VANISHING RESULTS FOR THE COHOMOLOGY OF
COMPLEX TORIC HYPERPLANE COMPLEMENTS

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Abstract: Suppose \( \mathcal{R} \) is the complement of an essential arrangement of toric hyperplanes in the complex torus \((\mathbb{C}^*)^n\) and \( \pi = \pi_1(\mathcal{R}) \). We show that \( H^\ast(\mathcal{R}; A) \) vanishes except in the top degree \( n \) when \( A \) is one of the following systems of local coefficients: (a) a system of nonresonant coefficients in a complex line bundle, (b) the von Neumann algebra \( \mathcal{N}_\pi \), or (c) the group ring \( \mathbb{Z}_\pi \). In case (a) the dimension of \( H^n \) is \( |e(\mathcal{R})| \) where \( e(\mathcal{R}) \) denotes the Euler characteristic, and in case (b) the \( n^{th} \) \( \ell^2 \) Betti number is also \( |e(\mathcal{R})| \).

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1. Introduction

A complex toric arrangement is a family of complex subtori of a complex torus \((\mathbb{C}^*)^n\). The study of such objects is a relatively recent topic. Different versions of these arrangements, also known as toral arrangements, have been introduced and studied in works of Lehrer \([16],[17]\), Dimca-Lehrer \([11]\), Douglass \([12]\), Looijenga \([18]\), and Macmeikan \([20],[21]\).

The foundation of the topic can be traced to the paper \([10]\) by De Concini and Procesi. There the main objects are defined and the cohomology of the complement of a toric arrangement is studied. An explicit goal of \([10]\) is to generalize the theory of hyperplane arrangements. (For an extensive account of the work of De Concini and Procesi see \([9]\).)

The next step is the work of Moci, in particular his papers \([22],[23]\), and \([24]\), developing the theory with a special focus on combinatorics. In \([25]\) Moci and the second author study the homotopy type of the

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complement of a special class of toric arrangements which they call \textit{thick}. In [3] d’Antonio and Delucchi generalize results in [25] to a wider class of toric arrangements which they call \textit{complexified} because of structural affinity with the case of hyperplane arrangements. They also prove that complements of complexified toric arrangements are minimal (see [4]).

In this paper we generalize to toric arrangements a well known result for affine arrangements: vanishing conditions for the cohomology of the complement $M(A)$ of an arrangement $A$ with coefficients in a complex local system $A$. Necessary conditions for $H^k(M(A); A) = 0$ if $k \neq n$, i.e., for the cohomology to be concentrated in top dimension, have been determined by a number of authors, including Kohno [15], Esnault, Schechtman and Viehweg [13], Davis, Januszkiewicz, and Leary [5], Schechtman, Terao and Varchenko [28], and Cohen and Orlik [2]. In particular, in [28] (see also [2]) it is proved that the cohomology of the complement $M(A)$ of an arrangement with coefficients in a complex local system is concentrated in top dimension provided certain \textit{nonresonance} conditions for monodromies are fulfilled for a certain subset of edges (i.e., intersections of hyperplanes) that are called \textit{denses}.

In order to generalize the above results we use techniques developed by the first author in a joint work with Januszkiewicz, Leary, and Okun, [5], [6], [7], [8]. One considers an open cover of the complement $M$ by “small” open sets each homeomorphic to the complement of a central arrangement. In the cases of nonresonant rank one local coefficients or $\ell^2$ coefficients, the $E_1$ page of the resulting Mayer-Vietoris spectral sequence is nonzero only along the bottom row, where it can be identified with the simplicial cochains with constant coefficients on a pair $(N(U), N(U_{\text{sing}}))$, which is homotopy equivalent to $(\mathbb{C}^n, \Sigma)$ where $\Sigma$ is the union of all hyperplanes in the arrangement. (The simplicial complex $N(U)$ is the nerve of an open cover of $\mathbb{C}^n$ and $N(U_{\text{sing}})$ is a subcomplex.)

It follows that the $E_2$ page can be nonzero only in position $(l, 0)$. One also can prove that for an affine hyperplane arrangement of rank $l$ only the $l$th $\ell^2$-Betti number of the complement $M$ can be nonzero and that it is equal to the rank of the reduced $(l-1)$-homology of $\Sigma$ (cf. [5]). Similarly, with coefficients in the group ring, $\mathbb{Z}\pi$, for $\pi = \pi_1(M)$, $H^*(M; \mathbb{Z}\pi)$ is nonzero only in degree $l$ (cf. [6]). We generalize all three of these vanishing results to the toric case in Theorems 5.1, 5.2 and 5.3.

