

ASSOCIATIVE AND LIE ALGEBRAS OF QUOTIENTS

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Abstract

In this paper we examine how the notion of algebra of quotients for Lie algebras ties up with the corresponding well-known concept in the associative case. Specifically, we completely characterize when a Lie algebra Q is an algebra of quotients of a Lie algebra L in terms of the associative algebras generated by the adjoint operators of L and Q respectively. In a converse direction, we also provide with new examples of algebras of quotients of Lie algebras and these come from associative algebras of quotients. In the course of our analysis, we make use of the notions of density and multiplicative semiprimeness to link our results with the maximal symmetric ring of quotients.

Introduction

In recent years, there has been a trend to extend notions and results of algebras of quotients of associative algebras to the non-associative setting. This has been achieved by a number of authors, see e.g. [14], [18], [9], [16], [1].

One of the leitmotifs for carrying out this process is the fact that, in the associative case, rings of quotients allow a deeper understanding of certain classes of rings. Thus it is to be expected that a similar role will be played by their non-associative siblings.

In this paper we shall be concerned with Lie algebras and their algebras of quotients. These were introduced by the second author in [18], following the original pattern of Utumi [19] (see below for the precise definitions). This approach prompts the question of whether there is a

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relationship between the associative and Lie algebras of quotients, beyond the formal analogy of the definitions. There are at least two ways to analyse this. On the one hand, any Lie algebra L gives rise to an associative algebra $A(L)$, which is generated by the adjoint operators given by the bracket in L . If $L \subseteq Q$ are Lie algebras such that Q is an algebra of quotients of L , it is then natural to ask whether $A(Q)$ is, in some sense, an algebra of quotients of $A(L)$. Of course, one has to circumvent the fact that $A(L)$ might not be well related to $A(Q)$ (it might not be even a subalgebra of $A(Q)$). In order to deal with this it is natural to consider the subalgebra $A_Q(L)$ of $A(Q)$ generated by the adjoint operators coming from elements in L , or the subalgebra A_0 of $A(Q)$ whose elements map L into L . This contains, and may not coincide with, $A_Q(L)$.

We prove that, if L and Q are Lie algebras such that Q is an algebra of quotients of L , then $A(Q)$ is a left quotient algebra of A_0 . This is one of the main themes of Section 2 and is accomplished in Corollary 2.9. The exact relationship between the property of Q being a quotient algebra of L and the corresponding property in terms of $A(Q)$ is of a more technical nature and is established in Theorem 2.8. This result is also important for the study of dense extensions of Lie algebras (see below).

Starting from the other endpoint, we may consider an extension of (semiprime) associative algebras $A \subseteq Q$ such that Q is a quotient algebra of A and $Q \subseteq Q_s(A)$ (the Martindale symmetric algebra of quotients). By considering the corresponding natural Lie structures on both A and Q , it is natural to ask whether $Q^{(-)}$ is a Lie algebra of quotients of $A^{(-)}$. This was proved to be the case by the second author in [18, Proposition 2.16] if moreover $Z(Q) = 0$. The general situation is much more intricate and thus requires a deeper analysis. Our main result in this direction uses arguments that go back to Herstein and asserts that $Q^{(-)}/Z(Q^{(-)})$ is a Lie quotient algebra of $A^{(-)}/Z(A^{(-)})$ whenever $A \subseteq Q \subseteq Q_s(A)$ (see Theorem 2.12). As a consequence we obtain in Corollary 2.14 that $[Q^{(-)}, Q^{(-)}]/Z([Q^{(-)}, Q^{(-)}])$ is a quotient algebra of $[A^{(-)}, A^{(-)}]/Z([A^{(-)}, A^{(-)}])$, under the same assumptions on A and Q .

Going back to the relationship between a Lie algebra L and the associative algebra $A(L)$, we examine in Section 3 the behaviour of this association under extensions. More concretely, if $L \subseteq Q$ are Lie algebras, we consider when the natural correspondence $A(L) \rightarrow A(Q)$ given by the change of domain is an algebra map. This amounts to requiring that the extension $L \subseteq Q$ is dense in the sense of Cabrera (see [5]).

The first examples of dense extensions were given in the context of multiplicatively semiprime algebras. Recall that an algebra A (associative or not) is said to be multiplicatively semiprime if A and the multiplication algebra $M(A)$ (generated by the right and left multiplication operators of A together with the identity) are both semiprime (see [6], [7], [3], [4] among others). Roughly, an extension $A \subseteq B$ of algebras is dense if every non-zero element in $M(B)$ remains non-zero when restricted to A . Cabrera proved in [5] that every essential ideal of a multiplicatively semiprime algebra is dense.

By using our techniques, we are able to find new and significant instances where dense extensions naturally appear. For example, if $A \subseteq Q \subseteq Q_{\max}^l(A)$ are associative algebras, where $Q_{\max}^l(A)$ is the maximal left quotient ring of A , then this is a dense extension, and the corresponding extensions of Lie algebras $A^{(-)}/Z(A) \subseteq Q^{(-)}/Z(Q)$ and $[A^{(-)}, A^{(-)}]/Z([A^{(-)}, A^{(-)}]) \subseteq [Q^{(-)}, Q^{(-)}]/Z([Q^{(-)}, Q^{(-)}])$ are also dense (see Lemma 3.4 and Proposition 3.5).

Our considerations finally lead to the conclusion that, if $L \subseteq Q$ is a dense extension of Lie algebras with Q multiplicatively semiprime, then the associative algebras A_0 and $A(Q)$ are indistinguishable under the formation of the maximal symmetric ring of quotients (in the sense of Schelter [17] and Lanning [13]), see Theorem 3.13.

1. Preliminaries

In this paper we will deal with algebras over an arbitrary unital and commutative ring of scalars Φ . In the case our algebras are associative, they need not be unital. First of all we will give some definitions and basic notation.

A Φ -module L with a bilinear map $[\cdot, \cdot]: L \times L \rightarrow L$, denoted by $(x, y) \mapsto [x, y]$ and called the *bracket* of x and y , is called a *Lie algebra* over Φ if the following axioms are satisfied:

- (i) $[x, x] = 0$,
- (ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (*Jacobi identity*),

for every x, y, z in L .

Every associative algebra A gives rise to a Lie algebra $A^{(-)}$ by considering the same module structure and bracket given by $[x, y] = xy - yx$.

Any element x of a Lie algebra L determines a map $\text{ad}_x: L \rightarrow L$ defined by $\text{ad}_x(y) = [x, y]$, which is a derivation of the Lie algebra L . We shall denote by $A(L)$ the associative subalgebra (possibly without identity) of $\text{End}(L)$ generated by the elements ad_x for x in L . This has

been referred to as the *multiplication ideal* in the literature (see, e.g. [4, p. 3493]).

An element x in a Lie algebra L is an *absolute zero divisor* if $(\text{ad}_x)^2 = 0$. This is equivalent to saying that $[[L, x], x] = 0$. The algebra L is said to be *non-degenerate* (or strongly nondegenerate according to Kostrikin) if it does not contain non-zero absolute zero divisors.

Given an ideal I in a Lie algebra L , we write $I^2 = [I, I]$, which is again an ideal. We say that L is *semiprime* if we have $I^2 \neq 0$ whenever I is a non-zero ideal. In other words, L has no abelian ideals. Clearly, non-degenerate Lie algebras are semiprime, while it is possible to find examples where the converse implication does not hold.

For a subset X of a Lie algebra L recall that the set

$$\text{Ann}(X) = \{a \in L \mid [a, x] = 0 \text{ for every } x \text{ in } X\}$$

is called the *annihilator* of X . This will be denoted by $\text{Ann}_L(X)$ when it is necessary to emphasize the dependence on L . If $X = L$, then $\text{Ann}(L)$ is called the *centre* of L and usually denoted by $Z(L)$. In the case that $L = A^{(-)}$ for an associative algebra A , then $Z(A^{(-)})$ agrees with the associative center $Z(A)$ of A . It is easy to check (by using the Jacobi identity) that $\text{Ann}(X)$ is an ideal of L when X is an ideal of L . Every element of $\text{Ann}(L)$ will be called a *total zero divisor*.

By an *extension of Lie algebras* $L \subseteq Q$ we will mean that L is a (Lie) subalgebra of the Lie algebra Q .

