A THEOREM ON SCHAUDER DECOMPOSITIONS IN BANACH SPACES

Miguel A. Ariño

Abstract. In this paper we prove that in a Banach space all Schauder decompositions are shrinking iff all Schauder decompositions are boundedly complete.

1. Definitions and preliminary results

A sequence \((x_n)_{n=1}^{\infty}\) in a Banach space \(X\) is called a Schauder basis if for every \(x \in X\) there exists a unique sequence \((a_n)_{n=1}^{\infty}\) in \(\mathbb{R}\) such that \(x = \sum_{n=1}^{\infty} a_n x_n\), and this series converges with respect to the norm of \(X\). A sequence \((y_n)_{n=1}^{\infty}\) is called a basic sequence if it is a basis of its closed linear span.

A Schauder decomposition of \(X\) is a sequence \((X_i)_{i=1}^{\infty}\) of closed subspaces of \(X\) such that for every \(x\) in \(X\) there exists a unique sequence \((x_i)_{i=1}^{\infty}\) with \(x_i \in X_i\) for all \(i\) and \(x = \sum_{i=1}^{\infty} x_i\). Every Schauder decomposition of \(X\) is related with a sequence of continuous projections \(P_n : X \to X\) defined by

\[ P_n(x) = P_n \left( \sum_{i=1}^{\infty} x_i \right) = \sum_{i=1}^{n} x_i \]

In all this paper, the linear span of an element \(x \in X\) is denoted by \([x]\) and the closed linear span of the subspaces \((X_i)_{i=n}^{m}\) \((1 \leq n < m \leq \infty)\) is denoted by \([X_i]_{i=n}^{m}\).

The following theorem characterizes the Schauder decompositions and it can be found in [5].
1. Theorem: Let $X$ be a Banach space and $(X_n)_{n=1}^\infty$ a sequence of closed subspaces of $X$. The following are equivalent:

i) $(X_n)_{n=1}^\infty$ is a Schauder decomposition of $X$.

ii) There exists a sequence $(P_n)_{n=1}^\infty$ of continuous projections $P_n: X \rightarrow [X_i]_{i=1}^n$ such that $P_n P_m = P_{\min(m,n)}$ and $\lim_{n \to \infty} P_n(x) = x$ for every $x$ in $X$.

iii) There exists a sequence $(P_n)_{n=1}^\infty$ of continuous projections $P_n: X \rightarrow [X_i]_{i=1}^n$ such that $P_n P_m = P_{\min(m,n)}$ and $(P_n)_{n=1}^\infty$ is uniformly bounded.

To $\sup_n \|P_n\|$ is called norm of the decomposition.

A Schauder decomposition $(X_n)_{n=1}^\infty$ in a Banach space $X$ is called **boundedly complete** if for every sequence $(x_n)_{n=1}^\infty$ with $x_n \in X_n$ such that $\sup_n \|\sum_{i=1}^n x_i\| < \infty$, the sequence $(\sum_{i=1}^n x_i)_{n=1}^\infty$ converges towards an element $x$ in $X$. And it is called **shrinking** if for every $x^* \in X^*$, $\lim_{n \to \infty} \|x^*\|_n = 0$, where

$$\|x^*\|_n = \sup\{|x^*(x)| \text{ with } x \in [X_i]_{i=n+1}^\infty \text{ and } \|x\| \leq 1\}.$$  

Boundedly complete and shrinking basis and basic sequences are defined in a similar way.

Singer (cf. [6]) has proved that in a Banach space all basic sequences are boundedly complete if and only if all basic sequences are shrinking. Afterwards Zippin (cf. [7] and [3]) proved a similar theorem for Schauder basis of $X$. Our purpose in this paper is to prove that in a Banach space all Schauder decompositions are boundedly complete iff all Schauder decompositions are shrinking.
If \( X \) is a locally bounded \( F \)-space, then there exists \( p (0 < p < 1) \) such that the topology of \( X \) is originated by a \( p \)-norm. In this case \( X \) is called \( p \)-Banach space (cf. [1] and [4]). Let \( X \) be a \( p \)-Banach space such that \( X \) separates points of \( X \) and let \( J: X \to X^{**} \) be the canonical imbedding of \( X \) into its bidual. We define in \( X \) the norm \( \| x \|^{**} \):

\[
\| x \|^{**} = \| J(x) \| \quad \text{if } x \in X.
\]

The Mackey topology of \( X \) is originated by this norm (cf. [2]) and it is called the Mackey norm of \( X \). The Mackey completion of \( X \) is denoted by \( \tilde{J}(X) \).

