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EQUIDISTRIBUTION AND THE RIEMANN HYPOTHESIS

H. L Hutson

Dedicated to the memory of Beth Lynn Tatarsky

Abstract

In this paper we demonstrate the relationship between uniform distribution modulo 1 of the sequence cp^α, p prime, and the zero free regions of the Riemann zeta function.

A classical theorem of Franel [**8**], [**5**] says that each digit in the j-th place after the decimal point in a table of natural numbers raised to the α-th power appears equally often on the average if $\alpha \in (0,1)$. This follows immediately from the uniform distribution modulo 1 of the sequence cn^α, $n = 1,2,3,\dots$ for each positive c [**8**], [**3**]. We revive this result by demonstrating the relationship between uniform distribution modulo 1 of the sequence cp^α, p prime, and the zero free regions of the Riemann zeta function.

To acquire insight into the sequence cp^α we consider $\Psi(x)$ which is defined by

$$\Psi(x) = \begin{cases} \displaystyle\sum_{p^\nu \le x} \ln(p) & x \text{ not a prime power} \\ \dfrac{\Psi(x+0)+\Psi(x-0)}{2} & \text{otherwise.} \end{cases}$$

We use von Mangoldt's important formula [**3**], [**6**]:

$$\Psi(x) = x - \zeta'(0)/\zeta(0) - 1/2\ln(1-x^2) - \sum_\rho \frac{x^\rho}{\rho}; \quad x > 1 \tag{1}$$

where the sum is over the complex zeros (ρ) of the zeta function $(\zeta(s))$ and is taken on the order of increasing $|\operatorname{im}(\rho)|$.

Our first result is from the folklore on the analytic theory of primes. It provides the first hint that the problem of equidistribution of a sequence of the form $\{f(p) - [f(p)]\}$ ($[x]$ denotes the integer part of x) is related to the zero free regions of $\zeta(s)$. In the following $e(x) = \exp(2\pi i x)$. We also let $\pi(x)$ denote the number of primes $\le x$.

Theorem 1.

If $f(x)$ is a real function with continuous derivative then

$$(2)\quad \frac{\sum_{p\le x} e(f(p))}{\pi(x)} = \frac{1}{x}\int_1^x e(f(t))\,dt - \sum_\rho \frac{1}{x}\int_1^x t^{\rho-1} e(f(t))\,dt + O\left(\frac{1}{\ln(x)}\right).$$

(The sum on the right extends over the complex zeros of $\zeta(s)$ and the terms are added in the order of increasing $|\operatorname{Im}(\rho)|$).

Proof:

Let

$$g(x) = \frac{e(f(x))}{\ln(x)}$$

and observe that

$$(3)\qquad \int_2^x g'(t)\Psi(t)\,dt = -\sum_{p^\nu \le x} g(p^\nu)\ln(p) + g(x)\Psi(x)$$

(to see this use integration by parts and Stieltjes integration).

One thus has

$$(4)\qquad \int_2^x g'(t)\Psi(t)\,dt = -\sum_{p^\nu\le x} \frac{e(f(p^\nu))}{\nu} + \frac{\Psi(x)e(f(x))}{\ln(x)}.$$

Define

$$ES_n(x) = \sum_{p<x} e(f(p^n))$$

and note that the sum on the sum on the right of (4) can be written as

$$-\sum_{n\ge 1} \frac{ES_n(x^{1/n})}{n}.$$

The prime number theorem shows that

$$\frac{-\sum_{n\ge 1} ES_n(x^{1/n})}{\pi(x)} = \frac{-ES_1(x)}{\pi(x)} + O\left(\frac{1}{\ln(x)}\right).$$

Thus we have

$$(5)\qquad \frac{ES_1(x)}{\pi(x)} = -\frac{1}{\pi(x)}\int_2^x g'(t)\Psi(t)\,dt + \frac{g(x)\Psi(x)}{\pi(x)} + O\left(\frac{1}{\ln(x)}\right).$$

