SOLVABLE GROUPS WITH MANY $BFC$-SUBGROUPS

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Abstract

We characterize the solvable groups without infinite properly ascending chains of non-$BFC$ subgroups and prove that a non-$BFC$ group with a descending chain whose factors are finite or abelian is a Černikov group or has an infinite properly descending chain of non-$BFC$ subgroups.

0. Introduction

In a series of papers Belyaev-Sesekin [2], Belyaev [3], Bruno-Phillips [4], [5], Kuzucuoglu-Phillips [12], Leinen-Puglisi [13], Asar [1], Leinen [14] have obtained the results on minimal non-$FC$ groups. In particular, in [2] are characterized the minimal non-$BFC$ groups, i.e. the non-$BFC$ groups in which every proper subgroup is $BFC$. Recall that a group $G$ is called a $BFC$-group if there is a positive integer $d$ such that no element of $G$ has more than $d$ conjugates. Due to the well known result of B. H. Neumann (see e.g. [16, Theorem 4.35]) the $BFC$-groups are precisely the groups with the finite commutator subgroups.

We say that a group $G$ satisfies the minimal condition on non-$BFC$ subgroups (for short Min-$BFC$) if for every properly descending series $\{G_n \mid n \in \mathbb{N}\}$ of subgroups of $G$ there exists a number $n_0 \in \mathbb{N}$ such that $G_n$ is a $BFC$-group for every integer $n \geq n_0$ and a group $G$ satisfies maximal condition on non-$BFC$ subgroups (for short Max-$BFC$) if there exists no infinite properly ascending series of non-$BFC$ subgroups in $G$. Every minimal non-$BFC$ group satisfies Min-$BFC$ and Max-$BFC$.

S. Franciosi, F. de Giovanni and Ya. P. Sysak [11] have investigated the locally graded groups with the minimal condition on non-$FC$ subgroups. In this paper we characterize the solvable groups satisfying Max-$BFC$ and Min-$BFC$, respectively. Namely, we prove the two following theorems.

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Key words. $BFC$-group, minimal non-$BFC$ group, maximal condition, minimal condition, solvable group.
Theorem 1. A solvable group $G$ satisfies Max-BFC if and only if it is of one of the following types:

(i) $G$ is a BFC-group;

(ii) $G = BU$ is a finitely generated group, where $B$ is a proper torsion normal subgroup of $G$, $U$ its polycyclic subgroup and $B(x)$ is either a BFC-subgroup or a finitely generated subgroup for every element $x$ of $U$;

(iii) $G = DU$ is a locally nilpotent-by-finite group with the torsion commutator subgroup $G'$, where $D$ is a normal divisible abelian $p$-subgroup, $U$ is a polycyclic subgroup, and if $(u)$ acts non-trivially on $D$ for an element $u$ of $U$, then $D$ is an indecomposable injective $\mathbb{Q}(u)$-module and $A(u)$ is a BFC-subgroup for every proper submodule $A$ of a $\mathbb{Z}(u)$-module $D$ with the action induced by the conjugation of $u$ on $D$.

Theorem 2. Let the group $G$ have a descending series whose factors are finite or abelian. If $G$ satisfies the minimal condition on non-BFC subgroups, then it is a BFC-group or a Černikov group.

Throughout this paper $p$ is a prime. For a group $G$, $Z(G)$ will always denote the centre of $G$, $G', G'', \ldots, G^{(n)}$ the terms of derived series of $G$, $\tau(G)$ the set of all torsion elements of $G$, $G^p = \langle g^p \mid g \in G \rangle$. In the sequel we will use the following notation:

- $\mathbb{Q}$ the rational number field;
- $\mathbb{F}_p$ the finite field with $p$ elements;
- $\mathbb{Q}_p$ the additive group of all rational numbers whose denominators are $p$-numbers;
- $\mathbb{Z}$ the additive group of all rational integers;
- $\mathbb{C}_p^\infty$ the quasicyclic $p$-group;
- $R(x)$ the group ring of a cyclic group $\langle x \rangle$ over a commutative ring $R$.

We will also use other standard terminology from [10] and [16].

