain class of examples for the aim of developing a machinery which enables to handle a specified collection of equations at one time.

In the present paper, we will be concerned with equations within a non-commutative infinite dimensional setting. To be more specific, they will be of the form

$$T_{\alpha, x_1, \ldots, x_n} = 0 \tag{1}$$

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where, for each 'parameter' α, $T_{\alpha,x_1,\dots,x_n}$ is a linear operator on a C^*-algebra A (with certain additional properties) and we are looking for elements $x_j \in A$ solving the equation (1) (or better, this system of equations). We will firstly collect some examples of questions which can be phrased in an equation such as (1), then describe a general tool to tackle them, and finally indicate solutions which yield answers to the questions listed. As a common feature, the questions in Section 2 lead to equations *in* a C^*-algebra, that is, we are looking for certain elements in a C^*-algebra solving the equation, while the conditions typically are formulated in terms of operators defined *on* the C^*-algebra. Needless to say that there are many more instances which can be settled by the proposed methods.

2. Examples

We have selected our examples from the following four classes of operators on C^*-algebras: derivations, completely positive operators, centralizing mappings, and generators of dynamical semigroups.

2.1. Derivations.

Let A be a C^*-algebra and δ a derivation on A, i.e. a linear mapping from A into itself satisfying Leibniz' rule $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in A$. Each derivation δ is automatically bounded whence it is meaningful and worthwhile to know under which circumstances δ is a compact operator, with respect to the norm or a weaker topology. Here, we ask when δ is *weakly compact*, that is, when does δ map the unit ball of A into a subset whose closure is compact with respect to the weak topology on A. (This is more closely related to the point of view taken in this paper than the *norm* compact case, which, however, can be treated similarly.)

Specialize to the case $A = B(H)$, the algebra of all bounded linear operators on some Hilbert space H. Since $B(H)$ is the second dual of $K(H)$, the closed ideal of all compact operators on H, and δ is continuous with respect to the $\sigma(B(H), K(H)^*)$-topology, δ coincides with $(\delta_|)^{**}$, the second adjoint of the restriction $\delta_|$ of δ to $K(H)$. It is well known that $\delta_|$ is weakly compact if and only if $(\delta_|)^{**}$ maps $K(H)^{**}$ into $K(H)$ [**15**, VI.4.2]. Moreover, by Gantmacher's theorem [**15**, VI.4.8], $\delta_|$ is weakly compact if and only if $(\delta_|)^{**}$ is weakly compact. Putting all this together yields that $\delta = (\delta_|)^{**}$ is weakly compact if and only if $\delta B(H) \subseteq K(H)$.

In the general case we have to replace $K(H)$ by the ideal $K(A)$ of all *compact elements* in A, and, using appropriate representations, we obtain the following, cf. [**23**, Theorem 2.7].

Proposition 1. *A derivation δ on a C^*-algebra A is weakly compact if and only if $\delta^{**}A^{**} \subseteq K(A)$.*

Again, $K(A)$ is δ-invariant and thus δ induces a derivation $\tilde{\delta}$ on the generalized Calkin algebra $A/K(A)$.

Corollary 2. *If δ is weakly compact, then $\tilde{\delta} = 0$.*

Suppose δ were *inner*, i.e. $\delta = \delta_a$, where $\delta_a(x) = xa - ax$, and the element a belonged to $K(A)$. Then, δ is weakly compact by [**41**, Theorem 3.1]. On the other hand, δ^{**} is always inner by Sakai's theorem. Therefore, the original question of weak compactness of δ leads to the following operator equation.

(1.1) Can $\tilde{\delta}_x = 0$ be solved in $K(A)$?

Going one step further we can ask a similar question for the product $\delta_1\delta_2$ of two derivations δ_1, δ_2 on A (which, in general, is no longer a derivation): when is $\delta_1\delta_2$ (weakly) compact? This question should be related to the Dunford-Pettis property of a commutative C^*-algebra which implies that T_1T_2 is a compact operator whenever T_1, T_2 are weakly compact on A. By similar arguments as above, it can be formulated in terms of operator equations as follows.

(1.2) Can $\tilde{\delta}_{x_1}\tilde{\delta}_{x_2} = 0$ be solved in $K(A)$?

Questions of this kind are studied in [**25**] and [**27**].

