Given a closed connected smooth $n$-dimensional manifold $M$, it is interesting to compare its span (i.e. the maximum number of linearly independent tangent vector fields) to its stable span (i.e. the maximum number $k$ such that $TM \oplus \mathbb{R}$ allows $k + 1$ linearly independent sections). The example of the $s$-sphere $S^n$ shows that these two numbers can differ dramatically: stable span $(S^n) = n$ while span $S^n$ is the Hurwitz-Radon number $h(n)$, e.g. $h(n) = 0$ for $n$ even, $h(n) = 1$ for $n \equiv 1(4)$, $h(n) = 3$ for $n \equiv 3(8)$ ... It has been known since long that for general $M^n$ often span$(M)$ equals either the stable span of $M$ or else the span $h(n)$ of the sphere $S^n$ of the same dimension (see e.g. [11] for the case when $M$ is stably parallelisable and [21] for many other cases), so it was widely believed that this should always hold. In our talk we disprove this conjecture. First we define an integer $s(M)$ (\gg \text{span}(S^n)) which can sometimes be calculated, e.g. $s(M) = 0$ iff $n$ is even; $s(M) = 1$ iff $n \equiv 1(4)$ and $w_1(M)^2 = 0$; $s(M) = 2$ iff $n \equiv 1(4)$, $w_1(M)^2 \neq 0$ but $w_1(M)^2 = yw_1(M) + y^2$ for some $y \in H^1(M; \mathbb{Z}_2)$; etc. Then we use the singularity approach to vectorfield problems (see [31]) to show that most of the time $s(M)$ is the correct alternative: the span of $M$ is equal either to $s(M)$ or to the stable span of $M$. As an example we exhibit an infinite family of manifolds of the form $M^n = P^r \times S^d$ such that span$(M)$ differs from stable span $(M)$. 
from \( \text{span} \ (S^n) \) and also from the span of the factor sphere \( S^d \), but \( \text{span}(M) = s(M) = 4 \). We give many other counterexamples to the conjecture mentioned above, including oriented ones.

(More details are given in the last section of [3]).

References

