Qualitative properties of stationary states of some nonlocal interaction equations

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Certifico que la present memòria ha estat realitzada per en Daniel Balagué Guardia, sota la direcció de José Antonio Carrillo de la Plata i José Alfredo Cañizo Rincón.
Bellaterra, abril de 2013.

Signat: Dr. José Antonio Carrillo de la Plata i Dr. José Alfredo Cañizo Rincón.

I a vegades ens en sortim, i a vegades ens en sortim.
I a vegades una tonteria de sobte
ens indica que ens en sortim.
I a vegades se'ns baixa La Verge i de sobte
ens revela que ens en sortim.

Manel
Fragment de la cançó *Captatio benevolentiae*

Yoda: *No more training do you require. Already know you, that which you need.*
Luke: *Then I am a Jedi.*
Yoda: *No. Not yet. One thing remains. Vader. You must confront Vader. Then, only then, a Jedi will you be. And confront him you will.*
Conversation between Master Yoda and Luke Skywalker.

George Lucas and Lawrence Kasdan,
Star Wars: Episode VI - Return of the Jedi

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Dues tires còmiques de PhD Comics estretament relacionades amb aquesta tesi...

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Introduction

Kinetic equations appear naturally when trying to model natural phenomena in physics or biology. They are typically a description of the average behavior of a large number of particles in the sense of statistical physics. Two fundamental classical models in kinetic theory are the Vlasov-Poisson equation, which models the time evolution of the distribution of electrons in a plasma [68, 112], or the classical Boltzmann equation for the behavior of molecules in rarefied gases [46, 123].

For a given particle system one can usually write a system of ordinary differential equations (ODEs) to model their behavior, usually equations based on Newton s laws. The equations typically describe the movement of each particle in time, which depends on the positions and the velocities of other particles. When taken together, these equations give us a microscopic description of the problem. Nevertheless, when the number of particles is large (as it happens when modeling any macroscopic part of the world) it is very di cult to follow the evolution of each particle, and another kind of framework is needed. A basic idea in kinetic theory is to think about the gas as a continuum and, instead of looking at each particle individually, to study the density of particles in space and velocity. It is often the case for physical particle systems that one can (at least formally) take a suitable limit when the number of particles is large and obtain a more manageable partial di erential equation (PDE) which describes the system in an averaged sense. In the case of electrons in a plasma, one can formally obtain the Vlasov-Poisson equation by a *mean-field limit* [68, 74]; in the case of particles in a rarefied gas, one can obtain the Boltzmann equation through the *Boltzmann-Grad limit* [46, 34, 113, 123]. In this process, one passes from a microscopic level of description to a PDE that we usually call a *mesoscopic* equation. One can also model these systems at di erent scales of detail by taking di erent limits. For example, one can obtain what we call a *macroscopic* description of a uid in which the density depends only on the position and time by considering a di erent limit of the same particle system used to derive the Boltzmann equation.

Nowadays, kinetic equations are used to study a wide variety of problems besides the classical ones. In more recent applications particles may represent di erent objects depending on the problem: polymers in materials science [33], animals in swarming [122, 38], cell movement in chemotaxis processes [29, 101], galaxies in astrophysics [112] or fuel particles in diesel engines [82].

However, it is not always possible to rigorously obtain mesoscopic or macroscopic equations as a limit of microscopic systems when the number of particles is large. This is the case of uid equations, where a rigorous microscopic derivation by Newton s laws is not known. The same happens with the Vlasov-Poisson equation where a classical problem is to determine the existence of a mean field limit for particle systems interacting through a Coulombian force [74]. In the case of the Boltzmann equation, the limit is known to hold rigorously for a short time, but no results are known globally in time. In a similar way, one may take scaling limits of mesoscopic PDEs (di usive or hydrodynamic limits) which lead to macroscopic equations, in which the main di erence is that one looks at the density depending only on the position in space (and thus the density in velocity is described only in an averaged sense). Typical terms that appear in macroscopic models are drift and/or di usion terms and these terms govern the macroscopic density of particles. The problem of showing that these limits can be taken rigorously is also a central one in kinetic theory [14, 13, 28, 123, 125, 69]. We have to take into account that when one changes scales, from the more detailed to the less detailed problem, from the microscopic to the macroscopic model, deep phenomena arise such as the passage from reversible to irreversible processes in the case of Boltzmann equation, and thus fundamental di culties have to be overcome in the rigorous study of these limits.

An important tool to study macroscopic models is the natural Lyapunov functional associated to the system, when this functional exists. Its defining property is that it is decreasing as a function of time along solutions of the PDE. Usually, these functionals are motivated by physics and, depending on the cases, they are given by the free energy, the entropy or the total energy. But it is not always the case: in some models there exist Lyapunov functionals that do not have any direct relation with physics [95, 96]. Further on we will give two examples of these Lyapunov functionals, one in interaction equations where the functional is motivated by physics, and one in fragmentation equations, where the functional comes from general properties of linear equations of a certain type. The first case is an instance of a relatively recent point of view in which our macroscopic equation has what is called a structure of a *gradient flow* with respect to the Lyapunov functional in the sense of the Wasserstein distance [77, 102, 124, 41, 42, 4]. In this sense, one can write macroscopic equations as a continuity equation in which the velocity field becomes minus the gradient of the variation of the Lyapunov functional with respect to the density. In the next section, we show an example of these gradient flow structures for a particular kind of interaction equations.

In order to find stationary states of the equation that are stable with respect to this dynamics, one can look for local minimizers of the interaction Lyapunov functional. To take advantage of this particular structure, we will use two different techniques. The first one is the so-called steepest descent method or the variational scheme for the Lyapunov functional. As an example applied to the Fokker-Plank equation, we refer to the Jordan-Kinderlehrer-Otto (JKO) scheme [77]. This method consists, roughly speaking, in choosing a time step and, at each step, to minimize the free energy functional plus the Wasserstein distance. There are several studies about the conditions that the interaction potential should satisfy for this method to converge. For instance, it was proven in [4] that for C^1 smooth potentials this method converges to weak solutions of the PDE, and one has global well-posedness and uniqueness for measures as initial data. In other cases, we may need to use a different strategy and we can use the comparison between the entropy and the entropy dissipation. This technique gives us stability of the stationary states proving convergence to the local minimizers and also gives us estimates on the rate of convergence to equilibrium. We will use this technique in Chapter 5.

Now, we summarize the results of this thesis. In Chapter 2, we start with an introduction to aggregation equations in the sense of [84], with repulsive-attractive radial interaction potential. For these aggregation models we make use of the gradient flow structure. We derive some existence results and convergence to spherical shell stationary states. We study radial ins/stability of these particular stationary states. Chapter 3 studies confinement properties of solutions of aggregation equations under certain conditions on the interaction potential. We show that solutions remain compactly supported in a large fixed ball for all times. We continue our research in aggregation equations in Chapter 4 where we characterize the dimensionality of local minimizers of the interaction energy.

Another problem that we study is the asymptotic behavior of growth-fragmentation models. In Chapter 5 we give estimates on asymptotic profiles and a spectral gap inequality for growth-fragmentation equations. These models are not a gradient flow of a particular energy functional. However, they have a Lyapunov functional that we use to prove exponentially fast convergence of solutions to the asymptotic profiles by showing an entropy - entropy dissipation inequality.

We finish this thesis with the results in Chapter 6, where we study two second order particle systems for swarming. We refer to these systems as individual based models (IBMs), which is the common language used in swarming. We prove the stability of two particular solutions: ock rings and mill rings. We relate the stability of these ring solutions of the second order models with the stability of the rings of a first order model, the discrete version of the aggregation equation of Chapter 2.

All these models share the common property of nonlocality and the existence of a Lyapunov functional. In the case of the interaction equations that we will consider at the beginning and the second order models in swarming, they have in common the nonlocal interaction term $(\nabla W * \ )$, where W is the interaction potential, and the density of particles in space. The case of the fragmentation equations is a bit di erent: they are integro-di erential equations, with the nonlocal term given by the fragmentation operator, an integral of a kernel against the density of particles.

Each chapter is presented in a self-contained way. The logical relation between the chapters is the following: Chapter 2 implies Chapters 3 and 4. Chapters 2 and 4 imply Chapter 6. Finally Chapter 5 is fully self-contained. The logical orders of reading this dissertation would be the order it is presented, or Chapter 5 and then Chapters 2, 3, 4 and 6.

Aggregation equations

Interaction equations can be interpreted as limits of systems with large number of particles. In this kind of systems, each particle not only interacts with its neighbors but also with particles far away. Nonlocal interaction equations that we are going to consider are of the form

$$\frac{\partial \mu_t}{\partial t} + \operatorname{div}(\mu_t v) = 0, \qquad v = -\nabla W * \mu_t, \tag{1}$$

where $\mu_t(x)$ is the probability measure of particles at a time t, W is the interaction potential and $v(t,x)$ is the velocity field of the particles at location $x \in \mathbb{R}^N$ at time t. We call the equation (1) the *aggregation equation.* All along this work we are going to suppose that W is radial, i.e., that $W(x) = k(|x|)$ and smooth enough away from the origin.

They are formal gradient ows with respect to the Wasserstein distance, see [41, 4, 124, 42], of the interaction energy

$$E_W[\mu_t] = \frac{1}{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} W(x-y) d\mu_t(x)\, d\mu_t(y). \tag{2}$$

As we will see below, this fact plays an important role in our analysis.

The interaction energy E_W arises in many contexts. In physical, biological, and materials science it is used to model particles or individuals effect on others via pairwise interactions. Given n particles located at $X_1, \ldots, X_n \in \mathbb{R}^N$, their discrete interaction energy is given by

$$E_W^n[X_1, \ldots, X_n] := \frac{1}{2n^2} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} W(X_i - X_j). \tag{3}$$

Then, one can think of equation (2) as a limit of (3) when $n \to \infty$.

In biological aggregation models [98, 97, 118, 16, 54], individuals in social aggregates (e.g., swarm, flock, school, or herd), materials science [56, 128, 111, 73, 126], particles, nano-particles, or molecules self-assemble in a way to minimize energies similar to (3) according to pairwise interactions generated by a repulsive- attractive potential (see the full set of references in Chapter 2).

Collective behavior is often observed in these applications, it usually stems from the fact that individuals attract each other in the long range in order to remain in a cohesive group, but they repulse each other in the short range not to be so close to other individuals to avoid collisions. To model these phenomena, one considers potentials of the form $W(x) = k(|x|)$ where $k : [0, +\infty) \to (-\infty, +\infty]$ is decreasing on some interval $[0, r_0)$ and increasing on $(r_0, +\infty)$. The function k may or may not have a singularity at $r = 0$. We will refer to such potentials as being *repulsive-attractive*. If the potential is purely attractive, particles will aggregate and collapse. But when the potential has both effects, repulsion and attraction, it is more likely that the solution converges to an stationary state.

The 1D case has been studied in a series of works [61, 62, 110]. It was shown that the behavior of solutions depends on the regularity of the interaction potential: for regular interaction, the solution converges to a sum of Dirac masses, whereas for singular repulsive potential, the solution converges to uniformly bounded densities.

Well-posedness

Now, we discuss some results about well-posedness of L^p-solutions. Global existence and uniqueness of L^p-solutions of equation (1) was studied in [22]. We denote by $\mathcal{W}^{m,p}(\mathbb{R}^N)$, $1 \le p \le \infty$ and $m \in \mathbb{N}$, the classical Sobolev spaces of L^p functions with derivatives up to order m in L^p and $\mathcal{P}^2(\mathbb{R}^N)$ the space of probability measures with bounded second moment.

Theorem 1 (L^p Well-posedness theory [22, Theorem 1])**.** *Consider* $1 \le q \le \infty$ *and* p *its Hölder conjugate. Suppose* $\nabla W \in \mathcal{W}^{1,q}(\mathbb{R}^N)$ *and* $\rho_0 \in L^p(\mathbb{R}^N) \cap$

$\mathcal{P}_2(\mathbb{R}^N)$. Then there exists a time T^* 0 and a nonnegative function $\in C([0,T^*], L^p(\mathbb{R}^N) \cap \mathcal{P}_2(\mathbb{R}^N)) \cap C^1([0,T^*], \mathcal{W}^{-1,p}(\mathbb{R}^N))$ such that (1) holds in the sense of distributions in $\mathbb{R}^N \times [0,T^*)$ with $d\mu_t = \ (t,x)\,dx$. Moreover, the function $t \to \| \ (t)\|^p_{L^p}$ is differentiable and satisfies

$$\frac{d}{dt}\{\| \ (t)\|^p_{L^p}\} = -(p-1) \int_{\mathbb{R}^N} (t,x)^p \mathrm{div}\ v(t,x)\ dx \qquad \forall t \in [0,T^*]. \tag{4}$$

Furthermore, if ess sup W $+\infty$, *then $t \to \| \ (t)\|^p_{L^p}$ does not grow faster than exponentially and $T^* = +\infty$.*

In this setting, uniqueness was proven using the ideas in [44]. The authors in [22] also prove global existence when W is bounded from above.

The repulsive-attractive potentials we will consider do not satisfy the condition $\nabla W \in \mathcal{W}^{1,q}(\mathbb{R}^N)$. In fact, we allow them to grow at infinity. For that reason in Theorem 2.2 in Chapter 2 we slightly extend the global-in-time well-posedness L^p- theory to repulsive-attractive potentials restricted to compactly supported initial data and, essentially, $\nabla W \in \mathcal{W}^{1,q}_{loc}$. Thanks to that, one can prove existence of solutions for all times for repulsive-attractive potentials and compactly supported initial data $_0 \in L^p(\mathbb{R}^N)$.

Moreover, if the initial data has more regularity, the solution will be more regular as well. Using classical analysis techniques it is not di cult to prove existence of classical solutions for initial data in $C^1([0,T] \times \mathbb{R}^N) \cap \mathcal{W}^{1,\infty}_{loc}(\mathbb{R}_+, \mathcal{W}^{1,\infty}(\mathbb{R}^N))$.

Weak measure solutions for the aggregation equation (1) with attractive potentials were constructed in [36, 35]. In this setting, measure initial data are allowed. Moreover, gradient- ow techniques are used to prove uniqueness and stability of solutions. One disadvantage is the restriction imposed on the potential W at the origin. The attractive potential W was required to be, at worst, Lipschitz singular at the origin, that is $W(x) \sim |x|$ with 1. Authors also prove well-posedness theory of measure solutions for -convex potentials with the same restriction at the origin.

In the L^p framework it is possible to consider potentials having a singularity of the form $|x|$ with $2-N$ and initial data in $L^1 \cap L^p$ with bounded second moment. In our results in Chapters 2 and 4, we consider potentials W with these properties.

Blow-up of solutions with initial data in $L^1 \cap L^p$, with 1 p ∞, occurs when the potential is fully attractive at $r = 0$. The density of particles collapses on itself to a Dirac delta function located at the center of mass. The question is to know when this phenomenon happens. In [18] the authors characterize this problem using Osgood conditions for well-posedness of the ODE characteristics. If the potential satisfies these conditions then blow-up occurs in infinite time whereas if they are not satisfied then it occurs in finite time.

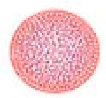
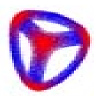
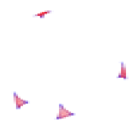
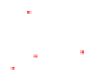
			
$= -0.5$ $= 5$	$= 0.5$ $= 23$	$= 1.25$ $= 15$	$= 2.5$ $= 5$

Table 1: Evolution of local minimizers in 3D when $\alpha > -1$ increases. The computations were done with $n = 2,500$ particles.

However, our potentials are not fully attractive. Since we allow repulsive-attractive potentials the dynamics and structure of stable stationary states can be much more complex. The authors in [81, 127, 126] perform a numerical study of the finite particle version of (1) and show that a repulsive-attractive potential can lead to very complicated patterns, as seen in Table 1. Further on, we will show that we can characterize the dimensionality of the local minimizers of the interaction energy studying the Laplacian of the potential generated by a density.

Radial ins/stability for nonlocal interaction equations

In Chapter 2 we study these nonlocal interaction equations with repulsive-attractive potentials. We focus on the study of the convergence of radially symmetric solutions toward spherical shells. A spherical shell is nothing but a uniform distribution on a sphere. We will denote the spherical shell of radius R by δ_R. We can explain the idea about the existence of these spherical shell stationary states with Figure 1.

The particle in black has a zone of repulsion determined by the black circle. Particles inside this circle (particles in blue) will interact with the black one with a repulsive force. But all the components of these forces except the radial direction will compensate exactly. We call F_r the total repulsive force in the radial direction. The same happens with particles on the red arc. These particles will interact with an attractive force. We denote by F_a the total attractive force in the radial direction. Then, by increasing or decreasing R, one expects to find an R^* for which $F_a + F_r = 0$.

Since the equation can be understood as a gradient ow of (2), the stable stationary states of (1) will be local minimizers of the interaction energy. We say that a probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ is a *steady state* of the nonlocal

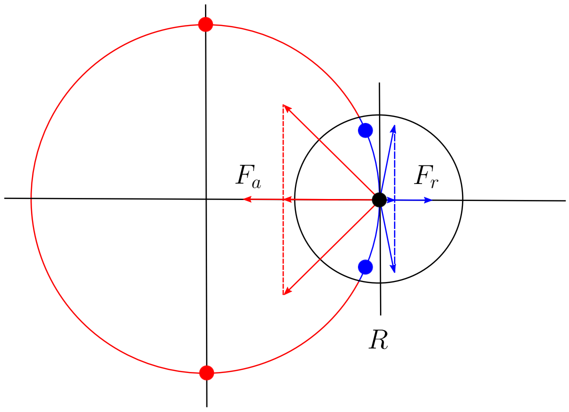


Figure 1: Sketch of the repulsive and attractive forces.

interaction equation of (1) if $(\nabla W * \mu)(x) = 0$ for all $x \in \text{supp}(\mu)$. This condition is equivalent to saying that the dissipation is zero. This can be seen if one formally computes the dissipation of the interaction energy:

$$\frac{d}{dt}E[\mu_t] = -\int_{\mathbb{R}^N} |v(t,x)|^2 \, d\mu_t(x).$$

To show results about stability of a uniform distribution of radius R (from now on, δ_R), it is convenient to write (1) in radial coordinates. Given $T > 0$, $C([0,T];\mathcal{P}^r(\mathbb{R}^N))$ denotes the set of continuous curves of radial measures where continuity is with respect to the weak-$*$ convergence. We say that $\mu \in C([0,T];\mathcal{P}^r(\mathbb{R}^N))$ is a radially symmetric solution of (1) if $\hat{\mu} \in C([0,T];\mathcal{P}([0,+\infty))$ satisfies the one dimensional conservation law:

$$\partial_t\hat{\mu} + \partial_r(\hat{\mu}\hat{v}) = 0, \tag{5}$$

$$\hat{v}(t,r) = \int_0^\infty \omega(r,\eta)d\hat{\mu}_t(\eta)\,, \tag{6}$$

in the distributional sense. In these new variables, the velocity field is not a convolution anymore, but it becomes an integral of the radial density $\hat{\mu}$ against the kernel $\omega(r,\eta)$, which depends on ∇W as follows:

$$\omega(r,\eta) = -\frac{1}{\sigma_N}\int_{\partial B(0,1)} \nabla W(re_1 - \eta y)\cdot e_1 \, d\sigma(y),$$

where the constant σ_N is the surface area of the unit sphere in $\mathbb{R}^N$, the vector e_1 is the first vector of the canonical basis of $\mathbb{R}^N$, $d\sigma$ denotes the volume element of the manifold where the integral is performed and $\mathbb{R}^2 = (0,+\infty)\times(0,+\infty)$.

A spherical shell of radius R will be a local minimizer of the interaction energy if the interaction potential W satisfies:

(C0) Repulsive-attractive balance: $\omega(R, R) = 0$,

(C1) Fattening stability: $\partial_1 \omega(R, R) \geq 0$,

(C2) Shifting stability: $\partial_1 \omega(R, R) + \partial_2 \omega(R, R) \geq 0$.

To summarize a bit these three conditions, **(C0)** just guarantees that a spherical δ_R is a critical point of the interaction energy. The second and the third give an idea about the stability of a δ_R and can be interpreted in terms of the energy. Suppose that W is such that $\omega \in C^1(\mathbb{R}^2_+)$. Then, if **(C1)** is not satisfied, i.e., $\partial_1 \omega(R, R) < 0$, we can find a better candidate as a minimizer of the interaction energy by splitting the spherical shell into two spherical shells. If condition **(C2)** is not satisfied, i.e., $\partial_1 \omega(R, R) + \partial_2 \omega(R, R) < 0$, then by increasing or decreasing the radius of the spherical shell we can decrease the energy. These results extend to nonradial cases. We can consider steady states $\overline{\mu}$ supported on C^2 hypersurfaces, see the results in Subsection 2.3.2 in Chapter 2.

We can also characterize the instability of the spherical shells in an L^p framework. To do that, we need to take into account that if the potential W is more singular than $|x|^{3-N}$ at the origin, the kernel $\omega \notin C^1(\mathbb{R}^2_+)$. Then, our instability result can be described as follows. Either if $\omega \in C^1(\mathbb{R}^2_+)$ and $\partial_1 \omega_1(R, R) < 0$, or $\omega \notin C^1(\mathbb{R}^2_+)$ and $\lim_{\substack{(r,\eta)\notin \mathcal{D} \\ (r,\eta)\to(R,R)}} \partial_1 \omega_1(r, \eta) = +\infty$, (i.e., if one of these two conditions is satisfied) then it is not possible for an L^p radially symmetric solution of (5)-(6) to converge weakly-$*$ as measures to δ_R as $t \to \infty$.

We give a brief definition of d_2 and d_∞ distances that we will use below. The precise definitions can be found in Chapter 1. The distance d_2 between two measures μ and ν is defined by

$$d_2^2(\mu, \nu) = \inf_{\pi \in \Gamma(\mu,\nu)} \left\{ \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^2 d\pi(x, y) \right\},$$

where $\Gamma(\mu, \nu)$ is the set of all transference plans π between $\mu \in \mathcal{P}(\mathbb{R}^N)$ and $\nu \in \mathcal{P}(\mathbb{R}^N)$ satisfying

$$\pi(A \times \mathbb{R}^N) = \mu(A) \quad \textit{and} \quad \pi(\mathbb{R}^N \times A) = \nu(A)$$

for all measurable sets $A \in \mathcal{B}(\mathbb{R}^N)$. The distance d_∞ is defined as

$$d_\infty(\mu, \nu) = \inf_{\pi \in \Gamma(\mu,\nu)} \sup_{(x,y)\in \mathrm{supp}(\pi)} |x - y| .$$

On the other hand, we can ensure that under *hypotheses of minimal regularity* **(HMR)**, we refer to Section 2.4 of Chapter 2 for more details, one has stability in the following sense:

Theorem 2 (Stability for local perturbations)**.** *Assume* $\omega \in C^1(\mathbb{R}^2_+)$ *and that* δ_R *is a stationary solution to* (5)-(6)*. Let us assume that* **(C1)** *and* **(C2)** *are satisfied with strict inequality:*

$$\partial_1\omega(R,R) \quad 0 \qquad \text{and} \qquad \partial_1\omega(R,R) + \partial_2\omega(R,R) \quad 0\,. \tag{7}$$

Then there exists $_0$ $\quad 0$ *such that if the initial data* $\mu_0 \in \mathcal{P}_2^r(\mathbb{R}^N)$ *satisfies* $\mathrm{supp}(\hat{\mu}_0) \quad [R - \ _0, R + \ _0]$*, and for any solution to* (5)-(6) *with initial data satisfying* **(HMR)** *we get*

$$d_2(\hat{\mu}_t, \delta_R) \quad Ce^{- \ t},$$

for any $0 \qquad - \max\left(\partial_1\omega(R,R),\ \frac{d}{dR}\omega(R,R)\right)$ *for suitable* C*.*

To give an intuition of above results and Theorem 2 we can look at Figure 2. For a stable spherical shell, one has that the velocity field generated by a δ_R is decreasing at the stable radius. If we think about particles, this means that if they are close to the spherical shell they are pushed towards it. On the contrary, if the spherical shell is unstable, we can observe that the velocity field generated by a δ_R is increasing near R. If particles are close to the unstable spherical shell, they will be pushed away from it.

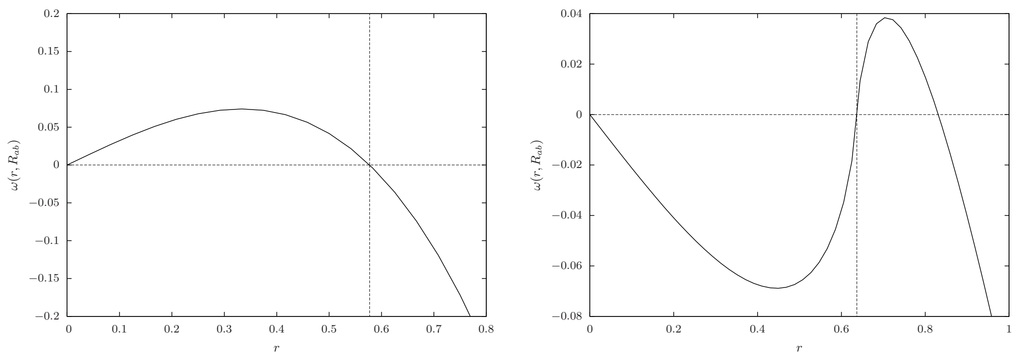


Figure 2: Left: Velocity eld generated by a δ_R of radius $R_{ab} = \frac{\sqrt{3}}{3}$, with interaction potential $W(x) = \frac{|x|^4}{4} - \frac{|x|^2}{2}$ in $\mathbb{R}^2$. Right: Velocity eld generated by a δ_R of radius $R_{ab} \sim 0.6366$, with interaction potential $W(x) = \frac{|x|^2}{2} - |x|$ in $\mathbb{R}^2$. In both gures, the vertical line marks the position of the radius of δ_R.

