Let $G(p)$ be the space of all equivariant automorphisms of a principal $G$-bundle $p : E \to B$, topologized as a subspace of $M(E,E)$, the space of maps from $E$ to itself. Composition of automorphisms gives $G(p)$ a group structure and indeed, $G(p)$ is a topological group. The topological group $G(p)$ has been used quite frequently in connection with certain problems of Theoretical Physics; for example, it appears in the Feynman approach to Quantum Mechanics as the group of all gauge transformations of a smooth principal $G$-bundle $p$, with $G$ a Lie group. In these problems, it is necessary on several occasions to know more about the space $G(p)$ or about certain of its homotopy groups (see [8]). Clearly, if $p$ is a trivial $G$-bundle over a space $B$, then $G(p)$ is homeomorphic to the space $M(B,G)$. In general, if $f \in G(p)$ and $x \in E$, because $G$ acts effectively and transitively on fibres there is a unique $g \in G$ such that $f(x) = gx$. This gives rise to a homeomorphism $\theta$ from $G(p)$ to the space $M_G(E,G)$ of all maps $\varphi$ from $E$ to $G$, such that $\varphi(gx) = g\varphi(x)g^{-1}$ for all $g \in G$ and all $x \in E$. In practice, a difficult space to deal with. Note that if $G$ is abelian, $\theta : G(p) \to M(B,G)$ [6]. This is a better result but, of course, it is too limited. A more general result was obtained by D.H. Gottlieb in 1972 [5]; if $BG$ is the classifying space for $G$, $k : B \to BG$ is the classifying map for the principal $G$-bundle $p : E \to B$ and $M(B,BG,k)$ is the path-component of $M(B,BG)$ containing $k$, then
Proposition 1—$G(p) = \Omega M(B, BG; k)$ ($w = \text{weak homotopy equivalence}$).

As for other types of fibrations, probably the first result along the lines of Proposition 1 was also obtained by Gottlieb [4]. To describe it, we must recall the following classification theorem due to A. Dold: let $E^F(B)$ be the set of all fibre-homotopy equivalence classes of Hurewicz fibrations over a path-connected CW-complex $B$ and with fibres of the homotopy type of a fixed space $F$; then, there is a CW-complex $B_m$ such that the functors $E^F$ and $(B_m)$ of CW into Set are naturally equivalent (here $[X,Y]$ represents the set of all homotopy classes of maps from $X$ into $Y$; see [3], Corollary 16.9).

Proposition 2—If $p : E \to B$ is a Hurewicz fibration with fibre $F$, $B$ is a path-connected CW-complex and $k : B \to B_m$ is the classifying map, the space $G(p)$ of all self-fibre homotopy equivalences of $p$ is such that

$$\pi_0(G(p)) = \pi_0(\Omega M(B, B_m; k)).$$

The purpose of this note is to report results of a joint work with P. Booth, P. Heath and C. Morgan, concerning the study—in a unified fashion—of the homotopy type and certain homotopy groups of the space $G(p)$, where $p$ is an object of an arbitrary category of fibrations over CW-complexes. Proofs will be given elsewhere.

The main examples of categories of fibrations we have in mind are the following (note that all fibrations considered have a path-connected CW-complex as a base space).

(i) Dold fibrations with fibres of the homotopy type of a fixed space $F$ (we define a Dold fibration as a fibration satisfying the Weak Covering Homotopy Property [2]):

(ii) Hurewicz fibrations with fibres of the homotopy type of a fixed space $F$.
(III) principal $G$-bundles, $G$ a topological group:

(IV) smooth principal $G$-bundles, $G$ a Lie group:

(V) vector bundles with fibres isomorphic to a fixed vector space $V$:

(VI) fibre bundles with fibre $F$, corresponding to a given effective action of a compact topological group $G$:

(VII) principal $H$-fibrations with fibres of the homotopy type of a strictly associative $H$-space with strict identity (see [11], Ex.3).

All these categories have in common the fact that each has a Universal Object $(E_0, p_0, B_0)$ from which one deduces a Classification Theorem of Dold's type; furthermore, in each one of these examples, $(E_0, p_0, B_0)$ also satisfies another type of universality which we shall describe later on and which plays a crucial role in our considerations.