In recent work [27], Papadima and Suciu generalize the result in [2] to arbitrary minimal CW-complex, i.e., a complex having as many $k$-cells as the $k$-th Betti number. It would be very interesting to decide if the
complement of toric arrangement also could be minimal. In this case Theorem 5.1 would be a consequence of minimality.

Our paper begins with a review of some background about toric and affine arrangements. Then, in Section 3, we give a brief account of open covers by “small” convex sets. In Section 4 we recall basic definitions on systems of local coefficients. Finally in Section 5 we prove that the cohomology of the complement of a toric arrangement with coefficient in a local system $A$ vanishes except in the top degree when $A$ is a nonresonant local system, the von Neumann algebra $\mathcal{N}\pi$ or the group ring $\mathbb{Z}\pi$.

2. Affine and toric hyperplane arrangements

**Affine hyperplanes arrangements.** A hyperplane arrangement $\mathcal{A}$ is a finite collection of affine hyperplanes in $\mathbb{C}^n$. A subspace of $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$. Denote by $L(\mathcal{A})$ the poset of subspaces, partially ordered by inclusion, and let $\overline{L}(\mathcal{A}) := L(\mathcal{A}) \cup \{\mathbb{C}^n\}$. An arrangement is central if $L(\mathcal{A})$ has a minimum element. Given $G \in L(\mathcal{A})$, its rank, $\text{rk}(G)$, is the codimension of $G$ in $\mathbb{C}^n$. The minimal elements of $L(\mathcal{A})$ form a family of parallel subspaces and they all have the same rank. The rank of an arrangement $\mathcal{A}$ is the rank of a minimal element in $L(\mathcal{A})$. $\mathcal{A}$ is essential if $\text{rk}(\mathcal{A}) = n$.

The singular set $\Sigma(\mathcal{A})$ of $\mathcal{A}$ is the union of hyperplanes in $\mathcal{A}$. The complement of $\Sigma(\mathcal{A})$ in $\mathbb{C}^n$ is denoted $M(\mathcal{A})$.

**Toric arrangements.** Let $T = (\mathbb{C}^*)^n$ be a complex torus and let $\Lambda = \text{Hom}(T, \mathbb{C}^*)$ denote the group of characters of $T$. Then $\Lambda \cong \mathbb{Z}^n$. A character is primitive if it is a primitive vector in $\Lambda$. Given a primitive character $\chi$ and an element $a \in \mathbb{C}^*$ put

$$H_{\chi,a} = \{t \in T \mid \chi(t) = a\}.$$  

The subtorus $H_{\chi,a}$ is a toric hyperplane. A finite subset $X \subset \Lambda \times \mathbb{C}^*$ defines a toric arrangement,

$$\mathcal{T}_X := \{H_{\chi,a}\}_{(\chi,a) \in X}.$$  

The projection of $X$ onto the first factor is denoted $p(X)$ and is called the character set of $\mathcal{T}_X$. (Thus, $p(X) := \{\chi \mid (\chi, a) \in X\}$.) The singular set, $\Sigma_X$, is the union of toric hyperplanes in the arrangement. Its complement, $\mathcal{T} - \Sigma_X$, is denoted $\mathcal{R}_X$. The intersection poset $L_X$ is the set of nonempty intersections of toric hyperplanes and $\overline{L}_X = L_X \cup \{T\}$. $\overline{L}_X$ is partially ordered by inclusion. The rank of the arrangement is the dimension of the linear subspace of $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $p(X)$. The arrangement is essential if its rank is $n$. 
Suppose $G \in L_X$. Choose a point $x \in G$. The tangential arrangement along $G$ is the arrangement $\mathcal{A}_G$ of linear hyperplanes which are tangent to the complex toric hyperplanes containing $G$ (i.e., all hyperplanes of the form $T_x(H_{\chi,a})$ where $T_x(G) \subset T_x(H_{\chi,a})$). It is a central hyperplane arrangement of rank equal to $n - \dim G$.

