THE MULTIPLICATIVE STRUCTURE
OF $K(n)^*(BA_4)$

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Abstract

Let $K(n)^*(-)$ be a Morava $K$-theory at the prime 2. Invariant theory is used to identify $K(n)^*(BA_4)$ as a summand of $K(n)^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2)$. Similarities with $H^*(BA_4; \mathbb{Z}/2)$ are also discussed.

Introduction

Let $G$ be a finite group, and let $N$ and $C$ denote respectively the normalizer and the centralizer of a $p$-Sylow subgroup $H$ of $G$.

For a large family of cohomology theories including the Brown-Peterson cohomology $BP^*(-)$ and Morava $K$-theories $K(n)^*(-)$, the author described $h^*(BG)$ when $H$ is cyclic [3], and discussed the case “$p$-rank $(H) < 3$” in [5], proving in particular that $h^*(BG)$ is generated as $h^*$-module by at most two elements if $|N : C|$ divides $p - 1$.

Results in this paper show that the condition above is really necessary, in fact we have

**Theorem 0.1.** Let $K(n)^*(-)$ be a Morava $K$-theory at the prime 2. $K(n)^*(BA_4)$ restricts to those elements in

$$K(n)^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2) \cong K(n)^*[x, y]/(x^{2^n}, y^{2^n})$$

which algebraically depend on

$$\bar{\sigma} = x^2 + y^2 + xy + \nu_n(x^{2^{n-1}+1}y^{2^{n-1}} + x^{2^{n-1}}y^{2^{n-1}+1}),$$

$$\bar{\tau}_1 = x^3 + y^3 + x^2y + \nu_n(x^{2^{n-1}-1}y^{2^{n-1}+2}),$$

$$\bar{\tau}_2 = x^3 + y^3 + xy^2 + \nu_n(x^{2^{n-1}+2}y^{2^{n-1}}).$$

This paper has several motivations. The knowledge of $K(n)^*(BA_4)$ could help to have explicit formulæ for the $K(n)^*$-Dickson classes. Furthermore, similarities among $H^*(BA_4; \mathbb{Z}/2)$ and $K(n)^*(BA_4)$ suggest to study $K(n)^*(BA_m)$ — whose rank as $K(n)^*$-module can be calculated [6] — to get information on $H^*(BA_m; \mathbb{Z}/2)$ which is not entirely known for $m \geq 16$ (see [1] for the cohomology of several alternating groups).

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1. Preliminaries. $H^*(BA_4)$

From now on $V$ will denote the group $\mathbb{Z}/2 \times \mathbb{Z}/2$, and $H^*(-)$ ordinary cohomology with coefficients in $\mathbb{Z}/2$.

In [7], the authors describe $H^*(PSL_2\mathbb{F}_q)$ for any odd $q$: they first calculate the cohomology of the generalized quaternion group $Q_{2^n+1}$ of order $2^{n+1}$, and then use the diagram

$$
\begin{array}{cccc}
\mathbb{Z}/2 & \rightarrow & SL_2\mathbb{F}_q & \rightarrow & PSL_2\mathbb{F}_q \\
\downarrow & & \uparrow i & & \uparrow j \\
\mathbb{Z}/2 & \rightarrow & Q_{2^n+1} & \rightarrow & D_n
\end{array}
$$

where $D_n$ is the dihedral group of order $2^n$, rows are fibrations, and $i$ and $j$ are inclusions of 2-Sylow subgroups. Nevertheless, we show in this section that the special case

$$PSL_2\mathbb{F}_3 \cong A_4$$

can be approached in a more direct way.

The alternating group $A_4$ is the central term of the short exact sequence of groups

$$0 \rightarrow V \rightarrow A_4 \rightarrow \mathbb{Z}/3 \rightarrow 0,$$

therefore for any mod 2 cohomology theory $h^*(-)$, $h^*(BA_4)$ is isomorphic to the ring of invariants $[h^*(BV)]^{\mathbb{Z}/3}$ under the action determined by the map $h^*(B\phi)$ induced by an automorphism $\phi$ of order 3 in Aut($V$). On $H^*(BV) \cong \mathbb{F}_2[x, y]$ the action of a generator of $\mathbb{Z}/3 \leq GL(V)$ is

$$x \stackrel{\alpha_3}{\rightarrow} y \quad \text{and} \quad y \stackrel{\alpha_3}{\rightarrow} x + y.$$

Consider now the map $\Phi$ from $\mathbb{F}_2[x, y]$ to itself which maps any element $c$ to the sum

$$\Phi(c) = c + \alpha_H(c) + \alpha_H^2(c);$$
\( \Phi \) is commonly known as norm map. It is easy to see that

\[
\text{Im} \Phi = \left[ H^\ast(BV) \right]^{\mathbb{Z}/3};
\]

furthermore the image of \( \Phi \) restricted to the set of monomials generates \([H^\ast(BV)]^{\mathbb{Z}/3}\) regarded as graded \( \mathbb{F}_2 \)-vector space.

