A STABILITY RESULT
ON MUCKENHOUPT’S WEIGHTS

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Abstract

We prove that Muckenhoupt’s $A_1$-weights satisfy a reverse Hölder inequality with an explicit and asymptotically sharp estimate for the exponent. As a by-product we get a new characterization of $A_1$-weights.

1. Introduction and statement of results

Muckenhoupt’s weights are important tools in harmonic analysis, partial differential equations and quasiconformal mappings. The self-improving property of Muckenhoupt’s weights is probably one of the most useful results in the field. The surprising fact that the weights are more regular than they seem to be a priori was observed already by Muckenhoupt [16]. The same phenomenon was studied by Gehring in [6] where he introduced the concept of reverse Hölder inequalities and proved that they improve themselves. Later Coifman and Fefferman [3] showed that Muckenhoupt’s weights are exactly those weights which satisfy a reverse Hölder inequality. Since then reverse Hölder inequalities have had a vast number of applications in modern analysis. An excellent source for all the mentioned results and other properties of Muckenhoupt’s weights is the monograph [7].

We are interested in a stability question related to Muckenhoupt’s $A_1$-class and reverse Hölder inequalities. Suppose that $w : \mathbb{R}^n \to [0, \infty]$ is a locally integrable function satisfying Muckenhoupt’s $A_1$-condition,

\begin{equation}
\frac{1}{|B|} \int_B w(x) \, dx \leq c_w \text{ess inf}_{x \in B} w(x),
\end{equation}

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for all balls $B \subset \mathbb{R}^n$ with the constant $c_w \geq 1$ independent of the ball $B$. Here $|B|$ is the volume of $B$. If $w$ belongs to Muckenhoupt’s class $A_1$, we denote $w \in A_1$; the smallest constant $c_w$ for which (1.1) holds is called the $A_1$-constant of $w$.

Condition (1.1) can be expressed in terms of the Hardy-Littlewood maximal function, defined by

$$
\mathcal{M}w(x) = \sup_B \frac{1}{|B|} \int_B w(y) \, dy,
$$

where the supremum is over all balls $B \subset \mathbb{R}^n$ containing the point $x$. It is easy to see [7, p. 389] that (1.1) is equivalent to the requirement that

$$
\mathcal{M}w(x) \leq c_w w(x)
$$

almost everywhere with exactly the same $c_w$ as in (1.1).

It is clear that (1.1) imposes a serious restriction on the function. If the $A_1$-constant is one, then

$$
0 \leq \frac{1}{|B|} \int_B \left( w(y) - \operatorname{ess inf}_{x \in B} w(x) \right) \, dy \leq \operatorname{ess inf}_{x \in B} w(x) - \operatorname{ess inf}_{x \in B} w(x) = 0
$$

and hence $w$ is constant. We are interested in the regularity of $A_1$-weights as the constant tends to one. It is well-known that $A_1$-weights satisfy the reverse Hölder inequality

$$
(1.2) \quad \left( \frac{1}{|B|} \int_B w(x)^p \, dx \right)^{1/p} \leq c \frac{1}{|B|} \int_B w(x) \, dx,
$$

for some $p > 1$ and $c$ independent of the ball $B$. Using (1.2) and (1.1) we see that $w^p \in A_1$ and $w$ is locally integrable to power $p$. The question is: how large can $p$ be? If the $A_1$-constant is one, then the weight is essentially bounded and it seems reasonable to expect that the degree of the local integrability increases as the $A_1$-constant tends to one. Questions related to the stability of reverse Hölder inequalities have obtained considerable attention in the last two decades, see [1], [2], [9], [11], [12], [13], [14], [15], [17], [18], [19], [20] and [21].

Our contribution is twofold. First, we present a new and a simple method which gives an explicit and asymptotically optimal bound for $p$. Second, our proof leads to a new characterization of $A_1$-weights (Corollary 2.11) which may be of independent interest.

Now we are ready to present our main result.
1.3. Theorem. If \( w \in A_1 \) with the constant \( c_w \), then there is a constant \( \nu \) depending only on the dimension such that \( w \) satisfies the reverse Hölder inequality (1.2) whenever
\[
1 \leq p < 1 + \frac{\nu}{c_w - 1}.
\]

In the one-dimensional case we may take \( \nu = 1 \) in (1.4), see [2] and [11], but our proof generally yields a small \( \nu \). Our method also allows us to replace balls in the \( A_1 \)-condition by cubes. Observe that the bound (1.4) for the local integrability of the weight is arbitrarily large provided \( c_w \) is close enough to one.