Given a toric arrangement $\mathcal{T}_X$ of rank $l$, let $K_X$ denote the identity component of the intersection of all kernels in $p(X)$, i.e., $K_X$ is the identity component of

$$\bigcap_{\chi \in p(X)} \ker \chi = \{ t \in T \mid \chi(t) = 1, \forall \chi \in p(X) \}.$$ 

Put $T_X := T/K_X$. Thus, $K_X$ and $T_X$ are tori of dimensions $n - l$ and $l$, respectively. ($K_X \cong (\mathbb{C}^*)^{n-l}$ and $T_X \cong (\mathbb{C}^*)^l$.) Let $\Sigma_X$ denote the image of $\Sigma_X$ in $\overline{T}_X$. Since $T \rightarrow T/K_X$ is a trivial $K_X$-bundle, we have a homeomorphism of pairs,

$$(T, \Sigma) \cong K_X \times (\overline{T}_X, \overline{\Sigma}_X).$$

In other words, the arrangement in $T$ is just the product of the arrangement in $\overline{T}_X$ with the torus $K_X$. We call $\overline{T}_X$ the essentialization of $\mathcal{T}_X$.

So, it is not restrictive to consider essential toric arrangements.

**Lemma 2.1** (cf. [5, Proposition 2.1]). Suppose $\mathcal{T}_X$ is an essential toric arrangement on $T$ and $\Sigma = \Sigma_X$. Then $H_*(T, \Sigma)$ is free abelian and concentrated in degree $n$.

**Proof:** We follow the “deletion-restriction” argument in [5, Proposition 2.1] using induction on $\text{Card}(\mathcal{T}_X)$. Choose a toric hyperplane $H \in \mathcal{T}_X$. Let $\mathcal{T}' = \mathcal{T}_X - \{H\}$ and let $\mathcal{T}''$ be the restriction of $\mathcal{T}_X$ to $H$, i.e., $\mathcal{T}'' = \{ H \cap H' \mid H' \in \mathcal{T}_X \}$. Let $\Sigma'$ and $\Sigma''$ denote the singular sets of $\mathcal{T}'$ and $\mathcal{T}''$, respectively. Consider the exact sequence of the triple $(T, \Sigma, \Sigma')$,

$$(2) \rightarrow H_*(T', \Sigma') \rightarrow H_*(T, \Sigma) \rightarrow H_{*-1}(\Sigma, \Sigma') \rightarrow$$

There is an excision, $H_{*-1}(\Sigma, \Sigma') \cong H_{*-1}(H, \Sigma'')$. The rank of $\mathcal{T}'$ is either $n$ or $n - 1$, while the rank of $\mathcal{T}''$ is always $n - 1$. The argument breaks into two cases depending on the rank of $\mathcal{T}'$.

**Case 1:** the rank of $\mathcal{T}'$ is $n$. By induction, $H_*(T, \Sigma')$ and $H_*(H, H \cap \Sigma)$ are free abelian and concentrated in degrees $n$ and $n - 1$, respectively. So, (2) becomes

$$0 \rightarrow H_n(T, \Sigma') \rightarrow H_n(T, \Sigma) \rightarrow H_{n-1}(H, H \cap \Sigma') \rightarrow 0$$
and all other terms are 0. Therefore, $H_*(T, \Sigma)$ is concentrated in degree $n$ and $H_n(T, \Sigma)$ is free abelian.

Case 2: the rank of $T'$ is $n - 1$. Then the projection $T \to \bar{T}$ takes $H$ isomorphically onto $\bar{T}$ and the arrangement $T''$ on $H$ maps isomorphically to the arrangement $\bar{T}_X$ on $\bar{T}$. So, $(H, H \cap \Sigma') \cong (\bar{T}, \bar{\Sigma})$. By (1), $(T, \Sigma') \cong K_X \times (H, H \cap \Sigma') \cong \mathbb{C}^* \times (H, H \cap \Sigma')$. By the Künneth Formula, $H_*(T, \Sigma') \cong H_*(\mathbb{C}^*) \otimes H_*(H, H \cap \Sigma')$. So,

$$H_{n-1}(T, \Sigma') \cong H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') \quad \text{and}$$

$$H_n(T, \Sigma') \cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma');$$

moreover, the first isomorphism is induced by the inclusion $(H, H \cap \Sigma') \to (T, \Sigma')$. So, (2) becomes

$$0 \to H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') \to H_n(T, \Sigma) \to H_{n-1}(H, H \cap \Sigma')$$

$$\to H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma')$$

where the last map is an isomorphism. It follows that $H_{n-1}(T, \Sigma) = 0$ and that $H_n(T, \Sigma) \cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma')$, which, by inductive hypothesis, is free abelian. This proves the lemma.