1. Solvable groups with Max-BFC

In this section we study the solvable groups with the maximal condition on non-BFC subgroups.
Lemma 1.1. Let $G$ be a group satisfying Max-$\text{BFC}$ and $H$ its subgroup. Then:

(i) $H$ satisfies Max-$\text{BFC}$;
(ii) if $H$ is normal in $G$, then the quotient group $G/H$ satisfies Max-$\text{BFC}$;
(iii) if $H$ is a normal non-$\text{BFC}$ subgroup of $G$, then $G/H$ satisfies the maximal condition on subgroups.

Proof: Is immediate. \hfill \square

Lemma 1.2. Let $G$ be a group which satisfies Max-$\text{BFC}$. If $G$ contains a normal abelian subgroup $N$ with the quasicyclic quotient group $G/N$, then $G$ is a nilpotent group.

Proof: We prove this lemma by the same arguments as in the proof of Lemma 2.3 from [2]. Since $G/N$ is a quasicyclic $p$-group for some prime $p$,

$$G/N = \bigcup_{n=1}^{\infty} \langle \pi_n \rangle,$$

where $\pi_n^p = \pi_{n-1}$, $\pi_0 = N$. Put $A_n = \langle N, a_n \rangle$. Then $A_n \triangleleft G$, $A_n' \triangleleft G$ and by Lemma 1.1(iii) $A_n$ is a $\text{BFC}$-subgroup. Hence $A_n' \leq Z(G)$ and consequently

$$G' = \bigcup_{n=1}^{\infty} A_n' \leq Z(G),$$

as desired. \hfill \square

Lemma 1.3. If $G$ is a Černikov group with Max-$\text{BFC}$, then it is a $\text{BFC}$-group or the quotient group $G/G'$ is finite.

Proof: Assume that the quotient group $G' = G/G'$ is infinite and $G$ is not a $\text{BFC}$-group. Then by Theorem 21.3 of [10] $G' \cong D \times F$ is a direct product of the non-trivial divisible part $D$ and a reducible subgroup $F$. Let $D$ and $F$ be the inverse images of $D$ and $F$ in $G$, respectively. By Corollary 2.2 of [2] $G' = D'F'$. Since $G$ is not a $\text{BFC}$-group, $F'$ is a finite group. It is clear that $D \cong \mathbb{C}_p^\infty$ for some prime $p$ and $G$ has a normal $\text{BFC}$-subgroup $N$ with $G/N \cong \mathbb{C}_p^\infty$. By Theorem 1.16 of [7] $G = NZ(G)$ and so $G' = N'$, a contradiction with our assumption. The lemma is proved. \hfill \square

Proposition 1.4. If a group $G$ satisfies Max-$\text{BFC}$, then it is a $\text{BFC}$-group or the quotient group $G/G'$ is finitely generated.
Proof: As it is well known \( \mathcal{G} = G/G' = \mathcal{D} \times \mathcal{S} \) is a direct product of the divisible part \( \mathcal{D} = D/G' \) and a reducible subgroup \( \mathcal{S} = S/G' \).

(1) First, let \( \mathcal{D} \) be a non-trivial subgroup. Then \( \mathcal{S} \) and \( \mathcal{G}' \) are the BFC-subgroups. It is clear that \( \mathcal{G} \) is a BFC-group or \( \mathcal{D} \) is a quasicyclic group. We suppose that \( \mathcal{D} \cong \mathbb{Z}_{p^\infty} \). Let \( \mathcal{P} = F/G' \) be a \( p \)-basic subgroup of \( \mathcal{S} \). If \( \mathcal{P} \neq \mathcal{S} \), then \( \mathcal{G}/\mathcal{P} \) is a direct product of a quasicyclic \( p \)-subgroup and an infinite \( p \)-divisible abelian subgroup. By Lemma 2.2 of [2] and Lemma 1.1 \( G \) is a BFC-group.

Assume that \( \mathcal{F} = \mathcal{S} \). Then by Lemma 26.1 and Proposition 27.1 from [10] \( \mathcal{G}/\mathcal{F}' = D^* \times F^* \) is a direct product of a quasicyclic \( p \)-subgroup \( D^* \) and a \( p \)-subgroup \( F^* \) of exponent \( p \). Lemma 2.2 of [2] implies \( \mathcal{G}' = D'S' \). If \( \mathcal{F} \) is not a finitely generated subgroup, then in view of Lemma 1.1 \( D \) and \( G \) are the BFC-groups. Therefore we assume that \( \mathcal{F} \) is a finitely generated subgroup. Since \( F \) is a BFC-subgroup, \( |G':D'| < \infty \). By Lemma 1.2 \( D/G' \) is an infinite and \( D/D' \) is a Černikov group. This yields that \( D \) is a Černikov group. By Lemmas 1.1 and 1.3 \( D \) is a BFC-group and as a consequence \( G \) is the ones.