In order to unify these ideas we begin by taking a category $F$ with a distinguished object $F$ and a faithful underlying space functor $F \to K$, where $K$ is the convenient category of $k$-spaces, that is to say, $K$ is the image of Top under the functor $k : \text{Top} \to \text{Top}$ called the $k$-ification functor — obtained as a left Kan-extension of the imbedding $C \to \text{Top}$ over itself, where $C$ is the category of all compact Hausdorff spaces. It is also assumed that for any two objects $X, Y \in F$, $F(X, Y)$ is non-empty. We then define an $F$-space as a triple $(E, p, B)$ such that $B$ is a CW-complex, $E \in K$, $p : E \to B$ is a map in $K$ and, finally, for every $b \in B$, $E_b = p^{-1}(b) \in F$. An $F$-map $(f_1, f_0) : (E, p, B) \to (E', p', B')$ is given by two maps $f_1 : E \to E'$, $f_0 : B \to B'$ such that $p' f_1 = f_0 p$ and the restriction of $f_1$ to any fibre $E_b$ is a morphism of $F$. If $B = B'$ and $f_0 = 1_B$, an $F$-map $(f_1, 1_B)$ is said to be an $F$-map over $B$. An $F$-homotopy is an $F$-map $(H, h)$ such that $pH = h q X$. If $A = B$ and $h$ is the projection map, we have the notion of $F$-homotopy over $B$. An $F$-map $g : X \to E$ over $B$ is an $F$-homotopy equivalence if there exists an $F$-map $g' : E \to X$ over $B$ such that $gg'$ and $g'g$ are $F$-homotopic.
over \( B \) to the respective identity maps. We now once more restrict the category \( F \) by requiring that every morphism of \( F \) is an \( F \)-homotopy equivalence over a point.

We are now prepared to define formally what we intend for a category of fibrations relatively to a category \( F \).

**Definition** - A category of fibrations is a non-empty, full subcategory \( A \) of the category of \( F \)-spaces and \( F \)-maps such that:

1. \((F,c,x) \in A\), where \( x \) is a singleton space and \( c \) is the constant map;
2. If \((E,p,B) \in A, A \in CW \) and \( f: A \to B \) is a map, the pullback \((f^*(E), p_f, A) \in A\);
3. \( A \) is closed under \( F \)-isomorphisms over a fixed base space;
4. If \((E,p,B) \in A \), there is a numerable open covering \((U)\) of \( B \) such that, for every \( U \in (U) \), \( p : p^{-1}(U) \to U \) is \( F \)-homotopy equivalent to \( pr : U \times F \to U \).

As examples of categories of fibrations we quote the categories [I] to [VII] described earlier; we limit ourselves to define the category \( F \) in each case. For the examples numbered [I] and [III], \( F \) is the category of all spaces of the same homotopy type as \( F \) and all homotopy equivalences between these spaces. For [III] and [IV], \( F \) is the category whose objects are right \( G \)-spaces \( Y \) such that, for all \( y \in Y \), the function \( \tilde{y} : G \to Y \) defined by \( \tilde{y}(g)y = yg \) is a homeomorphism; its morphisms are \( G \)-maps. For [VI], \( F \) consists of all vector spaces isomorphic to \( V \) and all isomorphisms between such vector spaces. For [VII], we first assume that \( G \) acts effectively on the left of \( F \); then, we define \( F \) by taking for its objects all pairs \((X, \psi)\) such that \( X \) is a left \( G \)-space and \( \psi : F \to X \) is a homeomorphism of left \( G \)-spaces; the set of morphisms from \((X, \psi)\) to \((X', \psi')\) is given by

\[
F((X, \psi), (X', \psi')) = \{ \psi' g \psi^{-1} : g \in G \}
\]

with the obvious operation of composition. Finally, for [VIII], \( F \) is similar to that of [III] ([I], Example 3).

We continue as in [I] by defining, for two arbitrary \( F \)-spaces \((X, q, A)\) and
(Y,r,B), the functional space

\[ XXX = \bigcup_{a \in A} F(X_a, Y_b) \]

and the function

\[ q r : XXX \to A \times B, q r (f : X_a \to Y_b) = (a,b). \]

The topology of XXX is given as follows. Let \( Y^+ = Y U(\infty) \) be the k-ification of the topology defined by requiring that \( C \) is closed in \( Y^+ \) if either \( C = Y^+ \) or if \( C \) is closed in \( Y \). Now define the function \( f : XXX \to M(X,Y^+) \) by \( f (x) = f (x) \) if \( x \in X_a, f : X_a \to Y_b \) and \( f (x) = \infty \), otherwise. \( M(X,Y^+) \) is endowed with the compact open topology. Then we give XXX the k-ification of the initial topology with respect to \( f \) and \( q r \). In general, \( XXX, q r, A \times B \) is not an E-space; however, the following holds.