Complexified toric arrangements. In [3] d’Antonio-Delucchi consider the case of “complexified toric arrangements.” This means that for each $(\chi, a) \in X$, the complex number $a$ has modulus 1 (where $X \subset \Lambda \times \mathbb{C}^*$ is a set defining a toric arrangement $\mathcal{T}_X$). Let $T^{\text{cpt}} = (S^1)^n \subset \mathbb{C}^n$ be the compact torus. Then for each $H \in \mathcal{T}_X$, $H \cap T^{\text{cpt}}$ is a compact subtorus of $T^{\text{cpt}}$. The set of subtori, $\mathcal{T}_X^{\text{cpt}} := \{ H \cap S \mid H \in \mathcal{T} \}$, is called the associated compact arrangement.

Let $\Sigma^{\text{cpt}} := \Sigma_X \cap T^{\text{cpt}}$. We note that $(T, \Sigma_X)$ deformation retracts onto $(T^{\text{cpt}}, \Sigma^{\text{cpt}})$. Here are a few observations.

(i) The universal cover of $T^{\text{cpt}}$ is $\mathbb{R}^n$ (actually the subspace $i\mathbb{R}^n \subset \mathbb{C}^n$). Let $\pi: \mathbb{R}^n \to T^{\text{cpt}}$ be the covering projection. Then for each $H^{\text{cpt}} \in \mathcal{T}_X^{\text{cpt}}$, each component of $\pi^{-1}(H^{\text{cpt}})$ is an affine hyperplane and the collection of these hyperplanes is a periodic affine hyperplane arrangement in $\mathbb{R}^n$.

(ii) If $\mathcal{T}_X$ is essential, then $\Sigma^{\text{cpt}}$ cuts $T^{\text{cpt}}$ into a disjoint union of convex polytopes, called chambers (see [25]). The inverse images of these polytopes under $\pi$ give a tiling of $\mathbb{R}^n$. 
(iii) When $T_X$ is essential, it follows from (ii) that for $n \geq 2$, $\Sigma^{cpt}$ is connected and that for $n \geq 3$, $\pi_1(\Sigma^{cpt}) = \pi_1(T^{cpt})$.

(iv) It is easy to prove Lemma 2.1 in the case of a compact arrangement. We have an excision $H_*(T^{cpt}, \Sigma^{cpt}) \cong H_*(\bigsqcup (P_i, \partial P_i))$ where each chamber $P_i$ is an $n$-dimensional convex polytope. Hence, $H_*(T^{cpt}, \Sigma^{cpt})$ is concentrated in degree $n$ and is free abelian. Moreover, the rank of $H_*(T^{cpt}, \Sigma^{cpt})$ is the number of chambers.

(v) Let $\tilde{\Sigma}^{cpt}$ denote the inverse image of $\Sigma^{cpt}$ in $\mathbb{R}^n$ and let $\tilde{\Sigma}_X$ be the induced cover of $\Sigma_X$. Suppose $T_X$ is essential. Then $\tilde{\Sigma}^{cpt}$ cuts $\mathbb{R}^n$ into compact chambers. It follows that $\tilde{\Sigma}^{cpt}$ (and hence, $\tilde{\Sigma}$) is homotopy equivalent to a wedge of $(n-1)$-spheres.

3. Certain covers and their nerves

Equip the torus $T = (\mathbb{C}^*)^n$ with an invariant metric. This lifts to a Euclidean metric on $\mathbb{C}^n$ induced from an inner product. Hence, geodesics in $T$ lift to straight lines in $\mathbb{C}^n$ and each component of the inverse image of a subtorus of $T$ is an affine subspace of $\mathbb{C}^n$. A convex subset of $T$ means a geodesically convex subset. Thus, each component of the inverse image of a convex subset of $T$ is a convex subset of $\mathbb{C}^n$.