(2) Now let the divisible part \( \mathcal{D} \) is trivial. If \( \mathcal{F} = \mathcal{S} \), then the quotient group \( \mathcal{G}/\mathcal{F}' \) is finitely generated or \( \mathcal{G}/\mathcal{F}' \) is a direct product of infinitely many cyclic subgroups of order \( p \) in which case \( G \) is a BFC-group.

Therefore we assume that \( \mathcal{F} \neq \mathcal{S} \). If \( \mathcal{F} \) is not finitely generated, in the same manner as above we can prove that \( G \) is a BFC-group.

Let \( \mathcal{F} \) be a finitely generated subgroup.

(a) Assume that the quotient group \( \mathcal{G}_1 = \mathcal{G}/\mathcal{F} \) is non-torsion. Then there exists a subgroup \( \mathcal{F}_0 \) such that \( \mathcal{F} \leq \mathcal{F}_0 \leq \mathcal{G} \) and \( \mathcal{G}/\mathcal{F}_0 \) is torsion-free. As noted in [6] (see also [7, Chapter 2, §6]) \( \mathcal{G}/\mathcal{F}_0 \) contains a subgroup \( \mathcal{T}/\mathcal{T}_0 \) isomorphic to \( \mathbb{Z}_p \). If \( \mathcal{Z}/\mathcal{F}_0 \) is a subgroup of \( \mathcal{T}/\mathcal{T}_0 \), then \( \mathcal{T}/\mathcal{Z} \) is a quasicyclic \( p \)-group, and it follows that \( G \) has a normal BFC-subgroup \( X \) with \( G/X \cong \mathbb{Z}_{p^\infty} \). By Lemma 1.2 \( G_0 = G/XF^p \) is a nilpotent group and so by Lemma 26.1 and Proposition 27.1 from [10] \( G_0/G_0' = F_1 \times K_1 \) is a direct product of a finite \( p \)-subgroup \( F_1 \) and an infinite \( p \)-divisible abelian subgroup \( K_1 \). Let \( K_0 \) be an inverse image of \( K_1 \) in \( G_0 \). From what is proved above it follows that \( K_0 \) has a normal subgroup \( K^* \) with \( K_0/K^* \cong \mathbb{Z}_{p^\infty} \). If \( K^* \neq (K^*)^p \), then Theorem 1.16 of [7] yields that \( K_0/(K^*)^p = \mathcal{X} \times \mathcal{Y} \) is a direct product of a quasicyclic \( p \)-subgroup \( \mathcal{X} \) and some divisible \( p \)-subgroup \( \mathcal{Y} \). Since \( \mathcal{Y} \) is a non-trivial subgroup, it is infinite. Consequently \( G \) is a BFC-group. Therefore we suppose that \( K^* = (K^*)^p \). As above we can prove that \( K^* \) contains a \( G \)-invariant subgroup \( L \) with \( K^*/L \cong \mathbb{Z}_{p^\infty} \). Hence \( K_0/L \cong \mathbb{Z}_{p^\infty} \times \mathbb{Z}_{p^\infty} \) and so \( G \) is a BFC-group.
(b) Let \( G_1 = \overline{G}/\overline{F} \) be an infinite torsion \( p' \)-group. Then without loss of generality we can assume that \( G_1 \) is an infinite \( q \)-group for some prime \( q \) different from \( p \). By \( B \) we denote a basic subgroup of \( G_1 \). If \( B = G_1 \), then the quotient group \( G/G' \) is finitely generated or \( B \) is an infinitely generated subgroup in which case \( G \) is a BFC-group.

