A GEOMETRICAL CHARACTERIZATION OF REFLEXIVITY IN BANACH SPACES

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Summary: The main result in this paper is the equivalence, for any Banach space \( B \), between

(i) 'Every normalized basic sequence \((a_n)_{n \in \mathbb{N}}\) in \( B \) is weakly null',

and

(ii) 'For every normalized basic sequence \((a_n)_{n \in \mathbb{N}}\) in \( B \),

\[ a_1 \in \overline{\text{span}} \left( a_n - a_{n+1} \right)_{n \in \mathbb{N}} \] .

Pelczyński proved that (i) characterizes the fact of \( B \) being reflexive. So, the same holds for (ii) and we have a "geometrical" characterization of reflexivity.

We finish quoting some equivalent version of the above result.

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Key words: Reflexive Banach spaces, basic sequences, sequence of differences.
1. Previous Concepts.

Let $B$ denote a Banach space and $K$ its scalar field, $\mathbb{N}$, the set of natural numbers, $[\ldots]$ "closed linear span", and $f = (a_n)_{n \in \mathbb{N}}$ be a linearly independent sequence of vectors in $B$.

Call $K(f) = \bigcap_{n \in \mathbb{N}} [a_n, a_{n+1}, \ldots]$ (kernel of $f$) and

$$K_s(f) = \{K(f') : f' is a subsequence (infinite) of f\}$$ (strict kernel of $f$).

$f$ is normalized if $\|a_n\| = 1$ ($n \in \mathbb{N}$).

$f$ is basic if there is a unique sequence of scalars $(\lambda_n)_{n \in \mathbb{N}}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n a_n,$$

for every $x \in [f]$.

The sequence $(a_n - a_{n+1})_{n \in \mathbb{N}}$ is called sequence of differences of $f$.

$f$ is said to be weakly convergent to $x \in B$ if $\lim f(a_n) = f(x)$, for every $f \in B^*$ (dual of $B$).

$f$ is said to be minimal if there exists a sequence $(a_n^*)$ in $[I]$ with $a_n^*(a_m) = \delta_{nm}$ (Kronecker indices), and uniformly minimal if it also verifies $\sup_n \|a_n\| \cdot \|a_n^*\| < \infty$.

2. The main result.

The result leans on the following two lemmas:

**Lemma 1**: Every subsequence $f'$ of a given sequence $f = (a_n)_{n \in \mathbb{N}}$ has zero strict kernel if and only if the normalized sequence $f_N = (a_n/\|a_n\|)_{n \in \mathbb{N}}$ has no subsequence weakly convergent to some vector distinct from zero.

**Proc.**: See $|T|$, p. 172.
Lemma 2: Let $f = (a_n)_{n \in \mathbb{N}}$ be a minimal sequence with zero kernel. Let $x \in \mathcal{H}$ such that the set $S_x = \{ k \in \mathbb{N} : a_k^*(x) \neq 0 \}$ is infinite. We note $S_x = (p_n)_{n \in \mathbb{N}}$. Then

$$x \in K \left( \sum_{h=1}^{n} a_h^*(x) a_{p_h} p_{h_n} \right) \text{ if and only if the sequence } \left( \sum_{h=1}^{n} a_h^*(x) a_{p_{h_n}} p_{h_n} \right) \text{ is weakly convergent to } x.$$

Proof: (See [I-T]). It follows from Lemma 1 and the third Fréchet's axiom of convergence (see $|X|$).

Now, we finally have the

Theorem: Let $B$ be a Banach space. Then the following statements are equivalent:

1. $B$ is reflexive,
2. Every normalized basic sequence $(a_n)_{n \in \mathbb{N}}$ in $B$ is weakly convergent to zero,
3. Every normalized basic sequence $(a_n)_{n \in \mathbb{N}}$ in $B$ verifies

$$a_1 \in \left[ a_n - a_{n+1} ; n \in \mathbb{N} \right].$$

Proof: In $|P|$ has been proved that (i) is equivalent to (ii).

(ii) implies (iii) is obvious, considering

$$a_1 - a_n = \sum_{i=1}^{n-1} (a_i - a_{i+1}).$$

(iii) implies (ii):

Suppose that for every normalized basic sequence $f = (a_n)_{n \in \mathbb{N}}$, $a_1 \in \left[ a_n - a_{n+1} ; n \in \mathbb{N} \right]$. 


Notice that \( a_1 \in \left[ a_n - a_{n+1} ; n \in \mathbb{N} \right] \) if and only if \( a_1 \in K((a_1 - a_n)_n) \)
(see, for instance, \(|\mathbb{R}|, \text{proposition 2.2}\))

Take \( (p_n)_{n \in \mathbb{N}} \) a subsequence of \( \mathbb{N} \), with \( p_1 = 1 \). By hypothesis, the
sequence \( (a_{p_n})_{n \in \mathbb{N}} \) also verifies \( a_1 \in \left[ a_{p_n} - a_{p_{n+1}} ; n \in \mathbb{N} \right] \), so, it
follows that \( a_1 \in K((a_1 - a_n)_{n \in \mathbb{N}}) \).

Now, applying lemma 2 to \( a_1 \) and \( (a_n - a_{n+1})_{n \in \mathbb{N}} \), we have that \( (a_n)_{n \in \mathbb{N}} \)
is weakly convergent to \( a_1 \), and therefore \( (a_n)_{n \in \mathbb{N}} \) is weakly convergent
to zero.

3. Equivalent versions.

In \(|\text{CH-I}| \) (preprint of this paper) the following equivalent versions
of the theorem are given:

(iv) \( \left[ a_n ; n \in \mathbb{N} \right] = \left[ a_n - a_{n+1} ; n \in \mathbb{N} \right] \), for every normalized basic
sequence \( (a_n)_{n \in \mathbb{N}} \) in \( B \),

(v) Let \( (a_n)_{n \in \mathbb{N}} \) be a normalized basic sequence in \( B \). Then, its
sequence of differences cannot be uniformly minimal,

(vi) For every normalized basic sequence \( (a_n)_{n \in \mathbb{N}} \) in \( B \),

\( \left[ a_n - a_{n+1} ; n \in \mathbb{N} \right] \) cannot be an hyperplane in \( \left[ a_n ; n \in \mathbb{N} \right] \).

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4. References

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