We remark that using factorization results of [10] and [4], our method gives similar estimates for Muckenhoupt’s \( A_p \)-weights as well. In the one-dimensional case this has been studied by Neugebauer [17].

2. Characterization of \( A_1 \)-weights

We begin by showing that every \( A_1 \)-weight can be approximated by smooth \( A_1 \)-weights.

2.1. Lemma. Suppose that \( w \in A_1 \) with the constant \( c_w \) and let \( \varphi \in C^\infty_0(\mathbb{R}^n) \), \( \varphi \geq 0 \) with \( \int_{\mathbb{R}^n} \varphi \, dx = 1 \). Then \( w \ast \varphi \in A_1 \) with the constant \( c_w \).

Proof: A direct calculation gives
\[
\frac{1}{B(x,r)} \int_{B(x,r)} w \ast \varphi(y) \, dy = \frac{1}{B(x,r)} \int_{B(x,r)} \int_{\mathbb{R}^n} w(y-z) \varphi(z) \, dz \, dy
\]
\[
= \int_{\mathbb{R}^n} \varphi(z) \frac{1}{B(x-z,r)} \int_{B(x-z,r)} w(y) \, dy \, dz
\]
\[
\leq c_w \int_{\mathbb{R}^n} \varphi(z) \, \text{ess inf}_{y \in B(x-z,r)} \, w(y) \, dz
\]
\[
= c_w \int_{\mathbb{R}^n} \varphi(z) \, \text{ess inf}_{y \in B(x,r)} \, w(y-z) \, dz
\]
\[
\leq c_w \, \text{ess inf}_{y \in B(x,r)} \int_{\mathbb{R}^n} w(y-z) \varphi(z) \, dz
\]
\[
= c_w \, \text{ess inf}_{y \in B(x,r)} w \ast \varphi(y).
\]

This completes the proof. \( \blacksquare \)

We record a well-known covering theorem.
2.3. BESICOVITCH’S COVERING THEOREM. Suppose that $E$ is a bounded subset of $\mathbb{R}^n$ and that $\mathcal{B}$ is a collection of balls such that each point of $E$ is a center of some ball in $\mathcal{B}$. Then there exists an integer $N \geq 2$ (depending only on the dimension) and subcollections $\mathcal{B}_1, \ldots, \mathcal{B}_N \subset \mathcal{B}$ of at most countably many balls such that the balls $B_{i,j}$, $j = 1, 2, \ldots,$ in each family $\mathcal{B}_i$, $i = 1, 2, \ldots, N$, are pairwise disjoint and

$$E \subset \bigcup_{i=1}^{N} \bigcup_{j=1}^{\infty} B_{i,j}.$$  

For the proof of Besicovitch’s covering Theorem we refer to [5, Theorem 1.1]. Some estimates for the constant $N$ are obtained in [8].

Now we show that $A_1$-weights satisfy a reverse Chebyshev inequality. This observation is a crucial ingredient in the proof of Theorem 1.3. For short we denote

$$E_\lambda = \{x \in \mathbb{R}^n : w(x) > \lambda\}, \quad \lambda > 0,$$

throughout the paper.

2.4. Lemma. Let $B \subset \mathbb{R}^n$ be a ball and suppose that $w : \mathbb{R}^n \to [0, \infty]$ is an $A_1$-weight with the constant $c_w$. Then there is a constant $\eta$, depending only on the dimension, so that

$$(2.5) \quad \int_{E_\lambda \cap B} w(x) \, dx \leq (c_w + \eta(c_w - 1)) \lambda |E_\lambda \cap B|,$$

whenever $\text{ess inf}_{x \in B} w(x) \leq \lambda < \infty$.

Proof: Fix a ball $B \subset \mathbb{R}^n$. Suppose first that $w$ is a continuous $A_1$-weight with the constant $c_w$ and that $\lambda \geq \text{inf}_{x \in B} w(x)$. Then $E_\lambda$ is open and for every $x \in E_\lambda$ we take the ball $B(x, r_x)$ where $r_x$ is the distance from $x$ to the boundary of $E_\lambda$. Let $\mathcal{B} = \{B(x, r_x) : x \in E_\lambda \cap B\}$. The radii of the balls in $\mathcal{B}$ are bounded, because $\overline{B} \setminus E_\lambda \neq \emptyset$. By Besicovitch’s covering Theorem, there are families $\mathcal{B}_i = \{B_{i,j} : j = 1, 2, \ldots\}$, $i = 1, 2, \ldots, N$, of countably many balls, chosen from $\mathcal{B}$, such that