2.2. Completely positive operators.

Recall that a linear mapping T on a C^*-algebra A is said to be *completely bounded* if the norms $\|T_n\|$ of the canonical extensions T_n of T to the matrix algebras $M_n(A)$ over A are all bounded by some real number, and T is *completely positive* if all T_n are positive operators on $M_n(A)$. The prototypes of completely bounded operators are the *elementary operators* given concretely as mappings of the form

$$S : x \mapsto \sum_{j=1}^{n} a_j x b_j \qquad \text{with } x \in A,\ a_1, \ldots, a_n,\ b_1, \ldots, b_n \in M(A),$$

where $M(A)$ denotes the multiplier algebra of A. This is justified by the representation theorem for completely bounded operators and the fact that certain completely bounded operators can be approximated

by elementary operators, cf. [12]. A natural question in this context is: what does a completely positive elementary operator S look like? Although this is involving *inequalities*, we immediately are led to an operator equation.

Denote by $M_{a,b}$ the (*two-sided*) *multiplication* $x \mapsto axb$. If $S = \sum_{j=1}^{n} M_{a_j,b_j}$ is positive, it is hermitian-preserving from which

$$\sum_{j=1}^{n} M_{a_j,b_j} = \sum_{j=1}^{n} M_{b_j^*,a_j^*}$$

follows. As a result we are to consider the following operator equation.

(1.3) Which elements x_j, $y_j \in M(A)$ solve $\sum_{j=1}^{n} M_{x_j,y_j} = 0$?

This question has emerged to be not only an example, but of fundamental significance for our approach, cf. [28].

2.3. Centralizing mappings.

Let R be a ring. An additive mapping $F : R \to R$ is *centralizing* if, for every $x \in R$, we have $[x, F(x)] = xF(x) - F(x)x \in Z(R)$, the center of R. In many cases, the existence of certain centralizing mappings yields commutativity criteria for R. For example, if R is a prime ring, then R is commutative if there is a non-zero centralizing derivation on R [38, Theorem 2], see also [30], or if there is a non-identical centralizing automorphism on R [31, Theorem]. In the context of operator algebras, there are analogues of these results as follows.

Proposition 3. *There is no non-zero centralizing derivation on a C^*-algebra.*

This seems to be a folklore extension of Singer's classical result that there are no non-zero derivations on commutative C^*-algebras. In fact, if δ is a centralizing derivation on a C^*-algebra A, it easily follows that $\delta A \subseteq Z(A)$. Hence, the restriction $\delta_|$ of δ to $Z(A)$ vanishes so that $\delta^2 = 0$. The identity

$$2\,\delta(x)y\delta(x) = \delta^2(xyx) - x\delta^2(yx) - \delta^2(xy)x + x\delta^2(y)x \qquad (x, y \in A)$$

therefore yields $M_{\delta(x),\delta(x)} = 0$ for all $x \in A$, whence $\delta = 0$.

The case of automorphisms requires some more work and was first studied by Miers.

Proposition 4. [**32**, Theorem 5] *Let α be a centralizing *-automorphism on a von Neumann algebra A. There is a central projection $e \in A$ such that $\alpha(e) = e$, $\alpha_{|Ae} = \mathrm{id}_{Ae}$ and $A(1-e)$ is commutative.*

Whether this result remains true for arbitrary (not necessarily *-preserving) automorphisms was answered only recently by Brešar, who also obtained a general structure theorem for centralizing mappings on von Neumann algebras as follows.

Proposition 5. [**8**, Theorem 2.1] *Let F be a centralizing additive mapping on a von Neumann algebra A. Then there exist an element $c \in Z(A)$ and an additive mapping $\zeta : A \to Z(A)$ such that $F = L_c + \zeta$.*

Here and in the sequel, we will denote by L_a the *left multiplication* $x \mapsto ax$ and by R_a the *right multiplication* $x \mapsto xa$.

We will now reformulate both the assumption as well as the conclusion in terms of operator equations. This will enable us to obtain an extension of Brešar's result to arbitrary C^*-algebras in Section 4.

Observe at first that every centralizing additive mapping F on a C^*-algebra A is in fact *commuting*, i.e. $[x, F(x)] = 0$ for all $x \in A$ [**9**, Proposition 3.1]. Replacing x by $x + y$ therefore gives

$$[x, F(y)] + [y, F(x)] = 0 \qquad (x, y \in A)$$

or equivalently,

$$\delta_{F(y)} - \delta_y F = 0 \qquad \text{for all } y \in A. \tag{2}$$

Secondly, if $F = L_c + \zeta$ where A is a C^*-subalgebra of a C^*-algebra B with centralizer $C_B(A)$, $c \in C_B(A)$ and $\zeta : A \to C_B(A)$, then $[x, F(y)] = [x, cy] + [x, \zeta(y)] = [x, cy]$ for all $x, y \in A$. Hence

$$[x, F(y) - cy] = 0 \qquad (x, y \in A)$$

or equivalently,

$$\delta_{F(y)-cy} = 0 \qquad \text{for all } y \in A. \tag{3}$$

Conversely, if $c \in C_B(A)$ satisfies (3), then $\zeta = F - L_c$ defines an additive mapping from A into $C_B(A)$. As a result we arrive at the following question.