The invariant of lowest positive degree in \( \mathbb{F}_2[x,y] \) is

\[
\sigma = \Phi(xy) = x^2 + xy + y^2.
\]

This element is actually the Dickson class known in literature as \( Q_{2,1} \) (see [9]).

The reader will find the relevant invariant theoretic computation in [2] to prove the algebraic dependence of every invariant on \( \Phi(xy), \Phi(x^2y), \Phi(xy^2) \). In fact we have the following proposition.

**Proposition 1.1.** As a graded ring, \( H^\ast(BA_4) \) is isomorphic to

\[
\mathbb{F}_2[\sigma, \tau_1, \tau_2]/R,
\]

where \( \deg \sigma = 2, \deg \tau_1 = \deg \tau_2 = 3 \), and \( R \) is the ideal generated by

\[
\sigma^3 + \tau_1^2 + \tau_1 \tau_2 + \tau_2^2.
\]

The proposition above can be restated in terms of pure invariant theory.

**Corollary 1.2.** Suppose that a \( \mathbb{Z}/3 \)-action on \( \mathbb{F}_2[x,y] \) is given by

\[
x \longrightarrow y \quad \text{and} \quad y \longrightarrow x + y.
\]

The ring of the invariants is a polynomial ring generated by

\[
\sigma = x^2 + y^2 + xy, \quad \tau_1 = x^3 + y^3 + x^2y, \quad \tau_2 = x^3 + y^3 + xy^2,
\]

quotiented by

\[
R = (\sigma^3 + \tau_1^2 + \tau_1 \tau_2 + \tau_2^2).
\]
2. The Morava $K$-theory of $BA_4$

We recall that Morava $K$-theory at the prime 2 is a complex oriented cohomology theory with coefficients

$$K(n)^*(\{pt\}) = \mathbb{F}_2[\nu_n, \nu_n^{-1}]$$

where $\deg \nu_n = -2(2^n - 1)$, and we have

$$K(n)^*(BV) \cong K(n)^*[x, y]/(x^{2^n}, y^{2^n})$$

where $\deg x = \deg y = 2$. As noticed in section 1, $K(n)^*(BA_4)$ is isomorphic to

$$[K(n)^*(BV)]^{\mathbb{Z}/3}$$

where the $\mathbb{Z}/3$-module structure is defined by the map $K(n)^*(B\phi)$, being $\phi$ a generator of $\mathbb{Z}/3 \leq \text{Aut}(V)$. The following lemma helps to give a concrete description of the $K(n)$-invariants.

**Lemma 2.1.** One of the two generators $\phi$ of $\mathbb{Z}/3 \leq \text{Aut}(V)$ acts as follows on $K(n)^*(BV)$:

$$\alpha_K \overset{\text{def}}{=} K(n)^*(B\phi) : x \rightarrow y \quad \text{and} \quad \alpha_K : y \rightarrow x + y + \nu_n x^{2^n-1} y^{2^n-1}.$$  

**Proof:** See [4].

The element $\alpha_K(y)$ is actually the formal sum of $x$ and $y$ with respect to the formal group law of mod 2 Morava $K$-theory

$$F_{K(n)}(x, y) \mod (x^{2^n}, y^{2^n}).$$

Consider now the norm map $\Psi$ defined as follows:

$$\Psi : c \in K(n)^*(BV) \mapsto c + \alpha_K(c) + \alpha_K^2(c) \in [K(n)^*(BV)]^{\mathbb{Z}/3}.$$  

The map $\Psi$ is obviously the analogue of $\Phi$ defined in section 1: it is surjective, and the invariants regarded as $\mathbb{F}_2$-vector space are spanned by the image of $\Psi$ restricted to monomials.

Notice also that we can equip

$$K(n)^*(BV) \cong K(n)^*[x, y]/(x^{2^n}, y^{2^n})$$

with a different $\mathbb{Z}/3$-module structure just by posing

$$\alpha_H(x) = y \quad \text{and} \quad \alpha_H(y) = x + y.$$  

Abusing notation, we shall use again $\Phi$ to denote the endomorphism defined on the generic element of $K(n)^*(BV)$ as follows:

$$c \mapsto c + \alpha_H(c) + \alpha_H^2(c).$$

We are ready now to prove our main result.
Theorem 2.2. \( K(n)^*(BA_4) \) restricts to those elements in \( K(n)^*(BV) \) which algebraically depend on
\[
\Psi(xy) = \bar{\sigma}, \quad \Psi(x^2y) = \bar{\tau}_1 \quad \text{and} \quad \Psi(xy^2) = \bar{\tau}_2.
\]