Let $L \subseteq Q$ be an extension of Lie algebras and let $A_Q(L)$ be the associative subalgebra of $A(Q)$ generated by $\{\text{ad}_x : x \in L\}$.

Recall that, given an associative algebra A and a subset X of A , we define the *right annihilator of X in A* as

$$\text{r. ann}_A(X) = \{a \in A \mid Xa = 0\},$$

which is always a right ideal of A (and two-sided if X is a right ideal). One similarly defines the *left annihilator*, which shall be denoted by $\text{l. ann}_A(X)$.

Lemma 1.1. *Let I be an ideal of a Lie algebra L with $Z(L) = 0$. Then $\text{Ann}_L(I) = 0$ if and only if $\text{r. ann}_{A(L)}(A_L(I)) = 0$.*

Proof: Suppose first $\text{Ann}_L(I) = 0$. Let $\mu \in \text{r. ann}_{A(L)}(A_L(I))$. Then $\text{ad}_y \mu = 0$ for all y in I . In particular, if $x \in L$ we get $0 = \text{ad}_y \mu(x) = [y, \mu(x)]$, and this implies that $\mu(x) \in \text{Ann}_L(I) = 0$. Hence $\mu = 0$.

Conversely, suppose $\text{r. ann}_{A(L)}(A_L(I)) = 0$. If $x \in \text{Ann}_L(I)$, $y \in I$, $z \in L$, then $\text{ad}_y \text{ad}_x z = [y, [x, z]] = [[y, x], z] + [x, [y, z]] = 0$, so $\text{ad}_x \in \text{r. ann}_{A(L)}(A_L(I)) = 0$. Since by assumption $Z(L) = 0$ we obtain $x = 0$. $\square$

For a subset X of an associative algebra A , denote by $\text{id}_A^l(X)$, $\text{id}_A^r(X)$ and $\text{id}_A(X)$ the left, right and two sided ideal of A , respectively, generated by X . When it is clear from the context, the reference to the algebra where these ideals sit into will be omitted.

Lemma 1.2. *Let I be an ideal of a Lie algebra L and suppose that Q is an overalgebra of L . Then $\text{id}_{A_Q(L)}^l(A_Q(I)) = \text{id}_{A_Q(L)}^r(A_Q(I)) = \text{id}_{A_Q(L)}(A_Q(I))$.*

Proof: The proof follows once we notice that, given x in L and y in I , we have $\text{ad}_x \text{ad}_y = \text{ad}_{[x,y]} + \text{ad}_y \text{ad}_x$. $\square$

Lemma 1.3. *Let I be an ideal of a Lie algebra L and suppose that Q is an overalgebra of L . Write $\text{id}(A(I))$ to denote the ideal of $A_Q(L)$ generated by $A_Q(I)$. Then*

- (i) $\text{r. ann}_{A_Q(L)}(\text{id}(A(I))) = \text{r. ann}_{A_Q(L)}(A_Q(I))$.
- (ii) $\text{l. ann}_{A_Q(L)}(\text{id}(A(I))) = \text{l. ann}_{A_Q(L)}(A_Q(I))$.

Proof: To see (i), it is enough to prove that $\text{r. ann}_{A_Q(L)}(A_Q(I)) \subseteq \text{r. ann}_{A_Q(L)}(\text{id}(A(I)))$ because the converse inclusion is obvious. Let $\lambda \in \text{r. ann}_{A_Q(L)}(A_Q(I))$. By Lemma 1.2 we know that, if $\mu \in \text{id}(A(I))$ there exist a natural number n , elements $x_{1,i}, \dots, x_{r_i,i}$ in L , and $y_{1,i}, \dots, y_{s_i,i}$ in I with $0 \leq r_i \in \mathbb{N}$ for all i and $\emptyset \neq \{s_1, \dots, s_n\} \subseteq \mathbb{N}$, such that $\mu = \sum_{i=1}^n \text{ad}_{x_{1,i}} \cdots \text{ad}_{x_{r_i,i}} \text{ad}_{y_{1,i}} \cdots \text{ad}_{y_{s_i,i}}$. Since $\text{ad}_{y_{s_i,i}} \lambda = 0$, we see that $\mu \lambda = 0$.

A similar argument establishes (ii). $\square$

2. Algebras of quotients: The Lie and associative cases

Inspired by the notion of ring of quotients for associative rings given by Utumi in [19], the second author introduced in [18] the notion of algebra of quotients of a Lie algebra. We now recall the main definitions and some results.

Let $L \subseteq Q$ be an extension of Lie algebras. For any q in Q , denote by ${}_L(q)$ the linear span in Q of q and the elements of the form $\text{ad}_{x_1} \cdots \text{ad}_{x_n} q$, where $n \in \mathbb{N}$ and $x_1, \dots, x_n \in L$. In particular, if $q \in L$, then ${}_L(q)$ is just the ideal of L generated by q .

We say that Q is *ideally absorbed into* L if for every nonzero element q in Q there exists an ideal I of L with $\text{Ann}_L(I) = 0$ such that $0 \neq [I, q] \subseteq L$.

Definitions 2.1 ([18, Definition 2.1 and Proposition 2.15]). Let $L \subseteq Q$ be an extension of Lie algebras. We say that Q is an *algebra of quotients of L* (or also that L is a *subalgebra of quotients of Q*) if the following equivalent conditions are satisfied:

- (i) Given p and q in Q , with $p \neq 0$, there exists x in L such that $[x, p] \neq 0$ and $[x, {}_L(q)] \subseteq L$.
- (ii) Q is ideally absorbed into L .

If given a non-zero element q in Q there exists x in L such that $0 \neq [x, q] \in L$, then Q is said to be a *weak algebra of quotients* of L .

A Lie algebra L has an algebra of quotients if and only if it has no nonzero total zero divisors, or, equivalently, $\text{Ann}(L) = 0$ (see [18, Remark 2.3]).

Remark 2.2. If Q is a weak algebra of quotients of L , then $Z(Q) = Z(L) = 0$. Indeed, given a non-zero element q in Q , there exists x in L such that $0 \neq [x, q] \in L$. Then $q \notin Z(Q)$. If $q \in L$, then the same argument shows that $q \notin Z(L)$ and so $Z(Q)$ and $Z(L)$ have no non-zero elements.

Lemma 2.3. *Let $L \subseteq Q$ be an extension of Lie algebras such that Q is a weak algebra of quotients of L , and let I be an ideal of L . If $\text{Ann}_L(I) = 0$ then $\text{r.ann}_{A(Q)}(A_Q(I)) = 0$.*

Proof: Assume that $\text{Ann}_L(I) = 0$. We first note that $\text{Ann}_Q(I) = 0$ since Q is a weak algebra of quotients of L (and applying [18, Lemma 2.11]). Now let $\mu \in \text{r.ann}_{A(Q)}(A_Q(I))$. Then $\text{ad}_y \mu = 0$ for every y in I . If $q \in Q$, we then have that $0 = \text{ad}_y \mu(q) = [y, \mu(q)]$. This says that $\mu(q) \in \text{Ann}_Q(I) = 0$, and so $\mu = 0$. $\square$

Lemma 2.4. *Let $L \subseteq Q$ be an extension of Lie algebras, and let $x_1, \dots, x_n, y \in L$. Then we have, in $A(Q)$:*

$$\text{ad}_{x_1} \cdots \text{ad}_{x_n} \text{ad}_y = \text{ad}_y \text{ad}_{x_1} \cdots \text{ad}_{x_n} + \sum_{i=1}^n \text{ad}_{x_1} \cdots \text{ad}_{[x_i, y]} \cdots \text{ad}_{x_n}.$$

In particular, if I is an ideal of L and $x_1, \dots, x_n \in I$, then

$$\text{ad}_{x_1} \cdots \text{ad}_{x_n} \text{ad}_y = \text{ad}_y \text{ad}_{x_1} \cdots \text{ad}_{x_n} + \alpha,$$

where $\alpha \in \text{span}\{\text{ad}_{z_1} \cdots \text{ad}_{z_n} \mid z_i \in I\}$.