For global stability results, it is necessary to have some control on the tails of $\omega(r,R)$. These results can be found in Section 2.4.

One interesting example, for its simplicity, is the case when the potential is a sum of powers. More precisely, if $W(x)$ is of the form:

$$W(x) = -\frac{|x|^b}{b} + \frac{|x|^a}{a} \quad a \quad b \quad 2 - N. \tag{8}$$

For the case of $a \quad b \quad 2 - N$ we can give some information about radially symmetric solutions. If W is a power-law repulsive-attractive potential such as (8) and the initial data is regular enough, radially symmetric and compactly supported, then there exists a global in time radially symmetric solution for the radial problem (5)-(6). Furthermore, this solution is compactly supported and confined in a large ball for all times. In this case, the function $\omega(r, \eta)$ can be explicitly computed and allows us to find the exact value of the radius of the spherical shell in terms of the powers a and b by using special functions. What is more, it is also possible to draw a diagram for the ins/stability of a spherical shell in terms of the powers (see Figure 2.1 in Chapter 2).

Confinement properties for nonlocal interaction equations

In this section, we present results about confinement properties of solutions of (1) with compactly supported initial data. All these results can be found in Chapter 3.

We say that the nonlocal equation (1) has the confinement property if every solution μ_t with compactly supported initial data μ_0 is compactly supported for all times and its support remains in a fixed ball whose radius only depends on μ_0 and the interaction potential W. Assuming always well-posedness theory in [36, 35] for weak measure solutions of (1), we can study confinement properties of (1) by use of the associated particle ODE system:

$$\dot{x}_i = -\sum_{j \in Z(i)} m_j \nabla W(x_i - x_j), \quad i = 1, \ldots, n, \tag{9}$$

where $Z(i) = \{j \in \{1, \ldots, n\} : x_j(t) \neq x_i(t)\}$, and $x_i(0) \in \mathbb{R}^N$ for all $i = 1, \ldots, n$, $0 \quad m_i \quad 1$, and $\sum_i m_i = 1$. In addition, due to translation invariance, we assume that the center of mass is located at the origin.

Under technical assumptions on W (see [36, 35] or Section 3.2 for the details), we can approximate the solutions of (1) by solutions of (9) due to the fact that the mean-field limit holds. We will assume as in [10, 8] that W is radially symmetric and attractive outside some ball,

(NL-RAD) W is radial, i.e., $W(x) = w(|x|)$, and there exists R_a 0 such that $w'(r)$ 0 for r R_a.

Moreover, repulsion should not be too strong, more precisely,

$$C_W := \sup_{x \in B(0,R_a)\setminus\{0\}} |\nabla W(x)|$$

is bounded above. The following condition tells us how fast the attractive strength decays at infinity.

(NL-CONF) $\lim_{r\to\infty} w'(r)r = +\infty,$

The main idea to prove confinement is to couple the evolution of the third moment with the evolution of the particle in the system which is the furthest from the center of mass. This idea gives a nice proof when the mean field limit for measure initial data holds, but it is useless in the case of the Newtonian repulsion potential for $L^1 \cap L^\infty$ solutions. The key idea in the case of the locally Newtonian repulsive potential is that we have L^∞-uniform bounds for solutions. We do not know how to obtain these L^∞-estimates for potentials less singular than the Newtonian at the origin.

Nonetheless, numerical simulations show that our results for confinement are not sharp. We consider a potential behaving like $\log(\log(r))$ at infinity given by

$$w(r) = \begin{cases} \frac{r(r-\mathrm{e})}{\mathrm{e}^2} - 2\,\frac{r(r-\mathrm{e})^2}{\mathrm{e}^3} + \frac{19}{6}\,\frac{r(r-\mathrm{e})^3}{\mathrm{e}^4} & 0 \quad r \quad \mathrm{e}, \\ \log\left(\log\left(r\right)\right) & r \quad \mathrm{e}. \end{cases} \tag{10}$$

This potential satisfies $w(0) = w(\mathrm{e}) = 0$, it is C^3 in $(0, +\infty)$, and the repulsion at the origin is asymptotically like $-r$. The numerical experiments suggest that by increasing the number of particles, the radius of the support of the stationary state increases and stabilizes.

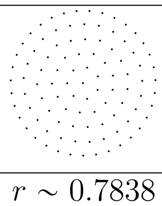
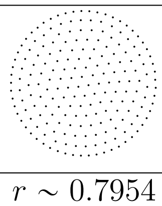
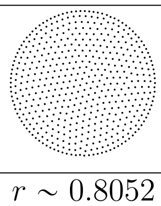
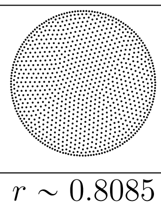
$n = 100$	$n = 200$	$n = 500$	$n = 1000$
			
$r \sim 0.7838$	$r \sim 0.7954$	$r \sim 0.8052$	$r \sim 0.8085$

Table 2: Stationary states and radius of their support as a function of the number of particles for the potential $w(r)$ given in equation (10).

Table 2 shows the stationary states as a function of the number of particles n for this potential. These numerical simulations, together with the evolution of the radius of the support both in time and as a function of the number of particles (see Chapter 3), indicate that even if the growth at infinity of the potential is less than $\log(r)$ there is still confinement. This behavior has been also observed in some particular cases of the Morse potential (exponentially decaying repulsive-attractive potentials, see Subsection 3.4.3 for the precise definition).

Dimensionality of local minimizers of the interaction energy

Given a Borel measurable function $W : \mathbb{R}^N \to (-\infty, +\infty]$ which is bounded from below, we recall that the interaction energy of a positive measure μ is given by

$$E_W[\mu] := \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)\, d\mu(x) d\mu(y)\,. \tag{11}$$

For a large number of particles, the continuum energy (11) is well approximated by the discrete energy (3) where $d\mu(x)$ is a general distribution of particles at location $x \in \mathbb{R}^N$. In fact, the continuum energy (11) of the discrete distribution $\frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$ reduces to (3).

We consider now the properties of the potential $W(x) = k(|x|)$. Since k has a global minimum at r_0, it is obvious that if we consider only two particles X_1 and X_2, in order to minimize $E_W^2[X_1, X_2]$, the two particles must be located at a distance r_0 from one another. Whereas the situation is simple with two particles, it becomes very complicated for a large number of particles. Recent works [81, 127, 64, 126, 80, 117, 110, 61, 62, 10] have shown that such repulsive-attractive potentials lead to the emergence of surprisingly rich geometric structures. Our aim is to understand how the dimensionality of these structures depends on the singularity of W at the origin.

Let us describe the main results of Chapter 4. Typically, the Laplacian of a repulsive-attractive potential will be negative in a neighborhood of the origin. We show that if

$$W(x) \sim -\frac{1}{|x|} \qquad \text{as } x \to 0 \tag{12}$$

for some 0 N, then the support of local minimizers of E_W has Hausdor dimension greater or equal to . We say that a locally integrable potential is -*repulsive* at the origin if there exist and C 0 such that

$$- \quad {}^0W(x) \quad \frac{C}{|x|} \qquad \text{for all} \qquad 0 \quad |x|$$
$$- \quad {}^0W(0) = +\infty.$$

Then one has the following result.

Theorem 3. *Suppose W satisfies* **(H1)** **(H4)** *(see below) and let μ be a compactly supported local minimizer of the interaction energy with respect to the topology induced by d_∞. If W is -repulsive at the origin, 0 N, then the Hausdorff dimension of the support of any part of μ is greater than or equal to .*

The precise hypotheses needed on W for this result to be true, are the following.

(H1) W is bounded from below.

(H2) W is lower semicontinuous (l.s.c.).

(H3) W is uniformly locally integrable: there exists M 0 such that

$$\int_{B(x,1)} W(y)dy \quad M \quad \text{for all} \quad x \in \mathbb{R}^N.$$

(H4) There exists C^* 0 such that

$$W(x) \quad C^* \quad \forall x \in \mathbb{R}^N \text{ and } \forall \ \in (0,1),$$

where W is the approximate Laplacian of a locally integrable function W,

$$- \quad W(x) := \frac{2(N+2)}{2} \left(W(x) - -_{B(0,\ )} W(x+y)dy \right),$$

and $-_{B(x_0,r)} f(x)dx$ stands for the average of f over $B(x_0, r)$.

The exponent appearing in (12) quantifies how repulsive the potential is at the origin. Therefore our result can be intuitively understood as follows: the more repulsive the potential is at the origin, the higher the dimension of the support of the local minimizers will be.

Potentials satisfying (12) have a singular Laplacian at 0 and we refer to them as *strongly repulsive at the origin.* The second main result is devoted to potentials which are *mildly repulsive at the origin*, that is potentials whose Laplacian does not blow up at the origin. To be more precise we show that if

$$W(x) \sim -|x|^b \qquad \text{as } x \to 0 \qquad \text{for some } b \geq 2 \tag{13}$$

then a local minimizer of the interaction energy cannot be concentrated on smooth manifolds of any dimension except 0-dimensional sets.

Theorem 4. *Let $W \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which is equal to $-|x|^b$ in a neighborhood of the origin. If $b \geq 2$ then a local minimizer of the interaction energy with respect to d_∞ cannot have a k-dimensional component for any $1 \leq k \leq N$.*

Note that this result suggests that local minimizers of the interaction energy of mildly repulsive potentials have zero Hausdorff dimension. However, we are currently unable to prove this stronger result.

Fragmentation equations

We consider in this section one example of linear models that have a Lyapunov functional for which we apply the entropy - entropy dissipation technique. In Chapter 5 we discuss fragmentation-drift equations. They are classical models for the evolution of a population of cells, in polymer physics for the size distribution of polymers, and other processes involving growth and fragmentation. These equations can be interpreted as continuum limits of discrete processes for the evolution of a probability distribution.

The type of fragmentation-drift equations that we are going to study have the form

$$\partial_t g_t(x) + \partial_x(\tau(x) g_t(x)) + \lambda g_t(x) = \mathcal{L}[g_t](x), \tag{14a}$$

$$(\tau g_t)(0) = 0 \qquad (t \geq 0), \tag{14b}$$

$$g_0(x) = g_{\mathrm{in}}(x) \qquad (x \geq 0). \tag{14c}$$

The function $g_t(x)$, which depends on the time $t \geq 0$, is the unknown and represents the density of the objects under study (cells or polymers) of size x at a given time t. The positive function τ represents the *growth rate*. The symbol $\mathcal{L}$ refers to the fragmentation operator and λ is the largest eigenvalue of the operator $g \mapsto -\partial_x(\tau g) + \mathcal{L}g$ acting on functions $g = g(x)$ depending only on x. We call $G(x)$ the associated eigenfunction. The fragmentation operator acts on a function $g = g(x)$ as

$$\mathcal{L}g(x) := \int_x^\infty b(y,x)g(y)\,dy - B(x)g(x)$$

where $b(y,x)$ is defined for $y \quad x \quad 0$, it is the fragmentation coe cient, and $B(x)$ is the total fragmentation rate of cells of size $x \quad 0$ given by

$$B(x) := \ {}_{0}^{x} \frac{y}{x} b(x,y)\,dy \qquad (x \quad 0).$$

For these equations, we give accurate asymptotic estimates for both functions $G(x)$ and $\ (x)$ where is the first eigenfunction of the dual operator defined by $\ \mapsto \ ' + \mathcal{L}^*(\ )$. The existence of these stationary profiles is proven using a truncated problem in $[0, L]$ and letting $L \to \infty$ using similar arguments as the ones in [55].

The dual eigenproblem is interesting because gives a conservation law for (14):

$$\ {}_{0}^{\infty} \ (x) g_t(x)\,dx = \ {}_{0}^{\infty} \ (x) g_{in}(x)\,dx = \mathrm{Cst} \quad (t \quad 0).$$

We define $\ := \ {}_{1}^{x} \frac{\ +B(y)}{\ (y)}\,dy$ and we assume that $B(x) \sim x\ ,\ \ (x) = x\ $ with $\ - \ +1 \quad 0$ (see Section 5.1.1 for more general hypothesis on B and $\ $). In Section 5.2 we provide some estimates for the stationary profiles G and $\ $:

- $G(x) \underset{x\to\infty}{\sim} Ce^{-\ (x)} x^{-\ }$.
- $G(x) \underset{x\to 0}{\sim} Cx^{\ -\ }$, if $\ \ 1$ or $G(x) \underset{x\to 0}{\sim} Cx^{\ -1}$ if $\ \ 1$.
- When $x \quad 1$, $C_1 x \quad \ (x) \quad C_2 x$ if $\ \ 0$ or $C_1 x^{\ -1} \quad \ (x) \quad C_2 x^{\ -1}$ if $\ \ 0$.
- $\ (x) \underset{x\to 0}{\sim} Ce^{-\ (x)}$.

The previous four points give us a summary about the estimates of the profiles and they show that G and are bounded above and below by functions of the same type. In addition, these estimates allow us to prove a spectral gap inequality. The *general relative entropy principle* [95, 96, 83, 30] applies. Denoting $u(x) := \frac{g(x)}{G(x)}$ and choosing the function $H(x) = (x-1)^2$ we define

$$H[g|G] := \ {}_{0}^{\infty} \ G(u-1)^2\,dx \tag{15}$$

$$D[g|G] := \ {}_{0}^{\infty} \ {}_{x}^{\infty} \ (x)G(y)b(y,x)(u(x)-u(y))^2\,dydx, \tag{16}$$

and we obtain that

$$\frac{d}{dt} H[g|G] = -D[g|G] \quad 0.$$

Moreover, it is possible to prove an entropy - entropy dissipation inequality. To compare the entropy with the dissipation we break $u(x) - u(y)$ into intermediate reactions, a similar argument as in [51, 30].

Theorem 5. *Assume that the coefficients satisfy* Hypotheses 1 6 *with one of the following additional conditions on the exponents* $_0$ *and* $_0$*:*

either $_0 \quad 1$,

or $_0 = 1$ *and* $_0 \quad 1 + \quad / \, _0$,

or $_0 \quad 1$ *and* $_0 \quad 2 - \quad _0$.

Consider also that we are in the case $\neq 0$. *Then the following inequality holds*

$$H[g|G] \quad CD[g|G], \tag{17}$$

for some constant $C \quad 0$ *and for any nonnegative measurable function* $g : (0, \infty) \to \mathbb{R}$ *such that* $g = 1$. *Consequently, if* g_t *is a solution of problem* (14) *the speed of convergence to equilibrium is exponential in the* L^2*-weighted norm* $\| \cdot \| = \| \cdot \|_{L^2(G^{-1} \; dx)}$, *i.e.,*

$$H[g_t|G] \quad H[g_0|G] \, e^{-Ct} \qquad \text{for } t \quad 0.$$

Second order models in swarming

As we explained at the beginning of this introduction, individual based models are useful to model collective behavior of individuals. In this case, we are going to use them in swarming. In Chapter 6, we study two particular examples of IBMs. The first one was proposed in [54, 47] for swarming:

$$\begin{cases} \dot{x}_j = v_j \\ \dot{v}_j = (\quad - \quad |v_j|^2) v_j + \dfrac{1}{N} \displaystyle\sum_{\substack{l=1 \\ l \neq j}}^{N} \nabla W(x_l - x_j) \end{cases}, \quad j = 1, \ldots, N \tag{18}$$

for $\;,\;\; 0$. Notice that the model has a self-propelled friction term that gives us an asymptotic speed for the individuals equal to $\sqrt{\;/\;}$. A pairwise interaction is given by the radial interaction potential W. In this section, we use N for the total number of particles instead of the dimension. Here we consider x_j, $v_j \in \mathbb{R}^2$. The second model that we are going to study is

$$\begin{cases} \dot{x}_j = v_j \\ \dot{v}_j = \frac{1}{N}\sum_{l=1}^{N} \frac{1}{(1+r^2)}(v_l - v_j) + \frac{1}{N}\sum_{\substack{l=1 \\ l\neq j}}^{N} \nabla W(x_l - x_j) \end{cases}, \quad j = 1, \ldots, N, \tag{19}$$

with 0. Again, $x_j, v_j \in \mathbb{R}^2$. However, the velocity v_j evolves according to the Cucker-Smale alignment term [48, 49, 72, 71, 37] instead of a self-propulsion e ect, and there is a pairwise interaction given by the radial interaction potential W.

In both cases we will consider a radial potential $W(x) = k(|x|)$. We focus our analysis on the linear stability of a particular type of solutions for these models: *flock rings*.

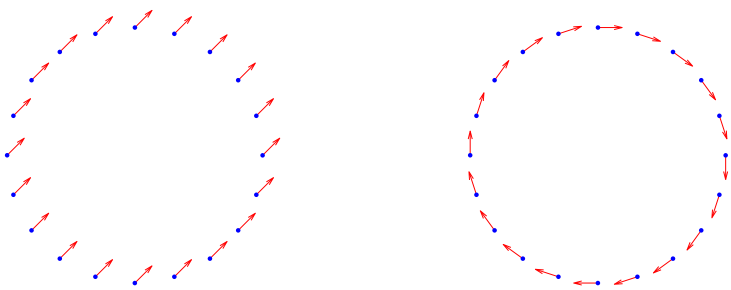


Figure 3: Flock and Mill ring solutions.

The idea is to study the stability of the system of ODEs by studying the eigenvalues of a suitable linearized system with restricted perturbations. We start with subtracting the asymptotic velocity in the systems to avoid the zero eigenvalue due to translations. Otherwise, the ock would always be linearly unstable because this zero eigenvalue generates a generalized eigenvector. Then we impose conservation of mean velocity. We characterize all cases in which the linearized system has eigenvalues with zero real part and their consequences. Finally, we can relate the instability of these second order models with the first order model

$$\dot{X}_j = \sum_{l=1}^{N} \nabla W\left(X_j - X_l\right), \quad j = 1, \ldots, N, \tag{20}$$

We summarize the results in the following theorem.

Theorem. *The linearized second order systems associated to* (18)-(19) *around the flock ring solution have an eigenvalue with positive real part if and only* *if the linearized first order system associated to* (20) *around the ring solution has a positive eigenvalue.*

We illustrate the results in a particular case of a power-law repulsive-attractive potentials

$$k(r) = \frac{r^a}{a} - \frac{r^b}{b}.$$

When one uses repulsive-attractive potentials, new configurations for the ock appear. For instance, individuals may concentrate on points (clustering), lines, or annuli (fattening). Table 3 shows three examples.

$a = 3, b = 2.5$	$a = 7, b = 1.5$	$a = 5, b = 0.5$

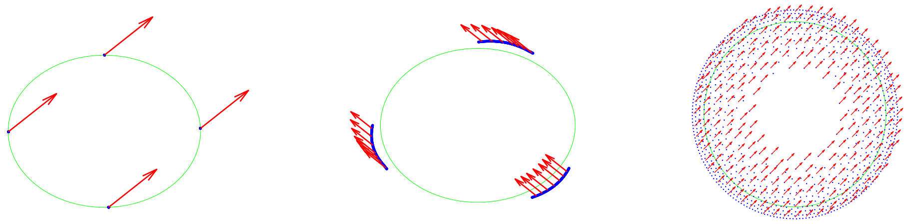


Table 3: Long time simulations with $N = 1000$ particles. Clustering, concentration on lines and fattening e ects.

These e ects can be explained due to the results on the dimensionality of the local minimizers in Chapter 4.

$\|u_0\| = 0.5$	$\|u_0\| = 5$	$\|u_0\| = 0.1$

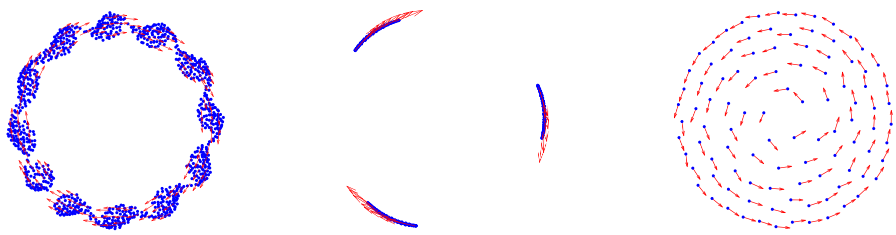


Table 4: Long time simulations with $N = 1000$ particles. First and second pictures have xed $a = 5$, $b = 0.5$. The third has $a = 4$ and $b = 0.001$.

Another type of solutions for these models are *mill rings*. Their stability was already studied in careful detail in [23]. Again, with repulsive-attractive potentials, new mill configurations appear. We numerically investigate the formation of fat mills due to the repulsive force. In addition, formation of clusters appears when varying the asymptotic speed. Some of these possible configurations are displayed in Table 4.

Agraïments

Quatre... Ja han passat quasi quatre anys des que vaig començar aquesta aventura. No la puc anomenar de cap altra manera. Realment ha estat una aventura.

Quan em varen donar la beca FPI, alguns dels meus companys (i que han anat marxant poc a poc a altres universitats o a l empresa) em van dir que viatjaria molt. Bé, els voldria dir que estaven molt equivocats... He viatjat moltíssim! Però deixem de banda els viatges per un moment.

El primer que vull fer és donar les gràcies al Prof. José A. Carrillo (entengui s Prof. com a Professor de l anglès) per donar-me l oportunitat de realitzar un doctorat, i també al Prof. José A. Cañizo per voler codirigir aquest treball. Sense ell tampoc hauria estat possible. Vull donar-los les gràcies per la seva ajuda. La seva direcció ha estat essencial per a poder realitzar aquest treball. També per ser-hi en els moments difícils, tan acadèmics com personals, per ajudar-me com a directors i com a amics. I també per ajudar-me a aprendre. Gràcies per tenir tanta paciència.

A part d ells, també vull donar les gracies a tots els col laboradors directes que han fet que aquest treball fos possible: Giacomo Albi, Pierre Gabriel, Thomas Laurent, Gaël Raoul, James von Brecht i Yao Yao. Gràcies per ajudar-me també. M ho he passat molt bé treballant amb vosaltres, i espero poder repetir aviat. I em deixo a la noia de Granada que viu al CRM i al noi que està a Los Angeles, tot i que el treball que tenim començat encara trigarà una mica (espero que no gaire) en acabar-se.

I també a tots aquells que m han ajudat durant aquesta aventura. Recupero els viatges i, un per un, intentaré no deixar-me a ningú.

Tot comença a finals de l estiu de 2009 a Barcelona, quan vaig iniciar els meus estudis del màster. Encara estaries a la UAB un any més i tu ets el primer que vaig conèixer. També has estat un dels meus guies en aquest camí. Parlar amb tu ha estat de gran ajuda i he aprofitat tots els consells que m has anat donant al llarg de tots aquests anys. Has estat com el germanet gran en aquesta aventura. I tampoc no m oblido de la meva germaneta. Va ser molt divertit compartir els dos primers anys de tesi amb tu. Les hores que ens vam arribar a passar matant-nos a estudiant per poder passar el màster. I les estones que vam compartir a la facultat. Gràcies als dos per ser-hi també (se us troba a faltar!).

I també recordo les noies del 212. Bé, les que estaven allà quan vaig començar el doctorat, algunes que coneixia, i algunes que no. Entre les que coneixia hi ha la noia que sempre estava anant d un lloc a l altre i aquella noia alta amb una mica d accent de les terres de ponent. Aquí vull donar les gràcies especialment a la segona. Gràcies pel teu suport i per la teva ajuda durant la primavera de 2010. Vas aconseguir que fos més senzill seguir endavant.

Ep! Que no m oblido de tu, eh?! Moltes gràcies per les xerrades que hem tingut, pel teatre, les teves magnífiques galetes, però per sobre de tot, pels teus consells.

Primera estada: Cambridge. Era la primera vegada que marxava tant de temps fora de casa. Entengui s casa com el lloc on viuen familiars i amics. Va ser força difícil. Aquí vull donar les gràcies a la noia matemàtica, fan del barça, amb la que vaig mirar alguns partits de futbol, a la noia de Granada que també estava d estada, al xicot alemany que ara està a l Imperial College de Londres i al noi italià que va venir dos mesos i que, entre d altres coses, em va ensenyar a fer pa. Ells van fer-ho una mica més senzill.

Als companys i amics del CRM, on he passat moltes hores també. Sobretot amb la noia de *Graná* i a l amic que vaig conèixer al banc. Però també a la noia de la UPC i al noi de la UB, i als que van provocar un con icte al correu electrònic del CRM per tenir el mateix nom. Amb ells també he compartit bones estones.

