DYNAMICS OF SYMMETRIC HOLOMORPHIC MAPS ON PROJECTIVE SPACES

KOHEI UENO

Abstract

We consider complex dynamics of a critically finite holomorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$, which has symmetries associated with the symmetric group $S_{k+2}$ acting on $\mathbb{P}^k$, for each $k \geq 1$. The Fatou set of each map of this family consists of attractive basins of superattracting points. Each map of this family satisfies Axiom A.

1. Introduction

For a finite group $G$ acting on $\mathbb{P}^k$ as projective transformations, we say that a rational map $f$ on $\mathbb{P}^k$ is $G$-equivariant if $f$ commutes with each element of $G$. That is, $f \circ r = r \circ f$ for any $r \in G$, where $\circ$ denotes the composition of maps. Doyle and McMullen [4] introduced the notion of equivariant functions on $\mathbb{P}^1$ to solve quintic equations. See also [11] for equivariant functions on $\mathbb{P}^1$. Crass [2] extended Doyle and McMullen’s algorithm to higher dimensions to solve sextic equations. Crass [3] found a good family of finite groups and equivariant maps for which one may say something about global dynamics. Crass [3] conjectured that the Fatou set of each map of this family consists of attractive basins of superattracting points. Although I do not know whether this family has relation to solving equations or not, our results will give affirmative answers for the conjectures in [3].

In Section 2 we shall explain an action of the symmetric group $S_{k+2}$ on $\mathbb{P}^k$ and properties of our $S_{k+2}$-equivariant map. In Sections 3 and 4 we shall show our results about the Fatou sets and hyperbolicity of our maps by using properties of our maps and Kobayashi metrics.

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2. \( S_{k+2} \)-equivariant maps

Crass [3] selected the symmetric group \( S_{k+2} \) as a finite group acting on \( P^k \) and found an \( S_{k+2} \)-equivariant map which is holomorphic and critically finite for each \( k \geq 1 \). We denote by \( C = C(f) \) the critical set of \( f \) and say that \( f \) is critically finite if each irreducible component of \( C(f) \) is periodic or preperiodic. More precisely, \( S_{k+2} \)-equivariant map \( g_{k+2} \) defined in Section 2.2 preserves each irreducible component of \( C(g_{k+2}) \), which is a projective hyperplane. The complement of \( C(g_{k+2}) \) is Kobayashi hyperbolic. Furthermore restrictions of \( g_{k+2} \) to invariant projective subspaces have the same properties as above. See Section 2.3 for details.

2.1. \( S_{k+2} \) acts on \( P^k \)

An action of the \((k+2)\)-th symmetric group \( S_{k+2} \) on \( P^k \) is induced by the permutation action of \( S_{k+2} \) on \( C^{k+2} \) for each \( k \geq 1 \). The transposition \((i, j)\) in \( S_{k+2} \) corresponds with the transposition \( u_i \leftrightarrow u_j \) on \( C^{k+2}_u \), which pointwise fixes the hyperplane \( \{ u_i = u_j \} = \{ u \in C^{k+2}_u \mid u_i = u_j \} \).

Here \( C^{k+2} = C^{k+2}_u = \{ u = (u_1, u_2, \ldots, u_{k+2}) \mid u_i \in C \text{ for } i = 1, \ldots, k+2 \} \).

The action of \( S_{k+2} \) preserves a hyperplane \( H \) in \( C^{k+2}_u \), which is identified with \( C^{k+1}_x \) by projection \( A: C^{k+2}_u \to C^{k+1}_x \),

\[
H = \left\{ \sum_{i=1}^{k+2} u_i = 0 \right\} \cong C^{k+1}_x \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}.
\]

Here \( C^{k+1} = C^{k+1}_x = \{ x = (x_1, x_2, \ldots, x_{k+1}) \mid x_i \in C \text{ for } i = 1, \ldots, k+1 \} \).