Proof: The second part of the conclusion follows immediately from the first one, and this is just a direct consequence of ad_y being a derivation in Q . $\square$

Let $L \subseteq Q$ be an extension of Lie algebras. Denote by A_0 the associative subalgebra of $A(Q)$ whose elements are those μ in $A(Q)$ such that $\mu(L) \subseteq L$. We obviously have the containments:

$$A_Q(L) \subseteq A_0 \subseteq A(Q).$$

In order to ease the notation in the next few results we will use $\tilde{I}$ to denote, for any ideal I of L , the two-sided ideal of $A_Q(L)$ generated by the elements of the form $\text{ad}_x : Q \rightarrow Q$ for x in I , i.e. $\tilde{I} = \text{id}_{A_Q(L)}(A_Q(I))$.

Lemma 2.5. *Let $L \subseteq Q$ be an extension of Lie algebras. Let I be an ideal of L and $q_1, \dots, q_n \in Q$ such that $[q_i, I] \subseteq L$ for every $i = 1, \dots, n$. Then, for $\mu = \text{ad}_{q_1} \cdots \text{ad}_{q_n}$ in $A(Q)$, we have that $\mu \cdot (\tilde{I})^n + (\tilde{I})^n \cdot \mu \subseteq A_0$ (where $(\tilde{I})^n$ denotes the n -th power of $\tilde{I}$ in the associative algebra $A_Q(L)$).*

Proof: Using Lemma 1.2, we have $(\tilde{I})^n = (A_Q(I))^n A_Q(L) + (A_Q(I))^n = A_Q(L)(A_Q(I))^n + (A_Q(I))^n$. Thus it is enough to prove that, for any $y = \text{ad}_{x_1} \cdots \text{ad}_{x_{n-1}} \text{ad}_{x_n}$, where $x_i \in I$, both μy and $y\mu$ belong to A_0 .

We will use induction on n . For $n=1$ we have $\text{ad}_x \text{ad}_q = \text{ad}_{[x,q]} + \text{ad}_q \text{ad}_x$ and since $[x, q] \in L$ we see that $\text{ad}_{[x,q]} \in A_0$. On the other hand, $\text{ad}_q \text{ad}_x(L) \subseteq \text{ad}_q(I) \subseteq L$, and so $\text{ad}_q \text{ad}_x \in A_0$.

Assume the result true for $n-1$. Now, by Lemma 2.4 we have

$$\begin{aligned} (\dagger) \quad \text{ad}_{x_1} \cdots \text{ad}_{x_n} \text{ad}_{q_1} \cdots \text{ad}_{q_n} &= (\text{ad}_{q_1} \text{ad}_{x_1})(\text{ad}_{x_2} \cdots \text{ad}_{x_n} \text{ad}_{q_2} \cdots \text{ad}_{q_n}) \\ &\quad + \sum_{i=1}^n \text{ad}_{x_1} \cdots \text{ad}_{[x_i, q_1]} \cdots \text{ad}_{x_n} \text{ad}_{q_2} \cdots \text{ad}_{q_n}. \end{aligned}$$

The first summand on the right side belongs to A_0 because, as proved before, $\text{ad}_{q_1} \text{ad}_{x_1} \in A_0$ and $\text{ad}_{x_2} \cdots \text{ad}_{x_n} \text{ad}_{q_2} \cdots \text{ad}_{q_n} \in A_0$ by the induction hypothesis. On the other hand, for each of the terms $\text{ad}_{x_1} \cdots \text{ad}_{[x_i, q_1]} \cdots \text{ad}_{x_n} \text{ad}_{q_2} \cdots \text{ad}_{q_n}$ we have that $x_i \in I$ and $[x_i, q_1] \in L$. Using Lemma 2.4 we may write this as:

$$\begin{aligned} \text{ad}_{[x_i, q_1]} \text{ad}_{x_1} \cdots \text{ad}_{x_{i-1}} \text{ad}_{x_{i+1}} \cdots \text{ad}_{x_n} \text{ad}_{q_2} \cdots \text{ad}_{q_n} \\ + \alpha \cdot \text{ad}_{x_{i+1}} \cdots \text{ad}_{x_n} \text{ad}_{q_2} \cdots \text{ad}_{q_n}, \end{aligned}$$

where $\alpha \in \text{span}\{\text{ad}_{z_1} \cdots \text{ad}_{z_{i-1}} \mid z_j \in I\}$. The induction hypothesis applies again to show that this belongs to A_0 . Hence $y\mu \in A_0$.

If we continue to develop in the expression $(\dagger)$, we get, for some α_0 in A_0 ,

$$\begin{aligned} \text{ad}_{q_1} \text{ad}_{q_2} \text{ad}_{x_1} \cdots \text{ad}_{x_n} \text{ad}_{q_3} \cdots \text{ad}_{q_n} \\ + \sum_{i=1}^n \text{ad}_{q_1} \text{ad}_{x_1} \cdots \text{ad}_{[x_i, q_2]} \cdots \text{ad}_{x_n} \text{ad}_{q_3} \cdots \text{ad}_{q_n} + \alpha_0. \end{aligned}$$

Using Lemma 2.4 we can write each term of the form

$$\text{ad}_{q_1} \text{ad}_{x_1} \cdots \text{ad}_{[x_i, q_2]} \cdots \text{ad}_{x_n} \text{ad}_{q_3} \cdots \text{ad}_{q_n}$$

as:

$$\begin{aligned} \text{ad}_{q_1} \text{ad}_{[x_i, q_2]} \text{ad}_{x_1} \cdots \text{ad}_{x_{i-1}} \text{ad}_{x_{i+1}} \cdots \text{ad}_{x_n} \text{ad}_{q_3} \cdots \text{ad}_{q_n} \\ + \text{ad}_{q_1} \cdot \alpha \cdot \text{ad}_{x_{i+1}} \cdots \text{ad}_{x_n} \text{ad}_{q_3} \cdots \text{ad}_{q_n}, \end{aligned}$$

where $\alpha \in \text{span}\{\text{ad}_{z_1} \cdots \text{ad}_{z_{i-1}} \mid z_j \in I\}$. Notice that

$$\text{ad}_{q_1} \text{ad}_{[x_i, q_2]} \text{ad}_{x_1} = \text{ad}_{q_1} \text{ad}_{[[x_i, q_2], x_1]} + \text{ad}_{q_1} \text{ad}_{x_1} \text{ad}_{[x_i, q_2]}.$$

Hence, using $[x_i, q_2] \in L$ and $x_i \in I$, we see that the first summand above belongs to A_0 . For the second summand, assuming that $\alpha = \text{ad}_{z_1} \cdots \text{ad}_{z_{i-1}}$, with $z_j \in I$, we have $(\text{ad}_{q_1} \text{ad}_{z_1}) \text{ad}_{z_2} \cdots \text{ad}_{z_{i-1}} \text{ad}_{x_{i+1}} \cdots \text{ad}_{x_n} \text{ad}_{q_3} \cdots \text{ad}_{q_n}$, which is also an element of A_0 . Continuing in this way, we find that

$$\text{ad}_{x_1} \cdots \text{ad}_{x_n} \text{ad}_{q_1} \cdots \text{ad}_{q_n} - \text{ad}_{q_1} \cdots \text{ad}_{q_n} \text{ad}_{x_1} \cdots \text{ad}_{x_n} \in A_0$$

and by what we have just proved, we see that $\text{ad}_{q_1} \cdots \text{ad}_{q_n} \text{ad}_{x_1} \cdots \text{ad}_{x_n} \in A_0$, as was to be shown. $\square$

Corollary 2.6. *Let $L \subseteq Q$ be an extension of Lie algebras. Let $\mu = \text{ad}_{q_1} \cdots \text{ad}_{q_n}$ be in $A(Q)$ (where $q_1, \dots, q_n \in Q$) and suppose that there exists an ideal I of L that satisfies $[q_i, I] \subseteq L$ for every $i = 1, \dots, n$. Then $\mu \widetilde{I}^n \subseteq A_0$ and $\widetilde{I}^n \mu \subseteq A_0$ (where I^n denotes the n -th power of I in the Lie algebra L).*

Lemma 2.7. *Let L be a semiprime Lie algebra. If I is an ideal of L with $\text{Ann}_L(I) = 0$, then $\text{Ann}_L(I^s) = 0$ for any $s \geq 1$. Any finite intersection of ideals with zero annihilator will also have zero annihilator.*

Proof: For semiprime Lie algebras, ideals with zero annihilator are exactly the essential ideals, by [18, Lemma 1.2 (ii)]. Hence the conclusion follows at once. $\square$

Theorem 2.8. *Let $L \subseteq Q$ be an extension of Lie algebras with L semiprime. Then the following conditions are equivalent:*

- (i) Q is an algebra of quotients of L ,
- (ii) $Z(Q) = 0$ and, if $\mu \in A(Q) \setminus \{0\}$, there is an ideal I of L with $\text{Ann}_L(I) = 0$ such that $\mu \widetilde{I} \subseteq A_0$, $0 \neq \widetilde{I} \mu \subseteq A_0$. If $\mu = \text{ad}_q$, then we also have $\mu \widetilde{I}(L) \neq 0$.

