Pub. Mat. UAB
Vol. 30 Nº 1 Maig 1986

STATE DIAGRAM FOR OPERATORS WITH NULL SPACE OR CONULL SPACE IN AN IDEAL OF BANACH SPACES

Teresa Alvarez

1.- Introduction

Let B be the class of all Banach spaces; the scalar field K is either the real field or the complex field. All operators acting between Banach spaces which appear in this article are supposed to be linear. For $X,Y \in B$, $\mathcal{L}(X,Y)$ is the space of all operators from X into Y, the class of all operators from X into Y with dense domain is denoted by $\mathcal{L}_D(X,Y)$, I_X denotes the identity operator on X, J_X is the embedding map of X into X'', and $X \underset{q}{\subset} Y$ means that X is a quotient space of Y. For $T \in \mathcal{L}(X,Y)$, D(T), N(T) and R(T) will denote the domain, null space and range of T respectively, and we also write CON(T): = Y/R(T), $\overline{\text{CON}}(T)$: = $Y/\overline{R(T)}$, while $\alpha(T)$, $\beta(T)$ and $\bar{\beta}(T)$ will denote the dimension of N(T), CON(T) and $\overline{\text{CON}}(T)$ respectively. We shall consider

$$\mathcal{NS}(X,Y) := \{T \in \mathcal{L}(X,Y) : T \text{ is normally solvable}\}$$
$$L(X,Y) := \{T \in \mathcal{L}(X,Y) : T \text{ is bounded}\}$$

Let A be an ideal of Banach spaces. For informations and notations about operator ideals and space ideals we refer to [5]. We consider the ideals, S, R or F, the ideals of all separable, reflexive or finite dimensional Banach spaces respectively.

Some notations will be used without explanation because their meaning is obvious.

In this paper we obtain a state diagram of a linear operator with dense domain between Banach spaces and its conjugate operator, and we prove that this diagram is complete.

2. "GENERALIZED" CLASSIFICATION OF (T,T'): STATE DIAGRAM

2.1. THEOREM. Let A be an ideal and $T \in \mathcal{L}_D(X,Y)$. Then:

(i) $\alpha(T') = \overline{\beta}(T)$, $\alpha(T) \leq \overline{\beta}(T')$; in general the inequality is strict. If, in addition $T \in \mathcal{NS}$, then $\alpha(T) = \beta(T')$.

(ii) Let A be a completely symmetric ideal, then:

(ii_1) $N(T') \in A$ if and only if $\overline{CON}(T) \in A$

(ii_2) $T \in \mathcal{NS}$: $N(T) \in A$ if and only if $CON(T') \in A$.

(ii_3) Suppose A surjective, if $\overline{CON}(T') \in A$ then $N(T) \in A$.

For arbitrary ideals, the properties are not valid, in general.

Proof. (i) It is an obvious consequence of the duality relations,

$\overline{CON}(T)' \simeq \overline{R(T)}^{\circ} = N(T')$,

$N(T)' = X'/N(T)^{\circ} \simeq (X'/\overline{R(T')})/(N(T)^{\circ}/\overline{R(T')}) \subset_q \overline{CON}(T')$

To see that, in general, the inequality $\alpha(T) \leq \overline{\beta}(T')$ is strict we define $T \in L(1)$ by $T(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = (0, \alpha_1, 2^{-1}\alpha_2, \ldots, n^{-1}\alpha_n, \ldots)$, $(\alpha_n) \in 1$. Its conjugate operator is $T'(\beta_n) = (n^{-1}\beta_{n+1})$, $(\beta_n) \in 1_{\infty}$. It is obvious that $N(T) = \{0\}$, $1_{\infty}/c_0 \subset_q \overline{CON}(T')$ since $R(T') \subset c_0$. Clearly $\overline{CON}(T') \notin R$ since if $1_{\infty}/c_0 \in R$ then c_0 has a subspace isomorphic to 1_{∞}; that contradicts $c_0 \in S$ and $1_{\infty} \notin S$.

(ii_1) It is suffices to notice that $N(T') \simeq \overline{CON}(T)'$ and that A is completely symmetric.

(ii_2) It $T \in \mathcal{NS}$ then $N(T)' \simeq CON(T')$.

(ii_3) Note that $N(T)' \subset_q \overline{CON}(T')$, A surjective and completely symmetric.