Thus the permutation action of \( S_{k+2} \) on \( C^{k+2}_u \) induces an action of \( "S_{k+2}" \) on \( C^{k+1}_x \). Here \( "S_{k+2}" \) is generated by the permutation action \( S_{k+1} \) on \( C^{k+1}_x \) and a \((k+1, k+1)\)-matrix \( T \) which corresponds to the transposition \((1, k+2)\) in \( S_{k+2} \),

\[
T = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \ldots & 1 \end{pmatrix}.
\]

Hence the hyperplane corresponding to \( \{ u_i = u_j \} \) is \( \{ x_i = x_j \} \) for \( 1 \leq i < j \leq k+1 \). The hyperplane corresponding to \( \{ u_i = u_{k+2} \} \) is \( \{ x_i = 0 \} \) for \( 1 \leq i \leq k+1 \). Each element in \( "S_{k+2}" \) which corresponds to some transposition in \( S_{k+2} \) pointwise fixes one of these hyperplanes in \( C^{k+1}_x \).
The action of “$S_{k+2}$” on $\mathbb{C}^{k+1}$ projects naturally to the action of “$S_{k+2}$” on $\mathbb{P}^k$. These hyperplanes on $\mathbb{C}^{k+1}$ projects naturally to projective hyperplanes on $\mathbb{P}^k$. Here $\mathbb{P}^k = \{x = [x_1 : x_2 : \ldots : x_{k+1}] \mid (x_1, x_2, \ldots, x_{k+1}) \in \mathbb{C}^{k+1} \setminus \{0\}\}$. Each element in the action of “$S_{k+2}$” on $\mathbb{P}^k$ which corresponds to some transposition in $S_{k+2}$ pointwise fixes one of these projective hyperplanes. We denote “$S_{k+2}$” also by $S_{k+2}$ and call these projective hyperplanes transposition hyperplanes.

2.2. Existence of our maps.

One way to get $S_{k+2}$-equivariant maps on $\mathbb{P}^k$ which are critically finite is to make $S_{k+2}$-equivariant maps whose critical sets coincide with the union of the transposition hyperplanes.

**Theorem 1 ([3]).** For each $k \geq 1$, $g_{k+3}$ defined below is the unique $S_{k+2}$-equivariant holomorphic map of degree $k+3$ which is doubly critical on each transposition hyperplane.

$$g = g_{k+3} = [g_{k+3,1} : g_{k+3,2} : \ldots : g_{k+3,k+1}] : \mathbb{P}^k \to \mathbb{P}^k,$$

where $g_{k+3,l}(x) = x_l \sum_{s=0}^{k} (-1)^s \frac{s+1}{s+3} x_l^s A_{k-s}$, $A_0 = 1$,

and $A_{k-s}$ is the elementary symmetric function of degree $k - s$ in $\mathbb{C}^{k+1}$.

Then the critical set of $g$ coincides with the union of the transposition hyperplanes. Since $g$ is $S_{k+2}$-equivariant and each transposition hyperplane is pointwise fixed by some element in $S_{k+2}$, $g$ preserves each transposition hyperplane. In particular $g$ is critically finite. Although Crass [3] used this explicit formula to prove Theorem 1, we shall only use properties of the $S_{k+2}$-equivariant maps described below.

2.3. Properties of our maps.

Let us look at properties of the $S_{k+2}$-equivariant map $g$ on $\mathbb{P}^k$ for a fixed $k$, which is proved in [3] and shall be used to prove our results. Let $L^{k-1}$ denote one of the transposition hyperplanes, which is isomorphic to $\mathbb{P}^{k-1}$. Let $L^m$ denote one of the intersections of $(k - m)$ or more distinct transposition hyperplanes which is isomorphic to $\mathbb{P}^m$ for $m = 0, 1, \ldots, k - 1$.

First, let us look at properties of $g$ itself. The critical set of $g$ consists of the union of the transposition hyperplanes. By $S_{k+2}$-equivariance,
g preserves each transposition hyperplane. Furthermore the complement of the critical set of g is Kobayashi hyperbolic.