All the above definitions for Banach spaces can be extended to \( p \)-Banach spaces.

2. Shrinking and boundedly complete Schauder decomposition.

**2. Lemma.** Let \( (X_n)_{n=1}^{\infty} \) be a Schauder decomposition of a Banach space \( X \) and let \( (P_n)_{n=1}^{\infty} \) be its sequence of projections. We suppose that each \( X_n \) admits a topological decomposition \( X_n = Y_n \oplus Z_n \). The following are equivalent:

i) \( (Y_1, Z_1, \ldots, Y_n, Z_n, \ldots) \) is a Schauder decomposition of \( X \).

ii) If \( A_n \) is the continuous projection from \( X_n \) into \( Y_n \), then \( \sup_n \| A_n \| < \infty \).

**Proof:** i = ii. If \( (Q_n)_{n=1}^{\infty} \) is the sequence of projections of \( (Y_1, Z_1, \ldots, Y_n, Z_n, \ldots) \), as \( A_n = Q_{2n-1} |_{X_n} \), the statement ii is proved.
and thus \( (Q_n)_{n=1}^{\infty} \) is a uniformly bounded sequence of projections which defines the decomposition \( (Y_n, Z_n)_{n=1}^{\infty} \) because of theorem 1. //

Remark that if any of the previous subspaces is 0, it must be taken away in the decomposition.

3. Corollary. Let \( (X_n)_{n=1}^{\infty} \) be a Schauder decomposition of a Banach space \( X \) and \( (X_n)_{n=1}^{\infty} \) a normalized sequence in \( X \) with \( x_n \in X_n \). For every \( n \), there exists an hyperplane \( W_n \) of \( X_n \) such that \( (\langle x_1, W_1 \rangle, \ldots, \langle x_n, W_n \rangle, \ldots) \) is a Schauder decomposition of \( X \).

Proof: As \( \|x_n\| = 1 \), we can define \( A_n(x) = \frac{1}{n} \langle x, x_n \rangle \), where \( u^*_n \in X^*_n \) and \( u^*_n(x_n) = \|u^*_n\| = 1 \). //

4. Lemma. Let \( X \) be a Banach space and a Schauder decomposition of the form \( (\langle y_1, W_1 \rangle, \ldots, \langle y_n, W_n \rangle, \ldots) \), where \( (y_n)_{n=1}^{\infty} \) satisfies \( \inf \|y_n\| = C > 0 \) and \( \sup \|n \sum_{i=1}^{n} y_i\| = M < \infty \). We define the sequence \( (v_n)_{n=1}^{\infty} \) by \( v_n = \sum_{i=1}^{n} y_i \). Then \( (\langle v_1, W_1 \rangle, \ldots, \langle v_n, W_n \rangle, \ldots) \) is a Schauder decomposition of \( X \).

Proof: Let \( (P_n)_{n=1}^{\infty} \) be the sequence of projections of \( (X_n)_{n=1}^{\infty} \) and let \( K \) be its norm. Each \( P_{2n} - P_{2n-1} \) (the projection over \( \langle y_n \rangle \)) is originated by a \( y_n^* \in X^* \) according to
\[
(P_{2n-1} - P_{2n-2})(x) = y_n^*(x)y_n \text{ if } x \in X.
\]

and thus

i) \(y_n^*(y_m) = \delta_{n,m}\)

ii) \(|y_n^*||y_n| \leq 2K\) for every \(n\), and

iii) \(y_n|w_m = 0\) for every \(n\) and \(m\).

As \(\inf_{n} ||y_n|| = C > 0\), from ii) we obtain that \(\sup_{n} ||y_n^*|| \leq 2K/C\).

Let \((v_n^*)_n=1\) be defined by \(v_n^* = y_n^*-y_{n+1}^*\). It is easy to check that \(v_n^*(v_m) = \delta_{n,m}\).