The intersection of an open convex subset of $T$ with the toric hyperplanes in $T_X$ is equivalent to an affine arrangement. An open convex subset $U \subset T$ is small (with respect to $T_X$) if this affine arrangement is central. In other words, $U$ is small if the following two conditions hold (cf. [5], [6]):

(i) $\{G \in \bar{L}(T_X) \mid G \cap U \neq \emptyset\}$ has a unique minimum element, $\text{Min}(U)$.

(ii) A toric hyperplane $H \in T_X$ has nonempty intersection with $U$ if and only if $\text{Min}(U) \subset H$.

If (i) and (ii) hold, then the arrangement in $U$ is equivalent to the tangential arrangement along $\text{Min}(U)$, which we denote by $A_{\text{Min}(U)}$. The intersection of two small convex open sets is also a small convex set; hence, the same is true for any finite intersection of such sets.

Let $U = \{U_i\}_{i \in I}$ be an open cover of $T$ by small convex sets, put $U_{\text{sing}} := \{U \in U \mid U \cap \Sigma_X \neq \emptyset\}$.

Given a nonempty subset $\sigma \subset I$, put $U_\sigma := \bigcap_{i \in \sigma} U_i$. The nerve $N(U)$ of $U$ is the simplicial complex defined as follows. Its vertex set is $I$ and a finite, nonempty subset $\sigma \subset I$ spans a simplex of $N(U)$ if and only if $U_\sigma$ is nonempty. We have the following lemma.
Lemma 3.1. Suppose $T_X$ is essential. $N(U)$ is homotopy equivalent to $T$ and $N(U_{\text{sing}})$ is a subcomplex homotopy equivalent to $\Sigma_X$. Moreover, $H_*(N(U), N(U_{\text{sing}}))$ is concentrated in degree $n$ and $H_n(N(U), N(U_{\text{sing}}))$ is free abelian.

Proof: $U_{\text{sing}}$ is an open cover of a neighborhood of $\Sigma_X$ which deformation retracts onto $\Sigma_X$. For each simplex $\sigma$ of $N(U)$, $U_{\sigma}$ is contractible (in fact, it is a small convex open set). By a well-known result (see [14, Corollary 4G.3 and Example 4G(4)]) $N(U)$ is homotopy equivalent to $T$ and $N(U_{\text{sing}})$ is homotopy equivalent to $\Sigma_X$. The last sentence of the lemma follows from Lemma 2.1. □

Definition 3.2. $\beta(T_X)$ is the rank of $H_n(N(U), N(U_{\text{sing}}))$. Equivalently, $\beta(T_X)$ is the rank of $H_*(T, \Sigma_X)$. It is not difficult to see that, for essential arrangements, $(-1)^n \beta(T_X) = e(T, \Sigma_X) = -e(\Sigma_X) = e(R_X)$, where $e(\cdot)$ denotes Euler characteristic.

4. Local coefficients

Generic and nonresonant coefficients. Consider an affine arrangement $A$. The fundamental group $\pi$ of its complement, $M(A)$, is generated by loops $a_H$ for $H \in A$, where the loop $a_H$ goes once around the hyperplane $H$ in the “positive” direction. Let $\alpha_H$ denote the image of $a_H$ in $H_1(M(A))$. Then $H_1(M(A))$ is free abelian with basis $\{\alpha_H\}_{H \in A}$. So, a homomorphism $H_1(M(A)) \rightarrow \mathbb{C}^*$ is determined by an $A$-tuple $\Lambda \in (\mathbb{C}^*)^A$, where $\Lambda = (\lambda_H)_{H \in A}$ corresponds to the homomorphism sending $\alpha_H$ to $\lambda_H$. Let $\psi_\Lambda: \pi \rightarrow \mathbb{C}^*$ be the composition of this homomorphism with the abelianization map $\pi \rightarrow H_1(M(A))$. The resulting rank one local coefficient system on $M(A)$ is denoted $A_\Lambda$.