Next, let us look at properties of g restricted to \( L^m \) for \( m = 1, 2, \ldots, k - 1 \). Let us fix any \( m \). Since g preserves each \( L^m \), we can also consider the dynamics of g restricted to any \( L^m \). Each restricted map has the same properties as above. Let us fix any \( L^m \) and denote by \( g|_{L^m} \) the restricted map of g to the \( L^m \). The critical set of \( g|_{L^m} \) consists of the union of intersections of the \( L^m \) and another \( L^{k-1} \) which does not include the \( L^m \). We denote it by \( L^{m-1} \), which is an irreducible component of the critical set of \( g|_{L^m} \). By \( S_{k+2}\)-equivariance, \( g|_{L^m} \) preserves each irreducible component of the critical set of \( g|_{L^m} \). Furthermore the complement of the critical set of \( g|_{L^m} \) in \( L^m \) is Kobayashi hyperbolic.

Finally, let us look at a property of superattracting fixed points of g. The set of superattracting points, where the derivative of g vanishes for all directions, coincides with the set of \( L^0 \)’s.

Remark 1. For every \( k \geq 1 \) and every \( m, 1 \leq m \leq k \), a restricted map of \( g_{k+3} \) to any \( L^m \) is not conjugate to \( g_{m+3} \).

2.4. Examples for \( k = 1 \) and 2.

Let us see transposition hyperplanes of the \( S_3\)-equivariant function \( g_4 \) and the \( S_4\)-equivariant map \( g_5 \) to make clear what \( L^m \) is. In [3] one can find explicit formulas and figures of dynamics of \( S_{k+2}\)-equivariant maps in low-dimensions.

2.4.1. \( S_3\)-equivariant function \( g_4 \) in \( \mathbb{P}^1 \).

\[
g_3(x_1 : x_2) = [x_1^3(-x_1 + 2x_2) : x_2^3(2x_1 - x_2)]: \mathbb{P}^1 \to \mathbb{P}^1,
\]
\[
C(g_3) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_2\} = \{0, 1, \infty\} \text{ in } \mathbb{P}^1.
\]
In this case “transposition hyperplanes” are points in \( \mathbb{P}^1 \) and \( L^0 \) denotes one of three superattracting fixed points of \( g_3 \).

2.4.2. \( S_4\)-equivariant map \( g_5 \) in \( \mathbb{P}^2 \).

\[
C(g_5) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_1 = x_2\} \cup \{x_2 = x_3\} \cup \{x_3 = x_1\} \text{ in } \mathbb{P}^2.
\]
In this case \( L^3 \) denotes one of six transposition hyperplanes in \( \mathbb{P}^2 \), which is an irreducible component of \( C(g_5) \). For example, let us fix a transposition hyperplane \( \{x_1 = 0\} \). Since \( g_5 \) preserves each transposition hyperplane, we can also consider the dynamics of \( g_5 \) restricted to \( \{x_1 = 0\} \).
We denote by \( g_5|_{\{x_1=0\}} \) the restricted map of \( g_5 \) to \( \{x_1 = 0\} \). The critical set of \( g_5|_{\{x_1=0\}} \) in \( \{x_1 = 0\} \cong \mathbb{P}^1 \) is
\[
C(g_5|_{\{x_1=0\}}) = \{[0 : 1 : 0], [0 : 0 : 1], [0 : 1 : 1]\}.
\]

When we use \( L^0 \) after we fix \( \{x_1 = 0\} \), \( L^0 \) denotes one of intersections of \( \{x_1 = 0\} \) and another transposition hyperplane, which is a superattracting fixed point of \( g_5|_{\{x_1=0\}} \) in \( \mathbb{P}^1 \). The set of superattracting fixed points of \( g_5 \) in \( \mathbb{P}^2 \) is
\[
\{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1], [1 : 1 : 0], [1 : 0 : 1], [0 : 1 : 1]\}.
\]

In general \( L^0 \) denotes one of intersections of two or more transposition hyperplanes, which is a superattracting fixed point of \( g_5 \) in \( \mathbb{P}^2 \).