(1.4) Suppose that F satisfies (2) for all $y \in A$. Is there an element $c \in C_B(A)$ for a 'suitable' C^*-algebra B containing A satisfying (3) for all $y \in A$?

Note that (3) precisely is a system of operator equations of the form (1) parametrized by all elements in A.

2.4. Generators of dynamical semigroups.

Let A be a unital C^*-algebra. A bounded hermitian-preserving linear operator $L : A \to A$ with $L(1) = 0$ is called *completely dissipative* if, for all $n \in \mathbb{N}$,

$$L_n(x^*x) \geq x^*L_n(x) + L_n(x^*)x \qquad (x \in M_n(A)).$$

These operators are the generators of norm-continuous one-parameter semigroups $(T_t)_{t\in\mathbb{R}_+}$ of unital completely positive operators T_t on A, which describe the irreversible dynamics of open quantum systems, or, equivalently, serve as transition operators of non-commutative Markov processes. In many concrete situations, they are built from two prototypes: the completely positive operators and the hermitian-preserving generalized inner derivations $\delta_{k,k^*} = R_k + L_{k^*}$. The converse question, when a given completely dissipative operator L can be decomposed into

$$L = \psi + \delta_{k,k^*} \tag{4}$$

with ψ completely positive from A into some possibly larger C^*-algebra B and $k \in B$ was first studied by Gorini, Kossakowski and Sudarshan [**18**] and Lindblad [**21**], and related to cohomological properties of A in [**22**] and [11]. If $A \subseteq B(H)$, then a decomposition (4) of L always exists with $\psi A \subseteq A''$ and $k \in A''$. In general, this decomposition will not be unique. The uniqueness problem can be reformulated in terms of an operator equation as follows. Suppose that

$$L = \psi_1 + \delta_{k_1,k_1^*} = \psi_2 + \delta_{k_2,k_2^*}$$

are two decompositions. Then, putting $a = k_1 - k_2$, we have

$$\delta_{a,a^*} + \psi_1 - \psi_2 = 0. \tag{5}$$

Thus, we may ask

(1.5) Under which conditions does (5) imply that $\delta_{a,a^*} = 0$?

A more general question would be which a in A'' solve the equation (5).

3. Devices

All the above equations (1.1) through (1.5) can be subsumed under the general form (1). To motivate our tools for solving them, let us

furthermore consider a special case of (1.3). Let $A = B(H)$ and $b \in A$ be given.

(1.3') Which $a \in A$ solve $M_{a,b} = 0$?

In our particular situation, the answer is quickly reached. If $M_{a,b} = 0$, then $axb\xi = 0$ for all $x \in A$ and $\xi \in H$. If $b = 0$, obviously all $a \in A$ are solutions. If $b \neq 0$, pick $\xi \in H$ with $b\xi \neq 0$ and note that $b\xi$ is cyclic for A, i.e. $Ab\xi = H$, and thus $a = 0$.

Clearly, this method only works in the presence of a Hilbert space on which A acts 'transitively enough', e.g. if A is irreducible. The algebraic method presented now works without underlying space.

It is convenient to rephrase (1.3) using the following concept. For every C^*-algebra A we let $\mathcal{E}\ell(A)$ be the algebra of all elementary operators on A. We define a surjective algebra homomorphism

$$\theta : M(A) \otimes M(A)^{op} \longrightarrow \mathcal{E}\ell(A), \quad \theta(a \otimes b) = M_{a,b} \tag{6}$$

where $M(A) \otimes M(A)^{op}$ denotes the algebraic tensor product of $M(A)$ with its opposite algebra. The problem now is to determine the kernel of θ. The following was proved in [**24**, Part I, Corollary 4.4].

(7) *θ is injective if and only if A is prime.*

Since primitive C^*-algebras are prime, it is tempting to use representation theory in order to approach the general case from the special one. However, as it emerged, there may be problems in putting the 'local' information together to obtain a 'global' picture. It seems advantageous to view the *prime* C^*-algebras as the building blocks, which results in regarding a C^*-algebra as a *semiprime* algebra rather than a *semisimple* one. In fact, similar techniques and results as those described below are available in the setting of semiprime rings.