Segona estada: Los Angeles. A l Andrea Bertozzi per convidar-me a passar uns mesos allí. Per donar-me la oportunitat de conèixer la gent del seu departament i amb els que van sorgir dues col laboracions (amb la Yao Yao i el James). I tinc al cap les rialles amb el meu germanet i l homenot dels Països Baixos. I pel temps que vaig poder passar amb *tu*.

Tercera estada: Londres. Aquesta és la més recent! Tot i que ha estat curteta, vaig poder conèixer a una noia italiana i a un noi francès amb els que vaig compartir moltes hores de dinar, despatx i algun cap de setmana. I també a la noia dels cabells arrissats de Rubí. Quina sorpresa saber que estaves a Londres!

I, evidentment, a tots els que han estat en aquestes tres llocs quan jo també hi era i que, per no repetir, no els (d)escric.

També a la gent de Granada, una ciutat que m agrada visitar de tant en tant pel seu encant.

Als meus excompanys de despatx dels barracons amb els que vaig passar tantes hores i que vaig molestar bastant. Pel temps que vaig compartir al màster i les discussions (de problemes matemàtics o de filosofia o d història) amb el nostre activista indignat que apareix a les notícies de la televisió. Al meu company de despatx, que no me n oblidi. Per cert, enhorabona pel premi! A la topòloga de cabells llargs i arrissats i a tots els que fan possible els Dissabtes de les matemàtiques de la UAB. I a tota la gent de secretaria i els informàtics pel que els he arribat a molestar i la paciència que han tingut (sobretot la ema jota). I a totes les persones del departament que m han ajudat durant tot aquest temps.

Per anar acabant, i no per això menys importants, a tota la meva família i amics, que són molts noms. Sempre hi han sigut i m han donat suport. I com que no em vull allargar més, no els (d)escric (ells ja saben qui són i segur que es donen per al ludits). Només vull mencionar el de Cal Rave, que quan em veu ja no em pregunta quan marxo sinó quan torno. Gràcies família!

I sobretot a *tu*, a qui dedico especialment l últim paràgraf. Per ajudar-me a moure m per Los Angeles la primera vegada que hi vaig anar tot sol. Per les estones (màgiques) compartides arreu on ens trobàvem. Pel temps que vas passar a Barcelona. Per la teva ràpida integració en el meu cercle d amistats i amb la meva família. Per ajudar-me ens els moments difícils durant els últims dos anys. Per la nova família que he conegut. I per voler compartir tot el temps que ens queda per endavant junts. Gràcies, Alethea.

I pels que hagueu llegit la introducció i hagueu arribat fins aquí.

Aquesta tesi tampoc hauria estat possible sense l ajuda dels següents projectes:

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MTM2011-27739-C04-02 DGI, Ecuaciones en Derivadas Parciales en Física, Biología Matemática y Geometría , Ministerio de Ciencia e Innovación.

2009-SGR-345 AGAUR Equacions diferencials en derivades parcials i aplicacions Grup de recerca reconegut i finançat de la Generalitat de Catalunya.

Chapter 1

Preliminaries in transport distances

We denote by $\mathcal{B}(\mathbb{R}^N)$ the family of Borel subsets of $\mathbb{R}^N$. Given a set $A \in \mathcal{B}(\mathbb{R}^N)$, its Lebesgue measure is denoted by $|A|$. We denote by $\mathcal{M}(\mathbb{R}^N)$ the set of (nonnegative) Borel measures on $\mathbb{R}^N$ and by $\mathcal{P}(\mathbb{R}^N)$ the set of Borel probability measures on $\mathbb{R}^N$. The support of $\mu \in \mathcal{M}(\mathbb{R}^N)$, denoted by $\operatorname{supp}(\mu)$, is the closed set defined by

$$\operatorname{supp}(\mu) := \{x \in \mathbb{R}^N : \mu(B(x, \varepsilon)) > 0 \text{ for all } \varepsilon > 0\}.$$

Let us give a brief self-contained summary of the main concepts related to distances between measures in optimal transport theory, we refer to [124, 67, 92] for further details. A probability measure γ on the product space $\mathbb{R}^N \times \mathbb{R}^N$ is said to be a transference plan between $\mu \in \mathcal{P}(\mathbb{R}^N)$ and $\nu \in \mathcal{P}(\mathbb{R}^N)$ if

$$\gamma(A \times \mathbb{R}^N) = \mu(A) \quad \text{and} \quad \gamma(\mathbb{R}^N \times A) = \nu(A) \tag{1.1}$$

for all $A \in \mathcal{B}(\mathbb{R}^N)$. If $\mu, \nu \in \mathcal{P}(\mathbb{R}^N)$, then

$$\Gamma(\mu, \nu) := \{\gamma \in \mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N) : \text{(1.1) holds for all } A \in \mathcal{B}(\mathbb{R}^N)\}$$

denotes the set of admissible transference plans between μ and ν. Informally, if $\gamma \in \Gamma(\mu, \nu)$ then $d\gamma(x, y)$ measures the amount of mass transferred from location x to location y. With this interpretation in mind note that $\sup_{(x,y)\in\operatorname{supp}(\gamma)} |x - y|$ represents the maximum distance that an infinitesimal element of mass from μ is moved by the transference plan γ. We will work with the ∞-Wasserstein distance d_∞ between two probability measures μ, ν defined by

$$d_\infty(\mu, \nu) = \inf_{\gamma\in\Gamma(\mu,\nu)} \sup_{(x,y)\in\operatorname{supp}(\gamma)} |x - y|, \tag{1.2}$$

which can take infinite values, but it is obviously finite for compactly supported measures. This distance induces a complete metric structure restricted to the set of probability measure with finite moments of all orders, $\mathcal{P}_\infty(\mathbb{R}^N)$, as proven in [67].

We remind that for $1 \quad p \quad \infty$ the distance d_p between two measures μ and   is defined by

$$d_p^p(\mu, \ ) = \inf_{\ \in \ (\ , \ )} \left\{ \ _{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^p d \ (x,y) \right\}.$$

Note that $d_p(\mu, \ ) \quad \infty$ for $\mu, \ \in \mathcal{P}_p(\mathbb{R}^N)$ the set of probability measures with finite moments of order p. Since $d_p(\mu, \ )$ is increasing as a function of $1 \quad p \quad \infty$, one can show that it converges to $d_\infty(\mu, \ )$ as $p \to \infty$. Since the distances are ordered with respect to p, it is obvious that the topologies are also ordered. More precisely, open sets for d_p are always open sets for d_∞, and thus, d_∞ induces the finest topology among d_p, $1 \quad p \quad \infty$. More properties of the distance d_∞ can be seen in [92].

Given $\mathcal{T} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ measurable, we say that   is the push-forward of μ through $\mathcal{T}$, $\ = \mathcal{T}\#\mu$, if $\ [A] := \mu[\mathcal{T}^{-1}(A)]$ for all measurable sets $A \quad \mathbb{R}^N$, equivalently

$$\ _{\mathbb{R}^N} \ (x)\, d \ (x) = \ _{\mathbb{R}^N} \ (\mathcal{T}(x))\, d\mu(x)$$

for all $\ \in C_b(\mathbb{R}^N)$. In case there is a map $\mathcal{T} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ transporting μ onto , i.e. $\mathcal{T}\#\mu = \ $, we immediately obtain

$$d_\infty(\mu, \ ) \quad \sup_{y \in \mathrm{supp}(\ )} |y - \mathcal{T}(y)| \ .$$

This comes from (1.2), by using the transference plan $\ _{\mathcal{T}} = (\mathbb{1}_{\mathbb{R}^N} \times \mathcal{T})\#\mu$.

Lemma 1.1. *Assume that $\mu, \mu \in \mathcal{P}(\mathbb{R}^N)$ are two convex combinations: $\mu = m_0\mu_0 + m_1\mu_1$ and $\mu = m_0\mu_0 + m_1\mu_1$, where μ_0 and μ_0 are supported in $B(x_0, \ )$ for some $x_0 \in \mathbb{R}^N$ and $\quad 0$. Then $d_\infty(\mu, \mu) \quad 2 \ $.*

Proof. Let $\ _1 \in \ (\mu_1, \mu_1)$ be the transport plan induced by the identity map, that is

$$\ _{\mathbb{R}^N \times \mathbb{R}^N} \ (x,y) d \ _1(x,y) = \ _{\mathbb{R}^N} \ (x,x) d\mu_1(x)$$

and let $\ _0 \in \ (\mu_0, \mu_0)$ be any transport plan between μ_0 and μ_0. Note that $\ = m_0 \ _0 + m_1 \ _1 \quad \ (\mu, \mu)$ and $\mathrm{supp}(\ ) = \mathrm{supp}(\ _0) \cup \mathrm{supp}(\ _1)$. Since $\ _1$ is supported on the diagonal we have $\sup_{(x,y) \in \mathrm{supp}(\ _1)} |x-y| = 0$. On the other hand, $\mathrm{supp}(\ _0) \quad \mathrm{supp}(\mu_0) \times \mathrm{supp}(\mu_0) \quad B(x_0, \ ) \times B(x_0, \ )$ and therefore $\sup_{(x,y)\in\mathrm{supp}(\ _0)} |x-y| \quad 2\ $. We conclude that $\sup_{(x,y)\in\mathrm{supp}(\ )} |x-y| \quad 2$ which implies $\inf_{\ \in\ (\ ,\ )} \sup_{(x,y)\in\mathrm{supp}(\ )} |x-y| \quad 2\ $. □

A measure $\ \in \mathcal{M}(\mathbb{R}^N)$ is said to be a part of μ if $\ (A) \quad \mu(A)$ for all $A \in \mathcal{B}(\mathbb{R}^N)$ and it is not identically zero. This terminology is justified by the fact that if $\ $ is a part of μ, then μ can be written $\mu = \ + \ $ for some $\ \in \mathcal{M}(\mathbb{R}^N)$ ($\ = \mu - \ $ to be more precise). We will say that a probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ can be decomposed as a convex combination of $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^N)$ if there exists $0 \quad m_0, m_1 \quad 1$ with $m_0 + m_1 = 1$ such that $\mu = m_0\mu_0 + m_1\mu_1$.

Let us introduce some notation related to the interaction potential energy. We denote by $B_W : \mathcal{P}(\mathbb{R}^N) \times \mathcal{P}(\mathbb{R}^N) \to (0, +\infty]$ the bilinear form defined by

$$B_W[\mu_1, \mu_2] := \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)\, d\mu_1(x) d\mu_2(y). \tag{1.3}$$

Obviously we have that $E_W[\mu] = B_W[\mu, \mu]$. Let us define the shortcut notation

$$T_W[\mu_1, \mu_2] := E_W[\mu_1] - 2B_W[\mu_1, \mu_2] + E_W[\mu_2] \tag{1.4}$$

which will often occur in several computations. For notational simplicity, we will drop the subscript for E_W, B_W, and T_W in detailed proofs while kept in the main statements.

Chapter 2

Nonlocal interactions by repulsive-attractive potentials: radial ins/stability

The contents of this chapter appear in:

> *D. Balagué, J. A. Carrillo, Aggregation equation with growing at infinity attractive-repulsive potentials. Proceedings of the 13th International Conference on Hyperbolic Problems, Series in Contemporary Applied Mathematics CAM 17, Higher Education Press, Volume 1, 136-147, 2012.* [7]
>
> *D. Balagué, J. A. Carrillo, T. Laurent, G. Raoul, Nonlocal interactions by repulsive-attractive potentials: radial ins/stability. To appear in Physica D.* [10]

2.1 Introduction

Nonlocal interaction equations are continuum models for large systems of particles where every single particle can interact not only with its immediate neighbors but also with particles far away. These equations have a wide range of applications. In biology they are used to model the collective behavior of a large number of individuals, such as a swarm of insects, a ock of birds, a school of fish or a colony of bacteria [97, 119, 120, 48, 49, 72, 45, 37, 32, 118, 16, 24, 12, 52, 27, 26, 25]. In these models individuals sense each other at a distance, either directly by sound, sight or smell, or indirectly via chemicals, vibrations, or other signals. Nonlocal interaction equations also arise in various contexts in physics. They are used in models describing the evolution of vortex densities in superconductors [129, 115, 116, 87, 3, 2, 89, 57, 90]. They also appear in the modeling of dynamics of agglomerating particles in two dimensions (with loose links to the one-dimensional sticky particles system) [99]. They also appear in simplified inelastic interaction models for granular media [15, 41, 121, 86]. Going back to biology, nonlocal interaction equations arise also in the modeling of the orientational distribution of F-actin filaments in cells [66, 78, 109].

In their simplest form, nonlocal interaction equations can be written as

$$\frac{\partial \mu}{\partial t} + \operatorname{div}(\mu v) = 0 \quad , \quad v = -\nabla W * \mu \tag{2.1}$$

where $\mu(t) = \mu_t$ is the probability measure of particles at time t, $W : \mathbb{R}^N \longrightarrow \mathbb{R}$ is the interaction potential and $v(t, x)$ is the velocity of the particles at time t and at location $x \in \mathbb{R}^N$. We will always assume that the interaction potential $W(x) = k(|x|)$ is radial and C^2- or C^3-smooth away from the origin, depending on the results. Typically the potentials we will consider have a singularity (not C^2-smooth) at the origin.

When the potential W is purely attractive, i.e. W is a radially symmetric increasing function, then the density of particles collapse on itself and converge to a Dirac Delta function located at the center of mass of the density. This Dirac Delta function is the unique stable steady state and it is a global attractor [36]. The collapse toward the Dirac Delta function can take place in finite time if the interaction potential is singular enough at the origin and several works have been recently devoted to the understanding of these singular measure solutions [20, 18, 36, 19].

In biological applications however, it is often the case that individuals attract each other in the long range in order to remain in a cohesive group, but repulse each other in the short range in order to avoid collision [98, 100]. This lead to the choice of a radially symmetric potential W which is first decreasing then increasing as a function of the radius. We refer to these type of potentials as repulsive-attractive potentials. Compared with the purely attracting case where solutions always converge to a single Delta function, nonlocal interaction equations with repulsive-attractive potentials lead to solutions converging to possibly complex steady states. As such, nonlocal interaction equations with repulsive-attractive potentials can be considered as a minimal model for pattern formation in large groups of individuals. They also arise in material sciences [56, 128, 111, 73, 126] where particles, nano-particles or molecules self-assemble according to pairwise interactions generated by a repulsive-attractive potential.

Whereas nonlocal interaction equations with purely attractive potential have been intensively studied there are still relatively few rigorous results about nonlocal interaction equations with repulsive-attractive potential. The 1D case has been studied in a series of works [61, 62, 110]. The authors have shown that the behavior of the solution depends highly on the regularity of the interaction potential: for regular interaction, the solution converges to a sum of Dirac masses, whereas for singular repulsive potential, the solution remains uniformly bounded. They also showed that combining a singular repulsive with a smooth attractive potential leads to integrable stationary states. Pattern formation in multi-dimensions have recently been studied in [81, 127]. In these two works, the authors perform a numerical study of the finite particle version of (2.1) and show that a repulsive-attractive potential can lead to the emergence of surprisingly complex patterns. To study these patterns they plug in (2.1) an ansatz which is a distribution supported on a surface. This give rise to an evolution equation for the surface. They then perform a linear stability analysis around the uniform distribution on the sphere and derive simple conditions on the potential which classify the different instabilities. The various instability modes dictate toward which pattern the solution will converge. They also check numerically that what is true for the surface evolution equation also holds for the continuum model (2.1). In other recent works [64, 63] the specific case where the repulsive part of the potential is the Newtonian potential is analyzed showing the existence of radially compactly supported integrable stationary states. They also study their nonlinear stability for radial solutions.

In this chapter we focus primarily on proving rigorous results about the convergence of radially symmetric solutions toward spherical shell stationary states in multi-dimensions.

Definition 2.1 (Spherical Shell)**.** *The spherical shell of radius R, denoted δ_R, is the probability measure which is uniformly distributed on the sphere $\partial B(0,R) = \{x \in \mathbb{R}^N : |x| = R\}$.*

Given a repulsive-attractive radial potential whose attractive force does not decay too fast at infinity, there always exists an $R > 0$ so that the spherical shell of radius R is a stationary state as it will be remarked below. One needs then to address the question of whether or not this spherical shell is stable. It is classical, see [4, 41, 124, 42], that the equation (2.1) is a gradient flow of the interaction energy

$$E[\mu] = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x-y) d\mu(x) d\mu(y) \tag{2.2}$$

with respect to the euclidean Wasserstein distance. Thus, stable steady states of (2.1) are expected to be local minimizers of the interaction energy. Simple energetic arguments will show that in order for the spherical shell of radius R to be a local minimum of the interaction energy, it is necessary that the radial potential W satisfies:

(C0) Repulsive-Attractive Balance: $\omega(R, R) = 0$,

(C1) Fattening Stability: $\partial_1\omega(R, R) \quad 0$,

(C2) Shifting Stability: $\partial_1\omega(R, R) + \partial_2\omega(R, R) \quad 0$,

where the function $\omega : \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ is defined by

$$\omega(r, \eta) = -\frac{1}{\sigma_N} \int_{\partial B(0,1)} \nabla W(re_1 - \eta y) \cdot e_1 \, d\sigma(y), \tag{2.3}$$

σ_N is the hypersurface area of the unit sphere in $\mathbb{R}^N$, e_1 is the first vector of the canonical basis of $\mathbb{R}^N$, $d\sigma$ denotes the volume element of the manifold where the integral is performed and $\mathbb{R}_+^2 = (0, +\infty) \times (0, +\infty)$. Condition **(C0)** simply guarantees that the spherical shell δ_R is a critical point of the interaction energy. We will see that if condition **(C1)** is not satisfied then it is energetically favorable to split the spherical shell into two spherical shells. Heuristically this indicate that the density of particles, rather than remaining on the sphere, is going to expand and occupy a domain in $\mathbb{R}^N$ of positive Lebesgue measure. If condition **(C1)** is not satisfied we will therefore say that the fattening instability" holds. It can be easily checked that if $\omega(R, R) = 0$, then $\partial_1\omega(R, R)$ is simply the value of the divergence of the velocity field on the sphere of radius R. So the fattening instability corresponds to an expanding velocity field on the support of the steady state. We will also see that if condition **(C2)** is not satisfied it is energetically favorable to increase or decrease the radius of the spherical shell. This instability will be referred as the shift instability .

We investigate nonlocal interaction equations with repulsive-attractive radial potentials. Such equations describe the evolution of a continuum density of particles in which they repulse (resp. attract) each other in the short (resp. long) range. We prove that under some conditions on the potential, radially symmetric solutions converge exponentially fast in some transport distance toward a spherical shell stationary state. Otherwise we prove that it is not possible for a radially symmetric solution to converge weakly toward the spherical shell stationary state. We also investigate under which condition it is possible for a non-radially symmetric solution to converge toward a singular stationary state supported on a general hypersurface. Finally we provide a detailed analysis of the specific case of the repulsive-attractive power-law potential as well as numerical results.

We now outline the structure of the chapter and describe the main results. In the preliminary section, section 2.2, we derive **(C0)** **(C2)** from an energetic point of view and we show that they correspond to avoiding the fattening and shift instability. We also study the regularity of the kernel ω defined by (2.3). A good understanding of the regularity of ω will be necessary for later sections. Some technical results of this regularity analysis need nontrivial aspects of di erential geometry and are relegated to an appendix for readiness. We also remind the reader of previous results from [22, 7] about well posedness of (2.1) in $L^p(\mathbb{R}^N)$ and set up the overall notation.

Section 2.3 is devoted to a detailed study of the fattening instability, both in the radially symmetric case and in the non-radially symmetric case. We first show that if condition **(C1)** is not satisfied then it is not possible for a radially symmetric L^p-solution to converge weakly-$*$ as measures toward a spherical shell stationary state. We then investigate singular stationary states supported on hypersurfaces which are not necessarily spheres. Such steady states have been observed in numerical simulations [81, 127]. We show that if the divergence of the velocity field generated by such stationary state is positive everywhere on their support, then it is not possible for an L^p-solution to converge toward the stationary state in the sense of the topology defined by d_∞. Here d_∞ stands for the infinity-Wasserstein distance on the space of probability measures (see next section for a definition). We also show that if the repulsive-attractive potential W is singular enough at the origin, for example $W(x) \sim -|x|^b/b$ as $|x| \to 0$ with b $3 - N$, then the potential is so repulsive in the short range that solutions can not concentrate on an hypersurface, and this is independent of how attractive is the potential in the long range. To be more precise we show that for potentials with such a strong repulsive singularity at the origin, L^p solutions can not converge with respect to the d_∞-topology toward singular steady states supported on hypersurfaces.

Whereas section 2.3 is devoted to instability results, section 2.4 is devoted to stability results. We show that if **(C0)** **(C2)** hold with strict inequalities, then a radially symmetric solution of (2.1) which starts close enough to the spherical shell in the d_∞ topology will converge exponentially fast toward it. Under additional assumptions on the potential we can also prove convergence with respect to the d topology, $\in [1, +\infty)$. In order for the stability results of section 2.4 to hold a certain amount of regularity on the solutions is necessary. Unfortunately weak L^p-solutions do not have this amount of regularity. This is why in section 2.5 we prove well posedness of classical C^1-solutions. This covers a gap in the existing literature which mostly considers weak solutions. The results of section 2.4 are true for this class of classical C^1-solutions. The aim of section 2.6 is to show examples of how to apply the general instability and stability theory in the case of power-law repulsive-attractive potentials:

$$W(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b} \qquad 2 - N \quad b \quad a. \tag{2.4}$$

For this family of potentials, conditions **(C0)** **(C2)** can be explicitly formulated in terms of a and b, therefore leading to an explicit bifurcation diagram for the stability of the spherical shell in $\mathbb{R}^N$. Finally in the last section, section 2.7, we perform numerical computations of radially symmetric solutions of (2.1) with power-law potential (2.4) and study their convergence toward spherical shell stationary state. Since a spherical shell is a highly singular function, it is challenging to perform such computations with traditional methods. This is why, following [70, 25, 43, 61, 62], rather than simulating (2.1) directly, we simulate the evolution of the inverse of the cumulative distribution of the radial measure associated to μ. Since the inverse of the cumulative distribution of a spherical shell is a constant function, this approach has the virtue of smoothing the dynamics and this provides us with a robust numerical scheme. Our numerical simulations indicate the possible existence of integrable radial stationary states stable under radial perturbations in the parameter area corresponding to the fattening instability for power-law repulsive-attractive potentials, an issue that will be analysed elsewhere. This has already been proven in the particular case of $a \quad b = 2 - N$ in [64, 63]. Finally, we draw the main conclusions of this chapter in the final section.

2.2 Preliminary section

2.2.1 Basic notations & radially symmetric formulation

As said in the introduction $\mu(t) = \mu_t \in \mathcal{P}(\mathbb{R}^N)$ denotes the probability measure of particles at time $t \quad 0$ and the integration against it is denoted by $d\mu_t(x)$. Whenever μ_t is absolutely continuous with respect to Lebesgue measure, we will denote its density by $(t,x) = {}_t(x)$ and thus, $d\mu_t(x) = (t,x)\,dx = {}_t(x)\,dx$ in that case. We will keep this notation to make clear when we work at the measure μ or at the density level. We will make use of the notation $\mathcal{P}_p(\mathbb{R}^N)$, $1 \quad p \quad \infty$, that stands for the space of probability measures with finite moments of order p. Moreover, in many instances we will need to quantify precisely the distance between probability measures (see Chapter 1 for the precise definitions).

Definition 2.2 (Radial Measures)**.** *We denote by $\mathcal{P}^r(\mathbb{R}^N)$ the space of radially symmetric probability measures. If $\mu \in \mathcal{P}^r(\mathbb{R}^N)$ then $\hat{\mu} \in \mathcal{P}([0,+\infty))$ is defined by*

$$\int_{r_1}^{r_2} d\hat{\mu}(r) = \int_{r_1<|x|<r_2} d\mu(x) \qquad \text{and} \qquad \int_0^{r_2} d\hat{\mu}(r) = \int_{0\leqslant|x|<r_2} d\mu(x)$$

for all $0 \leqslant r_1 \leqslant r_2$. We endow this space with the standard weak-$*$ topology.

Whenever $\mu \in \mathcal{P}^r(\mathbb{R}^N)$ is absolutely continuous with respect to Lebesgue, we denote its radial density by $\hat{\rho}(r)$ and thus, $\rho(x) = \hat{\rho}(|x|)$. Furthermore, the integration against is given by $d\hat{\mu}(r) = \hat{\rho}(r)\, r^{N-1}\, dr = d\mu(x) = \rho(x)\, dx$. Let us denote by $\mathcal{P}_2^r(\mathbb{R}^N)$ the set of radial probability measures with bounded second moment.