$$E_\lambda \cap B = \bigcup_{i=1}^{N} \bigcup_{j=1}^{\infty} B_{i,j} \cap B$$

and the balls in every $\mathcal{B}_i$, $i = 1, 2, \ldots, N$, are pairwise disjoint. We denote the union of the pairwise disjoint balls by

$$E_\lambda^{\equiv} = \bigcup_{j=1}^{\infty} B_{i,j}, \quad i = 1, 2, \ldots, N.$$
The balls $B_{i,j}$ touch the boundary of $E_\lambda$ and, since $w$ is continuous, using the $A_1$-condition we get

$$
(2.6) \quad \frac{1}{|B_{i,j}|} \int_{B_{i,j}} w(x) \, dx \leq c \inf_{x \in B_{i,j}} w(x) \leq c \lambda,
$$

$$
i = 1, 2, \ldots, N, \; j = 1, 2, \ldots
$$

The balls $B_{i,j}$ are not, in general, contained in $B$, but there is a constant $\gamma > 0$, depending only on the dimension, so that

$$
|B_{i,j} \setminus B| \leq \gamma |B_{i,j} \cap B|, \quad i = 1, 2, \ldots, N, \; j = 1, 2, \ldots
$$

To see this, let $B_{i,j}$ be the ball $B(x, r_x) \subset E_\lambda$ with $x \in E_\lambda \cap B$. Then by geometry, there is a ball $B(y, r_x/2) \subset B(x, r_x) \cap B$. This gives us the estimate

$$
|B(x, r_x) \setminus B| \leq |B(x, r_x)| = 2^n |B(y, r_x/2)| \leq 2^n |B(x, r_x) \cap B|.
$$

Hence we may take $\gamma = 2^n$.

By observing that $w(x) > \lambda$ for every $x \in B_{i,j}$ and recalling (2.6) we see that

$$
\int_{B_{i,j} \cap B} w(x) \, dx \leq c_w \lambda |B_{i,j} \cap B| + c_w |B_{i,j} \setminus B| - \int_{B_{i,j} \setminus B} w(x) \, dx
$$

$$
\leq c_w \lambda |B_{i,j} \cap B| + (c_w - 1) \lambda |B_{i,j} \setminus B|
$$

$$
\leq (c_w + \gamma (c_w - 1)) \lambda |B_{i,j} \cap B|,
$$

$$
i = 1, 2, \ldots, N, \; j = 1, 2, \ldots
$$

Since the balls in each $B_i, i = 1, 2, \ldots, N,$ are pairwise disjoint, we arrive at

$$
(2.7) \quad \int_{E_\lambda \setminus B} w(x) \, dx = \sum_{j=1}^{\infty} \int_{B_{i,j} \cap B} w(x) \, dx
$$

$$
\leq (c_w + \gamma (c_w - 1)) \lambda |E_\lambda \cap B|, \quad i = 1, 2, \ldots, N.
$$

Let $\mu$ be a measure. Then we use the elementary inequality

$$
(2.8) \quad \mu(E_\lambda \cap B) = \sum_{i=1}^{N} \mu(E_i \cap B) - \sum_{k=2}^{N} \mu(F_k \cap B),
$$
where

\[ F^k_\lambda = \bigcup_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}} (E^{i_1}_\lambda \cap \cdots \cap E^{i_k}_\lambda), \quad k = 2, 3, \ldots, N. \]

A simple computation using (2.8), (2.7) and the fact that \( w(x) > \lambda \) in \( F^k_\lambda \cap B, k = 2, \ldots, N \), gives

\[
\int_{E_\lambda \cap B} w(x) \, dx = \sum_{i=1}^N \int_{E^i_\lambda \cap B} w(x) \, dx - \sum_{k=2}^N \int_{F^k_\lambda \cap B} w(x) \, dx
\]

\[
\leq (c_w + \gamma(c_w - 1)) \sum_{i=1}^N |E^i_\lambda \cap B| - \lambda \sum_{k=2}^N |F^k_\lambda \cap B|
\]

(2.9)

\[
= (c_w + \gamma(c_w - 1)) \lambda |E_\lambda \cap B| + \lambda (1 + \gamma)(c_w - 1) \sum_{k=2}^N |F^k_\lambda \cap B|
\]

\[
\leq (c_w + \gamma(c_w - 1)) \lambda |E_\lambda \cap B| + (N-1)(1 + \gamma)(c_w - 1) \lambda |E_\lambda \cap B|
\]

\[
= (c_w + \eta(c_w - 1)) \lambda |E_\lambda \cap B|,
\]

where \( \eta = N \gamma + N - 1 \) and \( \lambda \geq \inf_{x \in B} w(x) \).