The ideal structure of a prime algebra is distinguished by the fact that *every* non-zero ideal is *essential*, i.e. intersects each other non-zero ideal non-trivially. This allows to "move around from one place to another" within the C^*-algebra without loss of information. For an arbitrary C^*-algebra A we therefore denote by $\mathcal{I}_e$ and $\mathcal{I}_{ce}$ the collections of all essential and all closed essential ideals of A, respectively. Note that these are directed downwards by inclusion, i.e. $I_1, I_2 \in \mathcal{I}_e$ implies that $I_1 \cap I_2 \in \mathcal{I}_e$.

For every semiprime ring R, the *multiplier ring* $M(R)$ is defined by its universal property that R is an essential ideal in $M(R)$ and there is

a unique extension $\bar{\rho}$ of the inclusion $\rho : R \to M(R)$ which makes the following diagram commutative, whenever R is an ideal in another ring S,

$$\begin{array}{ccc} R & \xrightarrow{\rho} & M(R) \\ \downarrow & \nearrow_{\bar{\rho}} & \\ S & & \end{array}$$

in other words, $M(R)$ is the (abstract) idealizer of R. Usually, $M(R)$ is constructed via double centralizers of R. Moreover, $\bar{\rho}$ is injective if and only if R is essential in S. Now, if $I, J \in \mathcal{I}_{ce}$ and $J \subseteq I$, then J will be an essential ideal in $M(I)$ whence, from the above, there is a unique injective *-homomorphism $\rho_{IJ} : M(I) \to M(J)$ making the following diagram commutative

$$\begin{array}{ccc} J & \xrightarrow{\rho_J} & M(J) \\ \downarrow & \nearrow_{\rho_{IJ}} & \\ M(I) & & \end{array}$$

We may describe ρ_{IJ} as "restricting the double centralizers". By means of this, we obtain a directed system $\{M(I); \rho_{IJ}, J \subseteq I\}$ of C^*-algebras and inclusions, and its algebraic direct limit $\operatorname{alg}\varinjlim M(I)$ along $\mathcal{I}_{ce}$ will be denoted by $Q_b(A)$ and called *the bounded symmetric algebra of quotients of* A. This is a pre-C^*-algebra with completion $Q_b(A)\widetilde{} = \varinjlim M(I)$ denoted henceforth by $M_{loc}(A)$ and called *the local multiplier algebra* of A.

For each $I \in \mathcal{I}_e$ let $P(I)$ denote the Pedersen ideal of $\bar{I}$ [**37**, 5.6]. Using the fact that $P(I)$ is *-invariant, belongs to $\mathcal{I}_e$, and that $P(I)P(J) = P(I) \cap P(J)$ for all $I, J \in \mathcal{I}_e$ we define $Q_s(A) = \operatorname{alg}\varinjlim M(P(I))$ along $\mathcal{I}_e$ and observe that this definition leads to the *symmetric algebra of quotients of* A as defined (slightly differently) in ring theory. It follows that $Q_b(A)$ embeds as a *-subalgebra into $Q_s(A)$ and is in fact the bounded part of $Q_s(A)$ [**2**, Theorem 1.3]. A stronger relation between $Q_b(A)$ and $Q_s(A)$ proved in [**3**, Theorem 2] is that $Q_s(A)$ is the central localization of $Q_b(A)$.

Remarks. The construction of $M_{loc}(A)$ was first performed by Pedersen [**36**] and Elliott [**16**] under the name of *essential multipliers*. They used it to study operator equations of the form

$$\delta = \delta_a, \qquad a \in M_{loc}(A), \tag{8}$$

and

$$\alpha = M_{u,u^*}, \qquad u \in M_{loc}(A) \text{ unitary}, \tag{9}$$

that is, to obtain innerness of derivations δ and *-automorphisms α in $M_{loc}(A)$. In particular, Pedersen proved that (8) always has a solution if A is separable [**36**, Proposition 2].

At about the same time, Kharchenko introduced the symmetric ring of quotients for semiprime rings and used it in particular in Galois theory [**19**], [**20**]. This theme was further pursued by Passman [**34**], [**35**], Montgomery [**33**], and others. It is to be seen in a long tradition going back to the 30's in investigating general *rings of quotients*, cf. [**40**]. The basic idea — to enlarge a given 'domain' by additional 'numbers' (='fractions', 'quotients') in order to be able to solve more equations — also serves as the motivation for our approach to operator equations.