Proof: (ii) $\Rightarrow$ (i): Let $q \in Q \setminus \{0\}$. Then $\mu = \text{ad}_q \neq 0$ since $Z(Q) = 0$. Let $\widetilde{I}$ be as in (ii), so it satisfies $\mu \widetilde{I}(L) \neq 0$ and $\mu \widetilde{I} \subseteq A_0$. Set

$$I_0 := \text{span}\{\alpha(x) \mid x \in L \text{ and } \alpha \in \widetilde{I}\}.$$

Then I_0 is an ideal of L such that $\text{Ann}_L(I_0) = 0$. Indeed, if $x, y \in L$ and $\alpha \in \widetilde{I}$, we have $[y, \alpha(x)] = (\text{ad}_y \alpha)(x)$ and $\text{ad}_y \alpha \in \widetilde{I}$. If now $[x, I_0] = 0$

for some x in L , then $\text{ad}_x \tilde{I}(L) = 0$. In particular, for y and z in I we have that $\text{ad}_x \text{ad}_y(z) = 0$, so $x \in \text{Ann}_L([I, I])$, which is zero by Lemma 2.7.

Finally, $0 \neq [q, I_0] = \text{ad}_q \tilde{I}(L) \subseteq L$.

(i) $\Rightarrow$ (ii): Since Q is an algebra of quotients of L , it is also a weak algebra of quotients of L , hence Remark 2.2 applies in order to obtain that $Z(Q) = 0$.

Next, let $\mu = \sum_{i \geq 1} \text{ad}_{q_{i,1}} \cdots \text{ad}_{q_{i,r_i}} \in A(Q) \setminus \{0\}$. We may of course assume that all $q_{i,j}$ are non-zero elements in Q . Set $s = \sum_{i \geq 1} r_i$. As Q is an algebra of quotients of L , there exists, for every i and j , an ideal $J_{i,j}$ of L such that $\text{Ann}_L(J_{i,j}) = 0$ and $0 \neq [J_{i,j}, q_{i,j}] \subseteq L$. By Lemma 2.7, the ideal $J = \bigcap_{i,j} J_{i,j}$ and hence also $I = J^s$ will have zero annihilator in L . Then $[q_{i,j}, I] \subseteq [q_{i,j}, J_{i,j}] \subseteq L$. By Corollary 2.6 and taking into account that $J^s \subseteq J^{r_i}$ for every i , we have that $\mu \tilde{I} \subseteq A_0$ and $\tilde{I} \mu \subseteq A_0$.

If $\tilde{I} \mu = 0$, then $\mu \in \text{r.ann}_{A(Q)}(A_Q(I))$ which is zero by Lemma 2.3.

Finally, suppose that $\mu = \text{ad}_q$, where $q \in Q \setminus \{0\}$. By [18, Lemma 2.13], Q is an algebra of quotients of I^2 . This implies that there exist elements y and z in I such that $0 \neq [q, [y, z]]$. But $[q, [y, z]] = \text{ad}_q \text{ad}_y z \in \text{ad}_q \tilde{I}(L)$. $\square$

Recall that an associative algebra S is said to be a *left quotient algebra* of a subalgebra A if whenever p and $q \in S$, with $p \neq 0$, there exists x in A such that $xp \neq 0$ and $xq \in A$. An associative algebra A has a left quotient algebra if and only if it has no total right zero divisors different from zero. (Here, an element x in A is said to be a *total right zero divisor* if $Ax = 0$.)

Corollary 2.9. *Let $L \subseteq Q$ be an extension of Lie algebras with L semi-prime. Suppose that Q is an algebra of quotients of L . Then $A(Q)$ is a left quotient algebra of A_0 .*

Proof: We will use the characterization of left quotient (associative) algebras given in [18, Lemma 2.14]. Let $\mu \in A(Q) \setminus \{0\}$ and let I be an ideal of L satisfying condition (ii) in Theorem 2.8. Set $J = A_0 \tilde{I} + \tilde{I}$, a left ideal of A_0 that satisfies $0 \neq J \mu \subseteq A_0$ (because $0 \neq \tilde{I} \mu \subseteq A_0$).

Since also $\text{Ann}_L(I) = 0$ we obtain from Lemma 2.3 that $\text{r.ann}_{A(Q)}(A_Q(I)) = 0$. This, together with the fact that $\tilde{I}$ contains $A_Q(I)$, implies that $\text{r.ann}_{A(Q)}(\tilde{I}) = 0$. Since $\text{r.ann}_{A_0}(J) \subseteq \text{r.ann}_{A(Q)}(\tilde{I})$ we get that also $\text{r.ann}_{A_0}(J) = 0$. This concludes the proof. $\square$

Remark 2.10. As established in the proof of the previous result, if $L \subseteq Q$ is an extension of Lie algebras with L semiprime, and if I is an ideal of L with $\text{Ann}_L(I) = 0$, then $A_0\tilde{I} + \tilde{I}$ is a left ideal of A_0 with zero right annihilator.

In a somewhat different direction, we analyse other instances where the notion of an algebra of quotients in the associative case is closely related to the one in the Lie case. If A is any semiprime (associative) algebra, we denote by $Q_{\max}^l(A)$ the maximal left quotient algebra of A and by $Q_s(A)$ the Martindale symmetric algebra of quotients of A .

Recall that a left ideal J of A is *dense* provided that, for any x in A we have $\text{r.ann}(Jx^{-1}) = 0$, where $Jx^{-1} = \{y \in A \mid yx \in J\}$, see, e.g. [12]. The maximal left quotient algebra of A may be thought as equivalence classes of pairs (J, f) , where J is a dense left ideal and $f: J \rightarrow A$ is a left A -module homomorphism. The equivalence is defined by declaring two such pairs as equivalent provided the corresponding homomorphisms agree on a suitable common domain. Under natural operations, the set of equivalence classes $[(J, f)]$ is a ring, isomorphic to $Q_{\max}^l(A)$.

Recall that $Q_s(A)$ can be characterized as those elements q in $Q_{\max}^l(A)$ for which there is an essential ideal I of A satisfying $Iq + qI \subseteq A$ (see, e.g. [2, Section 2.2]).

We remind the reader that if A is an associative algebra, $A^{(-)}$ denotes the Lie algebra which equals A as a module and has Lie bracket given by $[a, b] = ab - ba$.

It was proved by the second author in [18, Proposition 2.16] that, if $A \subseteq Q \subseteq Q_s(A)$ are associative algebras with A semiprime and $Z(Q) = 0$, then $Q^{(-)}$ is an algebra of quotients of $A^{(-)}$. With considerably more effort we shall establish an improvement of this result that provides with new examples of algebras of quotients of Lie algebras.

In order to prove Theorem 2.12 below, we need a lemma which is an adaptation of [10, Lemma 2] to our setting. We include the necessary changes in the statement and proof. Recall that an algebra A is *2-torsion free* provided that $2x = 0$ implies $x = 0$.

Lemma 2.11. *Let A be a semiprime 2-torsion free associative algebra and let Q be a subalgebra of $Q_s(A)$ that contains A . Let $q \in Q$ and assume that there is an essential ideal I of A such that $qI + Iq \subseteq A$ and $[q, [I, I]] = 0$. Then $[q, I] = 0$.*

Proof: We first note that Q is also 2-torsion free. For, if q is a non-zero element in Q and $2q = 0$, then, since Q is an algebra of quotients of A , there exists a in A such that aq is a non-zero element of A . But $2(aq) = 0$, which contradicts our assumption that A is 2-torsion free.

Next, define $d: A \rightarrow Q$ by $d(x) = [x, q]$. As in the proof of [10, Lemma 2], one verifies using 2-torsion freeness of Q that

$$d(x)(uv - vu) = 0, \text{ whenever } x \in A, \text{ and } u, v \in [I, I].$$

It also follows as in [10, Lemma 2] that the set

$$J = \{r \in A \mid d(x)r = 0 \text{ for all } x \text{ in } A\}$$

is an ideal of A and we have $[[I, I], [I, I]] \subseteq J$.