3. The Fatou sets of the \( S_{k+2} \)-equivariant maps

3.1. Definitions and preliminaries.

Let us recall theorems about critically finite holomorphic maps. Let \( f \) be a holomorphic map from \( \mathbb{P}^k \) to \( \mathbb{P}^k \). The Fatou set of \( f \) is defined to be the maximal open subset where the iterates \( \{f^n\}_{n \geq 0} \) is a normal family. The Julia set of \( f \) is defined to be the complement of the Fatou set of \( f \). Each connected component of the Fatou set is called a Fatou component. Let \( U \) be a Fatou component of \( f \). A holomorphic map \( h \) is said to be a limit map on \( U \) if there is a subsequence \( \{f^n\}_{n \geq 0} |_{U} \) which locally converges to \( h \) on \( U \). We say that a point \( q \) is a Fatou limit point if there is a limit map \( h \) on a Fatou component \( U \) such that \( q \in h(U) \). The set of all Fatou limit points is called the Fatou limit set. We define the \( \omega \)-limit set \( E(f) \) of the critical points by
\[
E(f) = \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} f^n(C).
\]

**Theorem 2** ([10, Proposition 5.1]). If \( f \) is a critically finite holomorphic map from \( \mathbb{P}^k \) to \( \mathbb{P}^k \), then the Fatou limit set is contained in the \( \omega \)-limit set \( E(f) \).

Let us recall the notion of Kobayashi metrics. Let \( M \) be a complex manifold and \( K_M(x, v) \) the Kobayashi quasimetric on \( M \),
\[
\inf\left\{|a| \mid \varphi : D \to M : \text{holomorphic}, \varphi(0) = x, D\varphi\left(a \frac{\partial}{\partial z}\right)_0 = v, a \in \mathbb{C}\right\}
\]
for \( x \in M, v \in T_x M, z \in D \), where \( D \) is the unit disk in \( \mathbb{C} \). We say that \( M \) is Kobayashi hyperbolic if \( K_M \) becomes a metric. Theorem 5 is a corollary of Theorem 3 and Theorem 4 for \( k = 1 \) and 2.
Theorem 3 (a basic result whose former statement can be found in [8, Corollary 14.5]). If $f$ is a critically finite holomorphic function from $\mathbb{P}^1$ to $\mathbb{P}^1$, then the only Fatou components of $f$ are attractive components of superattracting points. Moreover if the Fatou set is not empty, then the Fatou set has full measure in $\mathbb{P}^1$.

Theorem 4 ([5, theorem 7.7]). If $f$ is a critically finite holomorphic map from $\mathbb{P}^2$ to $\mathbb{P}^2$ and the complement of $C(f)$ is Kobayashi hyperbolic, then the only Fatou components of $f$ are attractive components of superattracting points.

3.2. Our first result.

Let us fix any $k$ and $g = g_{k+3}$. For every $m$, $2 \leq m \leq k$, we can apply an argument in [5] to a restricted map of $g$ to any $L^m$ because every $L^{m-1}$ is smooth and because every $L^m \setminus C(g|_{L^m})$ is Kobayashi hyperbolic. We shall use this argument in Lemma 1, which is used to prove Proposition 1.

Proposition 1. For any Fatou component $U$ which is disjoint from $C(g)$, there exists an integer $n$ such that $g^n(U)$ intersects with $C(g)$.

Proof: We suppose that $g^n(U)$ is disjoint from $C(g)$ for any $n$ and derive a contradiction by using Lemma 1 and Remark 3 below. Take any point $x_0 \in U$. Since $E(g)$ coincides with $C(g)$, $g^n(x_0)$ accumulates to $C(g)$ as $n$ tends to $\infty$ from Theorem 2. Since $C(g)$ is the union of the transposition hyperplanes, there exists a smallest integer $m_1$ such that $g^n(x_0)$ accumulates to some $L^{m_1}$. Let $h_1$ be a limit map on $U$ such that $h_1(x_0)$ belongs to the $L^{m_1}$. From Lemma 1 below, the intersection of $h_1(U)$ and the $L^{m_1}$ is an open set in the Fatou set of $g|_{L^{m_1}}$. We next consider the dynamics of $g|_{L^{m_1}}$. If there exists an integer $n_2$ such that $g^{n_2}(h_1(U) \cap L^{m_1})$ intersects with $C(g|_{L^{m_1}})$, then $g^{n_2}(h_1(U) \cap L^{m_1})$ intersects with some $L^{m_1-1}$. In this case we can consider the dynamics of $g|_{L^{m_1-1}}$. On the other hand, if there does not exist such $n_2$, then there exists an integer $n_2$ and a limit map $h_2$ on $h_1(U) \cap L^{m_2}$ such that the intersection of $h_2(h_1(U) \cap L^{m_2})$ and some $L^{m_2}$ is an open set in the $L^{m_2}$ from Remark 3 below. Thus it is contained in the Fatou set of $g|_{L^{m_2}}$. Here $m_2$ is smaller than $m_1$. In this case we can consider the dynamics of $g|_{L^{m_2}}$.