Let \( B \neq G_1 \). If \( B \) is not a finitely generated subgroup, then Lemma 26.1 and Proposition 27.1 of [10] give that \( G_1/B' \cong \overline{B} \times C_{q\infty} \), where \( \overline{B} \) is an infinite abelian \( q \)-subgroup of exponent \( q \), and this yields that \( G \) is a BFC-group. Therefore we assume that \( B \) is a finitely generated subgroup. Then without loss of generality let \( B = 1 \) and \( G_1 \cong C_{q\infty} \). We would like to prove that the commutator subgroup \( G' \) is torsion. Since the subgroup \( G'' \) is finite, without restricting of generality let \( G'' = 1 \). But then \( \overline{F} = F/\tau(G') \) is an abelian subgroup of \( \overline{G} = G/\tau(G') \) and from \( G_1 \cong \overline{G}/\overline{F} \) it follows that \( \overline{G} \) is an abelian group. This means that \( G' \) is a torsion subgroup. By Lemma 1.2 \( G/F' \) is a nilpotent group and it has the torsion commutator subgroup. So Corollary 3.3 of [2] yields that \( G/F' \) is a torsion group. Hence \( G \) is a torsion group and \( G \cong C_{q\infty} \times M \), where \( M \) is a finite subgroup, a contradiction with our assumption.

(c) Finally, if \( G_1 = \overline{G}/\overline{F} \) is a torsion group and it has a non-trivial \( p \)-subgroup, then without loss of generality we can assume that \( G_1 \) is a quasicyclic \( p \)-group. As in the line (b) this gives that \( G \) is a BFC-group. The proposition is proved.

Lemma 1.5. Let \( G = B \langle x \rangle \) be a product of a normal abelian torsion-free subgroup \( B \) and a cyclic subgroup \( \langle x \rangle \). If \( G \) satisfies Max-BFC, then it is either an abelian group or a polycyclic group.

Proof: If \( F \) is any finitely generated subgroup of \( B \), then \( \langle F, x \rangle \) is a polycyclic subgroup in \( G \) and \( \langle F, x \rangle = A(x) \) for some \( G \)-invariant subgroup \( A \) of \( B \). Assume that the quotient group \( G/A \) is not finitely generated. Then \( A(x) \) is a BFC-subgroup in view of Lemma 1.1 and consequently it is abelian. Therefore a non-polycyclic group \( G \) is abelian, as desired.

Lemma 1.6. If \( G \) is a solvable group satisfying Max-BFC, then one of the following conditions holds:

(i) \( G = BU \) is a finitely generated group, where \( B \) is a proper torsion normal subgroup of \( G \), \( U \) its polycyclic subgroup and \( B \langle x \rangle \) is either a BFC-subgroup or a finitely generated subgroup for every element \( x \) of \( U \);

(ii) \( G \) is a BFC-group;
(iii) \( G = DV \) is a product of a normal divisible abelian \( p \)-subgroup \( D \) and a polycyclic subgroup \( V \).

**Proof:** Suppose that \( G \) is not a BFC-group. Let \( n \) be the derived length of \( G \). Then there exists an integer \( k \) such that \( G^{(k-1)} \) is not a BFC-group, but \( G^{(k)} \) is a BFC-group, where \( 1 \leq k \leq n - 1 \) and \( G^{(0)} = G \). Proposition 1.4 implies that \( G^{(k-1)} = G^{(k)}U \) for some polycyclic subgroup \( U \). By Lemma 1.5 \( \overline{U} \leq G^{(k-1)} \), where \( \overline{G^{(k-1)}} = G^{(k-1)}/\tau(G^{(k)}) = G^{(k-1)}/\tau(U) \), and so \( \overline{U} = G^{(k-1)} \). This means that \( G^{(k-1)} = \tau(G^{(k)})U \). We denote \( \tau(G^{(k)}) \) by \( B \).

(a) First we assume that \( G \) is not a finitely generated group. Clearly that there is an element \( u \) of \( U \) such that \( H_1 = G^{(k)}\langle u \rangle \) is a non-BFC group. We would like to prove that \( H = B\langle u \rangle \) is the ones. Indeed, if \( H \) is a BFC-group, then the quotient group \( H_1/HG^{(k+1)} \) is a nilpotent group and by Theorem 2.26 of [10] and Proposition 1.4 it is finitely generated. But then \( H_1 \) (and consequently \( G \)) is also a finitely generated group, a contradiction. Hence \( H \) is a non-BFC group.