We now proceed to prove that A/J is semiprime and 2-torsion free. Denote by $\bar{x}$ the class modulo J of an element x in A . If $2\bar{x} = 0$, then $2x \in J$ and hence $2d(y)x = d(y)(2x) = 0$ for all y in A . Since Q is 2-torsion free this implies that $x \in J$, that is, $\bar{x} = 0$.

Now, let K be an ideal of A (containing J) for which $\bar{K}^2 = 0$, that is, $K^2 \subseteq J$. We then have that $d(y)K^2 = 0$ for any y in A .

By assumption, $Iq + qI \subseteq A$, and so $Id(y) + d(y)I \subseteq A$ and also $d(y)KI$ is a right ideal of A for any y in A . Then $(d(y)KI)^2 = d(y)KI d(y)KI \subseteq d(y)K^2 = 0$ and it follows by semiprimeness of A that $d(y)KI = 0$. Since I is an essential ideal in A and Q is an algebra of quotients of A , I has zero left annihilator in Q and therefore $d(y)K = 0$. By the definition of J , this shows that $K \subseteq J$, i.e. $\bar{K} = 0$, and thus A/J is semiprime.

The ideal $\bar{I} := (I + J)/J$ of A/J clearly satisfies $[[\bar{I}, \bar{I}], [\bar{I}, \bar{I}]] = 0$. Using [10, Lemma 1] twice we conclude that $\bar{I} \subseteq Z(A/J)$, that is, $[\bar{I}, \bar{A}] = 0$. Thus $[I, A] \subseteq J$ and therefore $d(y)[I, A] = 0$ for all y in A .

Let $K_1 = \{x \in A \mid x[I, A] = 0\}$, which is an ideal of A . By what we have just proved, $d(u) \in K_1$ whenever $u \in A$.

We now claim that $K_1 I^2 d(u) = 0$ for any u in I . To see this, take x in K_1 , v, w in I and compute that (using $Iq + qI \subseteq A$):

$$\begin{aligned} xvw d(u) &= xvw(uq - qu) = x(vwuq - uqvw + uqvw - vwqu) \\ &= x[vw, uq] + x(uqvw - vwqu) \\ &= 0 + x(uqvw - qvwu + qvwu - vwqu) \\ &= x[u, qvw] + x(qvw - vwq)u \\ &= 0 + x(qvw - wqv + wqv - vwq)u \\ &= x([qv, w] + [wq, v])u = 0, \end{aligned}$$

from which the claim follows.

Since I^2 is also essential in A we obtain that $d(u)K_1 = 0$. Altogether this implies that $d(u) \in K_1 \cap \text{l.ann}_A(K_1)$, which is zero as it is a nilpotent ideal in A . Thus $d(u) = 0$ for all u in I , that is, $[q, I] = 0$, as desired. $\square$

Theorem 2.12. *Let A be a semiprime 2-torsion free associative algebra and let Q be a subalgebra of $Q_s(A)$ that contains A . Then $A^{(-)}/Z(A)$ and $Q^{(-)}/Z(Q)$ are semiprime Lie algebras and $Q^{(-)}/Z(Q)$ is a (Lie) algebra of quotients of $A^{(-)}/Z(A)$.*

Proof: Let $\overline{K}$ be a Lie ideal of $A^{(-)}/Z(A)$ such that $[\overline{K}, \overline{K}] = 0$. Then $\overline{K}$ is the image of a Lie ideal K of $A^{(-)}$ via the natural map $A^{(-)} \rightarrow A^{(-)}/Z(A)$. The condition on $\overline{K}$ translates upstairs into $[K, K] \subseteq Z(A)$. By [10, Lemma 1] we have $K \subseteq Z(A)$, that is, $\overline{K} = 0$.

Since Q is also semiprime (e.g. [2, Lemma 2.1.9 (i)]) and 2-torsion free (by the first part of the proof of Lemma 2.11) the same argument applies to show that $Q^{(-)}/Z(Q)$ is semiprime.

By [18, Lemma 1.3 (i)], $Z(A) = Z(Q) \cap A$. Therefore the natural map

$$A^{(-)}/Z(A) \rightarrow Q^{(-)}/Z(Q)$$

is an inclusion and we shall identify $A^{(-)}/Z(A)$ with its image into $Q^{(-)}/Z(Q)$. We now prove that $Q^{(-)}/Z(Q)$ is ideally absorbed into $A^{(-)}/Z(A)$.

For an element x in Q , denote by $\overline{x}$ the class of x in $Q^{(-)}/Z(Q)$. Let $\overline{q}$ be a non-zero element in $Q^{(-)}/Z(Q)$. Since by assumption $Q \subseteq Q_s(A)$, there exists an essential ideal I of A such that $Iq + qI \subseteq A$.

We claim that the Lie ideal $\overline{I} = (I^{(-)} + Z(A))/Z(A)$ has zero annihilator in $A^{(-)}/Z(A)$. For x in A , if $\overline{x}$ satisfies $[\overline{x}, \overline{I}] = 0$, then $[x, I^{(-)}] = [x, I] \subseteq Z(A)$. Therefore $[[x, I], I] = 0$. Applying the Jacobi identity this yields $[x, [I, I]] = 0$. Therefore [10, Lemma 2] allows to conclude that $[x, I] = 0$. Note that A is an algebra of quotients of I since it is an essential ideal of A . Hence, using [18, Lemma 1.3 (iv)] we obtain $[x, A] = 0$. Thus $\overline{x} = 0$ in $A^{(-)}/Z(A)$.

Next, $[\overline{I}, \overline{q}] = \overline{[I^{(-)}, q]} \subseteq A^{(-)}/Z(A)$ because $Iq + qI \subseteq A$.

Finally, we need to see that $[\overline{I}, \overline{q}] \neq 0$. To reach a contradiction, suppose that $[I, q] \subseteq Z(A)$. Then $[[q, I], I] = 0$ and using the Jacobi identity we have $[q, [I, I]] = 0$. By Lemma 2.11 this implies $[q, I] = 0$, and a second use of [18, Lemma 1.3 (iv)] yields $[q, Q] = 0$, that is, $\overline{q} = 0$, which contradicts the choice of q . $\square$

We will now draw a consequence of Theorem 2.12. First we need a lemma.

Lemma 2.13. *Let $L \subseteq Q$ be an extension of Lie algebras with L semiprime and such that Q is an algebra of quotients of L . Then $[Q, Q]$ is an algebra of quotients of $[L, L]$.*

Proof: Let $\sum_{i=1}^n [x_i, y_i]$ be a non-zero element in $[Q, Q]$. Since Q is an algebra of quotients of L and L is semiprime, we may choose an ideal I of L with $\text{Ann}_L(I) = 0$ such that $[[x_i, y_i], I] \subseteq L$ for all i .

We know that $[I, I]$ also has zero annihilator in L , and from the inclusion

$$\text{Ann}_{[L, L]}([I, I]) \subseteq \text{Ann}_L([I, I])$$

we obtain $\text{Ann}_{[L, L]}([I, I]) = 0$. Hence, $\text{Ann}_Q([I, I]) = 0$ using [18, Lemma 2.11]. In particular, $[\sum_{i=1}^n [x_i, y_i], [I, I]] \neq 0$.

Moreover, for every z, t in I and each i , we have that

$$[[x_i, y_i], [z, t]] = [[[x_i, y_i], z], t] + [z, [[x_i, y_i], t]] \in [L, L],$$

using the Jacobi identity. $\square$

Corollary 2.14. *Let A be a semiprime 2-torsion free associative algebra and let Q be a subalgebra of $Q_s(A)$ that contains A . Then $[Q^{(-)}, Q^{(-)}]/Z([Q^{(-)}, Q^{(-)}])$ is a quotient algebra of $[A^{(-)}, A^{(-)}]/Z([A^{(-)}, A^{(-)}])$.*

Proof: First note that $Z([A, A]) \subseteq Z(A)$ by [10, Lemma 2], that is, $Z([A, A]) = Z(A) \cap [A, A]$. Hence

$$\begin{aligned} [A/Z(A), A/Z(A)] &= ([A, A] + Z(A))/Z(A) \cong [A, A]/(Z(A) \cap [A, A]) \\ &= [A, A]/Z([A, A]), \end{aligned}$$

and analogously for Q . The result is then obtained applying Theorem 2.12 and Lemma 2.13. $\square$

3. Multiplicatively semiprime Lie algebras and dense extensions

Given an extension $L \subseteq Q$ of Lie algebras, we have considered in the previous section the associative algebra $A(Q)$ and the subalgebra $A_Q(L)$ generated by the elements ad_x for x in L . It is natural to study the relationship between this algebra and the associative algebra $A(L)$.