Proof: Since \( K(n)^*(-) \) is \( 2(2^n - 1) \)-periodic we can look at classes in \( K(n)^*(BV) \) whose degree is between 2 and \( 2(2^n - 1) \). In this range, elements of type
\[
\nu_n^2 x^h y^k
\]
are necessarily zero, since either \( h \) or \( k \) is greater than \( 2^n \). It follows that for any monomial \( x^h y^k \in K(n)^*(BV) \) we have
\[
(1) \quad \Psi(\nu_n x^h y^k) = \nu_n \Phi(x^h y^k).
\]
An element \( c \in K(n)^*(BV) \) is invariant under \( \alpha_K \) if and only if \( \Psi(c) = c \), and supposing
\[
2 \leq t \leq 2(2^n - 1),
\]
we have
\[
c = p(x, y) + \nu_n q(x, y),
\]
where \( p(x, y) \) and \( q(x, y) \) are homogeneous polynomials of \( \mathbb{F}_2[x, y] \) of degree \( t \) and \( t + 2(2^n - 1) \) respectively. If \( \Psi(c) = c \), it follows from the considerations above that \( \Phi(p(x, y)) = p(x, y) \), and by Corollary 1.2 there exists a polynomial \( r_1 \) in three indeterminates such that
\[
r_1(\sigma, \tau_1, \tau_2) = p(x, y).
\]
Define now
\[
\Psi(xy) = \bar{\sigma}, \quad \Psi(x^2y) = \bar{\tau}_1 \quad \text{and} \quad \Psi(xy^2) = \bar{\tau}_2.
\]
The element
\[
c - r_1(\bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2) = \nu_n s(x, y)
\]
is invariant under \( \alpha_K \). Notice now that \( s(x, y) \) can be regarded as a polynomial in \( \mathbb{F}_2[x, y] \); it follows by (1) that \( s(x, y) \) is invariant under \( \alpha_H \), and again by Corollary 1.2 there exists a polynomial \( r_2 \) in three indeterminates such that
\[
r_2(\sigma, \tau_1, \tau_2) = s(x, y).
\]
We finally get
\[ c = r_1(\bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2) - \nu_n r_2(\bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2) \]
as we claimed.

Theorem 2.2 also gives some information on \( K(n)^*(BA_5) \). Notice in fact that 2-Sylow subgroups in \( A_5 \) are abelian, and a 2-Sylow normalizer in \( A_5 \) is isomorphic to \( A_4 \). It follows by a theorem in [8] that \( BA_4 \) and \( BA_5 \) are stably 2-homotopy equivalent. Hence the map induced by inclusion
\[ K(n)^*(BA_5) \longrightarrow K(n)^*(BA_4) \]
is an isomorphism.

**Remark 2.3.** The element
\[ \bar{\sigma}^3 + \bar{\tau}_1^2 + \bar{\tau}_1 \bar{\tau}_2 + \bar{\tau}_2^2 \]
is zero in \( K(n)^*(BA_4) \), as the analogous algebraic expression in \( \sigma, \tau_1, \tau_2 \) for ordinary cohomology. The relation above is not however of minimal positive degree: the element
\[ \nu_n^2 \bar{\sigma}^{2n} \]
is zero and has degree four.

It is known that the subring of \( H^{even}(BA_4) \) generated by Chern classes is proper (see, for example [10, p. 100]), and the reader could ask if \( \bar{\sigma}, \bar{\tau}_1, \bar{\tau}_2 \) are \( K(n) \)-Chern classes of suitable representations.

We recall that up to equivalence the group \( A_4 \) has just four distinct complex irreducible representations. Three of them are one-dimensional, and their restriction to \( V \) is trivial. The fourth one has instead non-trivial total Chern class in \( K(n)^*(BA_4) \), as the next proposition shows.

**Proposition 2.4.** Let \( \xi \) be a 3-dimensional irreducible representation of \( A_4 \). The restriction \( \xi|_V \) to the 2-Sylow subgroup \( V \) has Chern classes
\[ c_1(\xi|_V) = \nu_n \bar{\sigma}^{2^n-1}, \quad c_2(\xi|_V) = \bar{\sigma}, \quad c_3(\xi|_V) = \bar{\tau}_1 + \bar{\tau}_2 + \nu_n \bar{\sigma}^{2^n-1+1} \]
in \( K(n)^*(BV) \).

**Proof.** Let \( g_1 \) and \( g_2 \) be two generators in \( V \). Consider two one-dimensional representations \( \rho_1 \) and \( \rho_2 \) defined as follows
\[ \rho_1 : g_1 \mapsto -1 \quad \rho_1 : g_3 \mapsto 1 \]
for $i = 1, 2$. The transfer $\xi$ of $\rho_1$ to $A_4$ represents the equivalence class of the 3-dimensional irreducible representations of $A_4$; its restriction to $V$ is given by

$$\rho_1 \oplus \rho_2 \oplus (\rho_1 \otimes \rho_2).$$

It follows that the total Chern class $c.(\xi|_V)$ is equal to

$$(1 + x)(1 + y)(1 + x + y + \nu_n x^{n-1} y^{n-1}).$$

Hence the proposition follows. 

References