ON THE RANGE SPACE OF YANO'S 
EXTRAPOLATION THEOREM AND NEW 
EXTRAPOLATION ESTIMATES AT_INFINITY

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Abstract

Given a sublinear operator $T$ satisfying that $\|Tf\|_{L^p(\nu)} \leq \frac{C}{p-1}\|f\|_{L^p(\mu)}$, for every $1 < p \leq p_0$, with $C$ independent of $f$ and $p$, it was proved in [C] that

$$
\sup_{r > 0} \frac{\int_{N} N_T(y) dy}{1 + \log^+ r} \leq \int_{M} |f(x)|(1 + \log + |f(x)|) d\mu(x).
$$

This estimate implies that $T: L^\log L \rightarrow B$, where $B$ is a rearrangement invariant space. The purpose of this note is to give several characterizations of the space $B$ and study its associate space. This last information allows us to formulate an extrapolation result of Zygmund type for linear operators satisfying $\|Tf\|_{L^p(\nu)} \leq Cp\|f\|_{L^p(\mu)}$, for every $p \geq p_0$.

1. Introduction

In 1951, Yano (see [Y], [Z]) using the ideas of Titchmarsh in [T], proved that for every sublinear operator satisfying

$$
\left( \int_{N} |Tf(x)|^p d\nu(x) \right)^{1/p} \leq \frac{C}{p-1} \left( \int_{M} |f(x)|^p d\mu(x) \right)^{1/p},
$$

where $N$ and $M$ are two finite measure spaces, $T: L \log L(\mu) \rightarrow L^1(\nu)$ is bounded. If the measures involved are not finite, then an easy modification of the above proofs, shows that $T: L \log L(\mu) \rightarrow L^1_{\log}(\nu)$ and, in fact, $T: L \log L(\mu) \rightarrow L^1(\nu) + L^\infty(\nu)$.

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Quite recently, it was proved, in [C], that under a weaker condition on the operator $T$, namely that
\begin{equation}
\left( \int_{\mathcal{M}} |T\chi_A(x)|^p \, dv(x) \right)^{1/p} \leq \frac{C}{p-1} \mu(A)^{1/p},
\end{equation}
for every measurable set $A \subset \mathcal{M}$ and every $1 < p \leq p_0$, with $C$ independent of $A$ and $p$, we have that, there exists a positive constant $K$, such that
\begin{equation}
\sup_{r>0} \int_{\mathcal{M}} \frac{\lambda_T^r(y) \, dy}{1 + \log^+ r} \leq K \int_{\mathcal{M}} |f(x)|(1 + \log^+ |f(x)|) \, d\mu(x),
\end{equation}
where $\lambda_T^r$ is the distribution function of $Tf$ with respect to $\nu$, and $\mu$ and $\nu$ are two $\sigma$-finite measures. This estimate allows us, as we shall see in this note, to improve Yano’s theorem in the following sense: There exists a rearrangement invariant space $B(\nu) \subset L^1 + L^\infty$, $B(\nu) \neq L^1 + L^\infty$ and such that for every sublinear operator $T$ satisfying (1), we have that $T: L \log L(\mu) \to B(\nu)$.

Throughout this paper, a sublinear operator satisfying (1) shall be called Yano’s operator. From (2), it is very easy to see that if we define
\begin{equation}
B(\nu) = \{ f \text{ measurable}; \|f\|_{B(\nu)} < \infty \},
\end{equation}
where
\begin{equation}
\|f\|_{B(\nu)} = \inf \left\{ \alpha > 0; \sup_{r>0} \int_{\mathcal{M}} \frac{\lambda_T^r(y) \, dy}{1 + \log^+ \frac{1}{r}} \leq 1 \right\},
\end{equation}
then, every Yano’s operator satisfies
\begin{equation}
T: L \log L \to B(\nu)
\end{equation}
is bounded.

The purpose of this note is to study in detail the space $B(\nu)$, including the identification of its associate space.

This last information will allow us to formulate an extrapolation result of Zygmund type (see [Z, p. 119]) for linear operators satisfying
\begin{equation}
\|Tf\|_{L^p(\nu)} \leq C_p \|f\|_{L^p(\nu)},
\end{equation}
for every $p \nearrow \infty$.