Given an element μ in $A(L)$, we can of course think of this element in $A(Q)$ because of the inclusion $L \subseteq Q$. In order to distinguish this change of domains, we will use the notation μ^L and μ^Q . Thus, for example, given x in L we have ad_x^L and ad_x^Q , which differ in the use of the bracket in L and in Q respectively. With these considerations in mind it is desirable to have a well-defined map $\varphi: A(L) \rightarrow A(Q)$ given by $\mu^L \mapsto \mu^Q$. Whilst it is not guaranteed this can be done, if such a map exists then it is an injective algebra homomorphism and $\varphi(A(L)) = A_Q(L)$.

The condition expressed above is just a rephrasing of the density condition introduced by Cabrera in [5]. Specifically, for any algebra L (not necessarily associative) over our commutative ring of scalars Φ , let $M(L)$ be the subalgebra of $\text{End}_\Phi(L)$ generated by the identity map together with the operators given by right and left multiplication by elements of L . In the case of a Lie algebra L , note that $M(L)$ is nothing but the unitization of $A(L)$.

Following [5], given an extension of (not necessarily associative) algebras $L \subseteq Q$, the *annihilator of L in $M(Q)$* is defined by:

$$L^{\text{ann}} := \{\mu \in M(Q) \mid \mu(x) = 0 \text{ for every } x \in L\}.$$

If $L^{\text{ann}} = 0$, then L is said to be a *dense subalgebra* of Q , and we will say that $L \subseteq Q$ is a *dense extension of algebras*.

Lemma 3.1. *Let $L \subseteq Q$ be a dense extension of Lie algebras. Then $Z(L) \subseteq Z(Q)$, that is, $Z(L) = Z(Q) \cap L$.*

Proof: Suppose that $x \in L$ satisfies $[x, L] = 0$. This means that $\text{ad}_x(L) = 0$, and hence $\text{ad}_x = 0$ as an element of $M(Q)$. This implies that $[x, Q] = 0$, that is, $x \in Z(Q)$. $\square$

Lemma 3.2. *Let $L \subseteq Q$ be an extension of Lie algebras and suppose that $Z(Q) = 0$. Then the following conditions are equivalent:*

- (i) L is a dense subalgebra of Q .
- (ii) If $\mu(L) = 0$ for some μ in $A(Q)$, then $\mu = 0$.

Proof: Clearly, (i) implies (ii). Conversely, suppose that $\mu \in M(Q)$ satisfies $\mu(L) = 0$. If $\mu(p) \neq 0$ for some p in Q , then use $Z(Q) = 0$ to find a (non-zero) element q in Q satisfying $\text{ad}_q \mu(p) \neq 0$. But then $\text{ad}_q \mu(L) = 0$ and since $A(Q)$ is a two-sided ideal of $M(Q)$, we have that $\text{ad}_q \mu \in A(Q)$. Hence condition (ii) yields $\text{ad}_q \mu = 0$, a contradiction. $\square$

We will now present some examples of dense extensions.

Lemma 3.3. *Let $L \subseteq Q$ be a dense extension of Lie algebras. Then $[L, L] \subseteq [Q, Q]$ is also dense.*

Proof: Let $\mu \in M([Q, Q])$ and suppose that $\mu([L, L]) = 0$. Consider $\tilde{\mu} \in M(Q)$ such that $\tilde{\mu}|_{[Q, Q]} = \mu$. Then, for any x in L , we have $\tilde{\mu} \text{ad}_x(L) = 0$. Since $\tilde{\mu} \text{ad}_x \in M(Q)$ and $L \subseteq Q$ is a dense extension we obtain $\tilde{\mu} \text{ad}_x(Q) = 0$. This implies that $\tilde{\mu} \text{ad}_q(x) = -\tilde{\mu} \text{ad}_x(q) = 0$ for every q in Q and every x in L . A second use of the density hypothesis implies that $\tilde{\mu} \text{ad}_q(Q) = 0$, that is, $\mu([Q, Q]) = \tilde{\mu}([Q, Q]) = 0$. $\square$

Lemma 3.4. *Let A be a semiprime associative algebra. Then, for any subalgebra Q of $Q_{\max}^l(A)$ that contains A , the extension $A \subseteq Q$ is dense.*

Proof: Let $\mu \in M(Q)$ such that $\mu(A) = 0$. We may write $\mu(x) = \sum_{i=1}^n p_i x q_i$, with p_i, q_i in $Q_{\max}^l(A)$. Then $\mu(x)$ is a Generalized Polynomial Identity on A (in the terminology of [2, p. 227]), and so [2, Proposition 6.3.13] applies to conclude $\mu(Q_{\max}^l(A)) = 0$. Therefore $\mu(Q) = 0$. $\square$

In the following lemma we consider an extension $A \subseteq Q$ of associative algebras where Q is a left quotient algebra of A . We then have that $Z(A) = Z(Q) \cap A$ ([18, Lemma 1.3 (i)]) so that we may consider, as in Theorem 2.12, that $A^{(-)}/Z(A)$ is a subalgebra of $Q^{(-)}/Z(Q)$.

Proposition 3.5. *Let A be a semiprime 2-torsion free associative algebra and Q a subalgebra of $Q_{\max}^l(A)$ that contains A . Then $A^{(-)}/Z(A) \subseteq Q^{(-)}/Z(Q)$ and*

$$[A^{(-)}, A^{(-)}]/Z([A^{(-)}, A^{(-)}]) \subseteq [Q^{(-)}, Q^{(-)}]/Z([Q^{(-)}, Q^{(-)}])$$

are dense extensions of Lie algebras.

Proof: Denote by $\bar{q}$ the classes of elements in $Q^{(-)}/Z(Q)$. Let $\mu \in M(Q^{(-)}/Z(Q))$ such that $\mu(A^{(-)}/Z(A)) = 0$. Then there is $\mu' \in M(Q)$ such that $\mu'(q) = \mu(\bar{q})$ for every q in Q .

The condition that μ vanishes on $A^{(-)}/Z(A)$ means that $\mu'(A) \subseteq Z(Q)$. Suppose that $\mu'(Q)$ is not contained in $Z(Q)$. Then there are non-zero elements p and q in Q for which $(\lambda_q \mu' - \rho_q \mu')(p) = [q, \mu'(p)] \neq 0$, where λ_q and ρ_q stand for the left and right (associative) multiplication by q , respectively.

But then $\lambda_q \mu' - \rho_q \mu'$ is a non-zero element in $M(Q)$ and since the extension $A \subseteq Q$ is dense by Lemma 3.4 we get that $[q, \mu'(A)] = (\lambda_q \mu' - \rho_q \mu')(A) \neq 0$. This contradicts the fact that $\mu'(A) \subseteq Z(Q)$. This shows $A^{(-)}/Z(A^{(-)}) \subseteq Q^{(-)}/Z(Q^{(-)})$ is dense.

Finally, $[A^{(-)}, A^{(-)}]/Z([A^{(-)}, A^{(-)}]) \subseteq [Q^{(-)}, Q^{(-)}]/Z([Q^{(-)}, Q^{(-)}])$ is also dense by the first part of the proof and Lemma 3.3, using as in Corollary 2.14 that $Z([A, A]) = Z(A) \cap [A, A]$. $\square$

Following Cabrera and Mohammed [7], we say that an algebra L is *multiplicatively semiprime* (respectively, *prime*) whenever L and its multiplication algebra $M(L)$ are semiprime (respectively, prime). Observe that in this situation, and if L is a Lie algebra, then $A(L)$, being an ideal of $M(L)$, will also be a semiprime (respectively, prime) algebra.