Recall that $\delta_R \in \mathcal{P}^r(\mathbb{R}^N)$ stands for the spherical shell of radius R (see Definition 2.1). The velocity field at point x generated by a spherical shell of radius R is given by $v_R(x) = -(\nabla W * \delta_R)(x)$. Since W is radially symmetric, then by symmetry there exists a function $\omega(r,\eta)$ such that

$$v_R(x) = -(\nabla W * \delta_R)(x) = \omega(|x|\,, R)\frac{x}{|x|} \tag{2.5}$$

and one can easily check that this function ω is defined by (2.3), see [18] for more details. Note also that if $\mu \in \mathcal{P}^r(\mathbb{R}^N)$ then it can be written as a sum of spherical shells, $\mu = \int_0^{+\infty} \delta_\eta \, d\hat{\mu}(\eta)$, and we conclude that

$$-(\nabla W * \mu)(x) = -\int_0^{+\infty} (\nabla W * \delta_\eta)(x) d\hat{\mu}(\eta) = \int_0^{+\infty} \omega(|x|\,,\eta) d\hat{\mu}(\eta)\,\frac{x}{|x|}.$$

Given $T \geqslant 0$, $C([0,T];\mathcal{P}^r(\mathbb{R}^N))$ denotes the set of continuous curves of radial measures where continuity is with respect to the weak-$*$ convergence. We say that $\mu \in C([0,T];\mathcal{P}^r(\mathbb{R}^N))$ is a radially symmetric solution of (2.1) if $\hat{\mu} \in C([0,T];\mathcal{P}([0,+\infty))$ satisfies the one dimensional conservation law:

$$\partial_t \hat{\mu} + \partial_r(\hat{\mu}\hat{v}) = 0 \tag{2.6}$$

$$\hat{v}(t,r) = \int_0^{+\infty} \omega(r,\eta) d\hat{\mu}_t(\eta)\,, \tag{2.7}$$

in the distributional sense. We will now give conditions for the velocity field to be well-defined by studying the properties of the function ω.

2.2.2 Regularity of the function $\omega(r, \eta)$

Let us remind that we assume that W is radially symmetric and belongs to $C^2(\mathbb{R}^N \backslash \{0\})$. The function $\omega(r, \eta)$ defined by (2.3) is clearly C^1 away from the diagonal $\mathcal{D} = \{(r, r) : r \quad 0\}$. Moreover, the derivatives of ω are given by

$$\partial_1 \omega(r, \eta) = -\frac{1}{\sigma_N} \quad_{\partial B(0,1)} \frac{\partial^2 W}{\partial x_1^2}(re_1 - \eta y)\, d\sigma(y), \tag{2.8}$$

and

$$\partial_2 \omega(r, \eta) = \frac{1}{\sigma_N} \quad_{\partial B(0,1)} \nabla \left(\frac{\partial W}{\partial x_1} \right) (re_1 - \eta y) \cdot y\, d\sigma(y), \tag{2.9}$$

away from the diagonal. We need to investigate the behavior of ω on the diagonal. Let us make the following definition:

De nition 2.3 (Integrability on hypersurfaces)**.** *A radially symmetric function $g \in \mathcal{C}(\mathbb{R}^N \backslash \{0\})$ is said to be locally integrable on hypersurfaces if*

$$_{[0,1]^{N-1}} |g(\hat{x}, 0)| \ d\hat{x} \quad +\infty$$

where $\hat{x} = (x_1, \ldots, x_{N-1})$, or equivalently, if $\hat{g}(r) r^{N-2}$ is integrable on $(0, 1)$ with $g(x) = \hat{g}(|x|)$. By an abuse of notation, we sometimes say $\hat{g}(r)$ is integrable on hypersurfaces.

Lemma 2.1 (Regularity of the function ω)**.** *Let $W(x) = k(|x|)$ be a radially symmetric potential belonging to $C^3(\mathbb{R}^N \backslash \{0\})$.*

(i) If $k'(r)$ is locally integrable on hypersurfaces then $\omega \in C(\mathbb{R}^2_+)$.

(ii) If $k'(r)$, $k''(r)$, and $r^{-1} k'(r)$ are locally integrable on hypersurfaces then $\omega \in C^1(\mathbb{R}^2_+)$.

(iii) Suppose $\quad W$ is negative in a neighborhood of the origin. If $k'(r)$ is locally integrable on hypersurfaces but $\widehat{\quad W} = k'' + (N-1) r^{-1} k'$ is not, then for any $R \quad 0$,

$$\lim_{\substack{(r,\ ) \notin \mathcal{D} \\ (r,\ ) \to (R,R)}} \partial_1 \omega(r, \eta) = +\infty. \tag{2.10}$$

Before proving the above lemma, let us discuss the result. Obviously the regularity of the function ω depends only on the behavior of W at the origin. Assume for simplicity that in the neighborhood of the origin, the potential W is a power-law, that is $W(x) = k(|x|) = -|x|^b / b$ for all $x \in B(0, \ )$, where

b is possibly negative. Note that $k'(r) \quad 0$ for $r \qquad$ so the potential is repulsive in the short range. Lemma 2.1 then claims that ω is continuous if $b \quad 2-N$ and continuously di erentiable if $b \quad 3-N$. Statement (iii) says that if $2-N \quad b \quad 3-N$, then ω is continuous but its first derivative goes to $+\infty$ as (r,η) approaches the diagonal.

Finally, let us remark that(i) is sharp in the sense that the Newtonian potential $|x|^{2-N}$ is the critical one for the integrability on hypersurfaces. Precisely, Newton s Theorem asserts that the function ω associated to the Newtonian potential is discontinuous, it has a singularity, across the spherical shell. We now prove the Lemma:

Proof. Let us prove (i). The function $\omega(r,\eta)$ can be rewritten as

$$\omega(r,\eta) = \frac{1}{\sigma_N \eta^{N-1}} \quad {}_{\partial B(0,\ )} \; e_1 \cdot \nabla W(re_1 - y)\, d\sigma(y).$$

Seeing ω as a function of $x = re_1$ and η, we can apply Lemma 2.10 from the appendix with $\mathcal{M} \ := \partial B(0,\eta)$, $\quad (x) := (\sigma_N \eta^{N-1})^{-1}$, and $G(x) := e_1 \cdot \nabla W(x)$. Since $|G(x)|$ is bounded by $|k'(|x|)|$ which is locally integrable on hypersurfaces, we obtain that $\omega \in C(\mathbb{R}^2_+)$.

We now turn to the proof of (ii). It is simple to check that

$$\frac{\partial^2 W}{\partial x_i \partial x_j} = k''(r)\frac{x_i x_j}{r^2} + k'(r)\frac{\delta_{ij}}{r} - k'(r)\frac{x_i x_j}{r^3}$$

and then $|\frac{\partial^2 W}{\partial x_i \partial x_j}|$ is bounded by a radial function which is locally integrable on hypersurfaces given by a linear combination of $k''(r)$ and $r^{-1}k'(r)$. Moreover, it has the regularity needed in Lemma 2.10. We now rewrite the derivatives $\partial_1\omega(r,\eta)$ and $\partial_2\omega(r,\eta)$ in (2.8) and (2.9) as

$$\partial_1\omega(r,\eta) = -\frac{1}{\sigma_N \eta^{N-1}} \quad {}_{\partial B(0,\ )} \; \frac{\partial^2 W}{\partial x_1^2}(re_1 - y)\, d\sigma(y),$$
$$\partial_2\omega(r,\eta) = \frac{1}{\sigma_N \eta^{N}} \quad {}_{\partial B(0,\ )} \; \nabla\left(\frac{\partial W}{\partial x_1}\right)(re_1 - y)\cdot y\, d\sigma(y)\,.$$

The reader can easily check that Lemma 2.10 applies similarly as before, so that $\omega \in C^1(\mathbb{R}^2_+)$.

Finally we prove (iii). Taking the divergence of (2.5) we obtain:

$$(\text{div } v_R)(x) = -(\quad W * \delta_R)(x) = \partial_1\omega(|x|, R) + (N-1)\frac{\omega(|x|, R)}{|x|}, \qquad (2.11)$$

and therefore $\partial_1\omega(r,\eta)$ can be written:

$$\partial_1\omega(r,\eta) = -(\quad W * \delta\quad)(re_1) - (N-1)\frac{\omega(r,\eta)}{r}.$$

For 0 $\qquad r_0$, let $\quad \in C^\infty(\mathbb{R}_+)$ be a cut-o function, such that $\quad = 1$ on $[0, \ /2]$, and $\quad = 0$ on $[\ , \infty)$. Choose such that the function $- \quad W\ (x) := -\ (x)\ \ W(x)$ is nonnegative. Using Lemma 2.11 with η_1 and η_2 such that $\eta_1 \quad R \quad \eta_2$, and noting that $\mathrm{dist}(re_1, \partial B(0,\eta)) = |r-\eta|$, we find that

$$\lim_{\substack{(r,\ )\notin\mathcal{D}\\(r,\ )\to(R,R)}} -(\quad W \quad * \delta\quad)(re_1) = +\infty.$$

To conclude the proof, note that the functions $(r,\eta) \mapsto \frac{(r,\ )}{r}$ and $(r,\eta) \mapsto ([(1-\quad)\quad W] * \delta\quad)(re_1)$ are bounded in a neighborhood of (R,R). □

2.2.3 The two radial instabilities

In this subsection we exhibit some elementary calculations in order to understand under which conditions a spherical shell is a stable steady state. Rigorous results about stability and instability of spherical shell with respect to the transport distance will be provided in section 3 and 4. This subsection provides motivations for the rigorous results to come later.

De nition 2.4 (Steady states)**.** *A probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ is said to be a steady state of the nonlocal interaction equation* (2.1) *if*

$$-(\nabla W * \mu)(x) = 0 \qquad \text{for all } x \in \mathrm{supp}(\mu).$$

Let us remark that due to the gradient ow structure of the nonlocal equation (2.1), we have an additional dissipation in our problem. Actually, we can formally compute the evolution of the interaction energy functional $E[\mu]$ in (2.2) on a solution of (2.1) to get

$$\frac{d}{dt}E[\mu(t)] = - \quad_{\mathbb{R}^N} |v(t,x)|^2 \, d\mu_t(x)\,.$$

Therefore, a stationary state of (2.1) has to satisfy the condition in Definition 2.4, i.e., the dissipation of the interaction potential associated to a steady state has to be null.

We now show that if the attractive strength $k'(r)$ of a repulsive-attractive potential $W(x) = k(|x|)$ does not decay faster than $1/r^N$ as $r \to \infty$, then there exists a spherical shell steady state.

Lemma 2.2 (Existence of spherical shell steady states)**.** *Let $W(x) = k(|x|)$ be a radially symmetric potential belonging to $C^1(\mathbb{R}^N \backslash \{0\})$ and such that $k'(r)$ is locally integrable on hypersurfaces. Let us assume that the potential is repulsive-attractive in the following sense: there exists $R_a \quad 0$ such that*

$$k'(r) \quad 0 \text{ for } r \quad R_a, \qquad \text{and} \qquad k'(r) \quad 0 \text{ for } 0 \quad r \quad R_a.$$

Defining for $r \quad 2R_a$ the function

$$(r) := \inf_{r/2 \leqslant s \leqslant 2r} k'(s) \quad 0, \tag{2.12}$$

we will further assume that

$$\lim_{r \to \infty} r^N \quad (r) = +\infty.$$

Then there exists at least a $R \quad 0$ such that the spherical shell $\delta_R \in \mathcal{P}(\mathbb{R}^N)$ is a steady state to (2.1)*.*

Proof. Note that from (2.5) we directly obtain that a spherical shell $\delta_R \in \mathcal{P}(\mathbb{R}^N)$ is a steady state if and only if $\omega(R, R) = 0$, that is, if and only if condition **(C0)** holds. Since $k'(r)$ is locally integrable on hypersurface $\omega \in C(\mathbb{R}^2_+)$ due to Lemma 2.1. So the function $F(r) := \omega(r, r) \in C(\mathbb{R}_+)$. Using formula (2.3), we get

$$F(r) = \frac{1}{\sigma_N} \quad_{\partial B(0,1)} k'(r|y - e_1|) \frac{y - e_1}{|y - e_1|} \cdot e_1 \, d\sigma(y) \, .$$

Let us remark that $(y - e_1) \cdot e_1 \quad 0$ for all $y \in \partial B(0, 1)$, and thus for $2r \quad R_a$ we easily get $F(r) \quad 0$. It is enough to show that there exists $r \quad 2R_a$ such that $F(r) \quad 0$. In order to do this, we proceed as in [35, Proposition 2.2] and divide the integral in the definition of $F(r)$ into two sets: $A := \partial B(0, 1) \cap B(e_1, R_a/r)$ and its complementary set A^c. Note that the integrand is positive on A and negative on A^c. We will show that for r large enough the integral over the set A^c is greater in absolute value than the integral over the set A. It is easy to see that $B := \{y \in \partial B(0, 1) \text{ such that } 2|y - e_1| \quad 1\} \quad A^c$ and $B \neq \emptyset$ as soon as $r \quad 2R_a$.

We first estimate the integral

$$\left| \quad_A k'(r|y - e_1|) \frac{y - e_1}{|y - e_1|} \cdot e_1 \, d\sigma(y) \right| = \quad_A |k'(r|y - e_1|)| \frac{(e_1 - y) \cdot e_1}{|e_1 - y|} \, d\sigma(y).$$

Let $\quad$ be the angle between $e_1 - y$ and e_1 and note that for all $y \in A := \partial B(0, 1) \cap B(e_1, R_a/r)$ we have by the law of cosines

$$\frac{e_1 - y}{|e_1 - y|} \cdot e_1 = \cos \quad \frac{R_a}{2r}.$$

Using Lemma 2.8 from the Appendix with $\mathcal{M} = \partial B(0,1)$ we then obtain

$$
\begin{aligned}
\Big| \int_A k'(r|y-e_1|)\frac{e_1-y}{|e_1-y|}\cdot e_1\, d\sigma(y)\Big| &\leq \frac{R_a}{2r}\int_A |k'(r|y-e_1|)| d\sigma(y)\\
&\leq \frac{R_a}{2r}\int_0^{R_a/r} |k'(rs)|\ |\mathcal{M}\cap\{|y-e_1|=s\}|_{\mathcal{H}^{N-2}}\, ds\\
&\leq C\frac{R_a}{2r}\int_0^{R_a/r} |k'(rs)|\ s^{N-2} ds\\
&= C\frac{R_a}{2r^N}\int_0^{R_a} |k'(z)| z^{N-2} dz \leq \frac{C_1}{r^N},
\end{aligned}
$$

where we have used the fact that $k'(r)$ is integrable on hypersurfaces to obtain the last inequality.

Since the integrand is negative in A^c and since $B \subset A^c$ for $r \geq 2R_a$ we have:

$$
\int_{A^c} k'(r|y-e_1|)\frac{y-e_1}{|y-e_1|}\cdot e_1\, d\sigma(y) \leq \int_B k'(r|y-e_1|)\frac{y-e_1}{|y-e_1|}\cdot e_1\, d\sigma(y).
$$

Moreover, thanks to the law of cosines, $\frac{y-e_1}{|y-e_1|}\cdot e_1 = \cos(\pi-\theta) \leq -1/4$ for $y \in B$, and then using (2.12)

$$
\begin{aligned}
\int_B k'(r|y-e_1|)\frac{y-e_1}{|y-e_1|}\cdot e_1\, d\sigma(y) &\leq -\frac{1}{4}\int_B k'(r|y-e_1|)\, d\sigma(y)\\
&\leq -\frac{1}{4}\int_B d\sigma(y)\ \varphi(r) \leq -C_2\ \varphi(r).
\end{aligned}
$$

Condition (2.12) on $\varphi(r)$ implies that $C_2\ \varphi(r) \geq C_1/r^N$ for r large enough and therefore $F(r) < 0$ for r large enough. Then the continuity of F implies the existence of a radius $r > 0$ such that $F(r) = 0$. □

The following proposition gives some hints about the stability properties of the spherical shell steady states.

Proposition 2.1 (Instability modes by energy arguments)**.** *Assume that the radial interaction potential W is such that $\omega \in C^1(\mathbb{R}_+^2)$ and let δ_R be a steady state, that is,* **(C0)***: $\omega(R,R) = 0$.*

(i) If **(C1)** *is not satisfied, i.e., $\partial_1\omega(R,R) > 0$, then by splitting the spherical shell into two spherical shells we can decrease the energy. More precisely there exists $dr_0 > 0$ such that, given $0 < |dr| < dr_0$,*

$$
E[(1-\epsilon)\delta_R + \epsilon\delta_{R+dr}] < E[\delta_R]
$$

if   is small enough.

(ii) If **(C2)** *is not satisfied, i.e.,* $\partial_1\omega(R,R)+\partial_2\omega(R,R) \quad 0$*, then by increasing or decreasing the radius of the spherical shell we can decrease the energy. More precisely there exists* $dr_0 \quad 0$ *such that*

$$E[\delta_{R+dr}] \quad E[\delta_R]$$

for all $0 \quad |dr| \quad dr_0$.

Proof. Let us introduce the notations

$$E[\mu,\ ] := \frac{1}{2} {}_{\mathbb{R}^N\times\mathbb{R}^N} W(x-y)d\mu(x)d\ (y),$$

so that $E[\mu,\mu]=E[\mu]$, and

$$E(r,\eta) := E[\delta_r,\delta\ ] = \frac{1}{2}\frac{1}{\sigma_N^2} {}_{\partial B(0,1)\times\partial B(0,1)} W(rx-\eta y)\, d\sigma(x)d\sigma(y)\,.$$

Taking the derivative we get:

$$\begin{aligned}
\frac{\partial E}{\partial r}(r,\eta) &= \frac{1}{2}\frac{1}{\sigma_N^2} {}_{\partial B(0,1)\times\partial B(0,1)} \nabla W(rx-\eta y)\cdot x\, d\sigma(x)d\sigma(y)\\
&= \frac{1}{2}\frac{1}{\sigma_N} {}_{\partial B(0,1)} \left(\frac{1}{\sigma_N} {}_{\partial B(0,1)} \nabla W(rx-\eta y)d\sigma(y)\right)\cdot x\, d\sigma(x)\\
&= \frac{1}{2}\frac{1}{\sigma_N} {}_{\partial B(0,1)} \Big((\nabla W * \delta\ )(rx)\Big)\cdot x\, d\sigma(x) = -\frac{1}{2}\omega(r,\eta)\,.
\end{aligned}$$

Since $E(r,\eta)=E(\eta,r)$, the Hessian matrix of $E(r,\eta)$ is given by

$$H(r,\eta) = -\frac{1}{2}\begin{bmatrix} \partial_1\omega(r,\eta) & \partial_2\omega(r,\eta)\\ \partial_2\omega(\eta,r) & \partial_1\omega(\eta,r)\end{bmatrix}.$$

If δ_R is a steady state, i.e. $\omega(R,R)=0$, then $\nabla E(R,R)=0$,

$$E(R+dr,R) = E(R,R) - \frac{1}{4}\partial_1\omega(R,R)dr^2 + o(dr^2)\,, \tag{2.13}$$

and

$$E(R+dr,R+dr) = E(R,R) - \frac{1}{2}(\partial_1\omega(R,R)+\partial_2\omega(R,R))\, dr^2 + o(dr^2)\,. \tag{2.14}$$

The proof of (ii) follows directly from the Taylor expansion (2.14). By using the Taylor expansions (2.13) and (2.14) as well as the bilinearity of $E[\mu] = E[\mu, \mu]$:

$$
\begin{aligned}
E\Big[(1 - \ )\delta_{\partial B(0,R)} + \ \delta_{\partial B(0,R+dr)}\ ,\ (1 - \ )\delta_{\partial B(0,R)} + \ \delta_{\partial B(0,R+dr)}\Big] \\
=&(1 - \ )^2 E(R,R) + 2\ (1 - \ )E(R+dr,R) \\
&+ \ ^2 E(R+dr, R+dr) \\
=&E(R,R) - \frac{}{2}\partial_1\omega(R,R)dr^2 - \frac{^2}{2}\partial_2\omega(R,R))\ dr^2 + o(dr^2)
\end{aligned}
$$

from which (i) follows by taking and dr_0 small enough. □

The following elementary Lemma shows that the instability condition $\partial_1\omega(R,R) \quad 0$ (i.e. **(C1)** is not satisfied) simply means that the divergence of the velocity field generated by the spherical shell is positive on itself. Being the velocity field expanding", it makes sense that splitting the spherical shell into two reduces the energy as proven in Proposition 2.1.

Lemma 2.3 (Divergence of the velocity field)**.** *Assume the spherical shell δ_R is a steady state, i.e., condition* **(C0)***: $\omega(R,R) = 0$. Let v_R be the velocity field generated by δ_R, given by* (2.5)*. Then*

$$(\operatorname{div} v_R)(x) = \partial_1\omega(R,R) \qquad \textit{for all } x \in \partial B(0,R).$$

Proof. This is a direct consequence of (2.11) together with the fact that $\omega(R,R) = 0$. □

2.2.4 Well-posedness of L^p-solutions

We will denote by $\mathcal{W}^{m,p}(\mathbb{R}^N)$, $1 \quad p \quad \infty$ and $m \in \mathbb{N}$, the classical Sobolev spaces of L^p functions with derivatives up to order m in L^p. Global existence and uniqueness of L^p-solutions of equation (2.1) was established in [22, Theorem 1] under some conditions on the interaction potential W:

Theorem 2.1 (L^p-Well posedness theory)**.** *Consider $1 \quad q \quad \infty$ and p its Hölder conjugate. Suppose $\nabla W \in \mathcal{W}^{1,q}(\mathbb{R}^N)$ and $_0 \in L^p(\mathbb{R}^N) \cap \mathcal{P}_2(\mathbb{R}^N)$. Then there exists a time $T^* \quad 0$ and a nonnegative function $\in C([0,T^*], L^p(\mathbb{R}^N) \cap \mathcal{P}_2(\mathbb{R}^N)) \cap C^1([0,T^*], \mathcal{W}^{-1,p}(\mathbb{R}^N))$ such that* (2.1) *holds in the sense of distributions in $\mathbb{R}^N \times [0,T^*)$ with $d\mu_t = \ (t,x)\,dx$. Moreover, the function $t \to \| \ (t)\|^p_{L^p}$ is differentiable and satisfies*

$$\frac{d}{dt}\{\| \ (t)\|^p_{L^p}\} = -(p-1) \int_{\mathbb{R}^N} \ (t,x)^p \operatorname{div} v(t,x)\ dx \qquad \forall t \in [0,T^*]. \quad (2.15)$$

Furthermore, if ess sup $W \quad +\infty$, *then* $t \to \| \quad (t)\|_{L^p}^p$ *does not grow faster than exponentially and* $T^* = +\infty$.

We will refer to the solutions provided by the above theorem as L^p-*solutions*. One can find in [7] that the authors slightly extend the global-in-time well posedness L^p-theory to repulsive-attractive potentials under suitable conditions. We summarize the result in the following theorem.

Theorem 2.2 (Dealing with possibly growing at ∞ attractive potentials)**.** *Assume that* $W(x) = k(|x|)$ *is a radially symmetric repulsive-attractive potential,* $W(x) = (W_R + W_A)(x) = (k_R + k_A)(|x|) = k(|x|)$, $k \in C^2((0, +\infty))$, W_A *attractive (i.e.* $k'_A \quad 0$*), with* $\nabla W \in \mathcal{W}^{1,q}_{loc}(\mathbb{R}^N)$, $1 \quad q \quad \infty$, *and* W_R *compactly supported repulsive (*$k'_R \quad 0$*). Furthermore, assume that* k *satisfies:*

(i) $\exists \delta_1 \quad 0$ *such that* $k''(r)$ *is monotonic in* $(0, \delta_1)$.

(ii) $\exists \delta_2 \quad 0$ *such that* $rk''(r)$ *is monotonic in* $(0, \delta_2)$.

(iii) $D := \sup_{r \in (0,\infty)} |k'_R(r)| \quad \infty$.

(iv) There exists $m \quad 0$ *such that* $\frac{k_A(r)}{1+r^m}$ *is bounded and increasing.*

Then there exists a global in time solution for the equation (2.1) *in the sense of Theorem* 2.1 *for compactly supported initial data* $_0 \in L^p(\mathbb{R}^N)$, *which is compactly supported for all* $t \quad 0$.

Proof. Due to translational invariance we can assume without loss of generality that the center of mass is located at the 0 and the total mass is equal to one. To construct global solutions, we proceed by approximation of the potential. In fact, let us consider a mesa function with $0 \qquad 1$, i.e., a positive C^∞ radially decreasing function, such that $(r) = 1$ for $0 \quad r \quad 1$, decaying at infinity such that $(1 + r^m) \quad (r) = 1$ for $r \quad 1 + 1/\ $.

Now, we define $W_A(x) = \quad (|x|)W_A(x)$ for all $0 \qquad 1$ and $x \in \mathbb{R}^N$. Finally, we define the regularized potential as $W\ (x) = W_A(x) + W_R(x)$. It is straightforward to check that the regularized potential W satisfies conditions (i)-(iv), and thus the existence and uniqueness of solution for the equation (2.1) with the regularized potential and initial data compactly supported and $_0 \in L^p(\mathbb{R}^N)$ follows from [22], see our Theorem 2.1. Let us denote by $X\ (x_0, t)$, with $x_0 \in \mathbb{R}^N$ and $t \quad 0$, the particle maps associated to the regularized potential. Let us remark that the assumption (iv) allows us to keep the attractive character of the potential W_A.