Some years ago, in the work of Jawerth and Milman (see [JM1], [JM2]), the extrapolation theory was extended to the setting of compatible couples of Banach spaces. More recently, in [CM], the authors have developed a new abstract extrapolation method, where the range space (of the previous method) has been improved.
2. On the range space $B$

Let $B = B(\nu)$ be the space defined in (3). Observe that

$$\int_r^\infty \lambda_f(y) \, dy = \int_M P_r(\|f(x)\|) \, d\nu(x),$$

where $P_r(t) = (t - r)^+$. Therefore, the functional $\| \cdot \|_B$ is similar to a uniform (in $r$) Luxembourg norm. Since $P_r$ is a convex function, the fact that it is a norm is an easy exercise. However, that $B$ is a rearrangement invariant Banach function space is a consequence of the fact that $B$ is a maximal Lorentz space (see Theorem 2.4 below).

Our first result proves that $B \subset L^1 + L^\infty$ and that $B \neq L^1 + L^\infty$.

**Proposition 2.1.** For every $p > 1$, $B \subset L^1 + L^p$ with constant less than or equal to $Cp/(p - 1)$.

**Proof:** Let $f \in B$ such that $\|f\|_B = 1$. Then $\int_1^\infty \lambda_f(y) \, dy \leq C < \infty$ and hence, if we define $\overline{f} = f \chi_{\{|f| > 1\}}$, we have that

$$\|\overline{f}\|_1 = \lambda_f(1) + \int_1^\infty \lambda_f(y) \, dy \leq C.$$
Now, if we set $f = f - \bar{f}$ and take $p > 1$, then an integration by parts shows that
\[
\|f\|_p^p = p \int_0^\infty y^{p-1} \lambda_f(y) \, dy = p \int_0^1 y^{p-1} \lambda_f(y) \, dy
\]
\[
= p(p-1) \int_0^1 y^{p-2} \left( \int_y^1 \lambda_f(s) \, ds \right) \, dy
\]
\[
\leq p(p-1) \int_0^1 y^{p-2} \left( 1 + \log \frac{1}{y} \right) \, dy
\]
\[
= p(p-1) \left( \frac{1}{p-1} + \frac{1}{(p-1)^2} \right) = \frac{p^2}{p-1},
\]
from which the result follows. \(\square\)

Our next step is to give a different and useful characterization of the space $B$.

**Lemma 2.2.** For every $s > 0$,

(a) \[
\int_{\frac{1}{2} f^{**}(s)}^\infty \lambda_f(y) \, dy \leq \int_0^s f^*(t) \, dt,
\]

(b) \[
\int_0^s f^*(t) \, dt \leq 2 \int_{\frac{1}{2} f^{**}(s)}^\infty \lambda_f(y) \, dy.
\]

**Proof:** (a) Using that $\lambda_f = \lambda_{f^*}$ and Fubini’s theorem, we obtain that
\[
\int_{\frac{1}{2} f^{**}(s)}^\infty \lambda_f(y) \, dy = \int_0^\infty \left( f^*(t) - f^{**}(s) \right)_+ \, dt = \int_0^s \left( f^*(t) - f^{**}(s) \right)_+ \, dt
\]
\[
\leq \int_0^s f^*(t) \, dt.
\]

(b) By the distribution formula proved in [CS1], we have that
\[
\int_0^s f^*(t) \, dt = \int_0^\infty \min \left( \lambda_f(y), s \right) \, dy \leq \int_0^{\frac{1}{2} f^{**}(s)} s \, dy + \int_{\frac{1}{2} f^{**}(s)}^\infty \lambda_f(y) \, dy
\]
\[
= \frac{1}{2} \int_0^s f^*(t) \, dt + \int_{\frac{1}{2} f^{**}(s)}^\infty \lambda_f(y) \, dy,
\]
from which the result follows. \(\square\)
Lemma 2.3.

\[ \sup_{s > 0} \frac{\int_0^s f^* (t) \, dt}{1 + \log^+ \frac{s}{\log^+ f}} \lesssim \sup_{r > 0} \frac{\int_r^\infty \lambda_f (y) \, dy}{1 + \log^+ \frac{1}{r}}. \]

**Proof:** Given \( s > 0 \), we have, by Lemma 2.2(b), that

\[ \frac{\int_0^s f^* (t) \, dt}{1 + \log^+ \frac{s}{\log^+ f}} \leq 2 \frac{\int_{f^**}^{f^**} \lambda_f (y) \, dy}{1 + \log^+ \frac{1}{\log^+ f}} \leq 2 \sup_{r > 0} \frac{1 + \log^+ \frac{1}{f}}{1 + \log^+ \frac{1}{r}} \int_r^\infty \lambda_f (y) \, dy \]

and therefore the inequality \( \lesssim \) follows.