Theorem 1-If \( (X,q,A), (Y,r,B) \in A \) then, \( q r : XXX \to A \times B \) is a Dold fibration.

As we have mentioned before, each one of the categories described in the examples [I] to [VIII] has a (free)universal object \( (E_\infty, p_\infty, B_\infty) \). It also happens that in these examples, the Dold fibration \( (F \times E_\infty, c \times p_\infty, \times B_\infty) \) has a weakly contractible total space (i.e., for every non-negative integer \( n \), \( \pi_n (F \times E_\infty) = 0 \)); in this case, we say that \( (E_\infty, p_\infty, B_\infty) \) is Weakly Contractible Universal. We wish to observe, at this point, that if a category of fibrations has a weakly contractible universal object, then such object is also free universal; however the converse is not necessarily true ([I], Theorem 3.2 and Example 4).

For a given object \( (E,p,B) \) of the category of fibrations \( A \) let \( G(p) \) be the space of all F-homotopy equivalences of \( p \) into itself over \( B \), topologized as a subspace of \( M(E,E) \); notice that the composition of F-maps of \( p \) into \( p \) over \( B \) gives to \( G(p) \) a continuous product under which \( G(p) \) becomes an associative H-space with a strict two-sided unit defined by the identity morphism of \( p \) into itself.
Theorem 2—Let $A$ be a category of fibrations with a weakly contractible object $(E_\infty,p_\infty,E_\infty)$ and let $(E,p,B)$ be an arbitrary element of $A$; suppose that $k:B \to B_\infty$ is a classifying map for $(E,p,B)$. Then, there exists an $H$-map

$$
5: \Omega M(B,B_\infty;k) \to G(p)
$$

which is a weak homotopy equivalence.

Observe that the Dold fibration $F\oplus E_\infty \to B_\infty$ has fibre $F\oplus F$ and so, if $F\oplus F$ has the homotopy type of a CW-complex. $F\oplus E_\infty$ is contractible; this, in turn, will imply that the $H$-map $5$ of Theorem 2 is a homotopy equivalence. This is precisely the situation of Example [IV], since $G\oplus G$ is homeomorphic to $G$.

From now on, we shall assume for technical reasons that $(E,p,B)$ is an object of the category of fibrations $A$ which satisfies a strengthened version of axiom [4] in the definition of a category of fibrations, implying that if $(X,q,A)$ and $(Y,r,B)$ are objects of $A$ then $(XY,q;r,A\times B)$ is a Hurewicz fibration; furthermore, we suppose that $A$ has a weakly contractible universal object $(E_\infty,p_\infty,B_\infty)$. This is the case of examples [II] and [IV]. Let $F$ be the fibre of $(E,p,B)$ over a point $x \in B$ and define $G^1(p)$ to be the subspace of $G(p)$ of all $F$-homotopy equivalences of $p$ over itself over $B$ which extend the identity map $1_F:F \to F$. The space $G^1(p)$ has proved itself very useful in certain problems of Mathematical Physics, where $(E,p,B)$ is an object of the category [IV] (see [8]). 'We wish to observe that the relation between $G^1(p)$ and $G(p)$ is deeper than just the relation *subspace-space*; in fact,

Theorem 3—There is a Hurewicz fibration $G(p) \to F\oplus F$ with fibre $G^1(p)$ over $1_F$.

A result similar to Theorem 2 holds for $G^1(p)$; in what follows $\mathcal{M}(B,B_\infty,k)$ denotes the space of all based maps from $B$ to $B_\infty$. 

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Theorem 4—There is an $H$-map

$$\delta: \Omega M_{\pm}(B.B_{\infty};k) \to G^1(\phi)$$

which is a weak homotopy equivalence (or a strong homotopy equivalence if $F\times E_{\infty}$ is contractible).

Next, consider the Hurewicz fibration $F\times E_{\infty} \to B_{\infty}$ (with fibre $F\times F$ over $b = k(\ast) \in B_{\infty}$) and its long homotopy sequence

$$\cdots \to \Omega(F\times E_{\infty}) \to \Omega B_{\infty} \to F\times F \to F\times E_{\infty} \to B_{\infty},$$

because $F\times E_{\infty}$ is weakly contractible.

$$\beta: \Omega B_{\infty} \to F\times F$$

is a weak homotopy equivalence (strong homotopy equivalence if $F\times E_{\infty}$ is contractible). This fact is used to prove the following.