One now substitutes the right hand side of von Mangoldt's formula for $\Psi(t)$ (1) in the integral and then integrates term wise (this needs justification: see [**3**, page 162]) to get upon integration by parts and replacement of $\pi(x)$ by $\pi/\ln(x)$ (De la Vallee Poussin's error estimate in the prime number theorem [**4**] shows that this does no damage to the O term)

$$\frac{ES_1(x)}{\pi(x)} = \frac{e(f(x))(\Psi(x)-x)}{x} + e(f(x))\sum_\rho \frac{x^{\rho-1}}{\rho} +$$
$$+\frac{\ln(x)}{x}\int_2^x \frac{e(f(t))}{\ln(t)}\,dt - \sum_\rho \frac{\ln(x)}{x}\int_2^x \frac{t^{\rho-1}e(f(t))}{\ln(t)}\,dt + O\left(\frac{1}{\ln(x)}\right)$$

(several quantities have been absorbed into the O term). Again the above stated error term in the prime number theorem shows that the first two expressions do not change the asymptotic estimate. Finally one checks that all occurrences of ln in the integrals can be eliminated with out effecting the error. To see this observe that

$$\text{(6)}\qquad \int_2^x \left(\frac{\ln(x)}{\ln(t)} - 1\right) dt = \ln(x)Li(x) - x + O(1) = O\left(\frac{x}{\ln(x)}\right).$$

The result follows upon using two integrations by parts. The alteration of the lower limit of integration is just for convenience and causes no difficulty. ■

Note that if the limit as x increases of the first term of (2) is zero and if the limit of the sum can be evaluated term wise then the sequence $\{f(p) - [f(p)]\}$ is equidistributed. This follows from Weyl's criterion for uniform distribution modulo 1 ([**2**, page 108]) and from the fact that $Re(\rho) < 1$. Unfortunately the evaluation of the limit of the sum term wise is a delicate matter. Nonetheless we have:

Theorem 2.
If $\zeta(s)$ is non zero for $Re(s) \geq 1-2\alpha$, $0 < \alpha < 1/4$ then the sequence $\{cp^\alpha - [cp^\alpha]\}$ is equidistributed for all positive c and thus, under the stated condition, each digit in the j-th place after the decimal point in a table of the primes raised to the α-th power appears equally often in the average.

Proof: We show that the limit of the right hand side of formula (2) with $f(x) = cx^\alpha$ is zero. The first term yields to [**7**, page 96]. We have

$$\left|\frac{1}{x}\int_1^x e(ct^\alpha)\,dt\right| \leq \frac{k}{x^\alpha}$$

for an appropriate constant k. For the second term integrate by parts to obtain

$$-e(f(x))\sum_\rho \frac{x^{\rho-1}}{\rho} + \frac{e(f(1))}{x}\sum_\rho \frac{1}{\rho} + q\sum_\rho \frac{1}{x\rho}\int_1^x t^{\rho+\alpha-1} e(f(t))\,dt$$

where $q = \alpha 2\pi i c$. The limit of the first term is zero by the prime number theorem. The second expression disappears as x increases because the sum converges. (Recall the ordering of the terms). Now integrate the quantities in the remaining expression by parts to obtain

$$\sum_\rho \frac{x^{\rho+\alpha-1}e(f(x))}{\rho(\rho+\alpha)} - \frac{1}{x}\sum_\rho \frac{e(f(1))}{\rho(\rho+\alpha)} - \\ -\sum_\rho \frac{h}{\rho(\rho+\alpha)}\int_1^x t^{\rho+2\alpha-1}e(f(t))\,dt \tag{7}$$

where h is a constant. Under the stated condition on $\zeta(s)$ the absolute values of the terms in each sum are dominated by quantities which decrease to zero as x increases and which add to a finite value. (In the integrals use the trivial estimate). The final statement follows as in the proof of Franel's classical result. ■

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Department of Mathematics
The U.S. Merchant Marine Academy
Kings Point, N.Y. 11024
U.S.A.

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