We will show that we can control uniformly in the support of the solutions to the regularized problem in bounded time intervals. Consider any ball $B\ (0)$ containing the support of $\ _0$, i.e., supp$(\ _0)\quad B\ (0)$. Then, we claim that there exists $\delta\quad 0$ such that for any point $y \in$ supp$(\ _0)$ we have that

$$|X\ (y,t)|^2 \quad \sigma^2(t) := \sigma^2 e^t + D^2(e^t - 1), \quad t \in [0,\delta]. \tag{2.16}$$

To prove this fact, let us study the ow of any point $x_0 \in \partial B\ (0)$ and let us recall that the particle maps are defined by

$$\frac{d}{dt} X\ (x_0,t) = u\ (X\ (x_0,t),t), \qquad \text{with } u\ = -\quad * \nabla W\ ,$$
$$X\ (x_0,0) = 0,$$

where u is the velocity field. By choosing $\delta\quad 0$ small enough and using the continuity of the particle map $X\ (x_0,t)$ and the compactness of $\partial B\ (0)$, we can deduce that

$$0 \quad d_1 \quad |X\ (x_0,t)| \quad d_2, \quad \text{supp}(\quad(\cdot,t)) \quad B_{d_1}(0)\,,$$

for all $0\quad t\quad \delta$ and all $x_0 \in \partial B\ (0)$. Then, for all $t \in (0,\delta)$ we can compute

$$\frac{1}{2}\frac{d}{dt}|X\ (x_0,t)|^2 = -\quad_{\mathbb{R}^N} {k_A}'(|X\ - y|)\,\frac{X\ - y}{|X\ - y|}\cdot X\quad(y)dy$$
$$-\quad_{\mathbb{R}^N} k'_R(|X\ - y|)\,\frac{X\ - y}{|X\ - y|}\cdot X\quad(y)dy \tag{2.17}$$

where we have omitted the variables for clarity. Notice that ${k_A}'(\eta)\quad 0$, $|(X\ - y)/|X\ - y||= 1$ and $-k'_R(\eta)\quad 0$. Moreover, we have that $(X\ - y)\cdot X\quad 0$ for all $y \in$ supp$(\quad)$. This is due to the fact that supp$(\quad)\quad B_{d_1}(0)$ which implies that $|y|\quad |X\ |$ so $(X\ - y)\cdot X\ = |X\ - y|\,(|X\ | - |y|\cos\ )\quad 0$ where is the angle between $X\ - y$ and $X\ $. Then, we can get rid of the first term of the right-hand side in (2.17) to get

$$\frac{d}{dt}|X\ (x_0,t)|^2 \quad -2\quad_{\mathbb{R}^N} k'_R(|X\ - y|)\frac{X\ - y}{|X\ - y|}\cdot X\quad(y)dy$$
$$2D\,|X\ (x_0,t)| \tag{2.18}$$

which implies for all $0\quad t\quad \delta$ and all $x_0 \in \partial B\ (0)$ the following di erential inequality

$$\frac{d}{dt}|X\ (x_0,t)|^2 \quad D^2 + |X\ (x_0,t)|^2\ ,$$

leading to (2.16).

Now let us take $t = \delta$ and consider $\sigma_1 := \sigma(\delta)$. Our first claim is that supp$(\quad(\cdot,\delta))\qquad B\ _1(0)$ and thus, that we are in the same conditions as initially. To show that assume by contradiction that

$$\sup\{\|y\|\ ;\ y \in \text{supp}(\quad(\cdot,\delta))\} = \sigma_1$$

then, it should hold too that

$$\sup\{\|y\| \; ; \, y \in \mathrm{supp}(\quad(\cdot,\delta))\} = |X\;(x_0,\delta)| \tag{2.19}$$

for some $x_0 \in \partial B\;(0)$.

Let us denote by $X_{-1}(y,\delta)$ the inverse map to $X\;(y,\delta)$. From the uniform Lipschitz estimates for the particle maps and their inverses in [22], we have that given any $T \quad \delta$, there exists $K = K(T,\;)$ such that

$$|X\;(x_0,\delta) - y| \quad K\left|x_0 - X_{-1}(y,\delta)\right|$$

for all $x_0 \in \partial B\;(0)$ and $y \in \mathrm{supp}(\quad(\cdot,\delta))$. Since we obviously have $X_{-1}(y,\delta) \in \mathrm{supp}(\;_0)$, then

$$|X\;(x_0,\delta) - y| \quad K \mathrm{dist}\,(\partial B\;(0), \mathrm{supp}(\;_0)) \quad 0\,,$$

contradicting (2.19).

Therefore, we can repeat the process by choosing $\delta_1 \quad 0$ small enough to get

$$\frac{d}{dt}\,|X\;(x_1,t)|^2 \quad D^2 + |X\;(x_1,t)|^2\,,$$

for all $\delta \quad t \quad \delta_1$ and all $x_1 \in \partial B\;_1(0)$, from which (2.16) follows. This is due to the fact that

$$|X\;(x_1,t)|^2 \quad f(t)$$

for all $\delta \quad t \quad \delta_1$ where $f(t)$ solves $f'(t) = D^2 + f(t)$ for all $\delta \quad t \quad \delta_1$ with $f(\delta) = \sigma_1^2 = \sigma^2(\delta)$, and thus by uniqueness of this ordinary di erential equation we infer that $f(t) = \sigma^2(t)$ for all $\delta \quad t \quad \delta_1$. This finally shows that the procedure can be repeated at all times showing (2.16) for all $t \quad 0$.

Therefore, given any fixed time $T \quad 0$, we have shown a uniform in bound of the support of in $[0,T]$, i.e.

$$\mathrm{diam}\,(\mathrm{supp}(\quad(\cdot,t))) \quad 2\sigma(T) \quad \forall t \in [0,T]. \tag{2.20}$$

It is trivial then to check that for all $t \in [0,T]$ and all $x \in \mathrm{supp}(\quad(\cdot,t))$, we have

$$\int_{\mathbb{R}^N} \nabla W_A(x-y) \quad (y,t)dy = \int_{B_{\;(t)}(0)} \nabla W_A(x-y) \quad (y,t)dy$$

by choosing such that $2\sigma(T) \quad 1$. Therefore by uniqueness [44, 22], we see that $\;^1(x,t) = \;^2(x,t) := \;(x,t)$ for all $t \in [0,T]$, $x \in \mathbb{R}^N$ provided $2\sigma(T)\max(\;_1,\;_2) \quad 1$, and that is a solution to the Cauchy problem for (2.1) with initial data $\;_0$ in the interval $[0,T]$. Due to the uniqueness result in [44, 22], we can perform this procedure for arbitrary $T \quad 0$. □

Let us remark that the hypotheses of the previous theorem are not too restrictive if the repulsion is not too strong. All the hypotheses are met by radial potentials changing from repulsive to attractive only once with power-law behaviors both at the origin and at $+\infty$, i.e., $W(x) \simeq |x|^b$ as $|x| \to 0$ and $W(x) \simeq |x|^a$ as $|x| \to \infty$, satisfying $a \quad b \quad 1$. Under these conditions, the potential $W(x)$ has a unique minimum at $r = r_m \quad 0$ whose value can be assumed to be zero without loss of generality. In this case, the decomposition of the potential is essentially done by extracting the repulsive $(0 \quad r \quad r_m)$ and the attractive $(r \quad r_m)$ parts of W and extending them by zero to the right or to the left of r_m respectively. For instance, power-law potentials like (2.4) with $a \quad b \quad 1$ are included.

In Section 5-6, we will develop a well-posedness theory of classical solutions for potentials more singular than these ones which include all power-law repulsive-attractive potentials (2.4).

2.3 The fattening instability and dimensionality of the steady state

2.3.1 The radially symmetric case

This first subsection concerns radially symmetric solutions. We show that if the singularity of W at the origin is such that the kernel ω is C^1, and if condition **(C1)** is not satisfied, then a radially symmetric solution can not converge weakly-$*$ toward the spherical shell stationary state. We also show that the same result holds if the singularity of W at the origin is so strong that the kernel ω is not C^1 (and this is independent of how strong the attractive part of the potential is).

Theorem 2.3 (Instability of spherical shells: radially symmetric case)**.** *Let $W(x) = k(|x|)$ be a radially symmetric potential belonging to $C^3(\mathbb{R}^N \backslash \{0\})$ and such that $k'(r)$ is locally integrable on hypersurfaces (so that ω is continuous). Assume that the spherical shell δ_R is a steady state, that is,* **(C0)***: $\omega(R, R) = 0$, and that one of the two following hypotheses hold:*

(i) *$k''(r)$ and $r^{-1}k'(r)$ are locally integrable on hypersurfaces, so that ω is C^1, and $\partial_1\omega(R, R) \quad 0$.*

(ii) *W is negative in a neighborhood of the origin and is not locally inte-*

grable on hypersurfaces, in which case ω is not C^1 and

$$\lim_{\substack{(r,\eta)\notin\mathcal{D}\\ (r,\eta)\to(R,R)}} \partial_1\omega(r,\eta) = +\infty.$$

Then it is not possible for an L^p radially symmetric solution of (2.6)-(2.7) *to converge weakly-$*$ as measures to δ_R as $t \to \infty$.*

Proof. If conditions (ii) of the Theorem holds, then from (2.10) of Lemma 2.1, it is clear that there exists $\delta > 0$ such that

$$\forall \eta \in (R-\delta, R+\delta),\ r \to \omega(r,\eta) \text{ is strictly increasing in } (R-\delta, R+\delta). \tag{2.21}$$

Of course (2.21) also trivially holds if condition (i) of the Theorem is satisfied. We proceed by contradiction. Assume that $\rho(t,x)\,dx = d\mu_t(x)$ is an L^p radially symmetric solution which converges weakly-$*$ as measures to a spherical shell of radius R as $t \to \infty$.

Step 1. Assume first that μ_t converges toward δ_R not only weakly-$*$ as measures but also that the support of the radial solution $\hat\mu_t$ to (2.6) converges to the point $\{R\}$. Choose $T > 0$ such that $\operatorname{supp}(\hat\mu_t) \subset (R-\varepsilon_0, R+\varepsilon_0)$ for all $t \geq T$. Using the monotonicity property (2.21) we obtain that for $t \geq T$ and for $R-\varepsilon_0 < r_1 < r_2 < R+\varepsilon_0$

$$\hat v(t,r_2) - \hat v(t,r_1) = \int_{R-\varepsilon_0}^{R+\varepsilon_0} \left(\omega(r_2,\eta) - \omega(r_1,\eta)\right) d\hat\mu_t(\eta) \geq 0$$

where $\hat v$ is the velocity field in radial coordinate defined by (2.7). Therefore for all $t \geq T$ the function $r \to \hat v(t,r)$ is increasing on $(R-\varepsilon_0, R+\varepsilon_0)$. Let $r_1(t)$ and $r_2(t)$ be two solutions of the ODE $r_i'(t) = \hat v(t, r_i(t))$, $i = 1, 2$. Since

$$\frac{d}{dt}(r_2(t) - r_1(t))^2 = 2(r_2(t) - r_1(t))(\hat v(t, r_2(t)) - \hat v(t, r_1(t))) \geq 0\,,$$

we easily see that if for some time $t \geq T$, $r_1(t)$ and $r_2(t)$ are in $(R-\varepsilon_0, R+\varepsilon_0)$, then their distance increases. This contradicts the fact that the support of $\hat\mu_t$ is converging to the point $\{R\}$ as $t \to \infty$. Let us be more precise. Since $\hat\mu_T$ is supported in $(R-\varepsilon_0, R+\varepsilon_0)$ and is absolutely continuous with respect to the Lebesgue measure, there exists R_1 and R_2 in $(R-\varepsilon_0, R+\varepsilon_0)$, $R_1 \neq R_2$, such that

$$\int_0^{R_1} d\hat\mu_T(x) = 1/3 \qquad \text{and} \qquad \int_{R_2}^{\infty} d\hat\mu_T(x) = 1/3\,.$$

Consider the ODEs $r_i'(t) = \hat{v}(t, r_i(t))$, $r_i(T) = R_i$, $i = 1, 2$. Clearly $r_1(t)$ and $r_2(t)$ remain in $(R-\delta, R+\delta)$ for all $t \quad T$ (otherwise the support of μ_t would not stay in $(R - \ _0, R + \ _0)$). So $|r_2(t) - r_1(t)| \quad |R_2 - R_1|$ for all $t \quad T$ and the support of μ_t can not converge to the point $\{R\}$, which contradicts our assumption.

Step 2. Assume now that μ_t converges weakly toward δ_R but their supports do not converge to $\{R\}$. From the continuity of the function $\eta \to \omega(r, \eta)$ together with (2.7) it is clear that $\hat{v}(r, t)$ converges pointwise to $\omega(r, R)$ as $t \to \infty$. Since the support of $\hat{\mu}_t$ does not converge to the set $\{R\}$, there is a sequence of times $\{t_j\}_{j \in \mathbb{N}} \to \infty$ at which there is always non-zero mass in $(0, R - \ ) \cup (R + \ , +\infty)$. Since $\omega(R, R) = 0$, the monotonicity condition (2.21) implies that $\omega(r, R) \quad 0$ for all $r \in (R - \ , R)$ and $\omega(r, R) \quad 0$ for all $r \in (R, R + \ )$ as long as $\quad\ _0$. Because of the pointwise convergence of $\hat{v}$ there exists a time $T \quad 0$ such that for all $t \quad T$, $\hat{v}(t, R - \ ) \quad 0$ and $\hat{v}(t, R+ \ ) \quad 0$. So after this time T mass cannot enter the region $[R- \ , R+ \ ]$. This together with the existence of a time $t_j \quad T$ with j large enough for which there is some mass in the complementary of $[R - \ , R + \ ]$ contradict the weak convergence towards δ_R. □

Remark 2.1. *To clarify the result in Theorem 2.3, let us consider the case where the repulsive-attractive potential $W(x) = k(|x|)$ has its repulsive part described by a power-law. Let say, for example, that $k(r) = -r^b/b$, for all $r \quad 1$ and $k'(r) \quad 0$ for $r \quad 2$. If $2 - N \quad b \quad 3 - N$, then $\quad W$ is not locally integrable on hypersurfaces and therefore, according to (ii), whatever is the behavior of $W(x)$ for $|x| \quad 1$, L^p radially symmetric solutions can not converge toward the steady state. In other words if the repulsive singularity of the potential is equal to or stronger than $|x|^{3-N}$ then the potential is so repulsive in the short range that solution can not concentrate on a spherical shell, and this is independent of how attractive the potential is in the long range. On the other hand if $b \quad 3 - N$ then the kernel ω is C^1. In this case, the balance between the repulsive part and the attractive part of the potential dictates whether or not the spherical shell is an attractor: if $\partial_1 \omega(R, R) \quad 0$, then the repulsive part dominates and the spherical shell is not an attractor.*

Remark 2.2. *In Step 2 of this proof, since we are dealing with radially symmetric solutions, the problem is essentially one dimensional and the characteristics are ordered. This allows us to exclude the possibility of an L^p solution converging toward a spherical shell even if this convergence is very weak and the support of the solution does not converge. Let us remark that the convergence of the support of the solution to the singleton $\{R\}$ is equivalent in one dimension to the convergence of μ_t towards δ_R in the d_∞ sense. In* *the non radially symmetric case we will be only able to exclude convergence in d_∞.*

2.3.2 The non-radially symmetric case

In this subsection we consider non-radially symmetric solutions and we investigate whether it is possible for an L^p-solution to converge toward a steady state supported on an hypersurface which not necessarily a sphere. Indeed in numerical simulations [81, 127], it is observed that depending on the choice of the repulsive-attractive potential W, solutions of (2.1) can either converge to steady states which are smooth densities or to singular steady states which are measures supported on an hypersurface. We consider steady states μ of the form

$$\int_{\mathbb{R}^N} f(x)d\mu(x) = \int_{\mathcal{M}} f(x) \quad (x)d\sigma(x) \quad \forall f \in C(\mathbb{R}^N) \tag{2.22}$$

where $\mathcal{M}$ is a compact C^2 hypersurface and $d\sigma$ is the volume element on $\mathcal{M}$. Roughly speaking, we prove that if the the potential is as singular or more singular than $|x|^{3-N}$ at the origin, then it is not possible for an L^p-solution to converge toward such a steady state with respect to the d_∞-topology. We also prove that the same result holds if the potential is less singular than $|x|^{3-N}$, and if the divergence of the velocity field generated by such steady state is strictly positive on its support.

Theorem 2.4 (Instability: Nonradial case). *Let $W(x) = k(|x|)$ be a radially symmetric potential which belongs to $C^2(\mathbb{R}^N\backslash\{0\})$. Assume that $\lim_{r\to 0} \widehat{\Delta W}(r) = -\infty$ and that close to the origin $\widehat{\quad W}(r)$ is monotone. Let μ be a steady state of the form (2.22) with $\mathcal{M}$ being a compact C^2 hypersurface and let v be the velocity field generated by μ, that is $v = -\nabla W * \mu$. If one of the two condition holds:*

(i) W is locally integrable on hypersurfaces, $\in L^\infty(\mathcal{M})$ and

$$(\mathrm{div}\ v)(x) := -(\quad W * \mu)(x) \quad 0 \qquad \text{for all } x \in \text{supp } \mu,$$

(ii) W is not locally integrable on hypersurfaces and $(x) \quad _0 \quad 0$ for all $x \in \mathcal{M}$,

then it is not possible for an L^p solution of (2.1) to converge to μ with respect to the d_∞-topology as $t \to \infty$.

Before to prove this Theorem, let us make some remarks:

Remark 2.3. *According to Lemma* 2.3*, the result of Theorem* 2.3 *(i) of the previous subsection can be reformulated as follows: assume that the spherical shell δ_R is a steady state and let v_R be its velocity field. If*

$$(\mathrm{div}\ v_R)(x) = -(\quad W * \delta_R)(x) \quad 0 \qquad \text{for all } x \in \partial B(0,R)$$

then it is not possible for an L^p radially symmetric solution to converge weakly-$*$ as measures to δ_R as $t \to \infty$. So conditions (i) of Theorems 2.3 *and* 2.4 *are essentially the same. Similarly condition (ii) of both Theorems are also essentially the same. In this sense Theorem* 2.4 *can be seen as a generalization of Theorem* 2.3 *to the non-radially symmetric case.*

Remark 2.4. *The assumption* $\lim_{r\to 0} \widehat{\quad W}(r) = -\infty$ *simply guarantees that the potential W is strongly repulsive at the origin. The monotonicity of $\widehat{\quad W}(r)$ in a neighborhood of the origin is not essential to the proof and could be replaced by weaker hypotheses. But in practice all potentials of interest satisfy this monotonicity condition.*

Remark 2.5. *Part (ii) of the Theorem, roughly speaking, states that if the repulsive-attractive potential W is more singular than $|x|^{3-N}$ at the origin, then whatever is its attractive part, it is not possible for an L^p solution of* (2.1) *to converge with respect to the d_∞-topology toward a singular steady state supported on an hypersurface. So we see that the dimensionality of stable steady states depends on the degree of singularity of the potential. We can understand it too from the particle viewpoint: if the potential is very repulsive at the origin particles will beat the attractiveness and spread as seen in simulations* [81]*. For such potential with a strong repulsive singularity at the origin, steady states are expected to be absolutely continuous with respect to the Lebesgue measure. This result has already been shown in the power-law case $a \quad b = 2 - N$ in* [64, 63].

Theorem 2.4 is a direct consequence of the three Lemmas to follow.

Lemma 2.4 (Approximating the divergence of the velocity field)**.** *Let W be as stated in* Theorem 2.4 *and let μ be a compactly supported probability measure not belonging to L^p. Suppose there exists a Hölder continuous function $W \quad\quad W$ such that*

$$- \quad W * \mu \quad 0 \qquad \text{on } \mathrm{supp}(\mu)\,, \tag{2.23}$$

then it is not possible for an L^p solution of (2.1) *to converge to μ with respect to the d_∞-topology as $t \to \infty$.*

Proof. We proceed by contradiction. Let μ_t be an L^p solution such that $\lim_{t\to\infty} d_\infty(\mu_t,\mu) = 0$. We are going to show that there exists a $T \quad 0$ and an $\quad 0$ such that for all $t \quad T$

$$(\quad W * \mu_t)(x) \quad -\frac{\quad}{4} \qquad \text{for all } x \in \operatorname{supp}(\mu_t), \tag{2.24}$$

and combined with the equality (2.15):

$$\frac{d}{dt}\|\mu_t\|_{L^p}^p = (p-1) \int_{\mathbb{R}^N} (\quad W * \mu_t)\mu_t^p dx,$$

this guarantees that a subsequence μ_{t_n} converges weakly in L^p to an L^p function, which contradicts the assumption that $\mu \notin L^p$. Let us prove (2.24). Write

$$W * \mu_t = \quad W * (\mu_t - \mu) + \quad W * \mu + (\quad W - \quad W) * \mu_t\,. \tag{2.25}$$

Since $\quad W$ is continuous so is $\quad W * \mu$ and, therefore, (2.23) implies that there exists an $\quad 0$ and an open set $\quad$ containing the support of μ such that $\quad W * \mu \quad -\frac{\quad}{2}$ on $\quad$. Here we used that the $\operatorname{supp}(\mu)$ is a compact manifold. Note that since $\lim_{t\to\infty} d_\infty(\mu_t,\mu) = 0$ the support of μ_t will eventually be in $\quad$. To estimate the first term of (2.25), we consider $\mathcal{T}_t : \mathbb{R}^N \to \mathbb{R}^N$ a map pushing forward μ_t to μ, i.e. $\mathcal{T}_t\#\mu_t = \mu$. Then

$$\left\| \quad W * (\mu_t - \mu)(x)\right\|_{L^\infty(\mathbb{R}^N)} \quad \int_{\mathbb{R}^N} | \quad W(x-y) - \quad W(x - \mathcal{T}_t(y))| d\mu_t(y)$$

$$\int_{\mathbb{R}^N} c|y - \mathcal{T}_t(y))| \quad d\mu_t(y),$$

since this inequality is true for any map $\mathcal{T}_t$ pushing forward μ_t to μ,

$$\| \quad W * (\mu_t - \mu)\|_{L^\infty(\mathbb{R}^N)} \quad c\, d_\infty(\mu_t,\mu) \quad ,$$

so that for $t \quad T$ with T large enough,

$$\| \quad W * (\mu_t - \mu)\|_{L^\infty(\mathbb{R}^N)} \quad \frac{\quad}{4}.$$

Since $\quad W \qquad W$, the last term of (2.25) is negative, so that (2.25) follows. □

In order to conclude the proof we now need to show that under the hypotheses of the theorem there exists a Hölder continuous function $\quad W \qquad W$ satisfying (2.23). Define

$$W \ (x) := \begin{cases} W(x) & \text{if } |x| \\ W(\ e_1) & \text{if } |x| \end{cases} .$$

The function $W\ (x)$ is obviously Hölder continuous and, due to the monotonicity of W around the origin we have $W\quad W$ for small enough. We are left to show that $W\ *\mu\quad 0$ on the support of μ for small enough and this is done in the following two Lemmas.

Lemma 2.5 (Continuity in of the divergence of the velocity field)**.** *Let μ, $\mathcal{M}$ and W be as stated in* Theorem 2.4 *(i). Then $W\ *\mu$ converges uniformly on $\mathcal{M}$ toward $W*\mu$. Therefore there exists $\quad 0$ such that $W\ *\mu\quad 0$ on the support of μ.*

Proof. Since $\mathcal{M}$ is C^2 and compact, using Lemma 2.8 from the appendix, there exist constants $\delta, C_1, C_2\quad 0$ so that

$$C_1 \quad {}_0\ g(r)r^{N-2}dr \qquad {}_{\mathcal{M}\cap B(x,\ )}\ g(|x-y|)d\sigma(y) \quad C_2 \quad {}_0\ g(r)r^{N-2}dr$$

for all $x \in \mathcal{M}$, for all $\quad \delta$ and for all nonnegative function g locally integrable on hypersurfaces. Since $W(x)$ is radial and goes to $-\infty$ monotonically as $|x| \to 0^+$, we clearly have that $W(\ e_1) - \ W(x-y)\quad 0$ for all $y \in B(x,\ )$ if is small enough. Then we obtain that, for all $x \in \mathcal{M}$,

$$\begin{aligned}
&|(\ W\ *\mu)(x) - (\ W*\mu)\ (x)| \\
&\qquad = {}_{\mathcal{M}\cap B(x,\ )}\Big[\ W(\ e_1) - \ W(x-y)\Big]\ (y)d\sigma(y) \\
&\qquad\quad C_2\|\ \|_{L^\infty(\mathcal{M})}\ {}_0\Big[\ \widehat{W}(\ ) - \ \widehat{W}(r)\Big]r^{N-2}dr \\
&\qquad\quad C_2\|\ \|_{L^\infty(\mathcal{M})}\ {}_0\Big[-\ \widehat{W}(r)\Big]r^{N-2}dr
\end{aligned}$$

and we conclude using the fact that W is integrable on hypersurfaces. □

Lemma 2.6. *Let μ, $\mathcal{M}$ and W be as stated in* Theorem 2.4 *(ii). Then there exists $\quad 0$ such that $W\ *\mu\quad 0$ on the support of μ.*

Proof. Choose r_0 as in Lemma 2.8 and also small enough so that $\widehat{W}(r)\quad 0$ for all $r\quad r_0$. For $\quad r_0$ we then have

$$(\ W\ *\mu)(x) \qquad {}_{\leqslant|x-y|\leqslant r_0}\ W(x-y)d\mu(y) + {}_{|x-y|>r_0}\ W(x-y)d\mu(y). \tag{2.26}$$

Since W is bounded on $B(0, 2\,\mathrm{diam}(\mathcal{M}))\backslash B(0, r_0)$, the second term is uniformly bounded for $x \in \mathcal{M}$. We use Lemma 2.8 to estimate the first term of (2.26):

$$\underset{\leqslant|x-y|\leqslant r_0}{} W(x-y)d\mu(y) = \underset{\mathcal{M}\cap\{y:\ \leqslant|x-y|\leqslant r_0\}}{} \widehat{W}(|x-y|)\ (y)d\sigma(y)$$

$$\quad {}_0^{r_0} \widehat{W}(r)\,|\mathcal{M}\cap\partial B(x,r)|_{\mathcal{H}^{N-2}}\,dr \quad C\ {}_0^{r_0} \widehat{W}(r)r^{N-2}dr,$$

and since W is not locally integrable on hypersurface and $\widehat{W}$ 0 on $[0, r_0]$, the last integral goes to $-\infty$ as $\to 0$. Then, for 0 small enough, $W * \mu$ 0 on supp (μ). □

2.4 Stability for radial perturbations

In this section, we give su cient conditions for the stability under radial perturbations of δ_R stationary solutions in transport distances for the system (2.6)-(2.7).