Theorem 5—Suppose that all path-components of $M(B.B)$ (resp. $M_{\pm}(B.B_{\infty})$) have the same homotopy type. Then

$$G(\phi) \simeq_w M(B,F\times F) \quad (\text{resp. } G^1(\phi) \simeq_w M_{\pm}(B,F\times F))$$

(strong homotopy equivalence if $F\times E_{\infty}$ is contractible); furthermore, these weak (strong) equivalences preserve the $H$-space structures.

In connection to the previous theorem the reader should recall that if $B$ is an $H$-cogroup (e.g., $B$ is a suspension space) then the hypothesis of Theorem 5 hold for $M_{\pm}(B.B_{\infty})$ and if $B_{\infty}$ is an $H$-group (e.g., $B_{\infty} = BU.BO. BSp$) then these hypothesis hold for both $M(B.B_{\infty})$ and $M_{\pm}(B.B_{\infty})$.

Theorem 6—If $F\times F$ is $(n-1)$-connected ($n$ positive) and $\dim B = m < 2n$, then for $0 \leq i < 2n - m$

$$\pi_i(G(\phi)) = \pi_i(M(B,F\times F;\phi))$$

$$\pi_i(G^1(\phi)) = \pi_i(M_{\pm}(B,F\times F;\phi))$$
where \( c : B \to F \times F \) is the constant map to \( 1_F \).

We complete these notes with a few computations. If \((E,p,B)\) is a smooth principal \(Sp(1)\)-bundle and \(B\) is a manifold of dimension \(m < 5\), since \(Sp(1) \cong S^3\) and \(Sp(1) \times Sp(1) \cong Sp(1)\), theorem 6 shows that if \(0 < j < 5-m\),

\[
\pi_j(G(p)) = \pi_j(M(B,Sp(1)))
\]

and

\[
\pi_j(G^1(p)) = \pi_j(M(B,Sp(1))).
\]

If \(p\) is a smooth principal \(G\)-bundle over a sphere \(S^n\), \(n > 0\), then

\[ G^1(p) = M_G(S^n,G) \]

and thus the homotopy groups of \(G^1(p)\) are totally determined by the homotopy groups of \(G\), since, for every \(j > 0\), \(\pi_j(G^1(p)) = \pi_j(\mathbb{R})\). If \(p\) is a smooth principal \(U\)-bundle over a manifold \(B\), then \(G(p) = M(B,U)\) and \(G^1(p) = M(B,U)\); if, in particular, \(B = S^n\) and \(n > 0\), then

\[
\pi_j(G^1(p)) = \pi_j(U) = \begin{cases} 0, & \text{if } j = \text{even}, n = \text{even} \\ Z, & \text{if } j = \text{even}, n = \text{odd} \\ Z, & \text{if } j = \text{odd}, n = \text{even} \\ 0, & \text{if } j = \text{odd}, n = \text{odd} \end{cases}
\]

On the other hand, Theorem 2.2 of [7] shows that \(M(S^n,U) = U \times M_G(S^n,U)\) and so,

\[
\pi_j(G(p)) = \begin{cases} 0, & \text{if } j = \text{even}, n = \text{even} \\ Z, & \text{if } j = \text{even}, n = \text{odd} \\ Z \times Z, & \text{if } j = \text{odd}, n = \text{even} \\ 0, & \text{if } j = \text{odd}, n = \text{odd} \end{cases}
\]

Finally, we recall from the Milnor construction of universal bundles that each countable, connected CW-complex \(X\) can be viewed as the base space of a universal \(G\)-bundle (\(G\) is constructed from \(X\)); let us take \(X\) to be \(S^4\) and let \(k : S^4 \to S^4\) be a degree \(k\) function and let \((E_k, p_k, S^4)\) be the corresponding principal \(G\)-bundle. Then,
\[ \pi_2(G(p)) \cong \pi_2(M(S^4, S^4; k)) \cong \pi_3(M(S^4, S^4; k)) \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_{12}. \]

according to ([7], Lemma 3.10). Since \( G^1(p) \cong M(S^4, G) \) it follows that

\[ \pi_2(G^1(p)) \cong \pi_6(G) \cong \pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}. \]

Independently of \( k \).
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