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ON INDUCED MORPHISMS OF MISLIN GENERA

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Dedicated to my good friend Karl Gruenberg, in admiration and affection, on the occasion of his 65th birthday

Abstract

Let N be a nilpotent group with torsion subgroup TN, and let $\alpha : TN \to \tilde{T}$ be a surjective homomorphism such that $\ker\alpha$ is normal in N. Then α determines a nilpotent group $\tilde{N}$ such that $T\tilde{N} = \tilde{T}$ and a function α_* from the Mislin genus of N to that of $\tilde{N}$ if N (and hence $\tilde{N}$) is finitely generated. The association $\alpha \mapsto \alpha_*$ satisfies the usual functorial conditions. Moreover $[N, N]$ is finite if and only if $[\tilde{N}, \tilde{N}]$ is finite and α_* is then a homomorphism of abelian groups. If $\tilde{N}$ belongs to the special class studied by Casacuberta and Hilton (*Comm. in Alg.* **19(7)** (1991), 2051–2069), then α_* is surjective. The construction α_* thus enables us to prove that the genus of N is non-trivial in many cases in which N itself is not in the special class; and to establish non-cancellation phenomena relating to such groups N.

0. Introduction

Guido Mislin introduced and discussed in [**M**] the *genus* $\mathfrak{G}(N)$ of a finitely generated (fg) nilpotent group N. This consists of isomorphism classes of fg nilpotent groups M such that

(0.1) $$M_p \cong N_p, \text{ for all primes } p,$$

where M_p is the p-localization of M. By abuse we say that M belongs to $\mathfrak{G}(N)$. It was early known that $\mathfrak{G}(N)$ is not trivial, but systematic methods of calculating the set $\mathfrak{G}(N)$ and representing its elements were lacking.

Mislin himself in [**M**], and together with the present author in [**HM**], described an abelian group structure which could be introduced into $\mathfrak{G}(N)$ if N satisfied the condition that its commutator subgroup $[N, N]$

is finite; we call the class of such fg nilpotent groups $\mathfrak{N}_0$; moreover, $\mathfrak{G}(N)$ is then finite. However, this still did not permit any kind of systematic calculation of $\mathfrak{G}(N)$. Calculations were done for specific groups in [**H2**]. Later, Casacuberta and Hilton [**CH**] introduced a class of nilpotent groups $\mathfrak{N}_1 \subset \mathfrak{N}_0$, and calculated $\mathfrak{G}(N)$ for $N \in \mathfrak{N}_1$; they further showed how to modify N to realize any given element in $\mathfrak{G}(N)$. The nature of the groups in $\mathfrak{N}_1$ was further analysed in [**S**], [**HS1**] —indeed, the class is very strongly restricted— and, in [**S**], [**HS2**], the calculation of the genus was extended from N to N^k, the direct product of k copies of N, provided $N \in \mathfrak{N}_1$. A key result in this work is that, for $N \in \mathfrak{N}_1$, $\mathfrak{G}(N)$ can only be non-trivial if $FN = N/TN$ is cyclic, where TN is the torsion subgroup of N; recall that FN is commutative for $N \in \mathfrak{N}_0$.

A significant difficulty in attempting to calculate $\mathfrak{G}(N)$ is that $\mathfrak{G}$ lacks any kind of functoriality. We endeavor in this paper to go some way towards remedying this defect. Thus we suppose given a fg nilpotent group N and a surjective homomorphism $\alpha : TN \to \tilde{T}$, for some finite group $\tilde{T}$ which is, of course, necessarily nilpotent. Given the supplementary condition that $\ker\alpha$ is normal in N, we construct a fg nilpotent group $\tilde{N}$ such that $T\tilde{N} = \tilde{T}$ and a function $\alpha_* : \mathfrak{G}(N) \to \mathfrak{G}(\tilde{N})$. Moreover, $N \in \mathfrak{N}_0$ if and only if $\tilde{N} \in \mathfrak{N}_0$; and α_* is then a homomorphism. It is easy to see that $\alpha \mapsto \alpha_*$ satisfies the usual functoriality conditions. Further we show in Section 2 that if $\tilde{N} \in \mathfrak{N}_1$ then α_* is surjective; thus, in this case, considerable information is made available about $\mathfrak{G}(N)$, since we may calculate $\mathfrak{G}(\tilde{N})$.

A particular, and important, example of the construction is afforded by taking $\tilde{T}$ to be the abelianization of TN with α the abelianizing homomorphism. To avoid triviality we take FN cyclic. Then $\tilde{N}$ satisfies two of the three conditions for membership of $\mathfrak{N}_1$ (see below). Moreover, the third condition will be automatically satisfied if $\tilde{T}$ happens to be cyclic.

We also show in Section 2 that a non-cancellation result proved in [**CH**] for groups in $\mathfrak{N}_1$ extends to groups, which, in our sense above, lie over groups in $\mathfrak{N}_1$. That is, we obtain pairwise non-isomorphic groups $(L, M, \dots)$ in $\mathfrak{G}(N)$ such that $L \times C \cong M \times C \cong \cdots \cong N \times C$, where C is cyclic infinite.

In Section 3 we give a typical example of the application of the method, with explicit calculations.