Returning to the case where $T_X$ is a toric arrangement, for each simplex $\sigma$ in $N(\hat{U})$, let $A_\sigma := A_{\text{Min}(U_{\sigma})}$ be the corresponding central arrangement (so that $\hat{U}_\sigma \cong M(A_\sigma)$). Given $\Lambda_\sigma \in (\mathbb{C}^*)^{A_\sigma}$, put

$$\lambda_\sigma := \prod_{H \in A_\sigma} \lambda_H.$$  

Let $A_{\Lambda_T} \in \text{Hom}(H_1(R_X), \mathbb{C}^*)$ be a local coefficient system on $R_X$. The localization of $A_{\Lambda_T}$ on the open set $\hat{U}_\sigma$ has the form $A_{\Lambda_\sigma}$, where $\Lambda_\sigma$ is a $A_\sigma$-tuple in $\mathbb{C}^*$. We call $\Lambda_T$ generic if $\lambda_\sigma \neq 1$ for all $\sigma \in N(U_{\text{sing}})$. We call $\Lambda_T$ nonresonant if $\Lambda_\sigma$ is nonresonant in the sense of [2] for all $\sigma \in N(U_{\text{sing}})$ i.e., if the Betti numbers of $M(A_\sigma)$ with coefficients in $A_{\Lambda_\sigma}$ are minimal.
\(\ell^2\)-cohomology and coefficients in a group von Neumann algebra. For a discrete group \(\pi\), \(\ell^2\pi\) denotes the Hilbert space of complex-valued, square integrable functions on \(\pi\). There are unitary \(\pi\)-actions on \(\ell^2\pi\) by either left or right multiplication; hence, \(\mathbb{C}\pi\) acts either from the left or right as an algebra of operators. The associated von Neumann algebra \(N\pi\) is the commutant of \(\mathbb{C}\pi\) (acting from, say, the right on \(\ell^2\pi\)).

Given a finite CW complex \(Y\) with fundamental group \(\pi\), the space of \(\ell^2\)-cochains on the universal cover \(\tilde{Y}\) is equal to \(C^*(Y; \ell^2\pi)\), the cochains with local coefficients in \(\ell^2\pi\). The image of the coboundary map need not be closed; hence, \(H^*\big(Y; \ell^2\pi\big)\) need not be a Hilbert space. To remedy this, one defines the reduced \(\ell^2\)-cohomology \(H^\text{red}_*\big(Y; \ell^2\pi\big)\) to be the quotient of the space of cocycles by the closure of the space of coboundaries. The von Neumann algebra admits a trace. Using this, one can attach a “dimension,” \(\dim_{N\pi} V\), to any closed, \(\pi\)-stable subspace \(V\) of a finite direct sum of copies of \(\ell^2\pi\) (it is the trace of orthogonal projection onto \(V\)). The nonnegative real number \(\dim_{N\pi} H^p\big(Y; \ell^2\pi\big)\) is the \(p\)th \(\ell^2\)-Betti number of \(Y\).

A technical advance of Lück [19, Chapter 6] is the use local coefficients in \(N\pi\) in place of the previous version of \(\ell^2\)-cohomology. He shows there is a well-defined dimension function on \(N\pi\)-modules, \(A \rightarrow \dim_{N\pi} A\), which gives the same answer for \(\ell^2\)-Betti numbers, i.e., for each \(p\) one has that \(\dim_{N\pi} H^p\big(Y; N\pi\big) = \dim_{N\pi} H^p\big(Y; \ell^2\pi\big)\).

**Group ring coefficients.** Let \(Y\) be a connected CW complex, \(\pi = \pi_1(Y)\) and \(r: \tilde{Y} \rightarrow Y\) the universal cover. There is a well-defined action of \(\pi\) on \(\tilde{Y}\) and hence, on the cellular chain complex of \(\tilde{Y}\). Given the left \(\pi\)-module \(\mathbb{Z}\pi\), define the cochain complex with group ring coefficients

\[
C^*(Y; \mathbb{Z}\pi) := \text{Hom}_\pi(C_*(\tilde{Y}), \mathbb{Z}\pi).
\]

Taking cohomology gives \(H^*(Y; \mathbb{Z}\pi)\).

**5. The Mayer-Vietoris spectral sequence**

**Statements of the main theorems.** Suppose \(\mathcal{T}_X\) is an essential toric arrangement in \(T\) and \(\pi = \pi_1(\mathcal{R}_X)\).

**Theorem 5.1.** Let \(\Lambda_T\) be a generic \(X\)-tuple with entries in \(k^*\). Then \(H^*(\mathcal{R}_X; A_{\Lambda_T})\) is concentrated in degree \(n\) and

\[
\dim_k H^n\big(\mathcal{R}_X; A_{\Lambda_T}\big) = \beta(\mathcal{T}_X).
\]
Theorem 5.2. (cf. [7]). The $\ell^2$-Betti numbers of $\mathcal{R}_X$ are 0 except in degree $n$ and $\ell^2 b_n(\mathcal{R}_X) = \beta(T_X)$.