We define the sequence of projections by

\[
A_{2n}(x) = \sum_{k=1}^{n} (P_{2k} - P_{2k-1})(x) + \sum_{k=1}^{n} y_k^*(x)v_k
\]

\[
A_{2n+1}(x) = A_{2n}(x) + v_{n+1}^*(x)v_{n+1}.
\]

Because of the theorem 1 we only need to prove that \((A_n)^n=1\) is uniformly bounded, and, because of the last considerations, it shall be proved if we prove that \(\sup_{n} ||A_{2n}|| < \infty\):

\[
||A_{2n}(x)|| = ||P_{2n}(x) - \sum_{k=1}^{n} y_k^*(x)y_k + \sum_{k=1}^{n} (y_k^*(x) - y_{k+1}^*(x))(\sum_{i=1}^{k} y_i)|| \leq
\]

\[
\leq K ||x|| + \sum_{k=1}^{n} y_k^*(x)y_k + \sum_{k=1}^{n} (y_k^*(x) - y_{k+1}^*(x))y_k || \leq
\]

\[
\leq K ||x|| + ||y_{n+1}^*|| ||x|| ||\sum_{k=1}^{n} y_k|| \leq (K + \frac{2K}{C} M) ||x||.
\]
5. Lemma. Let $X$ be a Banach space and $(([y_1], W_1, ...), [y_n], W_n, ...)$ a Schauder decomposition of $X$, where $\{y_n\}_{n=1}^\infty$ satisfies $\sup_n \|y_n\| = M < \infty$. We define the sequence $([v_1], W_1, ...; [v_n], W_n, ...)$ by $v_1 = y_1$ and $v_n = y_n - v_{n-1}$. Then the following are equivalent:

i) $([v_1], W_1, ...; [v_n], W_n, ...)$ is a Schauder decomposition of $X$.

ii) There exists $x^* \in X^*$ such that

a) $x^*(y_n) = 1$ for every $n$

b) $x^*|_{W_m} = 0$ for every $m$

Proof: If, for every $n$, there is a continuous projection from $X$ into $[v_n]$ parallel to the other subspaces, the existence of $x^* \in X^*$ satisfying a) and b) is necessary. We suppose that there exists such $x^*$. We define $([y_n])_{n=1}^\infty$ as in the preceding lemma, and if we consider the sequence $v^*_n = x^* - \sum_{k=1}^{n-1} y_n^*$, the orthogonal relations $v^*_n(v_m) = \delta_{n,m}$ hold.

Let $(A_n)_{n=1}^\infty$ be a sequence of projections as in the preceding lemma. We must prove that $\sup_n \|A_n\| < \infty$. For every $m$, $x^*|_{W_m} = 0$, and hence $x^*(x) = \sum_{n=1}^\infty y_n^*(x)$ for every $x$ in $X$ and so $([v_n])_{n=1}^\infty$ converges weakly to $0$ and $\sup_n \|\sum_{k=1}^n y_k\| = M_1 < \infty$. Also $\sup_n \|v_n\| \leq 2M$, again we must only prove that $\sup_n \|A_{2n}\| < \infty$.

we have

$$\sum_{k=1}^n v_k^*(x)v_k = x^*(x) + \sum_{k=2}^n x^*(y_k - y_{k-1}) - \sum_{k=2}^n \left[ (\sum_{l=1}^{k-1} y_l^*)(x) \right] (y_k - y_{k-1}) =$$
\[ a_n - x^* (x) y_1 + x^* (x) (y_n - y_1) - \sum_{k=1}^{n-1} y_k^* (x) y_n + \sum_{k=1}^{n-1} y_k^* (x) y_k = \]

\[ = \sum_{k=1}^n y_k^* (x) y_k + x^* (x) y_n - \sum_{k=1}^n y_k^* (x) y_n, \]

and thus

\[ A_{2n} (x) = p_{2n} (x) + x^* (x) y_n - \sum_{k=1}^n y_k^* (x) y_n. \]

And finally:

\[ \| A_{2n} (x) \| \leq \| p_{2n} (x) \| + \| x^* \| \| y_n \| + \| \sum_{k=1}^n y_k^* \| \| x \| \| y_n \|. \]

Now we can prove the main result:

6. Theorem. Let \( X \) be a Banach space. The following statements are equivalent:

i) All Schauder decompositions of \( X \) are shrinking

ii) All Schauder decompositions of \( X \) are boundedly complete

Proof. i) \( \Rightarrow \) ii. Let \( (X_i)_{i=1}^n \) be a non boundedly complete Schauder decomposition of \( X \). There exists then a sequence \( (x_i)_{i=1}^n \) with \( x_i \in X_i \) such that \( \sup_n \sum_{i=1}^n x_i \) is not a Cauchy sequence, and thus, there exist \( \epsilon \) and a strictly increasing sequence \( (m_k)_{k=1}^\infty \) such that \( \epsilon < \| \sum_{i=m_k-1}^{m_k} x_i \| \leq 2 \) for every \( k \). We define \( Y_k = \{ X_i \}_{i=m_k-1+1}^{m_k} \) and \( y_k = \sum_{i=m_k-1+1}^{m_k} x_i \) if \( k \geq 1 \). \( \{ Y_k \}_{k=1}^\infty \) is a Schauder decomposition of \( X \) with \( y_k \in Y_k \). Because of the corollary 3, for each \( k \) there exists a hyperplane \( W_k \) of \( Y_k \) such that \( (\{ y_1 \}, W_1, \ldots \)
...,[y_n], W_n,...) is a Schauder decomposition. Because of the lemma 4, the sequence \( \{v_k^m\}_{k=1}^\infty \) defined by \( v_n = \sum_{i=1}^{n} y_i \) originates the Schauder decomposition \( \{v_1, W_1, ..., [v_n], W_n, ...\} \) which is not shrinking because \( y^*(v_k) = 1 \) for every \( k \geq 1 \).

ii = i. Let \( \{x_n\}_{n=1}^\infty \) be a non-shrinking Schauder decomposition of \( X \). There exist then \( x^* \in X \) with \( \|x^*\| = 1 \), \( \epsilon > 0 \), a strictly increasing sequence of índex \( \{m_k\}_{k=1}^\infty \) and a sequence \( \{y_k\}_{k=1}^\infty \) with \( y_k \in V_k = [x_n]_{i=1}^{m_k} \) such that:

\[
\begin{align*}
  &a) \ 1 \leq \|y_n\| \leq 1/\epsilon \\
  &b) \ x^*(y_n) = 1.
\end{align*}
\]

We can choose the hyperplane \( W_k = V_k \cap \ker x^* \) and using the lemma 5, if \( v_1 = y_1 \) and \( v_n = y_n - y_{n-1} \), then \( \{v_1, W_1, ..., [v_n], W_n, ...\} \) is a Schauder decomposition of \( X \) which is not boundedly complete because of

\[
\| \sum_{k=2}^{n} v_k \| = \| \sum_{k=2}^{n} (y_k - y_{k-1}) \| = \| y_n - y_1 \| \leq 2/\epsilon
\]

while

\[
\| v_k \| = \| y_k - y_{k-1} \| \geq \frac{1}{k} \| y_{k-1} \|
\]

where \( K \) is the norm of \( \{x_n\}_{n=1}^\infty \).

With certain modifications, this theorem has an extension to \( p \)-Banach spaces (if its dual separates points). The Mackey topology of this spaces plays an important role in this extension. We need before a definition: we shall say that a Schauder decomposition \( \{x_n\}_{n=1}^\infty \) in a \( p \)-Banach space is an almost boundedly complete decomposition if for every sequence \( \{x_n\}_{n=1}^\infty \) with \( x_n \in X \) such that

\[
\sup_n \| \sum_{k=1}^{n} x_k \| < \infty,
\]

the sequence \( \{ \sum_{k=1}^{n} x_k \}_{n=1}^\infty \) converges in \( (J(X), \| \cdot \|^*) \).
We must point out that if \((X_n)_{n=1}^\infty\) is boundedly complete then it is also almost boundedly complete. Almost boundedly complete basis of \(X\) are defined in a similar way.

7. Theorem. Let \(X\) be a \(p\)-Banach space. The following are equivalent:

i) All Schauder decompositions of \(X\) are shrinking.

ii) All Schauder decompositions of \(X\) are almost boundedly complete.

Proof: Similar to the proof of Theorem 6, and it can be found in [2]

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References


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Facultad de Matemáticas
Dpto. de Teoría de Funciones
Universidad de Barcelona
Gran Via, 585
Barcelona
SPAIN