SINGULAR MEASURES AND THE KEY OF G

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Abstract

We construct a sequence of doubling measures, whose doubling
constants tend to 1, all for which kill a $G_δ$ set of full Lebesgue
measure.

0. Introduction

A non-zero Borel measure $\nu$ is said to be doubling if there is a con-
stant $C \geq 1$ such that

\[
C^{-1} \leq \frac{\nu(I)}{\nu(J)} \leq C,
\]

whenever $I, J$ are adjacent intervals of the same length. We call the
smallest $C = C_\nu$ for which this condition holds, the doubling constant
of $\nu$. A measure is a multiple of Lebesgue measure if and only if its
doubling constant is 1.

It was shown in [BHM] that if $U \subset [0,1]^n$ is open and $|\partial U| = 0$,
then $\nu_n(U) \to |U|$ whenever $\nu_n$ is a sequence of probability measures on
$[0,1]^n$ whose doubling constants tend to 1. In particular, if $U$ is an open
subset of $[0,1]$ of full measure, then $\nu_n(U) \to 1$. We will show, amongst
other things, that there exists a $G_δ$ set $G$ in $[0,1]$ of full measure, and a
sequence $\nu_n$ of measures whose doubling constants tend to 1, yet $\nu_n(G) =
0$ for all $n$. We can even choose the measures to be “renormalizations”
of a single measure $\nu$ which “fit the gaps in $G$” as a key fits a lock.

We wish to thank the referee for drawing our attention to the paper
of Kakutani.

1. Definitions and basic results

There is an easy way, essentially due to Kahane [Kh], to generate
doubling measures. Let $Q$ consist of all intervals on $[0,1)$ of the

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form \(|m4^{-k}, (m + 1)4^{-k}\)|, where \(m, k\) are non-negative integers, and set \(Q(j)\) to be the subset of \(Q\) consisting of those intervals of length \(4^{-j}\).

For any \(I \in Q\) the four children are labeled \(I_0, I_1, I_2, I_3\), moving from left to right. Now consider

\[
H_I(x) = \begin{cases} 
1, & x \in I_1 \\
-1, & x \in I_2 \\
0, & \text{otherwise.}
\end{cases}
\]

The product \(\prod_{I \in Q}(1 + a_I H_I)\) converges weak-* to a doubling, probability measure \(\mu\), provided that \(\sup_{I \in Q} |a_I| < 1\). We call any such measure \(\mu\) a Kahane measure and write \(\|\mu\|_K = \sup_{I \in Q} |a_I|\). Furthermore, the doubling constant \(C_\mu\) tends to 1 as \(\|\mu\|_K\) tends to 0; in fact, if \(\|\mu\|_K \leq 1 - \epsilon\), then there is a constant \(c_\epsilon\), dependent only on \(\epsilon\), such that \(C_\mu \leq 1 + c_\epsilon \|\mu\|_K\) whenever for some \(\epsilon > 0\).

For our purposes it will be sufficient to consider Kahane measures for which all of the coefficients \(a_I\) at any given scale are equal and \(\|\mu\|_K \leq 1 - \epsilon\) for some \(\epsilon > 0\), which we assume to be fixed from now on. We denote this class of measures by \(M_\epsilon\), or simply \(M\). Then every measure in \(M\) is of the form \(\prod_{j=1}^{\infty}(1 + a_j R_j)\) where \(R_j = \sum_{I \in Q(j)} H_I\); it is convenient to introduce the notation \(c_j(\mu) \equiv a_j\). We will focus on those measures \(\mu \in M\) for which \(c_j(\mu) \to 0\) as \(j \to \infty\), and we label these \(M_0\).

For \(\mu \in M\) and \(n = 0, 1, 2, \ldots\) the measure \(\mu_n \in M\) henceforth denotes the element of \(M\) with \(c_j(\mu_n) = c_{j+n}(\mu)\), \(j \in \mathbb{N}\). The measures \(\mu_n\) are “renormalized” versions of \(\mu\); in fact, if \(S \subset [0,1)\) is a measurable set and \(f_S\) is the periodic function with period 1 whose restriction to \([0,1)\) is the characteristic function of \(S\), then \(\mu_n(S) = \int_0^1 f_S(4^nt) \, d\mu(t)\). Given \(\mu \in M_0\), it follows from the estimate in the last paragraph that the sequence of doubling constants \((C_{\mu_n})\) has limit 1. Thus every \(\mu \in M_0\) is optimally doubling at small scales in the sense that \(\nu = \mu\) satisfies (1) with \(C = C_{\mu_n}\) whenever \(I, J\) are adjacent intervals with \(|I| = |J| \leq 4^{-n}\).