Theorem 5.3 (cf. [6], [8]). $H^*(\mathcal{R}_X; \mathbb{Z}_\pi)$ vanishes except in degree $n$ and $H^n(\mathcal{R}_X; \mathbb{Z}_\pi)$ is free abelian.

Remark 5.4. Suppose $W$ is a Euclidean reflection group acting on $\mathbb{R}^n$ and that $\mathbb{Z}^n \subset W$ is the subgroup of translations. The quotient $W' := W/\mathbb{Z}^n$ is a finite Coxeter group. The reflection group $W$ acts on the complexification $\mathbb{C}^n$ and $W'$ acts on the torus $T = \mathbb{C}^n/\mathbb{Z}^n$. The image of the affine reflection arrangement in $\mathbb{C}^n$ gives a toric arrangement $T_X$. The quotient of the compact torus by $W'$ can be identified with the fundamental simplex $\Delta$ of $W$ on $\mathbb{R}^n$. (If $W$ is irreducible, then $\Delta$ is a simplex.) It follows that $\beta(T_X)$ is the order of $W'$ (i.e., the index of $\mathbb{Z}^n$ in $W$). So, in this case Theorem 5.2 is a special case of the main result of [7] and Theorem 5.3 is a special case of a result of [8, Theorem 4.1].

Lemma 5.5. Suppose $\mathcal{A}$ is a finite, central arrangement of affine hyperplanes. Let $\pi' = \pi_1(M(\mathcal{A}))$. Then

(i) (Cf. [28], [2], [5].) For any generic system of local coefficients $A$, $H^*(M(\mathcal{A}); A)$ vanishes in all degrees.

(ii) (Cf. [5].) $H^*(M(\mathcal{A}); N\pi')$ vanishes in all degrees. Hence, all $\ell^2$-Betti numbers are 0.

(iii) (Cf. [6].) If the rank of $\mathcal{A}$ is $l$, then $H^*(M(\mathcal{A}); \mathbb{Z}\pi')$ vanishes except in the top degree, $l$.

Proofs using the Mayer-Vietoris spectral sequence. The proofs of these three theorems closely follow the argument in [7], [5] and particularly, in [6]. For $\pi = \pi_1(\mathcal{R}_X)$, let $A$ denote one of the left $\pi$-modules in Section 4.

Let $\mathcal{U} = \{U_i\}$ be an open cover of $T$ by small convex sets. We may suppose that $\mathcal{U}$ is finite and that it is closed under taking intersections. For each $G \in L_X$, put

$$\mathcal{U}_G := \{U \in \mathcal{U} \mid \text{Min}(U) \leq G\},$$

$$\mathcal{U}_G^{\text{sing}} := \{U \in \mathcal{U} \mid \text{Min}(U) < G\} = \{U \in \mathcal{U}_G \mid U \cap \Sigma_X \cap G \neq \emptyset\}.$$

The open cover $\mathcal{U}$ restricts to an open cover $\widehat{\mathcal{U}} = \{U - \Sigma_X\}_{U \in \mathcal{U}}$ of $\mathcal{R}_X$. Any element $\widehat{U} = U - \Sigma_X$ of the cover is homotopy equivalent to the complement of a central arrangement $M(\mathcal{A}_{\text{Min}(U)})$. 
Suppose $N(U)$ is the nerve of $U$ and $N(U_G)$ is the subcomplex defined by $U_G$. Since $N(U_G)$ and $N(U_G^{\text{sing}})$ are nerves of covers of $G$ and $\Sigma_X \cap G$, respectively, by contractible open subsets, we have that for each $G \in \mathcal{L}(A)$,

$$(3) \quad H^* (N(U_G), N(U_G^{\text{sing}})) = H^* (G, \Sigma(\mathcal{T} X \cap G)).$$

For each $k$-simplex $\sigma = \{i_0, \ldots, i_k\}$ in $N(U)$, let $U_\sigma := U_{i_0} \cap \cdots \cap U_{i_k}$ denote the corresponding intersection.