Here, we will work with radial solutions with the following hypotheses of minimal regularity **(HMR)**: we assume that for any given $\mu_0 \in \mathcal{P}_2^r(\mathbb{R}^N)$, there exists $\mu \in AC([0,T], \mathcal{P}_2^r(\mathbb{R}^N))$, with $\mu_t = \mu_0$ for $t = 0$, such that

$$\hat{v}(t,r) = {}_0^{+\infty} \omega(r,\eta)d\hat{\mu}_t(\eta) \in L^2((0,T)\times\mathbb{R}^N))$$

for all T 0 and their corresponding radial measures $\hat{\mu}_t$ satisfy (2.6) in the weak distributional sense. Moreover, they satisfy that ${}_0^{\infty} r\,d\hat{\mu}_t(r)$ is an absolutely continuous function in time for which

$$\frac{d}{dt}\ {}_0^{\infty} r\,d\hat{\mu}_t(r) = {}_0^{\infty} \hat{v}(t,r)\,d\hat{\mu}_t(r) \tag{2.27}$$

holds a.e. t 0. Furthermore, if $\hat{\mu}_0$ is compactly supported, we assume that

$$r_1(t) = \min\{\mathrm{supp}\ (\hat{\mu}_t)\} \quad \text{and} \quad r_2(t) = \max\{\mathrm{supp}\ (\hat{\mu}_t)\},$$

are absolutely continuous functions with $\frac{d}{dt}r_i(t) = \hat{v}(t, r_i(t))$ a.e. t 0, $i = 1, 2$.

The existence theory developed in Section 2.5 ensures that smooth classical solutions satisfying **(HMR)** exist for $d\mu_0(x) = {}_0(x)\,dx$ with ${}_0 \in \mathcal{P}_2^r(\mathbb{R}^N) \cap \mathcal{W}^{2,\infty}(\mathbb{R}^N)$ initial data under suitable assumptions on the potential. Therefore, we assume in this section that our radial solutions satisfy (2.6)-(2.7) with ω given by (2.3) verifying suitable hypotheses specified in each result.

Theorem 2.5 (Stability for local perturbations)**.** *Assume $\omega \in C^1(\mathbb{R}^2_+)$ and that δ_R is a stationary solution to* (2.6)-(2.7)*. Let us assume that* **(C1)** *and* **(C2)** *are satisfied with strict inequality:*

$$\partial_1\omega(R,R) \quad 0 \qquad \text{and} \qquad \partial_1\omega(R,R)+\partial_2\omega(R,R) \quad 0\,. \tag{2.28}$$

Then there exists $_0$ 0 such that if the initial data $\mu_0 \in \mathcal{P}^r_2(\mathbb{R}^N)$ satisfies $\mathrm{supp}(\hat\mu_0)$ $[R- \ _0, R+ \ _0]$*, and for any solution to* (2.6)-(2.7) *with initial data satisfying* **(HMR)** *we get*

$$d_2(\hat\mu_t,\delta_R) \quad Ce^{-\ t},$$

for any $0 \qquad -\max\left(\partial_1\omega(R,R), \frac{d}{dR}\omega(R,R)\right)$ *for suitable C.*

Proof of the Theorem. Since we have assumed that the solutions to (2.6)-(2.7) satisfy the regularity conditions **(HMR)**, then $\ (t) := \mathrm{diam}\ (\mathrm{supp}(\hat\mu_t)) = r_2(t)-r_1(t)$, and

$$\ (t) := \ _0^\infty r\, d\hat\mu_t(r) - R\,,$$

are absolutely continuous function of $t \quad 0$. We will proceed by contradiction.

We define $T := \min\{t \quad 0;\ \ (t)+|\ (t)| \quad 4\ _0\}$ and let us assume that $T \quad \infty$ for all $_0 \quad 0$ close to 0. Note that $T \quad 0$ by continuity of $\ (t)+|\ (t)|$, since supp $(\hat\mu_0)$ $[R- \ _0, R+ \ _0]$ implies that

$$\ (0)+|\ (0)| \quad 2\ _0 + \ _{R-\ _0}^{R+\ _0} |r-R|\, d\hat\mu_0(r) \quad 3\ _0\,.$$

Now, for $t\in[0,T]$, $\mathrm{supp}(\hat\mu_t)$ $[R-4\ _0, R+4\ _0]$, since

$$\begin{aligned}|r_i(t)-R| &\quad |r_i(t)-(R+\ (t))| + |\ (t)| \quad (r_2(t)-r_1(t)) + |\ (t)| \\ &= \ (t)+|\ (t)| \quad 4\ _0\,,\end{aligned} \tag{2.29}$$

using that the center of mass $\ (t)+R$ is obviously in $[r_1(t), r_2(t)]$, for all $t \quad 0$ and the definition of T.

Then, for $t\in[0,T]$, Taylor expanding to order one and using that $\partial_1\omega$ is uniformly continuous on $[R-4\ _0, R+4\ _0]^2$ together with **(HMR)**, we get

$$\begin{aligned}\frac{d}{dt}\ (t) =& \frac{d}{dt}r_2(t) - \frac{d}{dt}r_1(t) \\ =& \hat v(t,r_2(t)) - \hat v(t,r_1(t)) \\ =& \ _0^\infty [\omega(r_2(t),\eta)-\omega(r_1(t),\eta)]\ d\hat\mu_t(\eta) \\ =& \ _0^\infty [\partial_1\omega(r_1(t),\eta)(r_2(t)-r_1(t)) + g(r_1(t),r_2(t),\eta)]\ d\hat\mu_t(\eta),\end{aligned}$$

where g satisfies

$$\lim_{|r_2-r_1|\to 0}\left(\sup_{\;\in[r_1,r_2]}\frac{|g(r_1,r_2,\eta)|}{|r_2-r_1|}\right)=0. \tag{2.30}$$

Since (2.30) is satisfied, the integral of g can be estimated as follows

$$\begin{aligned}{}_0^{\infty}\, g(r_1(t),r_2(t),\eta)\,d\hat\mu_t(\eta) &= {}_{r_1(t)}^{r_2(t)}\, g(r_1(t),r_2(t),\eta)\,d\hat\mu_t(\eta)\\ &= o(r_2(t)-r_1(t)) = o(\ \ (t)).\end{aligned}$$

Proceeding with the same argument as before using (2.30), we can estimate

$$\begin{aligned}\frac{d}{dt}\ \ (t) =&(r_2(t)-r_1(t))\ {}_0^{\infty}\,\partial_1\omega(r_1(t),\eta)\,d\hat\mu_t(\eta) + o(\ \ )\\ =&(r_2(t)-r_1(t))\ {}_0^{\infty}\Big[\partial_1\omega(R,R)+\big(\partial_1\omega(r_1(t),\eta)-\partial_1\omega(R,R)\big)\Big]\,d\hat\mu_t(\eta)\\ &+o(\ \ ).\end{aligned}$$

Since $\eta \in \mathrm{supp}(\hat\mu_t) = [r_1(t), r_2(t)]\quad [R-4\ _0, R+4\ _0]$ thanks to (2.29), we can then use the uniform continuity of $\partial_1\omega$ on $[R-4\ _0, R+4\ _0]^2$ to get:

$$\begin{aligned}|\partial_1\omega(r_1(t),\eta)-\partial_1\omega(R,R)|\quad & C|r_1(t)-R|+|\eta-R|\\ & C|r_1(t)-R|+|r_2(t)-R|,\end{aligned}$$

for any $\eta\in\mathrm{supp}(\hat\mu_t)$. We can then use (2.29) again giving

$$\frac{d}{dt}\ \ (t) = \partial_1\omega(R,R)(r_2(t)-r_1(t)) + o(\ \ ) + o(|\ \ |).$$

On the other hand, we can also estimate using (2.27)

$$\begin{aligned}\frac{d}{dt}\ \ (t) =&\ {}_0^{\infty}\,\hat v(r,t)\,d\hat\mu_t(r) = {}_0^{\infty}{}_0^{\infty}\,\omega(r,\eta)\,d\hat\mu_t(r)\,d\hat\mu_t(\eta)\\ =&\ {}_0^{\infty}{}_0^{\infty}\,[\omega(\eta,\eta)+\partial_1\omega(\eta,\eta)(r-\eta)]\,d\hat\mu_t(r)\,d\hat\mu_t(\eta) + o(\ \ ) + o(|\ \ |),\end{aligned}$$

where we have again used an argument as in (2.30) to estimate the rest term of the Taylor expansion, and we use it once again to obtain

$$\frac{d}{dt}\ \ (t) = \int_0^\infty\int_0^\infty \left[\omega(R,R) + \frac{d\omega}{dR}(R,R)(\eta - R) + \partial_1\omega(\eta,\eta)(r-\eta)\right] d\hat{\mu}_t(r)d\hat{\mu}_t(\eta)$$
$$+ o(\ \ ) + o(|\ \ |)$$
$$= \frac{d}{dR}\omega(R,R)\left(\int_0^\infty \eta\, d\hat{\mu}_t(\eta) - R\right)$$
$$+ \partial_1\omega(R,R)\left(\int_0^\infty r\, d\hat{\mu}_t(r) - \int_0^\infty \eta\, d\hat{\mu}_t(\eta)\right) + o(\ \ ) + o(|\ \ |)$$
$$= \left(\frac{d}{dR}\omega(R,R)\right)\ \ + o(\ \ ) + o(|\ \ |).$$

We now combine the estimates on and to get:

$$\frac{d}{dt}(\ \ +|\ \ |)(t) \quad \max\left(\partial_1\omega(R,R),\ \frac{d}{dR}\omega(R,R)\right)(\ \ +|\ \ |)(t) + o(\ \ +|\ \ |)\,. \tag{2.31}$$

Let us point out that all the $o(\ \ +|\ \ |)$-terms can be made uniformly small in the interval $[0,T]$ by taking $_0$ small by their definitions and using that $\mathrm{supp}(\hat{\mu}_t)\quad [R-4\ _0, R+4\ _0]$ in $[0,T]$. More precisely, let us take $\in (0, -\max\left(\partial_1\omega(R,R),\ \frac{d}{dR}\omega(R,R)\right))$. We can choose $_0\quad 0$ small enough for the rest terms of (2.31) to satisfy:

$$\frac{o(\ \ (t)+|\ \ (t)|)}{\ \ (t)+|\ \ (t)|}\quad \left|\max\left(\partial_1\omega(R,R),\ \frac{d}{dR}\omega(R,R)\right)\right| - \ \ , \tag{2.32}$$

for any (t), (t) since $(t)+|\ \ (t)|\quad 4\ _0$ for all $t\in[0,T]$ due to (2.29). Then (2.32) is satisfied for all $t\in[0,T]$, and thus,

$$\frac{d}{dt}(\ \ +|\ \ |)(t)\quad -\ \ (\ \ +|\ \ |)(t),$$

so that for $t\in[0,T]$,

$$(\ \ +|\ \ |)(t)\quad (\ \ +|\ \ |)(0)e^{-\ \ t}\,. \tag{2.33}$$

In particular, for any time $t\in[0,T]$, $(\ \ +|\ \ |)(t)\quad (\ \ +|\ \ |)(0)\quad 3\ _0$ and thus, using the continuity of $(\ \ +|\ \ |)(t)$ since $T\quad +\infty$ we can continue up to $T\quad T$ satisfying $(\ \ +|\ \ |)(t)\quad 4\ _0$ contradicting the definition of T. Thus, $T=\infty$ for small enough $_0$ and (2.33) then holds for all $t\quad 0$. Thanks to (2.29), this implies the exponential convergence of $d_2(\hat{\mu}_t,\delta_R)$ to 0:

$$d_2(\hat{\mu}_t,\delta_R)^2 = \int_0^\infty (r-R)^2 d\hat{\mu}_t(r)\quad \max\left(|r_1(t)-R|^2, |r_2(t)-R|^2\right)$$
$$(\ \ +|\ \ |)^2(t)\quad 3\ _0 e^{-\ \ t}$$

for all $t\quad 0$. □

Remark 2.6. *Lemma* 2.1 *gives sufficient conditions to get the assumed regularity* $\omega \in C^1(\mathbb{R}^2_+)$. *Previous Theorem holds for all radially symmetric potentials* $W(x) = k(|x|)$ *belonging to* $C^3(\mathbb{R}^N \backslash \{0\})$ *such that* $k''(r)$ *and* $r^{-1}k'(r)$ *are integrable on hypersurfaces. This applies also to the next result for non local perturbations.*

Remark 2.7. *The first part of condition* (2.28) *implies intuitively that the velocity field created by* δ_R *given by* $\omega(r, R)$ *is decreasing at* $r = R$ *and therefore, particles are pushed locally in space and in time towards radius* R *for small perturbations.*

From now on, we denote by $\ (t, \cdot)$ the pseudo-inverse of the distribution function of the radial measure $\hat{\mu}_t$, that is

$$(t,\ ) = \inf \left\{ r \in \mathbb{R}_+; \quad {}_0^r \, d\hat{\mu}_t \qquad \right\}. \tag{2.34}$$

then satisfies

$$\partial_t \ (t,\ ) = \hat{v}(t,\ (t,\ )) = \ {}_0^\infty \ \omega(\ (t,\ ), \eta) \, d\hat{\mu}_t(\eta). \tag{2.35}$$

Note that by the definition of $\ $,

$$\ {}_{[r_1,r_2]} \, d\hat{\mu}_t(\eta) = \ {}_{\{\ ; r_1 \leqslant \ (t,\ ) \leqslant r_2\}} \ d\ . \tag{2.36}$$

In the next theorem, we will work with solutions to system (2.6)-(2.7) satisfying **(HMR)** for which the pseudo-inverse of the distribution function is an absolutely continuous function on time satisfying (2.35) in the classical sense a.e. in t. Solutions obtained in Section 2.5 do satisfy these conditions.

Theorem 2.6 (Stability: Tail control)**.** *Assume* $\omega \in C^1(\mathbb{R}^2)$ *and that* δ_R *is a locally-stable stationary solution to* (2.6)-(2.7)*, that is,* $\omega(R, R) = 0$ *and the local stability condition* (2.28) *holds. Assume moreover that the velocity field associated to* δ_R *verifies*

$$\omega(r, R) \quad 0 \text{ on } (0, R), \qquad \omega(r, R) \quad 0 \text{ on } (R, \infty), \qquad \text{and} \qquad \partial_1\omega(0, R) \quad 0\,,$$

and the following long-range controls on the interaction potential: for some $\ \ 1$*, there exists* $\ \ 0$ *such that*

$$\omega(r, \eta) \quad \frac{1}{\ } - \ \ r \quad \text{for } (r, \eta) \in \mathbb{R}_+ \times [R - \ , R + \ ], \tag{2.37}$$

$$\sup_{[0,\ ]} |\partial_1 \omega(\cdot, \eta)| \quad \frac{1}{\ }(1 + \eta\ ), \tag{2.38}$$

$$|\omega(r, \eta)| \quad \frac{1}{\ }(1 + r\ )(1 + \eta\ ) \text{ for } (r, \eta) \in \mathbb{R}^2_+. \tag{2.39}$$

Then, for any solution to (2.6)-(2.7) *satisfying* **(HMR)** *and* (2.35) *with initial data* $\mu_0 \in \mathcal{P}_2^r(\mathbb{R}^N)$ *such that* $\hat{\mu}_0(\{0\}) = 0$*, and* $d\ \ (\hat{\mu}_0, \delta_R)$ *is small enough,*

$$\lim_{t\to\infty} d\ \ (\hat{\mu}_t, \delta_R) = 0.$$

Remark 2.8. *If we assume that the initial condition is compactly supported, then the long-range controls* (2.37), (2.38), (2.39) *on the interaction potential are not required anymore. Those are only necessary to control the behavior of the tail of the distribution and its interaction with the rest.*

Proof of the Theorem.

Step 1.- "Claim: Given $\hat{\mu} \in \mathcal{P}_2^r(\mathbb{R}^N)$*. If* $d\ \ (\hat{\mu}, \delta_R)$ *is small, then the associated velocity fields to* $\hat{\mu}$ *and* δ_R *share some confining properties":* For any $\quad 0$ small enough, thanks to our assumptions on ω, we can show that there exists $\quad 0$ such that if $d\ \ (\hat{\mu}, \delta_R) \quad$, then

$$\begin{cases} \hat{v}(r) \quad C_1\, r \quad 0 \text{ on } (0,\ \ ], \\ \hat{v}(r) \quad v_1 \quad 0 \text{ on } [\ \ , R -\ \ ], \\ \hat{v}(r) \quad -v_1 \text{ on } [R +\ \ , \infty), \end{cases} \tag{2.40}$$

where $\hat{v}(r)$ is the velocity field associated to $\hat{\mu}$ by (2.7). To prove the first inequality, notice that $d\ \ (\hat{\mu}, \delta_R) \quad$ implies that

$$\int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} d\hat{\mu}(\eta) \qquad ^{-\ \ /2} \int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} |\eta - R|\ \ d\hat{\mu}(\eta) \qquad ^{\overline{2}} \tag{2.41}$$

is small. We can then estimate the velocity field v for $0 \quad r \qquad\quad$:

$$\hat{v}(r) = \int_0^\infty \omega(r, \eta)\, d\hat{\mu}(\eta) = \int_0^\infty [\omega(0, \eta) + r\partial_1\omega(\ \ , \eta)]\ d\hat{\mu}(\eta)$$

$$= r \int_{[R-\sqrt{\ },R+\sqrt{\ }]} \partial_1\omega(\ \ , \eta)\, d\hat{\mu}(\eta) + r \int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} \partial_1\omega(\ \ , \eta) d\hat{\mu}(\eta).$$

Note that $\omega(0, \eta)$ is equal to zero by definition. We then use (2.38) to get the following estimate:

$$\left| \int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} \partial_1\omega(\ \ , \eta) d\hat{\mu}(\eta) \right| \qquad \int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} C(1 + \eta\ \ ) d\hat{\mu}(\eta)$$

$$C\left(\ ^{\overline{2}} + d\ \ (\hat{\mu}, \delta_R)\ \right) \quad C \quad ^{/2}.$$

Now, if is small enough and $r \in [0, \ ]$, then thanks to (2.41) and an argument as in (2.31) we conclude

$$\hat{v}(r) \quad r\partial_1\omega(0,R)\left(1 - \int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} d\hat{\mu}(\eta) - Cr^{\ \ 1/2}\right)$$
$$- Cr^{\ \ /2} \quad \frac{\partial_1\omega(0,R)}{2} r.$$

The second inequality (2.40) comes directly from assumption (2.39) and the continuity of ω: for $r \in [\ , R - \ ]$ and small enough,

$$\hat{v}(r) = \int_{[R-\sqrt{\ },R+\sqrt{\ }]} \omega(r,\eta)\, d\hat{\mu}(\eta) + \int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} \omega(r,\eta)\, d\hat{\mu}(\eta)$$
$$\left(\omega(r,R) - \sqrt{\ }\, \|\partial_2\omega\|_{L^\infty([\ ,R-\ ]\times[R-\sqrt{\ },R+\sqrt{\ }])}\right) \int_{[R-\sqrt{\ },R+\sqrt{\ }]} d\hat{\mu}(\eta)$$
$$- C \int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} (1 + \eta\ )\, d\hat{\mu}(\eta)$$
$$\frac{1}{2}\omega(r,R) - C^{\ \ /2},$$

where we have used (2.41). Since $\omega(\cdot, R)$ 0 on $(0, R)$, if 0 is small enough, $v(r)$ 0 on $[\ , R - \ ]$.

For the last inequality in (2.40), we can write the velocity field as

$$\hat{v}(r) = \int_{[R-\sqrt{\ },R+\sqrt{\ }]} \omega(r,\eta)\, d\hat{\mu}(\eta) + \int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} \omega(r,\eta)\, d\hat{\mu}(\eta) \qquad (2.42)$$

and estimate the second term of (2.42) using (2.39) and (2.41) to obtain

$$\left|\int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} \omega(r,\eta)\, d\hat{\mu}(\eta)\right| \quad C(1 + r\ ) \int_{[R-\sqrt{\ },R+\sqrt{\ }]^c} (1 + \eta\ )\, d\hat{\mu}(\eta)$$
$$C^{\ \ /2}(1 + r\ )\,.$$

Let us distinguish two cases. In the set $r \quad \left(\frac{1}{\ }\left(\frac{1}{\ } - 1\right)\right)^{1/}$ which is equivalent to $\frac{1}{\ } - \ r \quad -1$, we deduce that there exists C_1 0 such that $\frac{1}{\ } - \ r$ $-C_1(1 + r\ )$. We can then control the first term of (2.42) using (2.37) and

(2.41) to get

$$
\begin{aligned}
{}_{[R-\sqrt{\ },R+\sqrt{\ }]} \omega(r,\eta)\, d\hat{\mu}(\eta) \quad & \left(\frac{1}{\ } - \quad r \ \right) {}_{[R-\sqrt{\ },R+\sqrt{\ }]} d\hat{\mu} \\
& -C_1(1+r \ ) \left(1 - \quad {}_{[R-\sqrt{\ },R+\sqrt{\ }]^c} d\hat{\mu}\right) \\
& -\frac{C_1}{2}(1+r \ ).
\end{aligned}
$$

For $r \in I := \left[R + \quad , \left(\frac{1}{\ }\left(\frac{1}{\ } - 1\right)\right)^{1/} \ \right]$, we use the assumption that $\omega(\cdot, R) \quad 0$ on the compact interval I. By continuity of ω, we thus have that for $\quad 0$ small enough and $r \in I$,

$$
\max\left\{\omega(r,\eta);\ r \in I, \eta \in [R - \sqrt{\ }, R + \sqrt{\ }]\right\} := -C_2 \quad 0\,,
$$

and thus,

$$
\begin{aligned}
{}_{[R-\sqrt{\ },R+\sqrt{\ }]} \omega(r,\eta)\, d\hat{\mu}(\eta) \quad & -C_2 \ {}_{[R-\sqrt{\ },R+\sqrt{\ }]} d\hat{\mu} \\
& -\frac{C_2}{2} \quad -C_3(1+r \ ) \quad 0\,,
\end{aligned}
$$

for $r \in I$ and small enough using (2.41). Then, (2.42) becomes

$$
\hat{v}(r) \quad \left(-\min(C_1, C_3) + C\sqrt{\ }\right)(1+r \ ) \quad -C_4(1+r \ ),
$$

for any $r \quad R +$, if $\quad 0$ is small enough.