In the late 80's, $M_{loc}(A)$ was rediscovered independently by Ara [**2**], [**3**] and the author [**26**], [**29**] which then launched a joint research project on the structure and applications of local multipliers [**4**], [**5**], [**6**]; a comprehensive account of this is to be given in [**7**].

We will now compile some of the basic properties of $M_{loc}(A)$.

Proposition 6. *Let A be a C^*-algebra with local multiplier algebra $M_{loc}(A)$.*

(i) *A is commutative if and only if $M_{loc}(A)$ is commutative.*
(ii) *A is prime if and only if $M_{loc}(A)$ has trivial center.*
(iii) *For each $I \in \mathcal{I}_{ce}$ and each unitization B of A we have*

$$M_{loc}(I) = M_{loc}(A) = M_{loc}(B).$$

(iv) *Let $\check{A}$ be the primitive spectrum of A. If $\check{A}$ is discrete, then $M_{loc}(A) = M(A)$.*
(v) *If A is an AW^*-algebra, then $M_{loc}(A) = A$.*

From (7) and (ii) in the above proposition we see that the kernel of θ is closely related to the center $Z = Z(M_{loc}(A))$ of $M_{loc}(A)$. It is therefore important to analyse its structure. The following was proved in [**5**, Theorem 1 and Corollary 1] and can be viewed as a local version of the well-known Dauns-Hofmann theorem identifying the center $Z(M(A))$ of $M(A)$ with the algebra $C(\beta\check{A})$ of all continuous complex-valued functions on the Stone-Čech compactification $\beta\check{A}$ of $\check{A}$.

Proposition 7. *For every C^*-algebra A, the center Z of $M_{loc}(A)$ is an AW^*-algebra and can be identified with $C(\varprojlim \beta\check{I})$, where the inverse*

limit (in the category of compact spaces) is taken over all dense open subsets $\check{I}$ of $\check{A}$.

The key to this result is by observing that $Z = \overline{C_b}$, where $C_b = \operatorname{alg}\varinjlim Z(M(I))$, $I \in \mathcal{I}_{ce}$, is the center of $Q_b(A)$ and called *the bounded extended centroid of A*. This one takes the role of the *extended centroid* $C = \operatorname{alg}\varinjlim Z(M(P(I)))$, $I \in \mathcal{I}_e$, being of fundamental importance in ring theory. In analogy to the *central closure* AC we define the *bounded central closure* ${}^cA$ by ${}^cA = \overline{AC_b} = \overline{AZ}$. The nicest C^*-algebras in this framework are those which are *boundedly centrally closed*, that is ${}^cA = A$. They can be characterized as follows.

Proposition 8. *A is boundedly centrally closed if and only if $\check{A}$ is extremally disconnected.*

The fact that every von Neumann algebra is boundedly centrally closed (which follows in particular from Proposition 7 (v)) allows to incorporate the results on von Neumann algebras in our approach, and the fact that $M_{loc}(A)$ is boundedly centrally closed [**5**, Theorem 2] yields an important stability property.

It can be shown that every C^*-subalgebra B of $M_{loc}(A)$ containing both A and C_b has center $Z(B)$ equal to Z [**7**], and hence may be regarded as a Z-bimodule in a natural way. Applying this to ${}^cM(A)$, the bounded central closure of $M(A)$, we obtain from (6) an induced homomorphism

$$\theta_Z : {}^cM(A) \otimes_Z {}^cM(A)^{op} \longrightarrow \mathcal{EL}({}^cA), \quad \theta_Z(a \otimes_Z b) = M_{a,b},$$

where the tensor product is taken in the category of bimodules. Using the fact that A is boundedly centrally closed if and only if $M(A)$ is, we can now formulate the fundamental result yielding solutions to the operator equations listed in Section 2.

Theorem 9. [**7**] *For every C^*-algebra A, we have that*

$$\ker\theta = \{u \in M(A) \otimes M(A)^{op} \mid u_Z = 0\},$$

where u_Z is the canonical image of u in ${}^cM(A) \otimes_Z {}^cM(A)^{op}$. Therefore, if A is boundedly centrally closed, then θ_Z is injective.