For arbitrary ideals, the above results are not guaranteed; for example, if $D := \{X \in B: J_X X$ is complemented in $X''\}$, T_1, T_2 the null maps on 1, c_0 respectively, then

$N(T_1') = 1_{\infty} \notin S$, $CON(T_1) = 1 \in S$

$N(T_2') = 1 \in D$, $CON(T_2) = c_0 \notin D$,

$N(T_1) = 1 \in S$, $CON(T_1') = 1_{\infty} \notin S$,

$N(T_2) = c_0 \notin D$, $CON(T_2') = 1 \in D$.

We now introduce the following classification of $T \in \mathcal{L}_D(X,Y)$.

I : $\alpha(T) < \infty$.

II : $\alpha(T) = \infty$ and $N(T) \in A$.

III : $\alpha(T) = \infty$ and $N(T) \notin A$.

1 : $\bar{\beta}(T) < \infty$.

2 : $\bar{\beta}(T) = \infty$ and $\overline{CON}(T) \in A$.

3 : $\bar{\beta}(T) = \infty$ and $\overline{CON}(T) \notin A$.

By combining these possibilities we obtain nine different situations. This classification scheme may now be applied to the conjugate T' of T.

The properties of the (2.1) theorem on the language of the previous classification can be written as:

$T' \in I \Longleftrightarrow T \in 1$

$T \notin I \Longrightarrow T' \notin 1$

$T \in \mathcal{NS}$: $T \in I \Longleftrightarrow T' \in 1$

A completely symmetric: $T' \in III \Longleftrightarrow T \in 3$

A completely symmetric and surjective: $T \in III \Longrightarrow T' \in 3$

A completely symmetric and $T \in \mathcal{JNS}$: $T \in III \Longleftrightarrow T' \in 3$.

We shall proceed to construct a diagram. The shaded squares in the diagram correspond to states that are impossible by virtue of (2.1) theorem.

T' \ T	I_1	I_2	I_3	II_1	II_2	II_3	III_1	III_2	III_3
III_3	X	cs ó $\mathcal{JNS}$	$\mathcal{JNS}$	X	cs	cs+ $\mathcal{JNS}$	X	cs	
III_2	X	cs ó $\mathcal{JNS}$	$\mathcal{JNS}$	X	cs		X	cs	cs+$\mathcal{JNS}$ ó cs+s
III_1	X	cs		X	X	X	X	X	X
II_3	X	$\mathcal{JNS}$	cs ó $\mathcal{JNS}$	X	cs+ $\mathcal{JNS}$	cs	X		cs
II_2	X	$\mathcal{JNS}$	cs ó $\mathcal{JNS}$	X		cs	X	cs+$\mathcal{JNS}$ ó cs+s	cs
II_1	X		cs	X	X	X	X	X	X
I_3	$\mathcal{JNS}$	X	X	cs+ $\mathcal{JNS}$	X	X		X	X
I_2	$\mathcal{JNS}$	X	X		X	X	cs+$\mathcal{JNS}$ ó cs+s	X	X
I_1		X	X	X	X	X	X	X	X

cs : Impossible if A is completely symmetric

cs + $\mathcal{NS}$: Impossible if A is completely symmetric and $T \in \mathcal{NS}$

cs + s : Impossible if A is completely symmetric and surjective.

$\mathcal{NS}$: Impossible if $T \in \mathcal{NS}$

We analyse if the diagram is complete, so, we prove that a procedure to construct state examples of L(X,Y) in the Taylor-Haldberg classification, introduced in 1962 by Goldberg and Thorp [2], is valid for our classification.

If $E_1, E_2 \in B$ then the map

$A : h \in (E_1 \times E_2)' \longrightarrow (h_1, h_2) \in E_1' \times E_2'$ where $h_1(x_1) := h(x_1, 0)$, $h_2(x_2) := h(0, x_2)$, $x_i \in E_i$, $i = 1,2$ is an isomorphic, thus we can identify $E_1' \times E_2'$ to $(E_1 \times E_2)'$ through the formula

($) $(h_1, h_2)(x_1, x_2) := h(x_1, x_2) := h_1(x_1) + h_2(x_2) \in K$ where $(x_1, x_2) \in E_1 \times E_2$.