Step 2.- "Claim: If μ_0 is close enough to δ_R, then $\hat{\mu}_t$ satisfies (2.40) *at all times."* Let $(t, \ )$ the associated pseudo-inverse function associated to $\hat{\mu}_t$ by (2.34). We assume that $\hat{\mu}_0$ satisfies

$$
d \ (\delta_R, \hat{\mu}_0) \qquad .
$$

For any $\quad 0$, we can estimate $| \ (0, \sqrt{\ }) - R|$ as follows:

$$
\begin{aligned}
d \ (\delta_R, \hat{\mu}_0) \quad d_1(\delta_R, \hat{\mu}_0) = & \ {}_0^1 \ | \ (0, \ ) - R|\, d \\
\max\left({}_0^{\sqrt{\ }} | \ (0, \ ) - R|\, d \ , \ {}_{\sqrt{\ }}^1 | \ (0, \ ) - R|\, d \right) & .
\end{aligned}
$$

Since is not decreasing, if $(0,\sqrt{\ })$ R, then $|\ (0,\ )-R|$ $|\ (0,\sqrt{\ })-R|$ for $\in[0,\sqrt{\ }]$. If $(0,\sqrt{\ })$ R, then $|\ (0,\ )-R|$ $|\ (0,\sqrt{\ })-R|$ for $\in[\sqrt{\ },1]$, so that

$$\min\left(\sqrt{\ }|\ (0,\sqrt{\ })-R|,(1-\sqrt{\ })|\ (0,\sqrt{\ })-R|\right),$$

which provides the estimate $|\ (0,\sqrt{\ })-R|$ $\sqrt{\ }$. Similarly, $|\ (0,1-\sqrt{\ })-R|$ $\sqrt{\ }$, so that

$$(0,[\sqrt{\ },1-\sqrt{\ }])\quad [R-\sqrt{\ },R+\sqrt{\ }]. \tag{2.43}$$

Let us define $(t):=\ (t,1-\sqrt{\ })-\ (t,\sqrt{\ })$ and

$$(t):=\frac{1}{1-2\sqrt{\ }}\int_{\sqrt{\ }}^{1-\sqrt{\ }}(t,\ )\,d\ -R\,.$$

Notice that for $\in[\sqrt{\ },1-\sqrt{\ }]$,

$$|\ (t,\ )-R|\qquad (t)+\quad (t). \tag{2.44}$$

For 0 $_0$, we define

$$T:=\min\{t\in[0,\ ];\quad (t)+|\quad (t)|\qquad _0 \text{ or } d\ (\delta_R,\hat{\mu}_t)\qquad _0\}\,.$$

Thanks to (2.43), T 0 by continuity for 0 $\min(\ ,\ _0)$ small enough. We will show that there exists 0 such that $T=+\infty$. Assume by contradiction that T ∞. By definition of T, we have $d\ (\delta_R,\hat{\mu}_t)$ $_0$ for $t\in[0,T]$.

Thus, $\hat{\mu}_t$ satisfies (2.40) for $t\in[0,T]$, $v(t,\cdot)$ is positive on $[0,R-\ ]$ and negative on $[R+\ ,\infty)$, and then (2.35) implies that $\in[0,1]$, $|\ (t,\ )-R|$ $\max(|\ (0,\ )-R|,\ )$. In particular, by (2.41) we get

$$\int_{[\sqrt{\ },1-\sqrt{\ }]^c}|\ (t,\ )-R|\ d\qquad \int_{[\sqrt{\ },1-\sqrt{\ }]^c}[|\ (0,\ )-R|\ +\quad]\,d\qquad C\sqrt{\ }. \tag{2.45}$$

For $t\in[0,T]$, we deduce that

$$\begin{aligned}\frac{d}{dt}\ (t)=&\hat{v}(t,\ (t,1-\sqrt{\ }))-\hat{v}(t,\ (t,\sqrt{\ }))\\ =&\int_{[\ (t,\sqrt{\ }),\ (t,1-\sqrt{\ })]}\left[\omega(\ (t,1-\sqrt{\ }),\eta)-\omega(\ (t,\sqrt{\ }),\eta)\right]d\hat{\mu}_t(\eta)\\ &+\int_{[\ (t,\sqrt{\ }),\ (t,1-\sqrt{\ })]^c}\left[\omega(\ (t,1-\sqrt{\ }),\eta)-\omega(\ (t,\sqrt{\ }),\eta)\right]d\hat{\mu}_t(\eta).\end{aligned}$$

The first term can be estimated as it has been done for (t) in the proof of Theorem 2.5. To estimate the second term, we use (2.39), (2.44), and (2.45) to conclude that

$$\left| \int_{[\ (t,\sqrt{\ }),\ (t,1-\sqrt{\ })]^c} \left[\omega(\ (t,1-\sqrt{\ }),\eta) - \omega(\ (t,\sqrt{\ }),\eta)\right] d\hat{\mu}_t(\eta) \right|$$
$$\frac{C}{\ }\left[1+\min\left(\ (t,\sqrt{\ })\ ,\ \ (t,1-\sqrt{\ })\ \right)\right] \int_{[\ (t,\sqrt{\ }),\ (t,1-\sqrt{\ })]^c} (1+\eta\ )\, d\hat{\mu}_t(\eta)$$
$$\frac{C}{\ }[1+o(\ \ +|\ \ |)]\left[C\sqrt{\ } + \int_{[\sqrt{\ },1-\sqrt{\ }]^c} |\ (t,\ )-R|\ d\ \right]$$
$$C\sqrt{\ } + o(\ \ +|\ \ |)\,.$$

The same can be done for , and we obtain

$$\frac{d}{dt}(\ \ +|\ \ |)(t) \quad \max\left(\partial_1\omega(R,R),\ \frac{d}{dR}\omega(R,R)\right)(\ \ +|\ \ |)(t)$$
$$+\,o(\ \ +|\ \ |) + C\sqrt{\ }\,.$$

As it has been done in the proof of Theorem 2.5, $_0$ can be chosen small enough such that this implies for $t \in [0,T]$, that

$$\frac{d}{dt}(\ \ +|\ \ |)(t) \quad -\ \ (\ \ +|\ \ |)(t) + C\sqrt{\ }\,,$$

where $:= \frac{1}{2}\left|\max\left(\partial_1\omega(R,R),\ \frac{d}{dR}\omega(R,R)\right)\right|$. Then, for $t \in [0,T]$,

$$(\ \ +|\ \ |)(t) \quad \max\left((\ \ +|\ \ |)(0), \frac{C}{\ }\sqrt{\ }\right) \quad C\sqrt{\ }\,,$$

and

$$d\ (\delta_R,\hat{\mu}_t) \quad [(\ \ +|\ \ |)(t)]\ + \int_{[\sqrt{\ },1-\sqrt{\ }]^c} |\ (t,\ )-R|\ d$$
$$\max\left((\ \ +|\ \ |)(0), \frac{C}{\ }\sqrt{\ }\right) + C\sqrt{\ } \quad C\sqrt{\ }\,,$$

due to (2.43) and (2.45).

If 0 is small enough, this implies that for $t \in [0,T]$, $(\ \ +|\ \ |)(t)$ $\frac{_0}{2}$, and $d\ (\delta_R,\hat{\mu}(t))$ $\frac{_0}{2}$. By a contradiction argument similar to the one used in the proof of Theorem 2.5, this shows that if 0 is small enough, then $T = +\infty$, and (2.40) is satisfied at all times.

Step 3.- "Claim: Asymptotic convergence of $\hat{\mu}_t$ to δ_R:" Since (2.40) is satisfied for all t 0, $\hat{v}(t,r)$ $C_1 r$ on $[0, \ ]$, and then $(t, \ )$ $(0, \ )e^{C_1 t}$ if $(t, \ )$, due to (2.35). We can thus estimate, using (2.36):

$$
\begin{aligned}
{}_0 (1+r \ )\, d\hat{\mu}_t(r) \quad (1+R \ ) {}_0 \ d\hat{\mu}_t(r) &= (1+R \ ) {}_{\{ \ ; \ (t, \ ) \leqslant \ \}} d \\
(1+R \ ) {}_{\{ \ ; \ (0, \ ) \leqslant \ e^{-C_1 t}\}} d \ &= (1+R \ ) {}_0^{e^{-C_1 t}} d\hat{\mu}_0(r).
\end{aligned} \tag{2.46}
$$

Since (2.40) is satisfied, we claim that $\hat{v}(t,r)$ v_1 on $[\ , R - \ ]$, and then $(t, \ ) \in [0, R - \ ]$ for t $\frac{R}{v_1}$ implies that $(t - \frac{R}{v_1}, \ )$.

To see this, we make use of (2.40) to get $\hat{v}(t,\cdot)$ 0 on $[0, R - \ ]$. If $(t, \ ) \in [0, R - \ ]$, $(\cdot, \ )$ is thus increasing on $[0,t]$. If $(t - \frac{R}{v_1}, \ )$, then $([t - \frac{R}{v_1}, t], \ )$ $[\ , R - \ ]$

$$
R \qquad (t, \ ) - \ (t - \frac{R}{v_1}, \ ) = {}_t^{t-\frac{R}{v_1}} \hat{v}(\sigma, \ (\sigma, \ ))\, d\sigma \quad \frac{R}{v_1} v_1,
$$

which is absurd, thus $(t - \frac{R}{v_1}, \ )$ as desired. We can then estimate, for t $\frac{R}{v_1}$, using (2.36),

$$
\begin{aligned}
{}_0^{R-} (1+r \ )\, d\hat{\mu}_t(r) \quad (1+R \ ) {}_0^{R-} d\hat{\mu}_t(r) \\
&= (1+R \ ) \Big({}_0 \ d\hat{\mu}_t(\ ) + {}^{R-} d\hat{\mu}_t(\ ) \Big) \\
&= (1+R \ ) \Big({}_{\{ \ ; \ (t, \ ) \in [0, \ ]\}} d \ + {}^{R-} d\hat{\mu}_t(\ ) \Big) \\
&\quad (1+R \ ) \Big({}_{\{ \ ; \ (0, \ ) \leqslant \ e^{-C_1 t}\}} d \ + {}_{\{ \ ; \ (t - \frac{R}{v_1}, \ ) \leqslant \ \}} d \Big) \\
&= (1+R \ ) {}_0^{e^{-C_1 t}} d\hat{\mu}_0(r) + (1+R \ ) {}_0 \ d\hat{\mu}_{t-R/v_1}(r).
\end{aligned}
$$

Now, using a similar argument as in (2.46) since $(t - \frac{R}{v_1}, \ )$ in the last integral, we get

$$
{}_0^{R-} (1+r \ )\, d\hat{\mu}_t(r) \quad 2(1+R \ ) {}_0^{e^{-C_1(t-R/v_1)}} d\hat{\mu}_0(r). \tag{2.47}
$$

Thanks to (2.40), if $(t, \ ) \quad R+ \ $, then

$$(t, \ ) = \ (0, \ ) + \int_0^t \hat{v}(s, \ (s, \ ))\, ds \qquad (0, \ ) - v_1 t\,.$$

In particular, $(t, \ ) \qquad (0, \ )$ and, thanks to (2.35), we get

$$\int_{R+}^{\infty} (1 + r \ )\, d\hat{\mu}_t(r) = \int_{\{ \ ; \ (t, \ ) \geqslant R+ \ \}} (1 + \ (t, \ ) \ )\, d \tag{2.48}$$

$$\int_{\{ \ ; \ (0, \ ) \geqslant R+ \ +v_1 t\}} (1 + \ (0, \ ) \ )\, d$$

$$= \int_{R+ \ +v_1 t}^{\infty} (1 + r \ )\, d\hat{\mu}_0(r). \tag{2.49}$$

Let $\quad 0$. Thanks to (2.47), (2.49), there exists $\quad 0$ such that for any $t \quad ,$

$$\int_{[R- \ ,R+ \ ]^c} (1 + r \ )\, d\hat{\mu}_t(r) \qquad \sqrt{\ }. \tag{2.50}$$

Then, in particular, for any $t \quad$, $\int_0^{R-} d\hat{\mu}_t(r) \quad \sqrt{\ }$ and $\int_{R+}^{\infty} d\hat{\mu}_t(r) \quad \sqrt{\ }$, that is $(t, [\sqrt{\ }, 1 - \sqrt{\ }]) \quad [R - \ , R + \ ]$ and $(\ + | \ |)(t) \quad 3$ due to (2.44). Now, for $t \quad$, with an argument similar to the one used in Step 2, we get

$$\frac{d}{dt}(\ + | \ |)(t) \quad \max\left(\partial_1\omega(R,R), \frac{d}{dR}\omega(R,R)\right)(\ + | \ |)(t)$$
$$+ o(\ + | \ |) + C\sqrt{\ }. \tag{2.51}$$

Since for any $t \quad$, $(t,\cdot)|_{[\sqrt{\ },1-\sqrt{\ }]}$ takes its values in the compact set $[R - \ , R + \ ]$ independent of , we can apply an argument similar to the one used in Step 2. Choose $\quad 0$ such that (2.51) implies

$$\frac{d}{dt}(\ + | \ |)(t) \quad - \ (\ + | \ |)(t) + C\sqrt{\ },$$

and then, there exists some $T \quad$ such that for $t \quad T$,

$$(\ + | \ |)(t) \quad 2\frac{C\sqrt{\ }}{\ }. \tag{2.52}$$

To conclude, we notice that thanks to (2.50) and (2.36),

$$\begin{aligned}
{}_{[\sqrt{\ },1-\sqrt{\ }]^c} | \ (t, \ ) - R| \ d \quad & \ {}_{[\sqrt{\ },1-\sqrt{\ }]^c} \max(\ , | \ (t, \ ) - R| \ ) \, d \\
& \ 2 \ \sqrt{\ } + {}_{\{ \ ; | \ (t, \ ) - R| \geqslant \ \}} | \ (t, \ ) - R| \ d \\
& \ 2 \ \sqrt{\ } + {}_{[R- \ , R+ \ ]^c} |r - R| \ d\hat{\mu}_t(r) \\
& \ C\sqrt{\ },
\end{aligned}$$

which, together with (2.52), implies that for $t \ \ T$,

$$\begin{aligned}
d \ (\hat{\mu}_t, \delta_R) \ &= {}_{[\sqrt{\ },1-\sqrt{\ }]} | \ (t, \ ) - R| \ d \ + {}_{[\sqrt{\ },1-\sqrt{\ }]^c} | \ (t, \ ) - R| \ d \\
& \ C\sqrt{\ }.
\end{aligned}$$

Since this is true for any $\ \ 0$, it shows that $d \ (\hat{\mu}_t, \delta_R) \to 0$ as $t \to \infty$. □

2.5 Existence theory

Existence and uniqueness of weak solutions for the aggregation equation in $\mathcal{P}_2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ have been given in [84, 17, 22]. Weak measure solutions to the the Cauchy problem for the aggregation equation (2.1) were given in [36] under the condition that the potential is smooth except possibly at the origin, the growth at infinity is no worse than quadratic, and the singularity at the origin of the derivative of the potential is not worse that Lipschitz. This section is aimed to give an existence theory of classical solutions for the aggregation equation.

Theorem 2.7 (Existence of classical solutions)**.** *Let W satisfy*

$$\nabla W \in L^1(\mathbb{R}^N), \quad D^2 W \in L^1(\mathbb{R}^N), \quad (\ W)_+ \in L^\infty(\mathbb{R}^N)$$

Then, for any initial data $\ _0 \in \mathcal{W}^{2,\infty}(\mathbb{R}^N)$, there exist a classical solution $\ \in C^1([0,T] \times \mathbb{R}^N) \cap \mathcal{W}^{1,\infty}_{loc}(\mathbb{R}_+, \mathcal{W}^{1,\infty}(\mathbb{R}^N))$ to (2.1)*. Moreover, if $\ _0 \in \mathcal{W}^{\ ,\infty}(\mathbb{R}^N)$ for $\ \in \mathbb{N}$, $\ \ 2$, then $\ \in C^1([0,T] \times \mathbb{R}^N) \cap \mathcal{W}_{loc}^{\ -1,\infty}(\mathbb{R}_+, \ \mathcal{W}^{\ -1}(\mathbb{R}^N))$. Furthermore, assuming in addition that $\ _0 \in L^1(\mathbb{R}^N)$ with bounded second moment, the solution is unique.*

Proof of the theorem. Step 1: A priori estimates. In this step we assume that the solution is smooth as needed. This assumption will be removed in the next step.

We consider first $x \in \mathbb{R}^N$ such that $(t,x) = \| (t)\|_\infty$. Then, $\nabla (t,x) = 0$, and

$$\begin{aligned} \partial_t (t,x) &= \nabla (t,x)(\nabla W *)(t,x) + (t,x)(W *)(t,x) \\ & (t,x)(W *)_+(t,x) \quad \|(W)_+\|_{L^\infty} (t,x), \end{aligned}$$

so that

$$\| (t)\|_\infty \quad \| _0\|_\infty e^{\|(W)_+\|_{L^\infty} t}. \tag{2.53}$$

Let now $K \in \mathbb{N}$, K , $i = (i_1, \ldots, i_N) \in \mathbb{N}^N$ and $x \in \mathbb{R}^N$ be such that $\sum_{j=1}^N i_j = K$, $|\partial_{i_1} \ldots \partial_{i_N} (t,x)| = \|\partial_{i_1} \ldots \partial_{i_N} (t)\|_\infty$, and we define

$$\left\|D^K \right\|_p = \sup\{\|\partial \|_{L^p}, |\sigma| \quad K\}, \quad 1 \quad p \quad \infty, K \quad ,$$

where $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{N}^N$. W.l.o.g. we suppose that $\partial_{i_1} \ldots \partial_{i_N} (t,x) \quad 0$ (to change the sign of this term, one just needs to replace the element e_1 of the basis of $\mathbb{R}^N$ by $-e_1$), and then,

$$\begin{aligned} \partial_t \partial_{i_1} \ldots \partial_{i_N} (t,x) &= \partial_{i_1} \ldots \partial_{i_N} \nabla_x \cdot ((\nabla W *))(t,x) \\ &= \sum_{k=0}^{K} \sum_{\substack{| |=k \\ \leqslant i}} (\partial \nabla (t,x)) \cdot (\partial ^c(\nabla W *)(t,x)) \\ &\quad + \sum_{k=0}^{K} \sum_{\substack{| |=k \\ \leqslant i}} (\partial (t,x))(\partial ^c(W *)(t,x)). \end{aligned}$$

Here, $\sigma \quad i$ denotes $\sigma_j \quad i_j$ for $j = 1, \ldots, N$ and $\sigma^c = \sigma - i$. Using that the term $k = K$ in the first sum is zero one obtains

$$\begin{aligned} \partial_t \partial_{i_1} \ldots \partial_{i_N} (t,x) =& \nabla (t,x) \cdot (\partial_{i_1} \nabla W) * (\partial_{i_2} \ldots \partial_{i_N} (t,x)) \\ &+ (t,x)(W * (\partial_{i_1} \ldots \partial_{i_N} (t,x))(t,x) \\ &+ (\partial_{i_1} \ldots \partial_{i_N} (t,x))(W *)(t,x) \\ &+ \sum_{k=1}^{K-1} \sum_{\substack{| |=k \\ \leqslant i}} (\partial \nabla (t,x))(\nabla W * \partial ^c)(t,x) \\ &+ \sum_{k=1}^{K-1} \sum_{\substack{| |=k \\ \leqslant i}} (\partial (t,x))(\nabla W * \nabla \partial ^c)(t,x), \end{aligned} \tag{2.54}$$

and then, we get the estimate:

$$
\begin{aligned}
\partial_t \partial_{i_1} \dots \partial_{i_N} \quad (t,x) \quad & \|\nabla \quad\|_\infty \|\partial_{i_1}\nabla W\|_{L^1} \|\partial_{i_2}\dots\partial_{i_N} \quad\|_\infty \\
& + \|\partial_{i_1}\dots\partial_{i_N} \quad\|_\infty \| \quad W\|_{L^1} \| \quad \|_\infty \\
& + \sum_{k=1}^{K-1} \sum_{\substack{| \quad |=k \\ \leqslant i}} \|\partial \quad \nabla \quad\|_\infty \|\nabla W\|_{L^1} \|\partial \quad ^c \quad\|_\infty \\
& + \sum_{k=1}^{K-1} \sum_{\substack{| \quad |=k \\ \leqslant i}} \|\partial \quad\|_\infty \|\nabla W\|_{L^1} \|\nabla\partial \quad ^c \quad\|_\infty \\
& \| \quad \|_\infty \| \quad W\|_{L^1} \|\partial_{i_1}\dots\partial_{i_N} \quad\|_\infty \\
& + \|\partial_{i_1}\nabla W\|_{L^1} \|\partial_{i_2}\dots\partial_{i_N} \quad\|_\infty \|\nabla \quad\|_\infty \\
& + C_K \|\nabla W\|_{L^1} \left\|D^{K-1} \quad\right\|_\infty^2 \left\|D^K \quad\right\|_\infty .
\end{aligned}
$$

An induction scheme on K initialized by (2.53) then provides the following exponential control for $K \in \{0, \dots, \quad\}$:

$$
\left\|D^K \quad\right\|_\infty \quad C_{1,K} e^{C_{2\ K} t} \tag{2.55}
$$

where $C_{1,K}, C_{2,K}$ only depend on K, $\|D^2 W\|_{L^1}$, $\|(\quad W)_+\|_{L^\infty}$, and $\| \quad _0\|_{\mathcal{W} \quad \infty}$. Coming back to (2.54) the following estimate on the time derivative follows:

$$
\left\|\frac{d}{dt} D^K \quad\right\|_\infty \quad C_{1,K} e^{C_{2\ K} t}. \tag{2.56}
$$

Finally, taking the derivative in t on (2.1) we get

$$
\begin{aligned}
\partial_t^2 \quad =& (\nabla W * \partial_t \quad) \cdot (\nabla W * \quad) + \nabla_x \quad \cdot (\nabla W * \partial_t \quad) \\
& + \quad (\quad W * \partial_t \quad) + \partial_t \quad (\quad W * \quad),
\end{aligned}
$$

from which it is easy to derive

$$
\left\|\frac{d^2}{dt^2} \quad\right\|_\infty \quad C e^{Ct} . \tag{2.57}
$$

Step 2: Construction of a solution trough an approximation problem. Let W be a smooth approximation of W, that is $W \quad \in \mathcal{W}^{2,\infty}(\mathbb{R}^N)$ such that $\|D^2 W \quad\|_{L^1}$, $\|(\quad W \quad)_+\|_{L^\infty}$ are uniformly bounded, and:

$$
\nabla W \quad \xrightarrow{\quad \to 0} \nabla W \quad \text{in} \quad L^1(\mathbb{R}^N).
$$

Thanks to [84, 22], there exists a classical solution $\quad \in \mathcal{W}^{1,\infty}_{loc}(\mathbb{R}_+, \mathcal{W}^{1,\infty}(\mathbb{R}^N))$ with initial data $\quad _0$ for each regular interaction potential W .

The estimate (2.55) provides a uniform bound on $\| \ \|_\infty$, $\|\nabla \ \|_\infty$ and $\|D^2 \ \|_\infty$. Since 2 then (2.56) implies that $\frac{d}{dt}$ and $\left(\frac{d}{dt}\nabla \ \right)$ are uniformly bounded for 0. Applying the Ascoli-Arzelá theorem, due to (2.55), (2.56) and (2.57) there exist limits for , ∂_t and ∇_x (where we have written instead of $_k$) on $C([0,T]\times B)$ for any compact subset B $\mathbb{R}^N$ and moreover the limits denoted by , ∂_t and ∇_x belong to $C([0,T];\mathcal{W}^{1,\infty}(\mathbb{R}^N))$. For the velocity field v we have that

$$
\begin{aligned}
|v \ (t,x) - v(t,x)| \quad & \int_{\mathbb{R}^N} |(\nabla W \ - \nabla W)(x-y)| \ (y)\, dy \\
&+ \int_{\mathbb{R}^N} |\nabla W(x-y)|\,| \ (y) - \ (y)|\, dy \\
&= (I) \ + (II) \ .
\end{aligned}
$$

For (I) one observes that

$$
|(I) \ | \quad \|\nabla W \ - \nabla W\|_{L^1} \| \ \|_\infty \to 0 \quad \text{as} \quad \to 0,
$$

and for (II) one has that that

$$
|\nabla W(x-y)|\,| \ (y) - \ (y)| \quad C\,|\nabla W(x-y)| \in L^1(\mathbb{R}^N).
$$

Then, by the dominated convergence theorem, $(II) \ \to 0$ as $\ \to 0$ and, as a consequence, $v \ (t,x)$ converges pointwise to $v(t,x)$ in $[0,T]\times\mathbb{R}^N$ for all T 0. The same reasoning is used to prove that $\nabla_x \cdot v \ \to \nabla_x \cdot v$ as $\ \to 0$. Thus, the regularized equation

$$
\frac{d}{dt} \ = \nabla \ \cdot (\nabla W \ * \ ) + (\ W \ * \ )
$$

passes to the limit and $\ \in C^1([0,T]\times\mathbb{R}^N) \cap \mathcal{W}^{1,\infty}([0,T],\mathcal{W}^{1,\infty}(\mathbb{R}^N))$.