We continue the same argument above. These reductions finally come to some $L^1$ and we use Theorem 3. One can find a similar reduction argument in the proof of Theorem 5. Consequently $g^n(x_0)$ accumulates to some superattracting point $L^0$. So there exists an integer $s$ such
that \( g^n \) sends \( U \) to the attractive Fatou component which contains the superattracting point \( L^0 \). Thus \( g^n(U) \) intersects with \( C(g) \), which is a contradiction. \( \square \)

**Remark 2.** Even if a Fatou component \( U \) intersects with some \( L^m \) and is disjoint from any \( L^{m-1} \), then the similar thing as above holds for the dynamics in the \( L^m \). In this case \( U \cap L^m \) is contained in the Fatou set of \( g|_{L^m} \) and there exists an integer \( n \) such that \( g^n(U \cap L^m) \) intersects with \( C(g|_{L^m}) \).

**Lemma 1.** For any Fatou component \( U \) which is disjoint from \( C(g) \) and any point \( x_0 \in U \), let \( h \) be a limit map on \( U \) such that \( h(x_0) \) belongs to some \( L^m \) and does not belong to any \( L^{m-1} \). If \( g^n(U) \) is disjoint from \( C(g) \) for every \( n \geq 1 \), then the intersection of \( h(U) \) and the \( L^m \) is an open set in the \( L^m \).

**Proof:** Let \( B \) be the complement of \( C(g) \). Since \( B \) is Kobayashi hyperbolic and \( B \) includes \( g^{-1}(B) \), \( g^{-1}(B) \) is Kobayashi hyperbolic, too. So we can use Kobayashi metrics \( K_B \) and \( K_{g^{-1}(B)} \). Since \( B \) includes \( g^{-1}(B) \),

\[
K_B(x,v) \leq K_{g^{-1}(B)}(x,v) \quad \text{for all } x \in g^{-1}(B), \quad v \in T_x\mathbb{P}^k.
\]

In addition, since \( g \) is an unbranched covering from \( g^{-1}(B) \) to \( B \),

\[
K_{g^{-1}(B)}(x,v) = K_B(g(x), Dg(v)) \quad \text{for all } x \in g^{-1}(B), \quad v \in T_x\mathbb{P}^k.
\]

From these two inequalities we have the following inequality

\[
K_B(x,v) \leq K_B(g(x), Dg(v)) \quad \text{for all } x \in g^{-1}(B), \quad v \in T_x\mathbb{P}^k.
\]

Since the same argument holds for any \( g^n \) from \( g^{-n}(B) \) to \( B \),

\[
K_B(x,v) \leq K_B(g^n(x), Dg^n(v)) \quad \text{for all } x \in g^{-n}(B), \quad v \in T_x\mathbb{P}^k.
\]

Since \( g^n \) is an unbranched covering from \( U \) to \( g^n(U) \) and \( B \) includes \( g^n(U) \) for every \( n \), a sequence \( \{K_B(g^n(x), Dg^n(v))\}_{n \geq 0} \) is bounded for all \( x \in U \), \( v \in T_x\mathbb{P}^k \). Hence we have the following inequality for any unit vectors \( v_n \) in \( T_{x_0} U \) with respect to the Fubini-Study metric in \( \mathbb{P}^k \),

\[
(1) \quad 0 < \inf_{|v|=1} K_B(x_0, v) \leq K_B(x_0, v_n) \leq K_B(g^n(x_0), Dg^n(x_0)v_n) < \infty.
\]

That is, the sequence \( \{K_B(g^n(x_0), Dg^n(x_0)v_n)\}_{n \geq 0} \) is bounded away from 0 and \( \infty \) uniformly.