Conversely, if \( \sup_{s > 0} \frac{\int_0^s f^* (t) \, dt}{1 + \log^+ \frac{s}{\log^+ f}} < \infty \), then necessarily \( f^** (+\infty) = 0 \), and hence, if \( r < \| f \|_{\infty} = \sup_s f^** (s) \), we have that \( 0 = f^** (+\infty) = \inf_s f^** (s) < r < \sup_s f^** (s) \) and by continuity, there exists \( s \) so that \( r = f^** (s) \). Then using Lemma 2.2(a),

\[ \frac{\int_r^\infty \lambda_f (y) \, dy}{1 + \log^+ \frac{1}{r}} = \int_{f^**}^{f^**} \lambda_f (y) \, dy \leq \frac{\int_0^s f^* (t) \, dt}{1 + \log^+ \frac{1}{f}}. \]

If \( r \geq \| f \|_{\infty} \), then \( \int_r^\infty \lambda_f (y) \, dy = 0 \) and the result follows immediately. \( \square \)

Given a concave function \( \varphi (t) \), we recall that the maximal Lorentz space is defined (see [BS, p. 69]) by

\[ \| f \|_{M(\varphi)} = \sup_{t > 0} \left( \varphi (t) f^** (t) \right), \]

and, for a positive locally integrable weight \( v \), the Lorentz space \( \Lambda^1 (v) \) is defined by

\[ \| f \|_{\Lambda^1 (v)} = \int_0^\infty f^* (t) v(t) \, dt. \]

**Theorem 2.4.** The space \( B \) coincides with the maximal Lorentz space \( M(\varphi) \) with equivalent norms, where \( \varphi (t) = t / (1 + \log^+ t) \).

**Proof:** Let \( \alpha > 0 \) satisfying

\[ \sup_{r > 0} \frac{\int_r^\infty \lambda_{f/\alpha} (y) \, dy}{1 + \log^+ \frac{1}{r}} \leq 1. \]
Then, by Lemma 2.3, there exists a positive constant $C$ so that
\[
\sup_{s>0} \frac{\int_0^s f'(t) \, dt}{1 + \log^+ \frac{s}{\int_0^s f'(t) \, dt}} \leq C,
\]
and thus, if $\Phi(t) = t/(1 + \log^+(1/t))$, we obtain that
\[
\sup_{s>0} s \Phi\left( \frac{f^{**}(s)}{\alpha} \right) \leq C.
\]
Consequently, for every $s > 0$, $f^{**}(s) \leq \alpha \Phi^{-1}(C/s)$ and, hence
\[
\alpha \geq \sup_{s>0} \frac{f^{**}(s)}{\Phi^{-1}(C/s)}.
\]
From this, the fact that $\Phi^{-1}(t) \approx t(1 + \log(1/t))$ and that this function satisfies the $\Delta_2$ condition, we conclude that
\[
\|f\|_B \geq \sup_{s>0} \frac{s}{1 + \log^+ s} f^{**}(s).
\]
The converse follows similarly. \hfill \Box

Remark 2.5. If $M$ is the Hardy-Littlewood maximal operator,
\[
Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy,
\]
where the supremum is taken over all cubes containing $x$, it is known (see for example [CS2]) that, for every $0 < \alpha < 1$, every function $f$ and $y > 0$,
\[
\frac{1}{y} \int_{\{x:|f(x)|>y\}} |f(x)| \, dx \leq 2\lambda_Mf(y)
\]
\[
\leq \frac{2}{(1-\alpha)y} \int_{\{x:|f(x)|>\alpha y\}} |f(x)| \, dx.
\]
Therefore, taking $\alpha = 1/2$,
\[
\int_r^\infty \lambda_{MF}(y) \, dy \lesssim \int_r^\infty \frac{1}{y} \left( \int_{\{x: |f(x)| > y/2\}} |f(x)| \, dx \right) \, dy
\]
\[
= \int_{\{|f(x)| > \frac{r}{2}\}} |f(x)| \left( \int_r^{2r|f(x)|} \frac{1}{y} \, dy \right) \, dx
\]
\[
= \int_{\{|f(x)| > \frac{r}{2}\}} |f(x)| \log \left( \frac{2|f(x)|}{r} \right) \, dx
\]
\[
= \int_{\mathbb{R}^n} |f(x)| \log^+ \left( \frac{2|f(x)|}{r} \right) \, dx.
\]
Similarly, if we now use the first inequality in (4), we obtain
\[
A := \sup_{r > 0} \frac{\int |f(x)| \log^+ \left( \frac{|f(x)|}{r} \right) \, dx}{1 + \log^+ \frac{r}{7}} \approx \sup_{r > 0} \int_r^\infty \lambda_{MF}(y) \, dy.
\]
Taking $r = 1$, we obtain that $A \geq \int |f(x)| \log^+ (2|f(x)|) \, dx$, and if $|f(x)| \leq 1$, we have, by dominated convergence theorem that
\[
A \geq \lim_{r \to \infty} \frac{\int |f(x)| \log^+ \left( 2r|f(x)| \right) \, dx}{1 + \log^+ (1/r)} \geq \int_{|f(x)| \leq 1} |f(x)| \, dx,
\]
and thus,
\[
\int_{\mathbb{R}^n} |f(x)| \left( 1 + \log^+ 2|f(x)| \right) \, dx \lesssim A.
\]
Since, obviously $A$ satisfies the converse inequality we conclude that
\[
\sup_{r > 0} \int_r^\infty \lambda_{MF}(y) \, dy \approx \int_{\mathbb{R}^n} |f(x)| \left( 1 + \log^+ |f(x)| \right) \, dx,
\]
and, therefore, the range space $B$ is optimal for the Hardy-Littlewood maximal operator in the following sense:

**Proposition 2.6.** If there exists a Banach space $E \subset L^0(\mathbb{R}^n)$, such that for every Yano’s operator $T$ on $L^0(\mathbb{R}^n)$, we have that $T: L \log L(\mathbb{R}^n) \to E$ is bounded, then
\[
\|Mf\|_E \lesssim \|Mf\|_B.
\]
In particular, if $E$ is a rearrangement invariant space, $\|f^*\|_E \lesssim \|f^*\|_B$.

Observe that if we were able to prove that $\|f^*\|_E \lesssim \|f^*\|_B$, we would have obtained the optimality of the range space $B$, in Yano’s theorem, in the setting of rearrangement invariant spaces.
3. Associate space of $B$ and extrapolation results at infinity

Given a Banach space $X$, the associate space $X^*$ is defined as the set of measurable functions $g$ so that
\[ \|g\|_{X^*} = \sup_f \frac{\int_X f(x)g(x) \, d\nu(x)}{\|f\|_X} < \infty. \]

If $X$ is a Banach function space, then by Lorentz-Luxemburg theorem (see [BS, p. 10]), $X = X^{**}$; that is, the associate of $X^*$ is $X$.

Also, if $X$ is a rearrangement invariant space and the measure $\nu$ is resonant, we have that
\[ \|g\|_{X^*} = \int_0^\infty f^*(t)g^*(t) \, dt. \]

In this section, we shall assume that the measure is resonant. In [Z, p. 119], it was proved that if $T$ is a linear operator so that
\[ \|Tf\|_{L^p(\nu)} \leq C\|f\|_{L^p(\mu)} \]
for $p$ big enough, $\mu(M) < \infty$ and $\nu(N) < \infty$, then
\[ T: L^\infty(\mu) \to L(\exp, \nu), \]
where
\[ L(\exp, \nu) = \left\{ f; \exists \lambda > 0, \int_N e^{\lambda|f(x)|} \, d\nu(x) < \infty \right\}. \]

Now, it $T$ satisfies (5) (and we shall say then that $T$ is a Zygmund’s operator), then the adjoint operator $T^*$ satisfies that
\[ \|T^*f\|_{L^{p'}(\nu)} \leq \frac{C}{p'} \|f\|_{L^{p'}(\nu)} \]
for $1 < p' \leq p_0$ and hence, $T^*$ is a Yano’s operator. Therefore,
\[ T^*: L\log L \rightarrow M(\varphi), \]
and we can deduce the following result.