The general case follows from a standard approximation argument using Lemma 2.1. Suppose that \( w \in A_1 \) with the constant \( c_w \). Let \( \varphi \in C^\infty_0(\mathbb{R}^n), \varphi \geq 0 \) with \( \int_{\mathbb{R}^n} \varphi \, dx = 1 \). We define \( w_\varepsilon = w * \varphi_\varepsilon \), where \( \varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon) \) and \( \varepsilon > 0 \). Lemma 2.1 shows that \( w_\varepsilon \) is a continuous \( A_1 \)-weight with the constant \( c_w \) for every \( \varepsilon > 0 \). Using (2.9) we see that

\[
\int_{\{w_\varepsilon > \lambda\} \cap B} w_\varepsilon(x) \, dx \leq (c_w + \eta(c_w - 1)) \lambda \inf_{x \in B} w_\varepsilon(x) \leq \lambda < \infty.
\]

Letting \( \varepsilon \to 0 \) we obtain (2.5). This completes the proof.

2.10. Remark. (1) Observe that the constant on the right side of (2.5) tends to one as \( c_w \) tends to one. On the other hand, it blows up as \( c_w \) increases.

(2) We also remark that inequalities of type (2.5) appear already in the proof of Theorem 4 in [3]. However, their approach does not seem to give the correct behaviour as \( c_w \) tends to one.

We observe that (2.5) gives a characterization of \( A_1 \)-weights.
2.11. Corollary. Suppose that \( w : \mathbb{R}^n \rightarrow [0, \infty] \) is a measurable function. Then \( w \in A_1 \) if and only if there is a constant \( c \), independent of the ball \( B \), so that

\[
\int_{E_\lambda \cap B} w(x) \, dx \leq c \lambda |E_\lambda \cap B|, \quad \text{ess inf}_{x \in B} w(x) \leq \lambda < \infty,
\]

for every ball \( B \subset \mathbb{R}^n \).

Proof: Lemma 2.4 shows that every \( A_1 \)-weight satisfies (2.12).

To see the reverse implication suppose that (2.12) holds and let \( B \) be a ball in \( \mathbb{R}^n \). Then

\[
\int_B w(x) \, dx = \int_{B \setminus E_\lambda} w(x) \, dx + \int_{E_\lambda \cap B} w(x) \, dx \\
\leq \lambda |B \setminus E_\lambda| + c \lambda |B \cap E_\lambda| \\
\leq c \lambda |B|, \quad \text{ess inf}_{x \in B} w(x) \leq \lambda < \infty.
\]

By inserting \( \lambda = \text{ess inf}_{x \in B} w(x) \) we get

\[
\frac{1}{|B|} \int_B w(x) \, dx \leq c \text{ess inf}_{x \in B} w(x),
\]

where the constant is independent of the ball and hence \( w \in A_1 \). \( \blacksquare \)

2.13. Remark. In the one-dimensional case we may take the constant in (2.12) equal to the \( A_1 \)-constant of \( w \), see [11].

Lemma 2.4 shows that \( w \) satisfies the assumptions of the following sharp version Muckenhoupt’s Lemma 4 in [16]. See also Lemma 2 in [2]. The proof of the following lemma can be found in [11], but we present it here for the sake of completeness.

2.14. Lemma. Suppose that \( w : \mathbb{R}^n \rightarrow [0, \infty] \) is a measurable function and let \( B \subset \mathbb{R}^n \) be a ball. If there are \( \alpha \geq 0 \) and \( c > 1 \) such that

\[
\int_{E_\alpha \cap B} w(x) \, dx \leq c \lambda |E_\alpha \cap B|, \quad \alpha \leq \lambda < \infty,
\]

then for every \( p, 1 < p < c/(c - 1) \), we have

\[
\int_{E_\alpha \cap B} w(x)^p \, dx \leq \frac{c}{c - p(c - 1)} \alpha^p |E_\alpha \cap B|.
\]
Proof: Let $\beta > \alpha$ and denote $w_\beta = \min(w, \beta)$. Then

$$\int_{\{w_\beta > \lambda\} \cap B} w(x) \, dx \leq c \lambda \{w_\beta > \lambda\} \cap B, \quad \alpha \leq \lambda < \infty.$$  