(1) Assume that \( B \) is an abelian \( \pi \)-subgroup for some set \( \pi \) of primes. If \( B = B_1 \times B_2 \) is a direct product of an infinite \( \pi_1 \)-subgroup \( B_1 \) and an infinite \( \pi_2 \)-subgroup \( B_2 \), where \( \pi_1 \) and \( \pi_2 \) are the disjoint subsets of \( \pi \) such that \( \pi = \pi_1 \cup \pi_2 \), then it is not difficulty to prove that \( H \) is a BFC-group, a contradiction. Thus \( \pi \) is a finite set and \( B = P \times S \), where \( P \) is an infinite \( p \)-subgroup for some prime \( p \in \pi \) and \( S \) is a finite \( p' \)-subgroup. Moreover \( P\langle u \rangle \) is a non-BFC group.

(2) If \( B \) is not necessary an abelian subgroup, then from the line (1) it follows that \( B/T \) is a divisible abelian \( p \)-group for some finite \( H \)-invariant subgroup \( T \). By Theorem 1.16 of [7] there exists a divisible abelian \( p \)-subgroup \( D \) of \( B \) such that \( D \leq Z(B) \) and \( B = DT \). Thus \( G = DV \), where \( V \) is a polycyclic subgroup.

(b) Now let \( G \) be a finitely generated group. Then \( G = BU \) for some polycyclic subgroup \( U \). Suppose that \( B(x) \) is not a BFC-group for some \( x \in U \). If \( B(x) \) is not finitely generated, then, as in the line (1) and (2), we can prove that \( B(x) = D_1V_1 \), where \( D_1 \) is a normal divisible \( p \)-subgroup, \( V_1 \) is a polycyclic subgroup and \( D_1 \leq B \). By Theorem of [2] \( B(x) \) contains a proper non-BFC subgroup \( K \). Since \( D_1K = D_1K/(D_1 \cap K) = D_1K \) and \( D_1 \) is a non-trivial divisible \( p \)-subgroup, we conclude that \( D_1K \) (and consequently \( G \)) contains an infinite properly ascending series of type

\[ K < K_1 < \cdots < K_n < \cdots , \]
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a contradiction. This means that $B(x)$ is a finitely generated subgroup. The lemma is proved.

Example 1.7. If $G = A \rtimes \langle t \rangle$, where $\langle t \rangle$ is an infinite cyclic subgroup, $A \cong \mathbb{C}_p^\infty$ and $a^t = a^{1+p} (a \in A)$, then $G$ satisfies Max-BFC.

If $D$ is a commutative Dedekind domain, $A$ right $D$-module, Spec($D$) the set of non-trivial prime ideals of $D$ and $P \in$ Spec($D$), then

$A_P = \{a \in A \mid aP^n = \{0\}$ for some positive integer $n = n(a) \in \mathbb{N}\}

is said to be the $P$-component of $A$, and $A$ is said to be a $D$-torsion module if

$A = \{a \in A \mid Ann(a) \neq \{0\}\}.$

Lemma 1.8. Let $G = A \rtimes \langle x \rangle$ be a semidirect product of a normal abelian subgroup $A$ of exponent $p$ and an infinite cyclic subgroup $\langle x \rangle$. If $G$ satisfies Max-BFC, then it is either a finitely generated group or a BFC-group.

Proof: It is clear that $A$ is a right $\mathbb{F}_p(x)$-module with the action determined by the conjugation of $x$ on $A$. Assume that $G$ is not neither a finitely generated group nor a BFC-group. Then $A$ is a $\mathbb{F}_p(x)$-torsion module and by Proposition 2.4 of [8, §8.2]

$A = \bigoplus_{P \in$Spec$(\mathbb{F}_p(x))} A_P$

is a module direct sum of its $P$-component $A_P$. Without loss of generality we can suppose that $|A : A_Q| < \infty$ for some $Q \in$ Spec($\mathbb{F}_p(x)$). Let $B$ be a basic submodule of $A_Q$. By our hypothesis $B = A_Q$. Since $B$ can be written as a direct product of two infinite $G$-invariant subgroup of infinite index, we obtain that $B \rtimes \langle x \rangle$ (and consequently $G$) is a BFC-group, a contradiction. The lemma is proved.