This result can be considerably strengthened using appropriate metric structures. Let $\mathcal{EL}(A)$ be endowed with the cb-norm, i.e. $\|S\|_{cb} =$

$\sup_n \|S_n\|$ for all $S \in \mathcal{EL}(A)$. Let ${}^cM(A) \otimes_Z {}^cM(A)^{op}$ be endowed with the *central Haagerup tensor norm* $\|\cdot\|_{Zh}$ defined by

$$\|u\|_{Zh} = \inf\left\{\left\|\sum_{j=1}^n a_j a_j^*\right\|^{1/2} \left\|\sum_{j=1}^n b_j^* b_j\right\|^{1/2} \mid u = \sum_{j=1}^n a_j \otimes_Z b_j\right\}^*$$

where the infimum is taken over all representations of u in ${}^cM(A) \otimes_Z {}^cM(A)^{op}$. Then we have

Theorem 10. [7] *For every C^*-algebra A,*

$$\theta_Z : \left({}^cM(A) \otimes_Z {}^cM(A)^{op}, \|\cdot\|_{Zh}\right) \longrightarrow \left(\mathcal{EL}({}^cA), \|\cdot\|_{cb}\right)$$

is an isometry.

Corollary 11. *θ_Z is an isometry for every von Neumann algebra.*

This last result was recently obtained in [10, Theorem 2.4], see also [39], for von Neumann algebras acting on separable Hilbert spaces using a number of non-trivial results on von Neumann subfactors as well as direct integral theory.

4. Solutions

In this final section we will outline answers to the questions raised in Section 2 exploiting the tools described in the previous section. As an immediate consequence of Theorem 9 we obtain the following answer to (1.3).

Theorem 12. *Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in M(A)^n$ be such that $\{b_1, \ldots, b_n\}$ is Z-independent. If $\sum_{j=1}^n M_{a_j,b_j} = 0$, then $a = 0$.*

Now the strategy to describe completely positive elementary operators is as follows, cf. [7]. If $S = \sum_{j=1}^n M_{a_j,b_j}$ is completely positive, we may without loss of generality assume that both $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are Z-independent. Then

$$\sum_{j=1}^n M_{a_j,b_j} = \sum_{j=1}^n M_{b_j^*,a_j^*}$$

*i.e., ${}^cM(A) \otimes_h {}^cM(A)^{op}$ inherits the operator space structure of ${}^cM(A) \otimes_h {}^cM(A)$.

together with Theorem 12 implies the existence of a self-adjoint matrix $\Lambda = (\lambda_{kj}) \in M_n(Z)$ such that

$$S = \sum_{k,j=1}^{n} \lambda_{kj} M_{b_k^*, b_j}. \tag{10}$$

Since Z is an AW^*-algebra by Proposition 7, Λ can be diagonalized by [**14**, Corollary 3.3], i.e. there is a unitary matrix $U \in M_n(Z)$ such that $U^*\Lambda U$ is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_n$. Hence, by putting $\tilde{b} = bU^* \in {}^cM(A)^n$ we can write S as

$$S = \sum_{j=1}^{n} \lambda_j M_{\tilde{b}_j^*, \tilde{b}_j}. \tag{11}$$

From the complete positivity of S we then conclude that $\lambda_j \geq 0$ for all $1 \leq j \leq n$ and hence, letting $c_j = \lambda_j^{1/2}\tilde{b}_j$, obtain the following answer to the question raised in 2.2.

Theorem 13. [**7**] *An elementary operator S on a C^*-algebra A is completely positive if and only if there are $c_1, \ldots, c_n \in {}^cM(A)$ such that $S = \sum_{j=1}^{n} M_{c_j^*, c_j}$.*

For prime C^*-algebras, this was obtained in [**24**, Part I, Theorem 4.10].

For simplicity, we stick to the prime case in answering the questions of Sections 2.1 and 2.4. If A is a prime C^*-algebra, then, by Theorem 12, $R_a + L_b = 0$ for some $a, b \in M(A)$ if and only if $a = -b \subset Z(M(A)) = \mathbb{C}1$. Suppose that δ is a weakly compact derivation on A. If $\delta = 0$, it clearly can be implemented by a compact element. If $\delta \neq 0$, then $\delta A \subseteq K(A)$ (Proposition 1) implies that $K(A) \neq \{0\}$ and thus A can be faithfully represented as an irreducible algebra on some Hilbert space H such that $K(A)$ becomes $K(H)$. By the arguments used in 2.1, we see that $\delta = \delta_a$ for some $a \in B(H)$ and $\tilde{\delta}_a = \delta_{\tilde{a}} = 0$ on the Calkin algebra $C(H) = B(H)/K(H)$. Since $C(H)$ is prime, $Z(C(H)) = \mathbb{C}1$ wherefore $\tilde{a} = \lambda 1$, equivalently, $a + \lambda 1 \in K(A)$. Consequently, we have the following.