We multiply both sides by $\lambda^{p-2}$ and integrate from $\alpha$ to $\infty$. This implies

$$\int_\alpha^{\infty} \lambda^{p-2} \int_{\{w_\beta > \lambda\} \cap B} w(x) \, dx \, d\lambda \leq c \int_\alpha^{\infty} \lambda^{p-1} \{w_\beta > \lambda\} \cap B \, d\lambda.$$  

Then we use the equality

$$(2.17) \quad \int_{E_\alpha \cap B} w(x)^p \, d\mu = p \int_\alpha^{\infty} \lambda^{p-1} \mu(E_\lambda \cap B) \, d\lambda + \alpha^p \mu(E_\alpha \cap B),$$

where $0 < p < \infty$, with $\mu$ replaced by $w \, d\mu$ and $p$ replaced by $p - 1$, to get

$$\int_{E_\alpha \cap B} w_\beta(x)^p \, dx \leq \int_{E_\alpha \cap B} w_\beta(x)^p - 1 w(x) \, dx$$

$$= (p - 1) \int_\alpha^{\infty} \lambda^{p-2} \int_{\{w_\beta > \lambda\} \cap B} w(x) \, dx \, d\lambda + \alpha^{p-1} \int_{E_\alpha \cap B} w(x) \, dx$$

$$\leq c(p - 1) \int_\alpha^{\infty} \lambda^{p-1} \{w_\beta > \lambda\} \cap B \, d\lambda + c \alpha^p |E_\alpha \cap B|.$$  

Next we estimate the first integral on the right side using (2.17) and find

$$\int_\alpha^{\infty} \lambda^{p-1} \{w_\beta > \lambda\} \, d\lambda \leq \frac{1}{p} \left( \int_{E_\alpha \cap B} w_\beta(x)^p \, dx - \alpha^p |E_\alpha \cap B| \right).$$

Hence we obtain

$$\int_{E_\alpha \cap B} w_\beta(x)^p \, dx \leq \frac{c(p - 1)}{p} \int_{E_\alpha \cap B} w_\beta(x)^p \, dx + \frac{c}{p} \alpha^p |E_\alpha \cap B|.$$  

Choosing $p > 1$ such that $c(p - 1)/p < 1$ and using the fact that all terms in the previous inequality are finite, we conclude

$$\int_{E_\alpha \cap B} w_\beta(x)^p \, dx \leq \frac{c}{c - p(c - 1)} \alpha^p |E_\alpha \cap B|.$$  

Finally, as $\beta \to \infty$, the monotone convergence theorem gives (2.16). This proves the lemma.  

2.18. Remark. Both the bound for $p$ and the constant in (2.16) are the best possible as is easily seen by taking $B$ to be the unit ball and $w : \mathbb{R}^n \to [0, \infty]$, $w(x) = |x|^{n/(c - 1)}$.  

3. Proof of Theorem 1.3

Let $B$ be a ball in $\mathbb{R}^n$ and suppose that $w \in \mathcal{A}_1$ with the constant $c_w$. Using (2.5) we see that

$$\int_{E_\lambda \cap B} w(x) \, dx \leq (c_w + \eta(c_w - 1)) \lambda |E_\lambda \cap B|, \quad \text{ess inf}_{x \in B} w(x) \leq \lambda < \infty,$$

where $\eta$ is the constant given by Lemma 2.4. This shows that $w$ fulfills the assumptions of Lemma 2.14 and from (2.16) we conclude that

$$\int_B w(x)^p \, dx = \int_{B \setminus E_\alpha} w(x)^p \, dx + \int_{B \cap E_\alpha} w(x)^p \, dx$$

$$\leq \alpha^p |B \setminus E_\alpha| + c \alpha^p |B \cap E_\alpha|$$

$$\leq c \alpha^p |B|,$$

whenever $\text{ess inf}_{x \in B} w(x) \leq \alpha < \infty$ and

$$1 \leq p < 1 + \frac{1}{(\eta + 1)(c_w - 1)}.$$

In particular, we get

$$\left( \frac{1}{|B|} \int_B w(x)^p \, dx \right)^{1/p} \leq c \frac{1}{|B|} \int_B w(x) \, dx.$$

The constant $c$ does not depend on $B$ and hence we may repeat the same reasoning in every ball $B$ and we see that $w$ satisfies the reverse Hölder inequality for every $p > 1$ such that (1.4) holds if we take $\nu = (\eta + 1)^{-1}$. This completes the proof of Theorem 1.3. □

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