We shall choose \( v_n \) so that \( Dg^n(x_0)v_n \) keeps parallel to the \( L^m \) and claim that \( Dh(x_0)v \neq 0 \) for any accumulation vector \( v \) of \( v_n \). Let \( h = \lim_{n \to \infty} g^n \) for simplicity. Let \( V \) be a neighborhood of \( h(x_0) \) and \( \psi \) a local coordinate on \( V \) so that \( \psi(h(x_0)) = 0 \) and \( \psi(L^m \cap V) \subset \{ y = (y_1, y_2, \ldots, y_k) \mid y_1 = \cdots = y_{k-m} = 0 \} \). In this chart there exists a
constant $r > 0$ such that a polydisk $P(0, 2r)$ does not intersect with any images of transposition hyperplanes which do not include the $L^m$. Since $\psi(g^n(x_0))$ converges to $0$ as $n$ tends to $\infty$, we may assume that $\psi(g^n(x_0))$ belongs to $P(0, r)$ for large $n$. Let $\{v_n\}_{n \geq 0}$ be unit vectors in $T_w P^k$ and $\{w_n\}_{n \geq 0}$ vectors in $T_{\psi(g^n(x_0))} C^k$ so that $w_n$ keep parallel to $\psi(L^m)$ with a same direction and

$$Dg^n(x_0)v_n = |Dg^n(x_0)v_n| D\psi^{-1}(w_n).$$

So we may assume that the length of $w_n$ is almost unit for large $n$. We define holomorphic maps $\varphi_n$ from $D$ to $P(0, 2r)$ as

$$\varphi_n(z) = \psi(g^n(x_0)) + rz v_n$$

for $z \in D$ and consider holomorphic maps $\psi^{-1} \circ \varphi_n$ from $D$ to $B$ for large $n$. Then

$$(\psi^{-1} \circ \varphi_n)(0) = g^n(x_0),$$

$$D(\psi^{-1} \circ \varphi_n) \left( \frac{|Dg^n(x_0)v_n|}{r} \frac{\partial}{\partial z} \right)_0 = Dg^n(x_0)v_n.$$

Suppose $Dh(x_0)v = 0$, then $Dg^n(x_0)v$ converges to $0$ as $n$ tends to $\infty$ and so does $Dg^n(x_0)v_n$. By the definition of Kobayashi metric we have that

$$K_B(g^n(x_0), Dg^n(x_0)v_n) \leq \frac{|Dg^n(x_0)v_n|}{r} \to 0$$

as $n \to \infty$.

Since this contradicts (1), we have $Dh(x_0)v \neq 0$. This holds for all directions which are parallel to $\psi(L^m)$. Consequently the intersection of $h(U)$ and the $L^m$ is an open set in $L^m$.

**Remark 3.** The similar thing as above holds for the dynamics of any restricted map. Thus even if a Fatou component $g^n(U)$ intersects with $C(g)$ for some $n$, the same result as above holds. Because one can consider the dynamics in the $L^m$ when $g^n(U)$ intersects with some $L^m$.

**Theorem 5.** For each $k \geq 1$, the Fatou set of the $S_{k+2}$-equivariant map $g$ consists of attractive basins of superattracting fixed points which are intersections of $k$ or more distinct transposition hyperplanes.

**Proof:** This theorem follows from Proposition 1 and Remark 2 immediately. Let us describe details. Take any Fatou component $U$. From Proposition 1 there exists an integer $n_k$ such that $g^{n_k}(U)$ intersects with $C(g)$. Since $C(g)$ is the union of the transposition hyperplanes, $g^{n_k}(U)$ intersects with some $L^{k-1}$. By doing the same thing as above for the dynamics of $g$ restricted to the $L^{k-1}$, there exists an integer $n_{k-1}$ such that $g^{n_k+n_{k-1}}(U)$ intersects with some $L^{k-2}$ from Remark 2. We
again do the same thing as above for the dynamics of $g$ restricted to the $L^{k-2}$.