For the convenience of the reader, we collect here the crucial facts about the class $\mathfrak{N}_1$. We assume $N \in \mathfrak{N}_0$ and refer to the extension

$$TN \rightarrowtail N \twoheadrightarrow FN. \tag{0.2}$$

Then $N \in \mathfrak{N}_1$ if

(i) TN is commutative;
(ii) (0.2) is a split extension for an action $\omega : FN \to \operatorname{Aut} TN$;
(iii) $\omega(FN)$ lies in the center of $\operatorname{Aut} TN$.

We then note that, in the presence of (i), condition (iii) is equivalent to

(iii)' for each $\xi \in FN$, there exists a positive integer u such that $\xi \cdot a = ua$, for all $a \in TN$.

To avoid a trivial genus, we assume FN cyclic, say, $FN = \langle \xi \rangle$. Let t be the order of $\omega(\xi)$ in $\operatorname{Aut} TN$. Then [**CH**], if $N \in \mathfrak{N}_1$,

$$\mathfrak{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\}. \tag{0.3}$$

Moreover, if $[m] \in (\mathbb{Z}/t)^*/\{\pm 1\}$, where m is prime to t, we may choose the isomorphism (0.3) so that the group N_m corresponding to m is obtained from N by introducing a new action ω_m of FN on TN, defined by

$$\omega_m(\xi) = \omega(\xi^m). \tag{0.4}$$

A final remark pertains to the general construction in Section 1. There is no need to insist that N be fg to carry out the construction. Thus Theorem 1.1 may be extended to yield a function α_* from the *extended genus* of N to the extended genus of $\tilde{N}$ (see [**H3**]).

1. The construction

Let $N \in \mathfrak{N}_{fg} \subset \mathfrak{N}$; that is, N is a fg nilpotent group. There is then a canonical exact sequence

$$TN \overset{i}{\rightarrowtail} N \overset{\pi}{\twoheadrightarrow} FN,\ TN = \text{ torsion subgroup of } N, \quad FN = \text{ torsionfree quotient} \tag{1.1}$$

Now let $\alpha : TN \twoheadrightarrow \tilde{T}$ be a surjection, so that $\tilde{T}$ is a finite nilpotent group. Assume that $\ker \alpha$ is normal in N; call this condition K. Then we know [**H1**] that we may embed (1.1) in a map of exact sequences

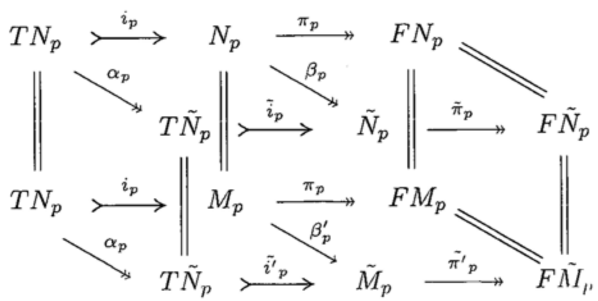
$$\begin{array}{ccccc} TN & \overset{i}{\rightarrowtail} & N & \overset{\pi}{\twoheadrightarrow} & FN \\ \alpha\downarrow & & \beta\downarrow & & \| \\ \tilde{T} & \overset{\tilde{i}}{\rightarrowtail} & \tilde{N} & \overset{\tilde{\pi}}{\twoheadrightarrow} & FN \end{array} \tag{1.2}$$

with $\tilde{N} \in \mathfrak{N}_{fg}$. Moreover, the LHS of (1.2) is a push-out in the category of groups; and, obviously, $F\tilde{N} = FN$, $T\tilde{N} = \tilde{T}$ —indeed, we will often write $T\tilde{N}$ for $\tilde{T}$. We now replace N by a nilpotent group M in the genus of N; we will assume, as we may, that $TM = TN$ and $M_p = N_p$ for all primes p. We claim that $\ker\alpha$ is normal in M under the natural embedding $\ker\alpha \subseteq TN = TM \subseteq M$. For $(\ker\alpha)_p$ is normal in M_p for all primes p, which shows that $\ker\alpha$ is normal in M. We thus have a commutative diagram

$$\begin{array}{ccccc} TN & \overset{i'}{\rightarrowtail} & M & \overset{\pi'}{\twoheadrightarrow} & FM \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta'}\downarrow & & \| \\ T\tilde{N} & \overset{\tilde{i}'}{\rightarrowtail} & \tilde{M} & \overset{\tilde{\pi}'}{\twoheadrightarrow} & FM \end{array} \tag{1.3}$$

Theorem 1.1. *The association* $M \mapsto \tilde{M}$ *defines a function* $\alpha_* : \mathfrak{G}(N) \to \mathfrak{G}(\tilde{N})$.