Proposition 1.9. If $G$ is a non-“finitely generated” non-BFC soluble group satisfying Max-BFC, then:

(1) $G$ is a locally nilpotent-by-finite group;
(2) $G = BU$ is a product of a normal divisible abelian $p$-subgroup $B$ and a polycyclic subgroup $U$;
(3) $B \langle u \rangle$ is a BFC-subgroup for an element $u \in U$ if and only if $u \in C_U(B)$;
(4) if $B \langle u \rangle$ is a non-BFC subgroup for some element $u \in U$, then $[B, \langle u \rangle] = B$;


(5) if $B\langle u \rangle$ is a non-BFC subgroup for some element $u \in U$, then $B$ is an indecomposable injective $Q\langle u \rangle$-module;

(6) if $B\langle u \rangle$ is a non-BFC subgroup for some $u \in U$, then $A\langle u \rangle$ is a BFC-subgroup for every proper $Z\langle u \rangle$-submodule $A$ of $B$, where the action is induced by the conjugation of $u$ on $B$;

(7) $G$ contains a normal subgroup $H$ of finite index in which every non-BFC subgroup is subnormal;

(8) $G'$ is a torsion subgroup of $G$.

Proof: (1) Is obvious.

(2) Follows from Lemma 1.6.

(3) Assume that $H = B\langle u \rangle$ is a BFC-subgroup for some element $u \in U$. If $u$ has a finite order, then $H/H'\langle u \rangle$ is a divisible group and by Theorem 1.16 of [7] $H = Z(H)H'\langle u \rangle$. Consequently $Z(H)$ is a subgroup of finite index in $H$ and $H$ is an abelian group.

Let $u$ be an element of infinite order. Since the subgroup $H'\langle u^s \rangle$ and the quotient group $B\langle u^s \rangle/(H'\langle u^s \rangle)'$ are nilpotent for some integer $s$, $B\langle u^s \rangle$ is a nilpotent group by Hall theorem [16, Theorem 2.27]. But then $B\langle u^s \rangle$ is an abelian group and therefore as proved above $H/(Z(H)\cap \langle u \rangle)$ is abelian. This yields that $H$ is an abelian group.

(4) If $B\langle u \rangle$ is a non-BFC subgroup for some element $u$ of $U$ and $[B, \langle u \rangle] \neq B$, then $T = [B, \langle u \rangle]\langle u \rangle$ is a BFC-subgroup. Since $B\langle u \rangle/T'$ is a nilpotent group, it is abelian, a contradiction.

(5) It is clear that $B$ is a right $Q\langle u \rangle$-module with the action induced by the conjugation of $u$ on $B$. Furthermore, $B$ is a divisible $Q\langle u \rangle$-module and therefore it is injective (see e.g. [11, Theorem 5.28]). By Theorem 2.5 of [15] $B$ has a decomposition as a module direct sum of indecomposable injective $Q\langle u \rangle$-submodules. Since $B\langle u \rangle$ satisfies Max-BFC, $B$ is an indecomposable module.

(6) Let $B\langle u \rangle$ be a non-BFC group and $A$ a proper submodule of a right $Z\langle u \rangle$-module $B$, where the action is induced by the conjugation of $u$ on $B$. By $F$ we denote a basic subgroup of $A$. If $A = F$, then $A\langle u \rangle$ is either a polycyclic group or a BFC-group in view of Lemmas 1.8 and 1.5. Therefore we assume that $F \neq A$. Since $B$ is an indecomposable $Q\langle u \rangle$-module, we conclude that $F$ is an infinite group. But then $A/A^p$ is also infinite and so $A\langle u \rangle/A^p$ is a BFC-group by Lemma 1.8. This yields that $A\langle u \rangle$ is the ones.
(7) If $V$ is a nilpotent subgroup of finite index in $U$ and $K$ is any non-BFC subgroup of $DV$, then $D \leq K$. Hence $K$ is a subnormal subgroup of $DV$.

(8) Is obvious. The proposition is proved.

Proof of Theorem 1: ($\Rightarrow$) Follows from Proposition 1.9.

($\Leftarrow$) Suppose that $K$ is a non-BFC subgroup of a non-BFC group $G$.

Let $G$ be a group of type (ii) and $BK = BK/B(B \cap K) = B \rtimes K$. Since $BK$ is a finitely generated subgroup, $S = (B \cap S) \rtimes K$, where $S$ is a subgroup of $BK$ which contains $K$, and $BK$ satisfies the maximal condition on normal subgroups by Theorem 5.34 of [10], we conclude that every properly ascending series of type $K \leq K_1 \leq \cdots < K_n \leq \cdots$ is finite. This means that $BK$ (and consequently $G$) satisfies Max-BFC.