As we have seen, under mild hypotheses (see Theorem 2.12 and Corollary 2.14) we have dense extensions of semiprime Lie algebras $A^{(-)}/Z(A) \subseteq Q^{(-)}/Z(Q)$ and

$$[A^{(-)}, A^{(-)}]/Z([A^{(-)}, A^{(-)}]) \subseteq [Q^{(-)}, Q^{(-)}]/Z([Q^{(-)}, Q^{(-)}]),$$

so that $Q^{(-)}/Z(Q)$ is a quotient algebra of $A^{(-)}/Z(A)$ and $[Q^{(-)}, Q^{(-)}]/Z([Q^{(-)}, Q^{(-)}])$ is a quotient algebra of $[A^{(-)}, A^{(-)}]/Z([A^{(-)}, A^{(-)}])$.

We remark that if A is a semiprime algebra over a field of characteristic not 2, then $A^{(-)}/Z(A)$ is multiplicatively semiprime (or even multiplicatively prime) in some important cases that are covered in [4, Corollary 2.4], but not in general (see [4, Theorem 2.1]). This contrasts with the case of $[A^{(-)}, A^{(-)}]/Z([A^{(-)}, A^{(-)}])$, which is always multiplicatively semiprime if A is semiprime ([4, Corollary 2.4]).

If, furthermore, our algebra A is endowed with an involution $*$, denote by K_A the set of all skew elements of A , that is, $K_A = \{x \in A \mid x^* = -x\}$. This is a Lie subalgebra of $A^{(-)}$, and it turns out that $K_A/Z(K_A)$ is multiplicatively semiprime or prime in various instances (see [4, Theorems 3.4, 3.6]). Again the situation is different for the Lie algebra $[K_A, K_A]/Z([K_A, K_A])$, which is always multiplicatively semiprime ([4, Theorem 2.3]). The above mentioned examples involving commutators are of great interest since they appear in Zelmanov's classification of simple M -graded Lie algebras over a field of characteristic at least $2d + 1$, where d is the width of M (see [20]). It is natural to expect that for algebras of the latter type $-K_A/Z(K_A)$ in the case where A has an involution— results analogous to Theorem 2.12 and Proposition 3.5 are available. This is indeed the case, as M. Cabrera and J. Sánchez Ortega have shown recently in [8].

Lemma 3.6. *Let $L \subseteq Q$ be a dense extension of Lie algebras. If Q is multiplicatively semiprime, then A_0 is semiprime.*

Proof: Let I be an ideal of A_0 with $I^2 = 0$. As in the proof of Theorem 2.8, let $I_0 = \text{span}\{\alpha(x) \mid \alpha \in I, x \in L\}$, which is clearly an ideal of L . For any μ in I , we evidently have $\mu(I_0) = 0$. It then follows from [5, Proposition 3.1] that $\mu(M(Q)(I_0)) = 0$ (where $M(Q)(I_0)$ is the set of finite sums of elements of $M(Q)$ applied to elements of I_0). This implies that $\mu M(Q)\mu(L) = 0$, and thus $\mu M(Q)\mu = 0$ since L is dense in Q . But $M(Q)$ is semiprime by hypothesis, so $\mu = 0$ and since μ was an arbitrary element of I , we get that $I = 0$, that is, A_0 is semiprime. $\square$

Remark 3.7. Given a dense extension of Lie algebras $L \subseteq Q$, where Q is multiplicatively semiprime, we have that L is also multiplicatively semiprime. This follows from [5, Proposition 2.2].

Proposition 3.8. *Let $L \subseteq Q$ be a dense extension of Lie algebras. Assume that Q is a multiplicatively semiprime algebra of quotients of L . Then $I \subseteq Q$ is a dense extension for every essential ideal I of L .*

Proof: We first observe that $Z(Q) = 0$ because Q is semiprime and hence Lemma 3.2 applies. Thus, let μ be in $A(Q)$ such that $\mu(I) = 0$, and by way of contradiction assume that $\mu \neq 0$. By Corollary 2.9 (which can be applied by virtue of Remark 3.7) $A(Q)$ is a left quotient algebra of A_0 and hence there exists λ in A_0 such that $0 \neq \lambda\mu \in A_0$. Since the extension $L \subseteq Q$ is dense, $\lambda\mu(L) \neq 0$, and since $\text{Ann}_L(I) = 0$, there is a (non-zero) element y in I such that $\text{ad}_y \lambda\mu(L) \neq 0$. Using now that $A(Q)$ has no total right zero divisors, we get that $A(Q) \text{ad}_y \lambda\mu \neq 0$, and this, coupled with the semiprimeness of $A(Q)$, implies that $A(Q) \text{ad}_y \lambda\mu A(Q) \text{ad}_y \lambda\mu \neq 0$. A second application of the fact that $L \subseteq Q$ is a dense extension yields $A(Q) \text{ad}_y \lambda\mu A(Q) \text{ad}_y \lambda\mu(L) \neq 0$. However, $\mu(I) = 0$ by assumption and thus [5, Proposition 3.1] implies that $\mu M(Q)(I) = 0$. But this is a contradiction, because of the containments

$$\mu A(Q) \text{ad}_y \lambda\mu(L) \subseteq \mu A(Q)([I, L]) \subseteq \mu A(Q)(I) \subseteq \mu M(Q)(I) = 0. \quad \square$$

Corollary 3.9 (cf. [4, Lemma 3.1]). *Let $L \subseteq Q$ be a dense extension of Lie algebras with Q multiplicatively prime. If Q is a quotient algebra of L , then $I \subseteq Q$ is a dense extension for any non-zero ideal I of L .*

Proof: This follows directly from the previous proposition once we realize that, in a prime Lie algebra, every non-zero ideal is essential. $\square$

Remark 3.10. Note that the proof of the corollary above also goes through if we only assume that Q is prime and multiplicatively semiprime.

Corollary 3.11. *Let $L \subseteq Q$ be a dense extension of Lie algebras. Assume that Q is a multiplicatively semiprime algebra of quotients of L . Then, for every essential ideal I of L , $\text{l.ann}_{A(Q)}(\tilde{I}) = 0$.*

Proof: Let $\mu \in \text{l.ann}_{A(Q)}(\tilde{I})$. Then, if $y \in I$ we have $\mu \text{ad}_y(I) = 0$. This implies that $\mu(I^2) = 0$. By Proposition 3.8 applied to the essential ideal I^2 of L , the extension $I^2 \subseteq Q$ is dense, and so $\mu = 0$. $\square$

We now tie up our results with the associative case by relating them to the maximal symmetric ring of quotients. This was first introduced by Schelter in [17] and systematically explored by Lanning in [13]. It has more recently come into prominence following [15]. In short, the maximal symmetric ring of quotients $Q_\sigma(R)$ of a ring R is the subring of $Q_{\max}^r(R)$ whose elements q satisfy $Jq \subseteq R$ for some dense left ideal J

of R . An alternative abstract characterization of $Q_\sigma(R)$ is given in [13, Proposition 2.1], as follows: Given a dense left ideal I and a dense right ideal J of R , we say that a pair of maps (f, g) , where $f: I \rightarrow R$ is a left R -homomorphism and $g: J \rightarrow R$ is a right R -homomorphism, is *compatible* provided that $f(x)y = xg(y)$ whenever $x \in I$ and $y \in J$.

Two sets of data (f, g, I, J) and (f', g', I', J') as above are *equivalent* if f and f' agree on some dense left ideal contained in $I \cap I'$ and similarly for g and g' . Denote by $[f, g, I, J]$ the equivalence class of (f, g, I, J) as above. Under natural operations, the set of such equivalence classes is a ring isomorphic to $Q_\sigma(R)$.

Proposition 3.12. *Let $L \subseteq Q$ be a dense extension of Lie algebras. Suppose that Q is a multiplicatively semiprime algebra of quotients of L . There is then an injective algebra homomorphism $\tau: A(Q) \rightarrow Q_\sigma(A_0)$.*

Proof: Observe first that L is (multiplicatively) semiprime by Remark 3.7. Next, given μ in $A(Q) \setminus \{0\}$, we have proved in Theorem 2.8 that there is an ideal I of L with $\text{Ann}_L(I) = 0$ such that $\mu\tilde{I} + \tilde{I}\mu \subseteq A_0$ (where $\tilde{I}$ is the ideal of $A_Q(L)$ generated by the elements of the form ad_x for x in I). By the observation made in Remark 2.10, the left ideal $A_0\tilde{I} + \tilde{I}$ has zero right annihilator in A_0 . By Corollary 3.11 we have that $\text{l.ann}_{A(Q)}(\tilde{I}) = 0$ and so the right ideal $\tilde{I}A_0 + \tilde{I}$ has zero left annihilator in A_0 . Since A_0 is semiprime by Lemma 3.6, the ideals $A_0\tilde{I} + \tilde{I}$ and $\tilde{I}A_0 + \tilde{I}$ are left and right dense in A_0 , respectively. Note that $(A_0\tilde{I} + \tilde{I})\mu \subseteq A_0$ and that $\mu(\tilde{I}A_0 + \tilde{I}) \subseteq A_0$. Hence, right and left multiplication by μ produce two homomorphisms $R_\mu: A_0\tilde{I} + \tilde{I} \rightarrow A_0$ and $L_\mu: \tilde{I}A_0 + \tilde{I} \rightarrow A_0$ which are left (respectively, right) A_0 -linear and compatible.