Proof: We have the commutative diagram (identifying FM_p with FN_p)



Now it is easy to prove that

$$\begin{array}{ccc} TN_p & \overset{i_p}{\rightarrowtail} & N_p \\ \downarrow{\scriptstyle\alpha_p} & & \downarrow{\scriptstyle\beta_p} \\ T\tilde{N}_p & \overset{\tilde{i}_p}{\rightarrowtail} & \tilde{N}_p \end{array}$$

is also a push-out in the category of groups. Thus we have a (unique) homomorphism $\kappa : \tilde{N}_p \to \tilde{M}_p$ such that $\kappa\beta_p = \beta'_p$ and $\kappa\tilde{i}_p = \tilde{i}'_p$. We claim that $\tilde{\pi}'_p\kappa = \tilde{\pi}_p$. For $\tilde{\pi}'_p\kappa\beta_p = \tilde{\pi}'_p\beta'_p = \pi_p = \tilde{\pi}_p\beta_p$ and $\tilde{\pi}'_p\kappa\tilde{i}_p = \tilde{\pi}'_p\tilde{i}'_p = 0 = \tilde{\pi}_p\tilde{i}_p$. Thus the diagram

$$\begin{array}{ccccc} T\tilde{N}_p & \overset{\tilde{i}_p}{\rightarrowtail} & \tilde{N}_p & \overset{\tilde{\pi}_p}{\twoheadrightarrow} & F\tilde{N}_p \\ \| & & \downarrow{\scriptstyle\kappa} & & \| \\ T\tilde{N}_p & \overset{\tilde{i}'_p}{\rightarrowtail} & \tilde{M}_p & \overset{\tilde{\pi}'_p}{\twoheadrightarrow} & F\tilde{M}_p \end{array}$$

commutes, showing that κ is an isomorphism. This proves that $\tilde{M} \in \mathfrak{G}(\tilde{N})$ and establishes the theorem. ■

The following "functorial" properties of the association $\alpha \mapsto \alpha_*$ are obvious.

Theorem 1.2. *(i)* Id : $TN \to TN$ *satisfies the condition* K *and* $\mathrm{Id}_* = \mathrm{Id}$.

(ii) If $\alpha : TN \twoheadrightarrow \tilde{T} = T\tilde{N}$ *satisfies condition* K *and* $\tilde{\alpha} : T\tilde{N} \twoheadrightarrow \tilde{\tilde{T}}$ *satisfies condition* K*, then* $\tilde{\alpha}\alpha$ *satisfies condition* K *and* $(\tilde{\alpha}\alpha)_* = \tilde{\alpha}_*\alpha_*$.

Proof: (i) is trivial. As to (ii), it suffices to remark that the existence of β in (1.2) guarantees that α satisfies condition K. Thus we superimpose diagrams to produce

$$\begin{array}{ccccc} TN & \rightarrowtail & N & \twoheadrightarrow & FN \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \| \\ T\tilde{N} & \rightarrowtail & \tilde{N} & \twoheadrightarrow & FN \\ {\scriptstyle\tilde{\alpha}}\downarrow & & {\scriptstyle\tilde{\beta}}\downarrow & & \| \\ T\tilde{\tilde{N}} & \rightarrowtail & \tilde{\tilde{N}} & \twoheadrightarrow & FN \end{array} \qquad (1.4)$$

and deduce, first, that $\tilde{\alpha}\alpha$ satisfies condition K and, second, that $(\tilde{\alpha}\alpha)_* = \tilde{\alpha}_*\alpha_*$. For, just as (1.3) was derived in similar manner to (1.2) so

(1.5)
$$\begin{array}{ccccc} TN & \rightarrowtail & N & \twoheadrightarrow & FM \\ \alpha\downarrow & & \beta'\downarrow & & \| \\ T\tilde{N} & \rightarrowtail & \tilde{M} & \twoheadrightarrow & FM \\ \tilde{\alpha}\downarrow & & \tilde{\beta}'\downarrow & & \| \\ T\tilde{\tilde{N}} & \rightarrowtail & \tilde{\tilde{M}} & \twoheadrightarrow & FM \end{array}$$

is derived in a similar manner to (1.4), and shows that

$$\tilde{\tilde{M}} = \tilde{\alpha}_*\alpha_*(M) = (\tilde{\alpha}\alpha)_*(M). \quad \blacksquare$$

We now make the further hypothesis that $N \in \mathfrak{N}_0$; this is equivalent to assuming that FN is commutative. Since $FN = F\tilde{N}$ it follows that $\tilde{N} \in \mathfrak{N}_0$, so that both $\mathfrak{G}(N)$, $\mathfrak{G}(\tilde{N})$ are finite abelian groups. (Notice that, in fact, $N \in \mathfrak{N}_0$ if and only if $\tilde{N} \in \mathfrak{N}_0$.) We then have

Theorem 1.3. *Suppose that $N \in \mathfrak{N}_0$. Then $\alpha_* : \mathfrak{G}(N) \to \mathfrak{G}(\tilde{N})$ is a homomorphism.*

Proof: Suppose that $K + L = M$ in $\mathfrak{G}(N)$. We continue to assume that

$$TK = TL = TM = TN.$$

Then, according to [**HM**], there exists an exhaustive pair $\varphi : N \to K$, $\psi : N \to L$, such that we may form the push-out (in $\mathfrak{N}$)

(1.6)
$$\begin{array}{ccc} N & \overset{\varphi}{\rightarrowtail} & K \\ \psi\downarrow & & \tau\downarrow \\ L & \overset{\sigma}{\rightarrowtail} & M \end{array}$$

We recall from [**HM**] that an *exhaustive pair* (φ, ψ) is defined by the requirements

(i) φ or ψ is a T-equivalence, where

$$T = T(N) = \{p | N \text{ has } p\text{-torsion}\};$$

and (ii) for all primes p, φ or ψ is a p-equivalence.