These reductions finally come to some $L^1$. That is, there exists integers $n_{k-2}, \ldots, n_2$ such that $g^{n_{k-2} + n_{k-3} + \cdots + n_2}(U)$ intersects with some $L^1$. From Theorem 3 there exists an integer $n_1$ such that $g^{n_1}(g^{n_{k-2} + n_{k-3} + \cdots + n_2}(U))$ contains some $L^0$. Hence $g^{n_{k-2} + n_{k-3} + \cdots + n_2}$ sends $U$ to the attractive Fatou component which contains the superattracting fixed point $L^0$ in $P^k$. 

4. Axiom A and the $S_{k+2}$-equivariant maps

4.1. Definitions and preliminaries.

Let us define hyperbolicity of non-invertible maps and the notion of Axiom A. See [6] for details. Let $f$ be a holomorphic map from $P^k$ to $P^k$ and $K$ a compact subset such that $f(K) = K$. Let $\hat{K}$ be the set of histories in $K$ and $\hat{f}$ the induced homeomorphism on $\hat{K}$. We say that $f$ is hyperbolic on $K$ if there exists a continuous decomposition $T_{\hat{K}} = E^u + E^s$ of the tangent bundle such that $D\hat{f}(E^u_x) \subset E^u_{f(x)}$ and if there exists constants $c > 0$ and $\lambda > 1$ such that for every $n \geq 1$,

$$|D\hat{f}^n(v)| \geq c\lambda^n|v| \quad \text{for all } v \in E^u \quad \text{and}$$

$$|D\hat{f}^n(v)| \leq c^{-1}\lambda^{-n}|v| \quad \text{for all } v \in E^s.$$

Here $|\cdot|$ denotes the Fubini-Study metric on $P^k$. If a decomposition and inequalities above hold for $f$ and $K$, then it also holds for $\hat{f}$ and $\hat{K}$. In particular we say that $f$ is expanding on $K$ if $f$ is hyperbolic on $K$ with unstable dimension $k$. Let $\Omega$ be the non-wandering set of $f$, i.e., the set of points for any neighborhood $U$ of which there exists an integer $n$ such that $f^n(U)$ intersects with $U$. By definition, $\Omega$ is compact and $f(\Omega) = \Omega$. We say that $f$ satisfies Axiom A if $f$ is hyperbolic on $\Omega$ and periodic points are dense in $\Omega$.

Let us introduce a theorem which deals with repelling part of dynamics. Let $f$ be a holomorphic map from $P^k$ to $P^k$. We define the $k$-th Julia set $J_k$ of $f$ to be the support of the measure with maximal entropy, in which repelling periodic points are dense. It is a fundamental fact that in dimension 1 the 1st Julia set $J_1$ coincides with the Julia set $J$. Let $K$ be a compact subset such that $f(K) = K$. We say that $K$ is a repeller if $f$ is expanding on $K$. 


Theorem 6 ([7]). Let f be a holomorphic map on \( P^k \) of degree at least 2 such that the \( \omega \)-limit set \( E(f) \) is pluripolar. Then any repeller for \( f \) intersects \( J_k \). In particular, \( J_k = \{ \text{repelling periodic points of } f \} \).

If \( f \) is critically finite, then \( E(f) \) is pluripolar. We need the following corollary to prove our second result.

Corollary 1 ([7]). Let \( f \) be the same as above. Suppose that \( J_k \) is a repeller. Then any repeller for \( f \) is a subset of \( J_k \).

4.2. Our second result.

Theorem 7. For each \( k \geq 1 \), the \( S_{k+2} \)-equivariant map \( g \) satisfies Axiom A.

Proof: We only need to consider the \( S_{k+2} \)-equivariant map \( g \) for a fixed \( k \), because argument for any \( k \) is similar as the following one. Let us show the statement above for a fixed \( k \) by induction. A restricted map of \( g \) to any \( L^1 \) satisfies Axiom A by using the theorem of critically finite functions (see [8, Theorem 19.1]). We only need to show that a restricted map of \( g \) to a fixed \( L^2 \) satisfies Axiom A. Then a restricted map of \( g \) to any \( L^m \), \( 3 \leq m \leq k \), is similar as for a restricted map of \( g \) to the \( L^2 \). Let us denote \( g|_{L^2} \), \( \Omega(g|_{L^2}) \), and \( L^2 \) by \( g \), \( \Omega \), and \( P^2 \) for simplicity.