The following result is a special case of a result of Kakutani [Kk, Corollary 1].

**Theorem A.** Let \(\mu, \nu \in M\), with \(a_j = c_j(\mu)\), \(b_j = c_j(\nu)\), for all \(j \in \mathbb{N}\). If \((a_j - b_j)_{j=1}^{\infty}\) lies in \(l^2\), the class of square summable sequences, then \(\mu \ll \nu \ll \mu\), otherwise \(\mu \perp \nu\).
In fact, when $\nu$ is Lebesgue measure and $(a_n) \in l^2$ above, more is true: $\mu$ lies in the Muckenhoupt class $A_\infty$, and in particular $\mu$ has density lying in $L^p([0, 1])$ for some $p > 1$; see [Bu] and [FKP].

Kakutani proves this result by careful analysis, but let us pause to prove the singularity part of this result using the Lyapunov version of Theorem B.

Theorem B. Suppose that $\{X_n\}_{n=1}^\infty$ is a sequence of independent random variables, and that the moments $E(X_n) = e_n$, $E(X_n-e_n)^2 = \sigma_n^2 \neq 0$, and $E|X_n-e_n|^3 = \tau_n^3$ are finite for each $n$. Let

$$s_n = \left(\sum_{i=1}^n \sigma_i^2\right)^{1/2}, \quad t_n = \left(\sum_{i=1}^n \tau_i^3\right)^{1/3}.$$

If $\lim_{n \to \infty} t_n/s_n = 0$, then $Y_n \equiv \sum_{i=1}^n (X_i - e_i)/s_n$ converges in distribution to the standard normal distribution.

In this paragraph we employ the notation of Theorem A. The functions $R_n$ are independent as random variables on $[0, 1]$ with respect to $\nu$, and so the functions $f_n = \log[1 + a_n R_n]/(1 + b_n R_n)$ are also independent. A little calculation with the power series expansion for $\log(1 + t)$ gives

$$E_\nu(f_n) \equiv e_n = -\frac{(a_n - b_n)^2}{4(1 - b_n^2)} + O(|a_n - b_n|^3),$$
$$E_\nu(f_n - e_n)^2 \equiv \sigma_n^2 = \frac{(a_n - b_n)^2}{2(1 - b_n^2)} + O(|a_n - b_n|^3),$$
$$E_\nu|f_n - e_n|^3 \equiv \tau_n^3 = \frac{|a_n - b_n|^3}{2} \cdot \frac{1 + b_n^2}{(1 - b_n^2)^2} + O(|a_n - b_n|^4).$$

Thus if $s_n, t_n$ are as in Theorem B, $\lim_{n \to \infty} |a_n - b_n| = 0$, and $(a_n - b_n)_{n=1}^\infty \notin l^2$, then $t_n^2 / \sum_{i=1}^n |a_i - b_i|^3$ and $s_n^2 / \sum_{i=1}^n (a_i - b_i)^2$ are bounded above and below by positive, finite constants that are independent of $n$. It is then routine to deduce that $\lim_{n \to \infty} t_n/s_n = 0$; one simply splits the sum at a point beyond which $|a_n - b_n|$ is very small and uses the estimate $|| | \cdot | e_i | \cdot | e_i |^3 | \cdot | e_i |^3 | \cdot | e_i |^3 |^{1/3}$. Thus Theorem B is applicable in the case $X_n = f_n$. Since $\sum_{i=1}^n e_i$ is much larger than $s_n$ for large $n$, it follows that $Y_n$ tends to $-\infty$ in $\nu$-measure and thus $\prod_{n=1}^\infty (1 + a_n R_n)/(1 + b_n R_n)$ converges in $\nu$-measure to the zero function. Set $\{P_n\}$ to be the partial products of this infinite product. We have

1These references only say that $\mu$ lies in dyadic $A_\infty$, but, since $\mu$ is a doubling measure, this implies that $\mu \in A_\infty$. 