However, examination of the proof of Theorem 2.3 of [**HM**] shows that we may assume that *both* φ *and* ψ *are* T*-equivalences*. For having constructed φ as a T-equivalence, we define

$$P = \{p | \varphi \text{ is not a } p\text{-equivalence}\}$$

and then, modifying the argument in [**HM**], construct ψ to be a $(P \cup T)$-equivalence.

With this strengthened sense of an exhaustive pair, we revert to (1.6). Then φ, ψ, when restricted to TN, are both isomorphisms, so we may suppose that both are identities on TN. We may then suppose that σ, τ are also identities on TN. Now let us factor out $\ker \alpha$ from each of K, L, M, N. Since $\ker \alpha \subseteq TN$, this gives rise to a commutative diagram

(1.7)
$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{\varphi}} & \tilde{K} \\ {\scriptstyle\tilde{\psi}}\downarrow & & \downarrow{\scriptstyle\tilde{\tau}} \\ \tilde{L} & \xrightarrow{\tilde{\sigma}} & \tilde{M} \end{array}$$

which is easily seen to inherit from (1.6) the property of being a push-out in $\mathfrak{N}$. Moreover, it is plain that $\tilde{\varphi}$, $\tilde{\psi}$ remain T-equivalences and that, for all primes p, $\tilde{\varphi}$ or $\tilde{\psi}$ is a p-equivalence. Since $T\tilde{N}$ is a quotient of TN it is plain that $T(\tilde{N}) \subseteq T(N)$, so that $\tilde{\varphi}$ and $\tilde{\psi}$ are $T(\tilde{N})$-equivalences and $(\tilde{\varphi}, \tilde{\psi})$ is an exhaustive pair. We conclude that

$$\tilde{K} + \tilde{L} = \tilde{M} \text{ in } \mathfrak{G}(\tilde{N}),$$

so that φ is a homomorphism. ∎

2. A special case

Since it has not yet proved possible to calculate $\mathfrak{G}(N)$ systematically for $N \in \mathfrak{N}_0$, it is not to be expected that we would have much success in trying to analyse the homomorphism α_* in the generality in which it has been introduced in the preceding section. However, we do find it possible to make some headway if we make the restrictive assumption that $\tilde{N} \in \mathfrak{N}_1$. We then prove

Theorem 2.1. *Let* $\alpha_* : \mathfrak{G}(N) \to \mathfrak{G}(\tilde{N})$ *be defined as in Section 1 and let* $\tilde{N} \in \mathfrak{N}_1$. *Then* α_* *is a surjective homomorphism.*

Proof: Since $\tilde{N} \in \mathfrak{N}_0$, it follows that $N \in \mathfrak{N}_0$ and α_* is a homomorphism. Now $\mathfrak{G}(\tilde{N}) = 0$ unless FN is cyclic [**S**], [**HS**]. Thus, to avoid triviality, we assume FN *cyclic*. Under this assumption, the top row of (1.2) splits for an action $\omega : FN \to \operatorname{Aut} TN$. Let $\sigma : FN \to N$ be a splitting ($\pi\sigma = 1$), so that, if $FN = \langle\xi\rangle$, then ω is given by

$$\omega(\xi)(a) = yay^{-1},\ a \in TN, \text{ where } y = \sigma(\xi). \tag{2.1}$$

We will often write $\xi \cdot a$ for $\omega(\xi)(a)$. We use $\beta\sigma : FN \rightarrowtail \tilde{N}$ to split the botton row of (1.2) and write $\tilde{\omega} : FN \to \operatorname{Aut} T\tilde{N}$ for the associated action. Note that $\tilde{\omega}$ is given by

$$\tilde{\omega}(\xi)(\alpha a) = \alpha(\omega(\xi)(a)), \quad a \in TN. \tag{2.2}$$

We write (2.2) more simply as

$$\xi \cdot \alpha a = \alpha(\xi \cdot a), \quad a \in TN. \tag{2.3}$$

Now let $\tilde{t}$ be the *height* of $\ker \tilde{\omega}$ in FN; that is, since FN is cyclic, $\tilde{t}$ is the order of $\tilde{\omega}(\xi)$ in $\operatorname{Aut} T\tilde{N}$. Then, by the main theorem of [**CH**],

$$\mathfrak{G}(\tilde{N}) \cong (\mathbb{Z}/\tilde{t})^*/\{\pm 1\}. \tag{2.4}$$

Moreover, we may choose the isomorphism (2.4) so that the group $\tilde{N}_m$, m prime to $\tilde{t}$, corresponding to $[m] \in (\mathbb{Z}/\tilde{t})^*/\{\pm 1\}$, is obtained from $\tilde{N}$ simply by replacing the action $\tilde{\omega}$ by a new action $\tilde{\omega}_m$, defined by

$$\tilde{\omega}_m(\xi)(\tilde{a}) = \tilde{\omega}(\xi^m)(\tilde{a}), \quad \tilde{a} \in T\tilde{N}. \tag{2.5}$$

Of course we have freedom in (2.4) to choose m within its given class $[m]$ without changing $\tilde{N}_m$. We will, in fact, choose *m to be a T'-number*, where $T = T(N)$ is the set of primes p such that N has p-torsion. To see that we can do this it suffices to notice that m is prime to $\tilde{t}$ so that, by Dirichlet's Theorem, the residue class $[m]$ contains primes not in T.