**Theorem 3.1.** If $T$ is a Zygmund operator then
\[ T: (M(\varphi))^* \rightarrow (L \log L)^*. \]

Now, the purpose of this section is to identify the two spaces appearing in Theorem 3.1 and conclude some endpoint estimate at $p = \infty$ for such operators. We emphasize that our measures are $\sigma$-finite and resonant but not necessarily finite.
**Proposition 3.2.** If \( \varphi(t) = \frac{t}{1 + \log^+ t} \), then
\[
(M(\varphi))^* = \Lambda^1 \left( \min(t^{-1}, 1) \right) \cap L^\infty.
\]

**Proof:** We have to compute
\[
\|g\|_{(M(\varphi))^*} = \sup_f \frac{\int_0^\infty f^*(t) g^*(t) \, dt}{\sup_{t > 0} t (1 + \log^+ t) f^{**}(t)}
= \sup_{t > 0} \int_0^\infty f^*(t) g^*(t) \, dt.
\]

Now, the last supremum was identified in [CPSS], where it was proved that
\[
\sup_{t > 0} \int_0^\infty f^*(t) g^*(t) \, dt \approx \sup_{t > 0} g^{**}(t) (1 + \log^+ t) + \int_1^\infty \frac{1}{t} g^*(t) \, dt,
\]
and since, for every \( t > 0 \),
\[
g^{**}(t)(1 + \log^+ t) \lesssim \|g\|_\infty + g^{**}(t) \left( \int_1^t \frac{ds}{s} \right) \lesssim \|g\|_\infty + \left( \int_1^\infty g^{**}(s) \frac{ds}{s} \right)
\]
\[
\lesssim \|g\|_\infty + \int_1^{\infty} g^{**}(s) \frac{ds}{s} \approx \|g\|_\infty + \int_1^{\infty} g^*(s) \frac{ds}{s},
\]
we obtain the result. \( \square \)

**Proposition 3.3.** We have that
\[
(L \log L)^* = M(\Psi),
\]
with \( \Psi(t) = 1/(1 + \log^+(1/t)) \).

**Proof:** In [BS, p. 243] it is proved that if \( \mu(M) = 1 \), then \( L \log L(\mu) = \Lambda^1(\log^+(1/t)) \) with equivalent norms. A slight modification of this result (see also [OP]), shows that, for a general measure space, \( L \log L(\mu) = \Lambda^1(1 + \log^+(1/t)) \). Then, using Theorem 2.12 in [CS1],
\[
\|g\|_{(L \log L)^*} = \sup_f \frac{\int_0^\infty f^*(s) g^*(s) \, ds}{\int_0^\infty f^*(s) (1 + \log^+(1/s)) \, ds}
= \sup_{r > 0} \frac{\int_0^r g^*(s) \, ds}{r (1 + \log^+(1/r))}
\]
\[
= \sup_{r > 0} \frac{g^{**}(r)}{r (1 + \log^+(1/r))}.
\]
\( \square \)
Therefore, we deduce, from Theorem 3.1, the following result:

**Corollary 3.4.** If $T$ is a Zygmund operator, then

$$
\sup_{t>0} \frac{(Tf)^*(t)}{(1 + \log^+ (1/t))} \lesssim \int_0^\infty f^*(t) \min \left( \frac{1}{t}, 1 \right) dt + \|f\|_\infty,
$$

equivalently

$$
\sup_{t>0} \frac{(Tf)^*(t)}{(1 + \log^+ (1/t))} \lesssim \int_0^\infty f^*(t) \min \left( \frac{1}{t}, 1 \right) dt + \|f\|_\infty,
$$

or

$$
\sup_{t>0} \frac{(Tf)^*(t)}{(1 + \log^+ (1/t))} \lesssim \int_1^\infty f^{**}(t) \frac{dt}{t} + \|f\|_\infty.
$$

**Proof:** The proof of the first part is an immediate consequence of Theorem 3.1 and Propositions 3.2 and 3.3. The second inequality follows also easily, since

$$
\sup_{t>0} \frac{(Tf)^*(t)}{(1 + \log^+ (1/t))} \approx \sup_{t>0} \frac{(Tf)^*(t)}{(1 + \log^+ (1/t))},
$$

and the last one can be deduced using that

$$
\int_1^\infty f^{**}(t) \frac{dt}{t} = \int_0^\infty f^*(t) \min \left( \frac{1}{t}, 1 \right) dt. 
$$

□

**Remark 3.5.** i) Observe that if $\mu(M) = \nu(N) = 1$, then, the above inequalities say (see [BS, p. 246]) that $T: L^\infty \rightarrow L(\exp)$ as proved in [Z, p. 119].

ii) Finally, let us just comment that obvious changes show that similar results can be obtained for sublinear operators satisfying

$$
\|Tf\|_p \leq C(p - 1)^{-\alpha} \|f\|_p,
$$

for $\alpha > 0$ and $p$ near 1, and, for linear operators such that

$$
\|Tf\|_p \leq C p^\alpha \|f\|_p,
$$

where again $\alpha > 0$ and $p$ near $\infty$.

**References**