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just seen that this sequence of functions converges to zero in $\nu$-measure. However, $P_N(x) = \mu(I_N(x))/\nu(I_N(x))$, where $I_N(x)$ is the unique element of $Q(N)$ containing $x$ and so, by the Radon-Nikodym theorem, \(\{P_N\}\) converges $\nu$-a.e. to the Radon-Nikodym derivative of $\mu$ with respect to $\nu$. Consequently, the Radon-Nikodym derivative is zero $\nu$-a.e., and so $\mu \perp \nu$ whenever $(a_j - b_j) \notin I^2$.

We are mainly interested in Theorem A when $\nu$ is Lebesgue measure. In this case if the sequence $(c_j(\mu))$ has limit zero but does not lie in $I^2$, then $\mu$ is a singular measure which is optimal doubling at small scales. The mere existence of such a measure may seem a little surprising and was only recently established (using different techniques) by Cantón [C] and Smith [S].

There is an obvious bijection, $A$, between $Q$ and the set of finite sequences whose terms lie in \(\{0, 1, 2, 3\}\). We will refer to $A(I)$ as the address of $I$. The $j$th term in the address is $A_j(I)$. For $I \in Q$, we let $E(I)$ consist of the union of the intervals $J \in Q$ for which $A_{2j}(J) = A_j(I)$ for all $j$. So the odd terms in $A(J)$ are arbitrary and the even terms are specified. If $I \in Q(j)$, $E(I)$ consists of $4^j$ elements of $Q(2j)$. For $n = 0, 1, 2, \ldots$ and $I \in Q$, $T_n(I)$ consists of those intervals $J \in Q$ for which $A_{n+j}(J) = A_j(I)$ for all $j$. So the first $n$ terms of $J$ are arbitrary and the remainder are specified. When $I \in Q(j)$, $T_n(I)$ consists of $4^n$ elements of $Q(j + n)$. Note that if $I$ and $J$ are disjoint, then $E(I)$ and $E(J)$ are disjoint, as are $T_n(I)$ and $T_n(J)$. For any set $B$ that is a union of disjoint elements $I$ of $Q$, we define $E(B)$ to be the union of the $E(I)$, and we define $T_n(B)$ similarly. It is easy to check that $|E(B)| = |B|$ and that $|T_n(B)| = |B|$. Let $\Sigma$ be the collection of subsets of $[0, 1]$ that are unions of elements of $Q(j)$. Any set $B \in \Sigma_m$ is said to be $j$-indifferent if whenever $B \supset I \in Q(m)$ and $J$ is one of the three elements of $Q(m)$ for which $A(J)$ and $A(I)$ differ only in the $j$th place, then $J \subset B$. Equivalently if $S(B)$ is the set of sequences of length $m$ given by $A(I)$ for each $I \in Q(m)$, $I \subset B$, then $B$ is $j$-indifferent precisely if $S(B)$ is measurable with respect to the $\sigma$-algebra generated by the sets

$$S_{k,l} = \{(a_i)_{i=1}^m : a_k = l\}, \quad 1 \leq k \leq m, \; k \neq j, \; l \in \{0, 1, 2, 3\}.$$  

The point of this definition is that if $B$ is $j$-indifferent, then $\mu(B)$ does not depend on the $c_j(\mu)$. In particular, if $B \in \Sigma_m$, then $E(B)$ is $j$-indifferent for all odd numbers $j$ and all even $j > 2m$, and $T_n(B)$ is $j$-indifferent for all $j \leq n$ and all $j > n + m$. 


2. Construction of $\mu$ and $G$

Our main result is as follows.

**Theorem 1.** There exists a measure $\mu \in \mathcal{M}_0$ on the interval $[0, 1)$ and a $G_\delta$ set $G$ contained in $[0, 1)$ which have the following properties:

(a) $\mu([0, 1)) = 1$, $|G| = 1$ and $\mu(G) = 0$.
(b) $\mu_n(G) = 1$ for all odd $n \in \mathbb{N}$ and $\mu_n(G) = 0$ for all even $n \in \mathbb{N}$.

Taking $\nu_n = \mu_{2n}$, we immediately get

**Corollary 2.** There exists a $G_\delta$ set $G$ in $[0, 1]$ of full measure and a sequence $\nu_n$ of probability measures on $[0, 1]$ whose doubling constants tend to 1 and for which $\nu_n(G) = 0$ for all $n$.