With such a choice of m, we show that $\tilde{N}_m$ may be represented as $\alpha_*(N_m)$ for a suitable group N_m in $\mathfrak{G}(N)$. We define N_m to be the semi-direct product of TN and FN for the action $\omega_m : FN \to \operatorname{Aut} TN$, given by

$$\omega_m(\xi)(a) = \omega(\xi^m)(a), \quad a \in TN. \tag{2.6}$$

We first show that $N_m \in \mathfrak{G}(N)$. Consider the diagram

$$\begin{array}{ccccc} TN & \rightarrowtail & N_m & \twoheadrightarrow & FN \\ \| & & & & \downarrow m \\ TN & \rightarrowtail & N & \twoheadrightarrow & FN \end{array} \tag{2.7}$$

where the endomorphism of FN is just $\xi \mapsto \xi^m$. Then (2.6) asserts that (2.7) satisfies the compatibility condition permitting us to complete it with $\varphi : N_m \to N$ to a commutative diagram. Now if $p \in T$ then $m : FN \to FN$ is a p-equivalence, so that $\varphi : N_m \to N$ is a p-equivalence. If $p \notin T$ then TN_p is the trivial group so both N and N_m are p-equivalent to FN and hence p-equivalent to each other. Thus $N_m \in \mathfrak{G}(N)$.

Finally we show that $\alpha_*(N_m) = \tilde{N}_m$. Consider the diagrams

$$\begin{array}{ccccc} TN & \rightarrowtail & N & \twoheadrightarrow & FN \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & \big\| \\ T\tilde{N} & \rightarrowtail & \tilde{N} & \twoheadrightarrow & FN \end{array}$$

(2.8)

$$\begin{array}{ccccc} TN & \rightarrowtail & N_m & \twoheadrightarrow & FN \\ {\scriptstyle\alpha}\big\downarrow & & & & \big\| \\ T\tilde{N} & \rightarrowtail & \tilde{N}_m & \twoheadrightarrow & FN \end{array}$$

Recall that we are writing "$\cdot$" to indicate the actions of FN on TN or $T\tilde{N}$ in the first diagram; let us write "$\circ$" for the actions of FN on TN or $T\tilde{N}$ in the second diagram of (2.8). Then (2.3) $\xi \cdot \alpha a = \alpha(\xi \cdot a)$, $a \in TN$ and (2.6) $\xi \circ a = \xi^m \cdot a$, $a \in TN$. Moreover, by (2.5), $\xi \circ \alpha a = \xi^m \cdot \alpha a$, $a \in TN$. But since $\xi \cdot \alpha a = \alpha(\xi \cdot a)$, it follows that $\xi^m \cdot \alpha a = \alpha(\xi^m \cdot a)$, whence

$$\alpha(\xi \circ a) = \alpha(\xi^m \cdot a) = \xi^m \cdot \alpha a = \xi \circ \alpha a, \quad a \in TN.$$

This, however, is precisely the compatibility condition guaranteeing the existence, in the second diagram of (2.8), of $\beta_m : N_m \to \tilde{N}_m$ making the diagram commutative. Then β_m must be surjective. This, however, guarantees that

$$\begin{array}{ccc} TN & \overset{i}{\rightarrowtail} & N_m \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta_m}\big\downarrow \\ T\tilde{N} & \overset{\tilde{i}_m}{\rightarrowtail} & \tilde{N}_m \end{array}$$

is a push-out in the category of groups and hence, by the uniqueness of push-outs, that $\tilde{N}_m = \alpha_*(N_m)$. ■

We now consider the groups $N_m \in \mathfrak{G}(N)$ constructed in the course of our proof of Theorem 2.1. We have immediately

Corollary 2.2. *Suppose $N_m \cong N_{m'}$. Then $m \equiv m' \bmod \tilde{t}$.*

For if $N_m \cong N_{m'}$ then $\tilde{N}_m \cong \tilde{N}_{m'}$. We use Corollary 2.2 to obtain a non-cancellation result. We need some preliminary lemmas, the first of which addresses Remark 1 of [**HM**, Section 4].

Lemma 2.3. *Let $N \in \mathfrak{N}_0$ and let $FZN = nZN$, where ZN is the center of N and $n = \exp TZN$. Let k be a T-number, where $T = T(N)$, and let $QN = N/kFZN$. Then QN is a finite group and $p \in T(QN)$ if and only if $p \in T$.*

Remark. In [**HM**] it was remarked that we achieved the same effect whether we defined n to be the exponent or the order of TZN; of course, in either case FZN is free abelian.