If $G$ is a group of type (iii), then it is clear that $K = (K \cap D)F$, where $F = \langle u_1, \ldots, u_t \rangle$ is some finitely generated subgroup. Assume that $K_i = (K \cap D)\langle u_i \rangle$ has the finite commutator subgroup $K_i'$ for all $i$ ($1 \leq i \leq t$). Since the subgroup $(K_1', \ldots, K_t')$ is a finitely generated and $(K_1', \ldots, K_t', F) = K_0 F$ for some finite $F$-invariant subgroup $K_0 \leq K \cap D$, $(K/K_0)' = (FK_0/K_0)'$ is a finite subgroup and therefore $K$ is a BFC-subgroup, a contradiction. Hence $(K \cap D)\langle u \rangle$ is non-BFC subgroup for some $u \in F$ and by our hypothesis $D = K \cap D \leq K$. The theorem is proved.

Corollary 1.10. A solvable group $G$ satisfies Max-BFC if and only if it is of one of the following types:

(i) $G$ is a BFC-group;

(ii) $G = BU$ is a finitely generated group, where $B$ is a proper torsion normal subgroup of $G$, $U$ its polycyclic subgroup and $B\langle x \rangle$ is either a BFC-subgroup or a finitely generated subgroup for every element $x \in U$;

(iii) $G = DU$ is a product of a normal divisible abelian $p$-subgroup $D$ and a polycyclic subgroup $U$ with $D \leq \bigcap \{H \mid H $ is a non-BFC subgroup of $G\}$.

2. Groups with Min-BFC

In this section we prove that a group which have a descending series with abelian or finite factors and satisfying Min-BFC is either a BFC-group or a Černikov group.

Lemma 2.1. If $G$ is a non-perfect group in which every proper normal subgroup is a BFC-subgroup, then $G$ is a BFC-group or $G = G'(x)$, where $x^{p^n} \in G'$ for some prime $p$ and some positive integer $n$. 
Proof: By Theorem 21.3 of [10] \( G = G' \times S \) is a direct product of the divisible part \( D \) and a reducible subgroup \( S \). Let \( B \) be a \( p \)-basic subgroup of \( S \). If \( B \) is not a finitely generated subgroup, then by Lemma 26.1 and Proposition 27.1 of [10] \( S/B = B_1 \times S_1 \) is a direct product of an infinite abelian subgroup \( B_1 \) of exponent \( p \) and a \( p \)-divisible subgroup \( S_1 \). By Corollary 2.1 of [2] \( G \) is a BFC-group.

Let \( B \) be a finitely generated subgroup. If \( D \) is a non-trivial subgroup or \( B = S \), then Lemma 26.1, Proposition 27.1 of [10] and Lemma 2.2 of [2] yield that \( G \) is a BFC-group. Finally, from \( B = S \) and \( D = 1 \) in view of Corollary 2.1 of [2] it follows that \( G \) is a BFC-group or \( G/G' \) is a cyclic \( p \)-group for some prime \( p \), as desired.

Proof of Theorem 2: Suppose that \( G \) is neither a BFC-group nor a Černikov group. Since \( G \) satisfies Min-BFC, we may invoke [9, Theorem 2.2] and obtain in this way that \( G \) is an FC-group. Choose

\[ G = G_0 \geq G_1 \geq \cdots \geq G_n \]

such that every \( G_i \) is not a BFC-subgroup (\( i = 0, \ldots, k - 1 \)), while every proper normal subgroup of \( G_n \) is a BFC-group. Since \( G \) has a descending series whose factors are finite or abelian, there exists a normal subgroup \( N \) in \( G_n \) such that \( G_n/N \) is finite or abelian. From Lemma 2.1, the quotient \( G_n/N \) is finite in both cases. Hence there exists a finite subset \( F \) of \( G_n \) such that \( G_n = NF \), and every element in \( G_n \) is of the form \( hf \) for suitable \( h \in N \), \( f \in F \). However, since \( G \) is an FC-group, every \( f \in F \) has just finitely many conjugates in \( G_n \). And for \( h \in N \) the number of conjugates of \( h \) in \( G_n \) is bounded by \( |G_n : N||N : C_N(h)| \) (note that \( N \) is a BFC-group). Hence \( G_n \) itself becomes a BFC-group. This contradiction shows that \( G \) must be a BFC-group or a Černikov group.

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References

Groups with Many BFC-Subgroups


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