If $\mu = 0$, we may then take $I = L$ and so $\tilde{I} = A_Q(L)$, which has zero right and left annihilator in $A(Q)$ (and hence in A_0).

This allows us to define a map

$$\tau: A(Q) \longrightarrow Q_\sigma(A_0)$$

by $\tau(\mu) = [R_\mu, L_\mu, A_0\tilde{I} + \tilde{I}, \tilde{I}A_0 + \tilde{I}]$. Note that, given any other pair of left and right dense ideals I' and J' of A_0 such that $I'\mu \subseteq A_0$ and $\mu J' \subseteq A_0$, we have that $\tau(\mu) = [R_\mu, L_\mu, I', J']$. It follows easily from this that τ is an algebra homomorphism.

In order to check that τ is injective, assume that $\tau(\mu) = 0$. This implies that there is a left dense ideal I' of A_0 with zero right annihilator in A_0 such that $I'\mu = 0$. Now, A_0 is a left quotient algebra of I' and, by

Corollary 2.9, $A(Q)$ is a left quotient algebra of A_0 . Thus $A(Q)$ is a left quotient algebra of I' and therefore $I'\mu = 0$ forces $\mu = 0$. $\square$

Theorem 3.13. *Let $L \subseteq Q$ be a dense extension of Lie algebras. Suppose that Q is a multiplicatively semiprime algebra of quotients of L . Then A_0 is semiprime and $Q_\sigma(A_0) = Q_\sigma(A(Q))$.*

Proof: The semiprimeness of A_0 follows from Lemma 3.6. Now, apply Proposition 3.12 and [13, Theorem 2.5] to obtain $Q_\sigma(A_0) = Q_\sigma(A(Q))$. $\square$

We close by exploring the possible converses to Corollary 2.9 in the presence of dense extensions of Lie algebras.

Definition 3.14. Given an extension of associative algebras $A \subseteq S$, we say that S is *right ideally absorbed into A* if for any q in $S \setminus \{0\}$ there is an ideal I of A with $\text{l.ann}_A(I) = 0$ and such that $0 \neq qI \subseteq A$. *Left ideally absorbed* can be defined analogously.

Observe that, in the definition above, we are requiring that the ideal I is two-sided rather than just a right ideal. In fact, in the latter case this would be equivalent to saying that S is a right quotient algebra of A (see [18, Lemma 2.14]).

Proposition 3.15. *Let $L \subseteq Q$ be a dense extension of Lie algebras with $Z(Q) = 0$. Suppose that $A(Q)$ is right ideally absorbed into A_0 . Then Q is an algebra of quotients of L .*

Proof: Let $q \in Q \setminus \{0\}$. Since $Z(Q) = 0$, we have that $\text{ad}_q \neq 0$. By hypothesis, there is an ideal I of A_0 such that $\text{l.ann}_{A_0}(I) = 0$ and $0 \neq \text{ad}_q I \subseteq A_0$. Set $I_0 = \{\alpha(x) \mid \alpha \in I, x \in L\}$, which is an ideal of L (see, e.g. the argument in the proof of Theorem 2.8). Moreover, $\text{Ann}_L(I_0) = 0$. Indeed, suppose that an element x in L satisfies $[x, I_0] = 0$. By definition, this means that $\text{ad}_x I(L) = 0$, and since the extension is dense we have that $\text{ad}_x I = 0$. Thus $\text{ad}_x \in \text{l.ann}_{A_0}(I) = 0$. By Lemma 3.1, $Z(L) = 0$ and so $x = 0$.

Finally, $0 \neq [q, I_0] = (\text{ad}_q I)(L) \subseteq A_0(L) \subseteq L$. $\square$

Corollary 3.16. *Let $L \subseteq Q$ be a dense extension of Lie algebras, with L semiprime and $Z(Q) = 0$. If $A(Q)$ is right ideally absorbed into A_0 , then $A(Q)$ is left ideally absorbed into A_0 .*

Proof: By Proposition 3.15, if $A(Q)$ is right ideally absorbed into A_0 we have that Q is an algebra of quotients of L . We then may apply Theorem 2.8 to achieve the conclusion. $\square$

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References

- [1] J. A. ANQUELA, E. GARCÍA, AND M. GÓMEZ LOZANO, Maximal algebras of Martindale-like quotients of strongly prime linear Jordan algebras, *J. Algebra* **280**(1) (2004), 367–383.
- [2] K. I. BEIDAR, W. S. MARTINDALE, III, AND A. V. MIKHALEV, “*Rings with generalized identities*”, Monographs and Textbooks in Pure and Applied Mathematics **196**, Marcel Dekker, Inc., New York, 1996.
- [3] J. C. CABELLO AND M. CABRERA, Structure theory for multiplicatively semiprime algebras, *J. Algebra* **282**(1) (2004), 386–421.
- [4] J. C. CABELLO, M. CABRERA, G. LÓPEZ, AND W. S. MARTINDALE, III, Multiplicative semiprimeness of skew Lie algebras, *Comm. Algebra* **32**(9) (2004), 3487–3501.
- [5] M. CABRERA, Ideals which memorize the extended centroid, *J. Algebra Appl.* **1**(3) (2002), 281–288.
- [6] M. CABRERA AND A. A. MOHAMMED, Extended centroid and central closure of the multiplication algebra, *Comm. Algebra* **27**(12) (1999), 5723–5736.
- [7] M. CABRERA AND A. A. MOHAMMED, Extended centroid and central closure of multiplicatively semiprime algebras, *Comm. Algebra* **29**(3) (2001), 1215–1233.
- [8] M. CABRERA AND J. SÁNCHEZ ORTEGA, Lie Quotients for Skew Lie algebras, Preprint (2005).
- [9] E. GARCÍA AND M. GÓMEZ LOZANO, Jordan systems of Martindale-like quotients, *J. Pure Appl. Algebra* **194**(1–2) (2004), 127–145.
- [10] I. N. HERSTEIN, On the Lie structure of an associative ring, *J. Algebra* **14** (1970), 561–571.
- [11] A. I. KOSTRIKIN, “*Around Burnside*”, Translated from the Russian and with a preface by James Wiegold, *Ergebnisse der Mathematik und ihrer Grenzgebiete* (3) **20**, Springer-Verlag, Berlin, 1990.
- [12] T. Y. LAM, “*Lectures on modules and rings*”, Graduate Texts in Mathematics **189**, Springer-Verlag, New York, 1999.
- [13] S. LANNING, The maximal symmetric ring of quotients, *J. Algebra* **179**(1) (1996), 47–91.

- [14] C. MARTÍNEZ, The ring of fractions of a Jordan algebra, *J. Algebra* **237**(2) (2001), 798–812.
- [15] E. ORTEGA, Rings of quotients of incidence algebras and path algebras, *J. Algebra* **303**(1) (2006), 225–243.
- [16] I. PANIELLO, Identidades polinómicas y álgebras de cocientes en sistemas de Jordan, Ph. D. Thesis, Universidad de Zaragoza (2004).
- [17] W. SCHELTER, Two-sided rings of quotients, *Arch. Math. (Basel)* **24** (1973), 274–277.
- [18] M. SILES MOLINA, Algebras of quotients of Lie algebras, *J. Pure Appl. Algebra* **188**(1–3) (2004), 175–188.
- [19] Y. UTUMI, On quotient rings, *Osaka Math. J.* **8** (1956), 1–18.
- [20] E. ZELMANOV, Lie algebras with a finite grading, *Math. USSR-Sb* **52** (1985), 347–385.

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