Proof of Lemma 2.3: Since $[N, N]$ is finite and N is fg nilpotent, N/ZN is finite. Also ZN is fg so $ZN/knZN$ is finite. Hence $N/knZN$ is finite. Now let $ZN = F \oplus TZN$, with F fg free abelian. Then $kFZN = knF$, so

(2.9) $$ZN/kFZN = F/knF \oplus TZN.$$

Also we have an exact sequence

(2.10) $$ZN/kFZN \rightarrowtail QN \twoheadrightarrow N/ZN.$$

From (2.9) we infer, for an arbitrary prime p,

ZN has p-torsion $\Rightarrow ZN/kFZN$ has p-torsion $\Rightarrow N$ has p-torsion.

Thus, from (2.10),

QN has p-torsion $\Rightarrow$ $ZN/kFZN$ or N/ZN has p-torsion $\Rightarrow$ N has p-torsion; and N has p-torsion $\Rightarrow$ ZN or N/ZN has p-torsion $\Rightarrow ZN/kFZN$ or N/ZN has p-torsion $\Rightarrow QN$ has p-torsion.

This completes the proof. ■

Lemma 2.4. *Let $N \in \mathfrak{N}_0$ with FN cyclic, $FN = \langle \xi \rangle$. Let t be the order of $\omega(\xi) \in \operatorname{Aut} TN$. Then t is a T-number, where $T = T(N)$.*

Proof: Certainly FZN is a free cyclic group. Suppose it is generated by (a, ξ^s), $a \in TN$. By conjugating with $(1, \xi)$ it is clear that $\xi \cdot a = a$. Let k be the order of a. Then $(a, \xi^s)^k = (1, \xi^{sk})$. Now, since t is the order of $\omega(\xi)$, we infer that $t | sk$.

We compute QN as in Lemma 2.3. We have

$$N = \langle TN, y \rangle, \text{ where } y = (1, \xi)$$

$$kFZN = \langle y^{sk} \rangle \text{ (we confuse additive and multiplicative notation here!)}$$

Thus, $QN = \langle TN, y | y^{sk} = 1 \rangle$.

When we abelianize QN we get generators from $(TN)_{ab}$, together with $\overline{y}$; and the only relation involving $\overline{y}$ is $\overline{y}^{sk} = 1$. Thus $sk | \exp QN_{ab}$, whence $t | \exp QN_{ab}$. Now since QN is a finite nilpotent group, $T(QN) = T(QN_{ab})$, so that, by Lemma 2.3,

$$T = T(N) = T(QN_{ab}). \tag{2.11}$$

Since $t | \exp QN_{ab}$, t is a $T(QN_{ab})$-number. Hence, by (2.11), t is a T-number. ■

Before stating our non-cancellation result, we observe that the invariant t provides us with a partial converse to Corollary 2.2. Thus we may prove

Theorem 2.5. *(i) $\tilde{t} | t$; (ii) if $m \equiv m' \bmod t$, then $N_m \cong N_{m'}$.*

Proof: (i) follows immediately from (2.3) and the fact that α is surjective.

As to (ii), observe first that $N_m \cong N_{-m}$; for we have the diagram

$$\begin{array}{ccccc} TN & \rightarrowtail & N_m & \twoheadrightarrow & FN \\ \| & & & & \downarrow{\scriptstyle -1} \\ TN & \rightarrowtail & N_{-m} & \twoheadrightarrow & FN \end{array}$$

satisfying the obvious compatibility condition, giving rise to an isomorphism $N_m \cong N_{-m}$. Further we have an actual equality between N_m and N_{m+qt} since $\xi^{m+qt} \cdot a = \xi^m \cdot a$, for all $a \in TN$. ■

We are now ready to enunciate our non-cancellation theorem; recall that we have constructed a group N_m in $\mathfrak{G}(N)$ for each m such that m is a T'-number prime to $\tilde{t}$; and that $N_m \cong N_{m'} \Rightarrow m \equiv \pm m' \bmod \tilde{t}$.

Theorem 2.6. *$N_m \times C \cong N \times C$, where C is cyclic infinite.*

Proof: Since m is a T'-number it follows from Lemma 2.4 that m is prime to t, the order of $\omega(\xi)$ in $\operatorname{Aut} TN$. Let $A = \begin{pmatrix} m & t \\ r & s \end{pmatrix}$ be a unimodular matrix over $\mathbb{Z}$; let $C = \langle \eta \rangle$ and interpret A as the automorphism of $FN \times C$ given by $\xi \mapsto \xi^m \eta^r$, $\eta \mapsto \xi^t \eta^s$. Consider the diagram

$$
(2.12) \qquad \begin{array}{ccccc} TN & \rightarrowtail & N_m \times C & \twoheadrightarrow & FN \times C \\ \| & & & & \downarrow A \\ TN & \rightarrowtail & N \times C & \twoheadrightarrow & FN \times C \end{array}
$$

We claim that (2.12) satisfies the compatibility condition. For C operates trivially on TN so we may write, for the top row of (2.12),

$$
(2.13) \qquad \xi \circ a = \xi^m \cdot a, \quad \eta \circ a = a, \quad a \in TN.
$$

and, for the bottom row of (2.12),

$$
(2.14) \qquad \eta \cdot a = a, \quad a \in TN.
$$

Moreover, each of $N_m \times C$, $N \times C$ is the semi-direct product for the given actions. Further

$$
\begin{aligned} A\xi \cdot a &= \xi^m \eta^r \cdot a = \xi^m \cdot a = \xi \circ a, \\ A\eta \cdot a &= \xi^t \eta^s \cdot a = a = \eta \circ a, \end{aligned}
$$

by (2.13) and (2.14). It follows that we may find

$$
\varphi : N_m \times C \rightarrowtail N \times C
$$

completing (2.12) to a commutative diagram. It is then clear that φ is an isomorphism. ■

Now to obtain an actual non-cancellation example, it suffices to find an example of the data of Theorem 2.1 in which $\tilde{t} \not\equiv 1, 2, 3, 4, 6$. In the next section we show, in fact, how to construct examples with *any* given $\tilde{t}$.