Let $r : \tilde{\mathcal{R}}_X \to \mathcal{R}_X$ be the universal cover. The induced open cover $\{r^{-1}(\hat{U})\}$ of $\tilde{\mathcal{R}}_X$ has the same nerve $N(\hat{U}) (= N(U))$. We have the Mayer-Vietoris double complex,

$$C_{i,j} := \bigoplus_{\sigma \in N^{(i)}} C_j (r^{-1}(\hat{U}_\sigma)),$$

where $N^{(i)}$ denotes the set of $i$-simplices in $N(U)$ (cf. [1, Chapter VII].) We get a corresponding double cochain complex,

$$(4) \quad E_{i,j}^0 := \text{Hom}_\pi (C_{i,j}, A),$$

where $\pi = \pi_1(\mathcal{R}_X)$. The filtration on the double complex gives a spectral sequence converging to the associated graded module for cohomology:

$$\text{Gr} H^m (\mathcal{R}_X; A) = E_\infty := \bigoplus_{i+j=m} E_{i,j}^\infty.$$

By first using the horizontal differential, there is a spectral sequence with $E_1$ page

$$E_{i,j}^1 = C^i (N(U); \mathcal{H}_j^1 (A))$$

where $\mathcal{H}_j^1 (A)$ is the coefficient system on $N(U)$ defined by

$$\sigma \mapsto H^j (\hat{U}_\sigma; A),$$

where $\hat{U}_\sigma \cong M(\mathcal{A}_{\text{Min}(U_\sigma)})$. For $A = A_T$ or $A = N\pi$ these coefficients are 0 for $G \neq T$. For $A = \mathbb{Z}\pi$, they are 0 for $j \neq \dim(G)$. Hence, in all cases, for any coface $\sigma'$ of $\sigma$, if $G' := \text{Min}(U_{\sigma'}) < G$, the coefficient homomorphism $H^j (M(\mathcal{A}_G); A) \to H^j (M(\mathcal{A}_{G'}); A)$ is the zero map.
Moreover, the $E_1$ page of the spectral sequence decomposes as a direct sum (cf. [8, Lemma 2.2]). In fact, for a fixed $j$, by using Lemma 5.5, we see that the $E^{i,j}_1$ term decomposes as

$$E^{i,j}_1 = \bigoplus_{G \in \mathcal{T}_X^{n-j}} C^i(N(U_G), N(U_G^{\text{sing}}; H^j(M(A_G); A)),$$

where we have constant coefficients in each summand. Hence, at $E_2$ we have

$$E^{i,j}_2 = \bigoplus_{G \in \mathcal{T}_X^{n-j}} H^i(N(U_G), N(U_G^{\text{sing}}; H^j(M(A_G); A))$$

(5)

$$= \bigoplus_{G \in \mathcal{T}_X^{n-j}} H^i(G, \Sigma_{\mathcal{X}} \cap G; H^j(M(A_G); A)),$$

where the second equation follows from (3).

When $A = A_{\Lambda_T}$ or $A = N\pi$, all summands vanish for $G \neq T$ and $j \neq 0$. So, we are left with $E^{n,0}_2 = H^n(T, \Sigma_X; A)$, which is isomorphic to the tensor product free abelian group of rank $\beta(T_X)$ with $A$. It follows that $H^*(\mathcal{R}_X; A)$ is concentrated in degree $n$ and that $\dim_{\mathbb{C}} H^n(\mathcal{R}_X; A_{\Lambda_T}) = \beta(T_X) = \dim_{N\pi} H^n(\mathcal{R}_X; N\pi)$. This proves Theorems 5.1 and 5.2.

Consider formula (5) for $A = \mathbb{Z}\pi$. By Lemma 2.1, $H^i(G, \Sigma_X \cap G)$ is concentrated in degree $\dim G = n - j$. Hence, $E^{i,j}_2$ is nonzero (and free abelian) only for $i + j = n$. It follows that the spectral sequence degenerates at $E_2$, i.e., $E_2 = E_\infty$. This proves Theorem 5.3.

Remark 5.6. Let us remark that the statement of Theorem 5.1 holds even if the local system $\Lambda_T$ is nonresonant or if it verifies the Schechtman, Terao and Varchenko nonresonance conditions in all small open convex sets, i.e. $\Lambda_\sigma$ verifies the nonresonance conditions in [28] for all $\sigma \in N(U_{\text{sing}})$. Indeed under these conditions Lemma 5.5 holds.

References