For $T_1 \in \mathcal{L}_D(X_1, Y_1)$, $T_2 \in \mathcal{L}_D(X_2, Y_2)$ it is possible identify $T_1' \times T_2'$ to $(T_1 \times T_2)'$ by using the ($) formula to consider $(T_1' h_1, T_2' h_2)$ as an element of $(X_1 \times X_2)'$. Also it is clear that if we define the product between two states of our classification by using the formula $(A_a, B_b) \times (C_c, D_d) := (\max(A,C)_{\max(a,c)}, \max(B,D)_{\max(b,d)})$ then the state of the operator $T_1 \times T_2$ is the product of the T_1 and T_2 states.

(2.2) THEOREM. The state diagram for (T,T') is complete.

Proof

(I_1, I_1) : Let T be the identity operator in X.

(I_1, I_2) : Let A = R, $(x_i)_{i \in I}$ a normalized Hamel basis of $l_2(N)$, $(e_i)_{i \in I}$ an orthonormal basis of $l_2(I)$. Define

$T: D(T) \subset l_2(I) \longrightarrow l_2(N)$

$e_i \longrightarrow Te_i := x_i$

where D(T) is the linear span of the e_i's. Clearly D(T) is dense in $l_2(I)$, $R(T) = l_2(N)$ and $N(T) = \{0\}$.

Let $(e_{n_k})_{k\in N} \subset (e_i)_{i\in I}$ be a sequence of different vectors and $x_m^n := \sum_{k=1}^{m} e_{n_k}/k^2 \in D(T)$, $m\in N$. Then $x_m^n \longrightarrow x^n = \sum_{k=1}^{\infty} e_{n_k}/k^2$, $Tx_m^n \longrightarrow y^n := \sum_{k=1}^{\infty} x_{n_k}/k^2 \in R(T)$ if $m \longrightarrow \infty$; hence there exists $z^n \in D(T)$ such that $Tz^n = y^n$, moreover $x^n - z^n \neq 0$ since $x^n \notin D(T)$, Consequently for $y\in D(T')$ we have that $\langle x^n - z^n, T'y\rangle = 0$ thus $x^n - z^n \in R(T')°$.

We can choose $(e_{n_k})_{k\in N} \subset (e_i)_{i\in I}$ disjoint sequences, thus for $n\in N$ we obtain $(x^n - z^n)_{n\in N} \subset R(T')°$; moreover, $x^n - z^n$ are linearly independent, hence dim $\overline{R(T')}° = \infty$. Clearly $l_2(I)/\overline{R(T')} \in R$.

(I_1, I_3) : Let A = R and T be the operator in (2.1) theorem

(I_2, II_1) : Let A be completely symmetric, $X\in F$, $Y\in A - F$, T the null map from X into Y.

(I_2, III_1) : Let A be non completely symmetric, $X\in F$, $Y\in A$, $Y' \notin A$, T the null map from X into Y

(I_3, II_1) : Let A be non completely symmetric, $X\in F$, $Y\notin A$, $Y'\in A$, T the null map from X into Y

(I_3, III_1) : Let A be completely symmetric, $M\subset X\notin A$, $M\in A$, $X/M\notin A$, T the inclusion from X into Y

(II_1, I_2) : In the example (I_2, II_1) it suffices to replace T by the conjugate operator.

(III_1, I_2) : In the example (I_2, III_1) it suffices to replace T by the conjugate operator

(III_1, I_3) : In the example (I_3, III_1) it suffices to replace T by the conjugate operator.

We can obtain the remaining allowed states by application of the previous procedure.

REFERENCES

[1] S.GOLDBERG. Unbounded linear operators. Mc Graw-Hill,(1966).

[2] S. GOLDBERG, E.O. THORP. The range as range space for compact operators. J. Reine Angew. Math., 211, (1962), 113-115

[3] G.J.O. JAMESON. Topology and normed spaces. Chapman and Hall, (1974).

[4] J. LINDENSTRAUSS, L. TZAFRIRI. Classical Banach spaces I. Springer-Verlag, (1977).

[5] A. PIETSCH. Operator ideals. North-Holland, (1980).

[6] A.E. TAYLOR, C.J.A. HALBERG. General theorems about a bounded linear operator and its conjugate, J. Reine Angew, Math., 198, (1957), 93-111.

Rebut el 14 de novembre del 1985

Teresa Alvarez

Departamento de Teoría de Funciones

Facultad de Ciencias

Universidad de Santander

Santander, ESPAÑA