3. Examples

We may apply Theorem 1.1 by factoring $[TN, TN]$, $[N, N] \cap TN$, TZN, $ZN \cap TN$ out of TN and N and letting α, β be the associated quotient maps. The first is especially interesting for then $T\tilde{N}$ is commutative, but $\tilde{N}$, in general, is not. If $N \in \mathfrak{N}_0$, we may apply Theorem 1.3; and we may further hope that $\tilde{N} \in \mathfrak{N}_1$ so that we can apply Theorem 2.1. If FN is cyclic we will only need to verify condition (iii) for membership of $\mathfrak{N}_1$ (see the Introduction), and, if $T\tilde{N}$ is also cyclic, condition (iii) is automatically verified.

We now give an example (or, rather, a family of examples) which gives rise to a group $\tilde{N}$ in $\mathfrak{N}_1$ (although $T\tilde{N}$ is not cyclic), and thus to the construction of non-trivial genera $\mathfrak{G}(N)$ for groups N in $\mathfrak{N}_0$, with TN non-commutative, and to explicit non-cancellation results, based on Corollary 2.2 and Theorem 2.6.

Given $\tilde{t}$, choose n and u such that (i) n is even; (ii) $p|n \Rightarrow p|u-1$, for all primes p; (iii) the order of $u \bmod n$ is $\tilde{t}$. Notice that (i) and (ii) imply that u is odd. As examples of possible choices for n and u, we have:

It $\tilde{t}$ is odd, say $\tilde{t} = p_1^{\ell_1} p_2^{\ell_2} \dots p_\lambda^{\ell_\lambda}$, choose

$$n = 2p_1^{\ell_1+1} p_2^{\ell_2+1} \dots p_\lambda^{\ell_\lambda+1}, \quad u = 1 + 2p_1 p_2 \dots p_\lambda;$$

if $\tilde{t}$ is even, say $\tilde{t} = 2^{\ell_1} p_2^{\ell_2} \dots p_\lambda^{\ell_\lambda}$, choose

$$n = 2^{\ell_1+2} p_2^{\ell_2+1} \dots p_\lambda^{\ell_\lambda+1}, \quad u = 1 + 4p_2 \dots p_\lambda.$$

Now set $TN = \langle x, y, z | x^2 = y^2 = z^{2n} = 1, [x,y] = z^n, [x,z] = [y,z] = 1\rangle$. Obviously TN is nilpotent of class 2. Let $FN = \langle \xi \rangle$ operate on TN by the rule

$$\xi \cdot x = x, \quad \xi \cdot y = y, \quad \xi \cdot z = z^u. \tag{3.1}$$

This clearly describes an automorphism of TN since u is prime to n by (ii) above and hence, being odd, prime to $2n$. Moreover, $z^{un} = z^n$, again because u is odd.

We claim that the action (3.1) is nilpotent. For we have $\Gamma^0_{FN} TN = TN$,

$$\begin{aligned} \Gamma^1_{FN} TN &= \langle z^{u-1}, z^n \rangle, \\ \Gamma^2_{FN} TN &= \langle z^{(u-1)^2}, z^{(u-1)n} \rangle = \langle (z^{(u-1)^2} \rangle, \\ \Gamma^3_{FN} TN &= \langle z^{(u-1)^3} \rangle, \dots, \end{aligned}$$

and thus, again by (ii) above, $\Gamma^k_{FN} TN = \{1\}$ for k sufficiently large. If, then, we form the semi-direct product N of TN and FN for this action, N is a nilpotent group and, indeed, $N \in \mathfrak{N}_0$.

Now $[TN, TN] = \langle z^n \rangle$. Thus we may factor out $[TN, TN]$ to form

$$\tilde{T} = (TN)_{ab} = \langle \tilde{x}, \tilde{y}, \tilde{z} | 2\tilde{x} = 2\tilde{y} = n\tilde{z} = 0 \rangle, \tag{3.2}$$

and, following the procedure of Section 1, we have the commutative diagram

$$\begin{array}{ccccc} TN & \rightarrowtail & N & \twoheadrightarrow & FN \\ \alpha\downarrow & & \beta\downarrow & & \| \\ T\tilde{N} & \rightarrowtail & \tilde{N} & \twoheadrightarrow & FN \end{array}, \quad T\tilde{N} = \tilde{T}. \tag{3.3}$$