We want to show that \( g|_{L^2} \) is hyperbolic on \( \Omega(g|_{L^2}) \) by using Kobayashi metrics. If \( g \) is hyperbolic on \( \Omega \), then \( \Omega \) has a decomposition to \( S \),

\[
\Omega = S_0 \cup S_1 \cup S_2,
\]

where \( i = 0, 1, 2 \) indicate the unstable dimensions. Since \( C(g) \) attracts all nearby points, \( S_0 \) includes all the \( L^0 \)'s and \( S_1 \) includes all the Julia sets of \( g|_{L^1} \). We denote by \( J(g|_{L^1}) \) the Julia set of \( g|_{L^1} \). Then \( g \) is contracting in all directions at \( L^0 \) and is contracting in the normal direction and expanding in an \( L^1 \)-direction on \( J(g|_{L^1}) \). Let us consider a compact, completely invariant subset in \( P^2 \setminus C \),

\[
S = \{ x \in P^2 \mid \text{dist}(g^n(x), C) \to 0 \text{ as } n \to \infty \}.
\]

By definition, we have \( J_2 \subset S_2 \subset S \). If \( g \) is expanding on \( S \), then it follow that \( S_0 = \cup L^0 \), \( S_1 = \cup J(g|_{L^1}) \). Moreover \( J_2 = S_2 = S \) holds from Corollary 1 (see Remark 4 below). Since periodic points are dense in \( J(g|_{L^1}) \) and \( J_2 \), expansion of \( g \) on \( S \) implies Axiom A of \( g \).
Let us show that $g$ is expanding on $S$. Because $f$ is attracting on $C$ and preserves $C$, there exists a neighborhood $V$ of $C$ such that $V$ is relatively compact in $g^{-1}(V)$ and the complement of $V$ is connected. We assume one of $L^1$’s to be the line at infinity of $\mathbb{P}^2$. By letting $B$ be $\mathbb{P}^2 \setminus V$ and $U$ one of connected components of $g^{-1}(\mathbb{P}^2 \setminus V)$, we have the following inclusion relations,

$$U \subset g^{-1}(B) \subset B \subset C^2 = \mathbb{P}^2 \setminus L^1.$$ 

Because $B$ and $U$ are in a local chart, there exists a constant $\rho < 1$ such that

$$K_B(x,v) \leq \rho K_U(x,v)$$

for all $x \in U, v \in T_xC^2$. In addition, since the map $g$ from $U$ to $B$ is an unbranched covering,

$$K_U(x,v) = K_B(g(x), Dg(v))$$

for all $x \in U, v \in T_xC^2$. From these two inequalities we have the following inequality

$$K_B(x,v) \leq \rho^n K_B(g^n(x), Dg^n(v))$$

for all $x \in S, v \in T_xC^2$. Consequently we have the following inequality for $\lambda = \rho^{-1} > 1$,

$$K_B(g^n(x), Dg^n(v)) \geq \lambda^n K_B(x,v)$$

for all $x \in S, v \in T_xC^2$. Since $K_B(x,v)$ is upper semicontinuous and $|v|$ is continuous, $K_B(x,v)$ and $|v|$ may be different only by a constant factor. There exists $c > 0$ such that

$$|Dg^n(x)v| \geq c\lambda^n |v|$$

for all $x \in S, v \in T_xC^2$. Thus $g$ is expanding on $S$ and satisfies Axiom A. 

Remark 4. Unlike the case when $k = 1$, it does not seem obvious that $S$ being a repeller implies $J_k = S$ when $k \geq 2$.

Remark 5. From [1, Theorem 4.11] and [9], it follows that the Fatou set of the $S_{k+2}$-equivariant map $g$ has full measure in $\mathbb{P}^k$ for each $k \geq 1$.

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References


Graduate School of Human and Environmental Studies
Kyoto University
Yoshida-Nihonmatsu-cho, Sakyo-ku
Kyoto 606-8501
Japan
E-mail address: ueno@math.h.kyoto-u.ac.jp

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