Proposition 14. *Let δ be a derivation on a prime C^*-algebra A. Then δ is weakly compact if and only if $\delta = \delta_a$ for some $a \in K(A)$.*

In fact, this result takes over verbatim to the case of a general C^*-algebra, which was first proved by Akemann and Wright [**1**, Theorem

3.3] using representation theory. As a result, a derivation δ is weakly compact if and only if the answer is "yes" in (1.1).

In a similar vein, $\delta_1\delta_2$ is weakly compact if and only if δ_1 or δ_2 is weakly compact, provided A is prime. Hence, $\delta_1\delta_2$ is weakly compact if and only if at least one of the x_i in (1.2) can be taken from $K(A)$. The formulation of the answer in the general case is somewhat more complicated, and we refer the reader for this (as well as for the norm compact case) to [**25**]. Note that

$$\delta_{x_1}\delta_{x_2} = M_{x_1x_2,1} - M_{x_1,x_2} - M_{x_2,x_1} - M_{1,x_2x_1}$$

and therefore (1.2) is closely related to (1.3) and a description of (weakly) compact elementary operators which was obtained in [**24**, Part II].

Specializing the above observation to the case $b = a^*$ we obtain that $\delta_{a,a^*} = 0$ if and only if $a = -a^* \in i\mathbb{R}1$ whenever A is prime. Using a slight elaboration of this we obtain the answer to (1.5).

Theorem 15. *Let A be a unital C^*-algebra and P a proper closed prime ideal of A. Let $L : A \to A$ be linear. Under the hypothesis $\psi A \subseteq P$, each two decompositions of L of the form $L = \psi + \delta_{k,k^*}$ with $\psi : A \to A$ completely positive and $k \in A$ only differ by an addition by δ_{c,c^*}, $c \in P$.*

Corollary 16. *Let A be a unital infinite dimensional prime C^*-algebra and $L : A \to A$. Then there is at most one decomposition $L = \psi + \delta_{k,k^*}$ with $k \in A$ and $\psi : A \to A$ completely positive and compact.*

These results are proved in [**17**]. Corollary 16 was first observed by Davies [**13**, Theorem 2] in the case $A = B(H)$.

We finally turn our attention to the structure of centralizing mappings of C^*-algebras and the questions raised in Section 2.3. Unlike in the other examples, there seems to be no direct connection with equations involving elementary operators such as (1.3). The following lemma indeed is the key observation which enables us to solve equation (3).

Lemma 17. *If F is an arbitrary mapping on a ring R such that $\delta_{F(y)} - \delta_y F$ maps R into some ideal J of R, then, for all $x, y, u, v \in R$, we have*

$$\big(M_{\delta_{F(y)}(x),\delta_u(v)} - M_{\delta_y(x),\delta_{F(u)}(v)}\big)R \subseteq J. \tag{12}$$

This result was obtained in [**8**, Lemma 2.2] for commuting additive mappings and $J = \{0\}$. Although we are dealing here with C^*-algebras

only, we give the proof in full generality as an illustration of the techniques and with a hope for future applications.

Proof: For all $y, z \in R$ we have

$$\begin{aligned}\delta_{yz} F &= (R_{yz} - L_{yz})F = (R_z R_y - L_y L_z)F \\ &= (R_z\delta_y + L_y\delta_z)F = R_z(\delta_y F) + L_y(\delta_z F)\end{aligned}$$

and hence

$$\delta_{F(yz)} - \delta_{yz}F = \delta_{F(yz)} - R_z(\delta_y F) - L_y(\delta_z F).$$

By assumption, it follows that

$$\big(\delta_{F(yz)} - R_z(\delta_y F) - L_y(\delta_z F)\big)R \subseteq J. \tag{13}$$

Observe that

$$\begin{aligned}&\big(\delta_{F(yz)} - R_z\delta_{F(y)} - L_y\delta_{F(z)}\big)(xu) - L_{xy}\delta_{F(z)}(u) - R_{zu}\delta_{F(y)}(x) \\ &\qquad + R_{uz}\delta_{F(y)}(x) + L_{yx}\delta_{F(z)}(u) \\ &= L_x\delta_{F(yz)}(u) + R_u\delta_{F(yz)}(x) - R_zL_x\delta_{F(y)}(u) - R_zR_u\delta_{F(y)}(x) \\ &\quad - L_yL_x\delta_{F(z)}(u) - L_yR_u\delta_{F(z)}(x) - L_xL_y\delta_{F(z)}(u) - R_uR_z\delta_{F(y)}(x) \\ &\qquad + R_zR_u\delta_{F(y)}(x) + L_yL_x\delta_{F(z)}(u) \\ &= L_x\big(\delta_{F(yz)} - R_z\delta_{F(y)} - L_y\delta_{F(z)}\big)(u) + R_u\big(\delta_{F(yz)} - R_z\delta_{F(y)} - L_y\delta_{F(z)}\big)(x) \\ &\qquad = L_x\big(R_z\delta_y F + L_y\delta_z F - R_z\delta_{F(y)} - L_y\delta_{F(z)}\big)(u) + j_1 \\ &\qquad\quad + R_u\big(R_z\delta_y F + L_y\delta_z F - R_z\delta_{F(y)} - L_y\delta_{F(z)}\big)(x) + j_2\end{aligned}$$