Now FN acts on $T\tilde{N}$ by

$$\xi \cdot \tilde{x} = \tilde{x}, \quad \xi \cdot \tilde{y} = \tilde{y}, \quad \xi \cdot \tilde{z} = u\tilde{z}, \tag{3.4}$$

so that

$$\xi \cdot \tilde{a} = u\tilde{a}, \text{ for all } \tilde{a} \in T\tilde{N}. \tag{3.5}$$

Moreover, $\exp T\tilde{N} = n$, so that $\tilde{N} \in \mathfrak{N}_1$ by (3.5) and

$$\mathfrak{G}(\tilde{N}) \cong (\mathbb{Z}/\tilde{t})^*/\{\pm 1\}, \tag{3.6}$$

by condition (iii). Thus

$$\alpha_* : \mathfrak{G}(N) \twoheadrightarrow (\mathbb{Z}/\tilde{t})^*/\{\pm 1\} \tag{3.7}$$

and $\mathfrak{G}(N)$ is a non-trivial group, provided that $\tilde{t} \neq 1, 2, 3, 4, 6$.

Now $u^{\tilde{t}} \equiv 1 \bmod n$. Thus $u^{2\tilde{t}} \equiv 1 \bmod 2n$, so that $t = 2\tilde{t}$ or $\tilde{t}$. Moreover, we may follow the procedure of Section 2 to construct N_m if m is prime to $\tilde{t}$ and a T'-number, where $T = T(N)$. Plainly $\exp TN = 2n$, so T consists of the prime divisors of n.

Let us now insist, for simplicity, as we clearly may, that $\tilde{t}$ and n have precisely the same prime divisors, except that $2|n$ even if $\tilde{t}$ is odd. Thus we can construct N_m if m is prime to $\tilde{t}$, with the additional condition that m is odd, even if $\tilde{t}$ is odd. We thus have

Theorem 3.1. *For a given $\tilde{t}$, choose (n, u) as above and construct the group N as described. Then there is a surjective homomorphism*

$$\alpha_* : \mathfrak{G}(N) \twoheadrightarrow (\mathbb{Z}/\tilde{t})^*/\{\pm 1\}.$$

We may also construct $N_m \in \mathfrak{G}(N)$ for any odd m prime to $\tilde{t}$, and

$$m \equiv \pm m' \bmod 2\tilde{t} \Rightarrow N_m \cong N_{m'} \Rightarrow m \equiv \pm m' \bmod \tilde{t}. \tag{3.8}$$

Moreover, $N_m \times C \cong N \times C$ for any odd m prime to $\tilde{t}$.

Finally, we become even more specific! Let $\tilde{t}$ itself be odd and choose (n, u) as follows (this modifies slightly our earlier example of a possible choice). Thus, if $\tilde{t} = p_1^{\ell_1} p_2^{\ell_2} \dots p_\lambda^{\ell_\lambda}$, choose

$$n = 2p_1^{\ell_1+1} p_2^{\ell_2+1} \dots p_\lambda^{\ell_\lambda+1}, \quad u = 1 + 4p_1 p_2 \dots p_\lambda. \tag{3.9}$$

The effect of this choice is that $t = \tilde{t}$, since the order of $u \bmod 2n$ is the same (i.e., $\tilde{t}$) as the order of $u \bmod n$. Thus, with the choice (3.9) —of course, other choices may have the same effect— we may improve (3.8) to

$$m \equiv \pm m' \bmod \tilde{t} \Leftrightarrow N_m \cong N_{m'}. \tag{3.8'}$$

Example 3.1. Let $\tilde{t} = 35$. Then, according to (3.9), we choose $n = 2450$, $u = 141$. Now $(\mathbb{Z}/35)^*/\{\pm 1\} \cong C_{12}$, its elements being [2], [4], [8], [16], [32], [29], [23], [11], [22], [9], [18], [1]. Thus, since we must take m odd, we have, as possible values of m,

$$m = 33, 31, 27, 19, 3, 29, 23, 11, 13, 9, 17, 1. \tag{3.10}$$

Each of these values of m yields, according to (3.8'), a group N_m in $\mathfrak{G}(N)$, no two of which are isomorphic. On the other hand all the groups $N_m \times C$, as m runs through the values of (3.10), are isomorphic.

Remark. It is easy to extend Theorem 2.1 to the study of $\mathfrak{G}(N^k)$, $k \geq 2$, where N^k is the direct product of k copies of N. For we recall from [**CH**] the surjective homomorphism $\rho : \mathfrak{G}(N) \twoheadrightarrow \mathfrak{G}(N^k)$, $N \in \mathcal{N}_0$, given by $\rho(M) = M \times N^{k-1}$. Plainly we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{G}(N) & \xrightarrow{\rho} & \mathfrak{G}(N^k) \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*^k} \\ \mathfrak{G}(\tilde{N}) & \xrightarrow{\rho} & \mathfrak{G}(\tilde{N}^k) \end{array} \tag{3.11}$$

so that, since α_* is surjective, so is α_*^k. Since we have calculated $\mathfrak{G}(\tilde{N}^k)$ for $\tilde{N} \in \mathfrak{N}_1$ [S], [**HS2**], we may extend the applications in this section from $\mathfrak{G}(N)$ to $\mathfrak{G}(N^k)$. We leave the details to the reader.

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