The oscillatory behaviour of $\mu_n(G)$ described in Theorem 1(b) is all the more remarkable since the measures $\mu_n$ are renormalized versions of a single measure $\mu$ whose doubling constants are tending to one. The idea is to construct $G$ from sets that are indifferent at odd levels $n$ (and thus treat such $\mu_n$ like Lebesgue measure), but which are concentrated in areas where $\mu_n$ is small whenever $n$ is even.

**Proof of Theorem 1:** Let $b$ be any number strictly between 0 and 1. Define $\nu_k$ to be the element of $\mathcal{M}$ whose coefficients are all $2^{-k}$. This measure is singular with respect to Lebesgue measure. It follows that for sufficiently large $n_k$, there exists $A_k \in \Sigma_{n_k}$ for which $|A_k| \geq 1 - b^k$ and $\nu_k(A_k) \leq b^k$. We can assume that the $n_k$ are increasing to $\infty$.

Divide the natural numbers into consecutive blocks $B_1, B_2, \ldots$ of length $2n_1, 2n_2, \ldots$. Set $a_j = 2^{-k}$ whenever $j$ is an even number in block $B_k$, and 0 otherwise. Define $\mu \in \mathcal{M}_0$ by the equations $c_j(\mu) = a_j$.

Now let $m_k = 2n_1 + \cdots + 2n_{k-1}$ for $k > 1$ and $m_1 = 0$. Thus $m_k$ is the total length of the blocks $B_1, \ldots, B_{k-1}$. Define $H_k$ to be $T_{m_k}(E(A_k))$. Then $H_k \in \Sigma_{m_k+2n_k}$ and is $j$-indifferent for all $j$ except even numbers larger than $m_k$ and no larger than $m_k + 2n_k$, i.e., all even numbers in $B_k$. Remove the endpoints of the intervals that make up $H_k$ to get an open set $U_k$. The sets $U_k$ and $H_k$ differ only by a countable number of points. Thus any doubling measure gives them the same measure (doubling measures on the line are non-atomic). Set $G_m = \bigcup_{k=m}^{\infty} U_k$ and $G = \bigcap_{m=1}^{\infty} G_m$. This set $G$ is a $G_\delta$ set.

We have $|H_k| = |A_k| \geq 1 - b^k$ for all $k$, hence $|G_m| = 1$ for all $m$, and $|G| = 1$. If $n$ is odd and $j$ is even, then $c_j(\mu_n) = 0$. But $H_k$ is $j$-indifferent for all odd $j$, so it follows that $\mu_n(H_k) = |H_k|$. As a result, $\mu_n(G) = 1$ whenever $n$ is odd.
The set $H_k$ is $j$-indifferent for all $j$ except even $j$ in $B_k$ and $c_j(\mu) = 2^{-k}$ for these exceptional integers. Thus $\mu(H_k) = \nu_k(A_k) \leq b^k$. Consequently, $\mu(G_m) \leq b^m(1 - b)^{-1}$ for all $m$, and so $\mu(G) = 0$.

Suppose $n - m$ is even. Then $c_j(\mu_n) = c_j(\mu_m)$ for “most” values of $j$ in the sense that for each $k$ the number of places where the coefficients of size $2^{-k}$ do not match up is bounded independently of $k$, indeed by $n - m$. It follows readily from Theorem A that $\mu_n \ll \mu_m \ll \mu_n$. In particular, $\mu_n(G) = 0$ for all even $n$.

Finally, we note two facts about the relationship between $\mu_n$ and $\mu_m$. First, if $n - m$ is odd, then one of $n$, $m$ is odd and the other is even. Thus one of the measures gives full measure to $G$, while the other gives $G$ zero measure. In particular, $\mu_n \perp \mu_m$. Secondly, when $n - m$ is even, the absolute continuity mentioned in the last paragraph of the proof can be strengthened: there exists a constant $C$, dependent only on $n - m$, such that $C^{-1}\mu_m(E) \leq \mu_n(E) \leq C\mu_m(E)$. It suffices to prove this last estimate for $E \in Q$, in which case the estimate follows from the fact, that $c_j(\mu_n) = c_j(\mu_m)$ for “most” values of $j$. We leave the details to the reader.

References


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