(with $j_1, j_2 \in J$ by (13))

$$\begin{aligned}&= L_x\big(R_z(\delta_y F - \delta_{F(y)}) + L_y(\delta_z F - \delta_{F(z)})\big)(u) + j_1 \\ &\qquad + R_u\big(R_z(\delta_y F - \delta_{F(y)}) + L_y(\delta_z F - \delta_{F(z)})\big)(x) + j_2 \in J\end{aligned}$$

since $(\delta_y F - \delta_{F(y)})R \subseteq J$ by assumption.

By (13) again, the first summand on the left hand side is in J too, from which we conclude that

$$L_{xy-yx}\,\delta_{F(z)}(u) + R_{zu-uz}\,\delta_{F(y)}(x) \in J,$$

equivalently,

$$\delta_y(x)\,\delta_{F(z)}(u) + \delta_{F(y)}(x)\,\delta_u(z) \in J \tag{14}$$

for all $x, y, z, u \in R$.

From

$$\delta_{F(z)}(u) + \delta_{F(u)}(z) \in J \qquad \text{for all } u, z \in R$$

and (14) it follows that

$$\delta_y(x)\,\delta_{F(u)}(z) - \delta_{F(y)}(x)\,\delta_u(z) \in J \qquad \text{for all } x, y, z, u \in R. \tag{15}$$

Replacing z by zv in (15) yields

$$\begin{aligned}&\delta_y(x)\,\delta_{F(u)}(zv) - \delta_{F(y)}(x)\,\delta_u(zv) = \\ &= \delta_y(x)z\delta_{F(u)}(v) + \delta_y(x)\delta_{F(u)}(z)v - \delta_{F(y)}(x)\delta_u(z)v - \delta_{F(y)}(x)z\delta_u(v) \in J\end{aligned}$$

which, together with (15), gives,

$$\delta_y(x)\,z\,\delta_{F(u)}(v) - \delta_{F(y)}(x)\,z\,\delta_u(v) \in J \tag{16}$$

for all $x, y, z, u, v \in R$. But (16) is nothing but the assertion. ■

As a consequence, every mapping satisfying (2) has the property that

$$M_{\delta_{F(y)}(x),\delta_u(v)} - M_{\delta_y(x),\delta_{F(u)}(v)} = 0 \qquad \text{for all } x, y, u, v \in A. \tag{17}$$

An elaboration of the solution to (1.3), the details of which are given in [6], then yields a family $\{c_x^y \mid x, y \in A\}$ of elements in C and a family $\{e_x^y \mid x, y \in A\}$ of projections in C_b such that

$$e_x^y\,\delta_{F(y)}(x) - c_x^y\,\delta_y(x) = 0 \qquad \text{for all } x, y \in A. \tag{18}$$

It is then the self-injectivity of C which allows to find $c \in C$ with $c\,c_x^y = c_x^y$, which finally has the property that

$$\delta_{F(y)}(x) - c\,\delta_y(x) = 0 \qquad \text{for all } x, y \in A,$$

that is, which solves (3). An additional argument is then needed to show that c can be found in C_b, that is, we obtain a solution to (1.4) in ${}^cA$.

We summarize this in the following statement.

Theorem 18. [6, Theorem 3.2] *Let $F : A \to A$ be a centralizing additive mapping on a C^*-algebra A. Then there are $c \in Z$ and an additive mapping $\zeta : A \to Z$ such that $F = L_c + \zeta$.*

Note that, by Proposition 6 (v), this is an extension of Brešar's result (Proposition 5). Under a natural condition, both c and ζ can be chosen uniquely.

5. Conclusion

We hope that the results described above may give some evidence that the local multiplier algebra can serve as a 'universe', in which operator equations on C^*-algebras, at least those of the form (1), can be solved by a unified method.

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