The propagation of the regularity follows from estimates (2.55) and (2.56). The proof of uniqueness follows from [44, 22]. □

Remark 2.9. *Under the assumptions on W in the previous theorem, the velocity field v is Lipschitz continuous both in space and time, then the characteristics are well defined:*

$$
\frac{d}{dt}X_t = -(\nabla W * \ )(t, X_t),
$$

and the solution is given by

$$
(t,x) = \ _0(X_t^{-1}) \det(DX_t^{-1}).
$$

Of course, a classical solution is a weak solution to (2.1) *with the measure* $d\mu_t(x) = \ (t,x)\,dx$. *Moreover, if* W *and* $_0$ *are radially symmetric, then the problem is invariant under rotations around the origin and the solution is radially symmetric at all times due to uniqueness. We also point out that if the solution is compactly supported then it remains of compact support for all times.*

2.6 The example of power-law repulsive-attractive potentials

The aim of this section is to show an example of how to apply the general instability and stability theory in the case of power-law repulsive-attractive potentials:

$$W(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b} \qquad 2 - N \quad b \quad a. \tag{2.58}$$

The condition $b \quad a$ ensures that the potential is repulsive in the short range and attractive in the long range. One can easily check that for these type of potentials $W \in L^\infty_{loc}(\mathbb{R}^N)$. The condition $2 - N \quad b$ ensures that the potential is in $\mathcal{W}^{1,q}_{loc}(\mathbb{R}^N)$ for some $1 \quad q \quad \infty$. Using algebraic computations, involving the Beta function, we give the conditions that the powers a and b should satisfy in order to apply the stability and instability theory, and we construct the bifurcation diagram for these powers. The main results of this section are the following:

Theorem 2.8 (Global existence of solutions for power-law repulsive-attractive potentials)**.** *Given* W *by* (2.58). *Assume* $_0 \in \mathcal{W}^{2,\infty}(\mathbb{R}^N)$ *is a radially symmetric compactly supported density. Then there exists a global in time radially symmetric classical solution for* (2.6)-(2.7) *in the sense of Theorem* 2.7. *Furthermore, the solution is compactly supported and confined in a large ball for all times.*

Theorem 2.9 (Sharp radial stability-instability for spherical shells)**.** *Assume that* W *is a power-law potential as in* (2.58). *Then, there exists a unique* $R_{ab} \quad 0$ *given by*

$$R_{ab} = \frac{1}{2}\left(\frac{\left(\frac{b+N-1}{2}, \frac{N-1}{2}\right)}{\left(\frac{a+N-1}{2}, \frac{N-1}{2}\right)}\right)^{\frac{1}{a-b}}$$

such that $\delta_{R_{ab}}$ *is stationary solution to* (2.1). *Moreover, the following properties hold:*

(i) If $2-N \quad b \quad 3-N$ *then* $\omega \in C(\mathbb{R}^2_+) \cap C^1(\mathbb{R}^2_+ \backslash \mathcal{D})$ *and for all* $(R, R) \in \mathcal{D}$ *we have*

$$\lim_{\substack{(r,\ )\to(R,R)\\ (r,\ )\notin\mathcal{D}}} \frac{\partial \omega}{\partial r}(r,\eta) = +\infty.$$

(ii) If $b \in \left(3-N, \frac{3a-Na-10+7N-N^2}{a+N-3}\right)$ *then* ω *is* $C^1(\mathbb{R}^2_+)$ *and*

$$\partial_1\omega(R_{ab}, R_{ab}) \quad 0.$$

(iii) If $b \in \left(\frac{3a-Na-10+7N-N^2}{a+N-3}, a\right)$ *then* ω *is* $C^1(\mathbb{R}^2_+)$ *and*

$$\partial_1\omega(R_{ab}, R_{ab}) \quad 0 \quad \text{and} \quad (\partial_1\omega + \partial_2\omega)(R_{ab}, R_{ab}) \quad 0.$$

As a consequence, if $b \in \left(2-N, \frac{3a-Na-10+7N-N^2}{a+N-3}\right)$ *then* $\delta_{R_{ab}}$ *is unstable in the sense of* Theorem 2.3 *and if* $b \in \left(\frac{3a-Na-10+7N-N^2}{a+N-3}, a\right)$ *then* $\delta_{R_{ab}}$ *is stable in the sense of* Theorem 2.6*.*

Remark 2.10. *Note that indeed for* $3-N \quad a$ *we have*

$$3-N \quad \frac{3a-Na-10+7N-N^2}{a+N-3} \quad a.$$

Remark 2.11. *In* [81] *the authors study the dynamic of a curve evolving in* $\mathbb{R}^2$ *according to the aggregation equation. They perform a linear stability analysis of the spherical shell steady state. They consider not only radially symmetric perturbations but also perturbations which break the symmetry of the spherical shell. The mode* $m = \infty$ *corresponds to a perturbation which preserves the symmetry of the spherical shell. Using a computation involving the Gamma function, they show that the mode* $m = \infty$ *is stable if and only if* $(a-1)(b-1) \quad 1$*. In order to prove* (ii) *we will perform similar type of computations involving special functions. Note that, when* $N = 2$

$$\frac{3a-Na-10+7N-N^2}{a+N-3} = \frac{a}{a-1}$$

so (ii) *is equivalent to* $(b-1)(a-1) \quad 1$ *and we recover the condition derived in* [81]*.*

As a summary of the stability and instability results for the $\delta_{R_{ab}}$ stationary state for power-law potentials we show the bifurcation diagram in Figure 2.1. For powers inside the region between $b = 2-N$ and the curve one has instability of $\delta_{R_{ab}}$. In the region above the curve one has stability of $\delta_{R_{ab}}$.

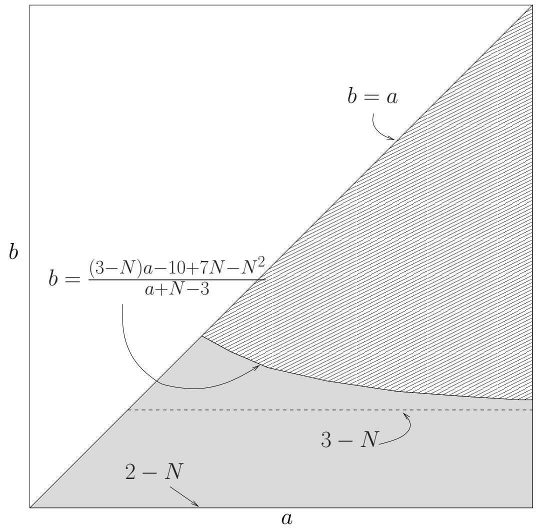


Figure 2.1: Bifurcation diagram. Stability and instability regions for $\delta_{R_{ab}}$.

2.6.1 Proof of Theorem 2.8

In order to prove the global existence theorem we need to introduce some notations. For potentials defined by (2.58), the kernel $\omega(r,\eta)$ defined in (2.3) becomes

$$\omega(r,\eta) = r^{b-1}\ \ _b(\eta/r) - r^{a-1}\ \ _a(\eta/r), \tag{2.59}$$

$$\ _a(s) = \frac{1}{\sigma_N} \quad_{\partial B(0,s)} \frac{(e_1 - sy)\cdot e_1}{|e_1 - sy|^{2-a}}\, d\sigma(y). \tag{2.60}$$

The properties of the function $\ _a(s)$ that we need are summarized in the following lemma and can be found in [53].

Lemma 2.7 (Properties of the function $\ _a(s)$)**.** *The function* $\ _a$ *is continuous with* $\ _a(0) = 1$ *and* $\lim_{s\to\infty} s^{2-a}\ \ _a(s) = \frac{N+a-2}{N}$.

The main di culty we have to cope with is the growth at infinity of the attractive part which restricts the range of direct application of Theorem 2.7.

Proof of Theorem 2.8. Due to translational invariance we can assume without loss of generality that the center of mass is located at zero. We write W as $W(x) = W_R(x) + W_A(x)$ where $W_A(x) = |x|^a/a$ is the attractive part and $W_R(x) = -|x|^b/b$ is the repulsive part. In addition, since W is radially symmetric, that is $W(x) = k(|x|)$, then we define $k(r) := k_R(r) + k_A(r)$. Finally, we write $\omega_A(r,\eta) = r^{a-1}\ \ _a(\eta/r)$ and $\omega_R(r,\eta) = r^{b-1}\ \ _b(\eta/r)$ with $\omega(r,\eta) = \omega_R(r,\eta) - \omega_A(r,\eta)$.

Step 1: A priori estimates on the support of ρ. Suppose that ρ is a smooth radially symmetric solution for the equation (2.1) with compactly supported initial data $\rho_0 \in \mathcal{W}^{2,\infty}(\mathbb{R}^N)$. Since the solutions are in $\mathcal{W}^{1,\infty}_{loc}(\mathbb{R}_+, \mathcal{W}^{1,\infty}(\mathbb{R}^N))$ and are compactly supported, the radial velocity field $\hat{v}$ is Lipschitz continuous in time and space and the characteristics are well defined. Thus $\hat{v}$ generates a C^1 flow map $r(t, r_0)$, $t \in [0, T]$, $r_0 \in \mathbb{R}_+$:

$$\frac{d}{dt} r(t) = \hat{v}(t, r(t, r_0)),$$
$$r(0, r_0) = r_0.$$

Let us define by $r_2(t)$ the characteristic curve starting at point $r_2(0) = \max\{\mathrm{supp}(\hat{\mu}_0)\}$. Then,

$$\frac{d}{dt} r_2(t) = \hat{v}(t, r_2(t)) = \int_0^\infty \omega(r_2(t), \eta)\, d\hat{\mu}_t(\eta) = \int_0^{r_2(t)} \omega(r_2(t), \eta)\, d\hat{\mu}_t(\eta)$$
$$= -\int_0^{r_2(t)} r_2(t)^{a-1} \psi_a(\eta / r_2(t))\, d\hat{\mu}_t(\eta) \tag{2.61}$$
$$+ \int_0^{r_2(t)} r_2(t)^{b-1} \psi_b(\eta / r_2(t))\, d\hat{\mu}_t(\eta), \tag{2.62}$$

where we have used the expression of ω given in (2.59). We remind that $\hat{\mu}_t$ denotes the measure with density $\hat{\rho}_t$, i.e., $d\hat{\mu}_t(r) = \hat{\rho}(t, r)\, r^{N-1}\, dr$. Using the properties of ψ_a in Lemma 2.7 in (2.61) we obtain the following inequality:

$$\frac{d}{dt} r_2(t) \le K_b r_2(t)^{b-1} - K_a r_2(t)^{a-1}, \tag{2.63}$$

where

$$\begin{cases} K_a = 1 & \text{and} \quad K_b = \psi_b(1), \quad \text{if } 2 \le b \le a, \\ K_a = 1 & \text{and} \quad K_b = 1, \quad \text{if } 2 - N \le b \le 2 \le a, \\ K_a = \psi_a(1) & \text{and} \quad K_b = 1, \quad \text{if } 2 - N \le b \le a \le 2. \end{cases}$$

Defining $R_{ab} := \left(\frac{K_a}{K_b}\right)^{\frac{1}{b-a}}$ and rewriting (2.63) as

$$\frac{d}{dt} r_2(t) \le r_2(t)^{a-1} (K_b r_2(t)^{b-a} - K_a)$$

one realizes that $r_2(t) \le \overline{R} := \max(r_2(0), R_{ab})$ which proves that the $\mathrm{supp}(\hat{\mu}_t)$ is bounded and contained in $B(0, \overline{R})$ for all times.

Step 2: Global existence. Given 0 1, consider (r) a $C^\infty(0,\infty)$ decreasing function with 0 1 such that $(r) = 1$ if 0 r 1/ and $(r) = 0$ if r $1 + 1/$. Define $f\ (r) := \ (r) \cdot k'_A(r) \in L^1(0,\infty)$, $k_A(r) := {}_0^r f\ (s)ds$ and $k\ (r) = k_R(r) + k_A(r)$. Now, the potential $W\ (x) = k\ (|x|)$ satisfies the hypotheses of Theorem 2.7 so we have existence and uniqueness of classical solution to (2.6)-(2.7) in $[0,T]$ with initial data ${}_0$. We denote by $\hat{\mu}_t$ the measure with density $\hat{}_t$.

Consider $r_2 = \max\{\mathrm{supp}(\hat{\mu}_0)\}$ and the characteristic curve $r_2(t)$ starting at point $r_2 = r_2(0)$. Computing the derivative with respect to time, one has

$$\begin{aligned}\frac{d}{dt}r_2(t) &= {}_0^\infty \omega\ (r_2(t),\eta)d\hat{\mu}_t(\eta) = {}_0^{r_2(t)} \omega\ (r_2(t),\eta)d\hat{\mu}_t(\eta)\\ &= - {}_0^{r_2(t)} \omega_A(r_2(t),\eta)d\hat{\mu}_t(\eta) + {}_0^{r_2(t)} r_2(t)^{b-1}\ {}_b(\eta/r_2(t))d\hat{\mu}_t(\eta)\\ &\quad C_b r_2(t)^{b-1},\end{aligned}$$

where we have split the kernel ω into its attractive and repulsive parts, and we have used that ω_A 0. The constant C_b depends on b. The last inequality leads us to

$$r_2(t) \quad \sigma(t) := (C_b(2-b)t + r_2^{2-b})^{\frac{1}{2-b}}, \tag{2.64}$$

which says that the solution exists at least up to time $T^* := \frac{1}{2}\min\{T, T_b\}$. where

$$T_b = \begin{cases} \frac{r_2(0)^{2-b}}{C_b(b-2)} & \text{if} \quad b \quad 2,\\ +\infty & \text{if} \quad b \quad 2.\end{cases}$$

In addition, (2.64) gives us a uniform estimate for the support of up to time T^*. Notice that for all t T^* and 0 such that $2\sigma(T^*)$ 1, then $\nabla W\ * \ {}_t = \nabla W * \ {}_t$ for all $x \in \mathrm{supp}(\ {}_t)$ and all $t \in [0,T^*]$. As a consequence ω given by (2.59) and ω associated to W by (2.3) are equal in the set $\{(r,\eta) \,|\, (r,\eta) \in \mathrm{supp}(\ {}_t)^2\}$ for all t T^*. Therefore, we can write:

$$\begin{aligned}\frac{d}{dt}r_2(t) &= - {}_0^{r_2(t)} \omega_A(r_2(t),\eta)d\hat{\mu}_t(\eta) + {}_0^{r_2(t)} r_2(t)^{b-1}\ {}_b(\eta/r_2(t))d\hat{\mu}_t(\eta)\\ &= - {}_0^{r_2(t)} \omega_A(r_2(t),\eta)d\hat{\mu}_t(\eta) + {}_0^{r_2(t)} r_2(t)^{b-1}\ {}_b(\eta/r_2(t))d\hat{\mu}_t(\eta)\\ &= - {}_0^{r_2(t)} r_2(t)^{a-1}\ {}_a(\eta/r_2(t))d\hat{\mu}_t(\eta)\\ &\quad + {}_0^{r_2(t)} r_2(t)^{b-1}\ {}_b(\eta/r_2(t))d\hat{\mu}_t(\eta),\end{aligned}$$

and we can use the a priori estimates developed in Step 1. Then, for all $t \quad T^*$ we can conclude that $r_2(t) \quad \overline{R}$. Now, let us take such that $2\,\overline{R} \quad 1$. Therefore $\nabla W \ast \ _t = \nabla W \ast \ _t$. For all $t \quad T^*$ in the support of $_t$. By uniqueness $_t =: \ _t$ for all $2\,\overline{R} \quad 1$ and it is a classical solution to (2.1) with potential W. Summarizing, we have shown the existence of solution in the time interval $[0, T^*]$ with $r_2(T^*) \quad \overline{R}$. Now, we can extend and repeat this argument for a time step $\ t := \frac{1}{2}\min\left(1, \overline{T}_b\right)$, where $\overline{T}_b = \frac{\overline{R}^{2-b}}{C_b(b-2)}$ if $b \quad 2$ or $\overline{T}_b = +\infty$ if $b \quad 2$, obtaining a solution up to time $T^* + \ t$ such that $r_2(t) \quad \overline{R}$ for all $t \in [0, T^* + \ t]$. Since $\ t$ is independent of the initial data and $\quad 0$, then we can extend the solution for all times.

Finally, the a priori estimates on the support of show that the support of the solution remains compact for all times. □

2.6.2 Proof the Theorem 2.9

It is first convenient to rewrite (2.60) as:

$$_a(s) = \frac{\sigma_{N-1}}{\sigma_N} \int_0 \frac{(1 - s\cos\ )(\sin\ )^{N-2}}{(1 + s^2 - 2s\cos\ )^{\frac{2-a}{2}}} d\ . \tag{2.65}$$

We recall that $\omega(r, \eta)$ is the velocity at r generated by $\partial B(0, \eta)$. So a δ_R with $R \quad 0$ is a steady state if and only if $\omega(R, R) = 0$, i.e.

$$R = R_{ab} = \left(\frac{_b(1)}{_a(1)}\right)^{\frac{1}{a-b}}, \tag{2.66}$$

where we have used (2.59).

Proof of the Theorem 2.9. The point (i) is a direct consequence of Lemma 2.3. Let us prove (ii). From Lemma 2.3, see also [53], it is clear that $\omega \in C^1(\mathbb{R}^2_+)$ and we have

$$\frac{\partial\omega}{\partial r}(R_{ab}, R_{ab}) = R_{ab}^{b-2}\Big[(b-1)\ _b(1) - \ _b'(1)\Big] - R_{ab}^{a-2}\Big[(a-1)\ _a(1) - \ _a'(1)\Big]. \tag{2.67}$$

After some algebra, one easily obtains from (2.66) and (4.9) that condition $\frac{\partial}{\partial r}(R_{ab}, R_{ab}) \quad 0$ is equivalent to

$$a - \frac{_a'(1)}{_a(1)} \quad b - \frac{_b'(1)}{_b(1)}. \tag{2.68}$$

Both $_a(1)$ and $_a'(1)$ can be expressed in terms of the Beta function. Recall that one of the expressions of the Beta function is:

$$(x, y) = 2\int_0^{\ /2} (\cos\ )^{2x-1}(\sin\ )^{2y-1} d\ . \tag{2.69}$$

We first compute $\ _a(1)$. Using (2.65):

$$\begin{aligned}
\frac{\sigma_N}{\sigma_{N-1}}\ {}_a(1) &= {}_0 \frac{(1-\cos\ )}{A(1,\ )^{2-a}}(\sin\ )^{N-2} d \\
&= 2^{\frac{a-2}{2}}\ {}_0 (1-\cos\ )^{a/2}(\sin\ )^{N-2} d \\
&= 2^{\frac{a-2}{2}}\ {}_0 (2\sin^2 \tfrac{}{2})^{a/2}(2\cos\tfrac{}{2}\sin\tfrac{}{2})^{N-2} d \\
&= 2^{a+N-3}\ {}_0 \left(\sin\frac{}{2}\right)^{a+N-2}\left(\cos\frac{}{2}\right)^{N-2} d \\
&= 2^{a+N-3}\ \left(\frac{a+N-1}{2}, \frac{N-1}{2}\right),
\end{aligned}$$

where we have used the fact that $A(1,\ ) = \sqrt{2(1-\cos\ )}$ and the identities $1-\cos\ = 2\sin^2\frac{}{2}$ and $\sin\ = 2\cos\frac{}{2}\sin\frac{}{2}$. We similarly compute

$$\begin{aligned}
\frac{\sigma_N}{\sigma_{N-1}}\frac{N-1}{(a-2)(a+N-2)}\ {}'_a(1) &= {}_0 \frac{(\sin\ )^N}{A(1,\ )^{4-a}} d \\
&= {}_0 \frac{(\sin\ )^N}{(2(1-\cos\ ))^{\frac{4-a}{2}}} d \\
&= {}_0 \frac{(2\cos\frac{}{2}\sin\frac{}{2})^N}{(2(2\sin^2\frac{}{2}))^{\frac{4-a}{2}}} d \\
&= 2^{N+a-4}\ {}_0 \left(\cos\frac{}{2}\right)^{N}\left(\sin\frac{}{2}\right)^{N+a-4} d \\
&= 2^{N+a-4}\ \left(\frac{a+N-3}{2}, \frac{N+1}{2}\right).
\end{aligned}$$

Note that since $a+N-3\ \ 0$ the Beta function is well defined. If we compute the quotient we obtain:

$$\frac{{}'_a(1)}{{}_a(1)} = \frac{1}{2}\frac{(a-2)(a+N-2)}{N-1}\frac{\left(\frac{a+N-3}{2}, \frac{N+1}{2}\right)}{\left(\frac{a+N-1}{2}, \frac{N-1}{2}\right)}.$$

At this point, we remind that $\ (z,t) = \frac{(z)\ (t)}{(z+t)}$. With this expression, the quotient can be simplified as

$$\frac{{}'_a(1)}{{}_a(1)} = \frac{1}{2}\frac{(a-2)(a+N-2)}{N-1}\frac{\left(\frac{N+1}{2}\right)\ \left(\frac{a+N-3}{2}\right)}{\left(\frac{a+N-1}{2}\right)\ \left(\frac{N-1}{2}\right)} = \frac{1}{2}\frac{(a-2)(a+N-2)}{a+N-3},$$

where we used that $\ (z+1) = z\ (z)$. Plugging the above expression into (2.68) and doing some algebra we deduce

$$(a+N-3)b^2 + (N^2-7N+10-a^2)b - (N^2-7N+10)a - (N-3)a^2\ \ 0.$$

The roots of the quadratic form are

$$b = a \quad \text{and} \quad b = \frac{3a - Na - 10 + 7N - N^2}{a + N - 3},$$

which gives (ii). To prove (iii) one just need to replace the sign by a in (2.68) to get the first condition. For the second condition of stability, we can easily compute $\frac{\partial}{\partial}$ from (2.59) and (4.9) to get

$$\left(\frac{\partial\omega}{\partial r} + \frac{\partial\omega}{\partial\eta}\right)(R_{ab}, R_{ab}) = R_{ab}^{b-2}(b-1) \quad _b(1) - R_{ab}^{a-2}(a-1) \quad _a(1)\,.$$

Now, using the definition of R_{ab} in (2.66), we finally obtain

$$\left(\frac{\partial\omega}{\partial r} + \frac{\partial\omega}{\partial\eta}\right)(R_{ab}, R_{ab}) = b - a \quad 0\,.$$

The stated instability and stability are direct applications of Theorem 2.3 and Theorem 2.6 respectively. □

Remark 2.12. *We want to point out that the general theory developed in the previous sections is still working in dimension $N = 1$ for even solutions which correspond to the radially symmetric solutions in higher dimensions. In the case $N = 1$, and for even solutions, the function $_a$, corresponding to $W(x) = \frac{|x|^a}{a}$ reads*

$$_a(s) = \frac{1}{2}\left[(1-s)\,|1-s|^{a-1} + (1+s)\,|1+s|^{a-1}\right].$$

One can easily check that the properties of the function $_a$ and ω for $W(x)$ as in (2.58) *in $N = 1$ are the same as in* Lemma 2.7 *and in* Theorem 2.9. *The radius is $R_{ab} = \frac{1}{2}$ whatever the powers are, see* (2.66). Theorem 2.9 *applies: if $b \in (1, 2)$ then we are in the instability case* (i) *and if $b \in [2, a)$ we are in the stability case. The curve which separates the instability and stability regions in Figure 2.1 degenerates and becomes the line $b = 2$.*

Moreover, in [61, 62] *the authors proved the existence of weak solutions and convergence, up to extractions of subsequences, of $(\cdot, t)$ for potentials like* (2.58) *with $a = 2$ and $b \in (0, 1]$. They also showed numerical simulations supporting the conjecture that the stationary state $\frac{1}{2}\left(\delta_{x=-1/2} + \delta_{x=1/2}\right)$ is unstable. Note that our* Theorem 2.9, *only applies in $N = 1$ for $1 \quad b \quad a$. Since global existence was proven in* [61, 62], *then our instability result also applies to $a = 2$, $b \in (0, 1]$ in $N = 1$.*

2.7 Numerical results

In this section, we illustrate the previous results and get some further conjectures for the instability cases. Our numerical code is based on the inverse distribution function in radial coordinates. As it was reminded in (2.35), the equation for the inverse distribution function reads

$$\frac{\partial}{\partial t}(t, \ ) = {}^{1}_{0}\ \omega(\ (t, \ ), \ (t, \ ))d\ . \tag{2.70}$$

A solution of (2.6) converges to a Dirac mass if and only if its pseudo inverse distribution becomes at.

Numerical codes based on (2.70) are then more stable when dealing with mass concentration. We will then use a backward Euler scheme in time coupled to a composite Simpson rule to approximate the integral term, and solve the resulting nonlinear system by the Newton-Raphson algorithm. Let us remark that the convergence of the semi-discrete backward Euler scheme is equivalent to the convergence of the JKO variational scheme for (2.1) (see [25, 43]). The convergence of the semi-discrete backward Euler scheme is therefore known under suitable conditions on the interaction potential, see [36] for details. All simulations are done for $N = 2$.

Test case, total concentration at the origin: $W(x) = \frac{|x|^2}{2}$

In this case, (2.70) reduces to $\frac{\partial}{\partial t}(t, \ ) + \ (t, \ ) = 0$. To test our scheme, we use this attractive potential for which the solution converges exponentially fast to a total concentration at zero, that is to $\ \equiv 0$, as seeing in Figure 2.2.

Stability Case for the Spherical Shell: $W(x) = \frac{|x|^4}{4} - \frac{|x|^2}{2}$.

In this case, we have a repulsive-attractive power-law potential with powers in the stability region of Figure 2.1. We thus expect that the mass will concentrate towards a spherical shell, thanks to the results of Theorem 2.9. The radius of the spherical shell can be computed using (2.66):

$$R_{ab} = \left(\frac{{}_2(1)}{{}_4(1)}\right)^{\frac{1}{2}} = \frac{\sqrt{3}}{3}.$$

For $b \ \ 2$ and both a and b integers, one can compute explicitly the expression for the velocity field $\omega(r, \eta)$, which is a polynomial function, in our case $\omega(r, \eta) = -r^3 - 2r\eta^2 + r$. The evolution of is shown in Figure 2.3. In Figure 2.3 we also plot the velocity field $r \mapsto \omega(r, R_{ab})$. Notice that $r \mapsto$

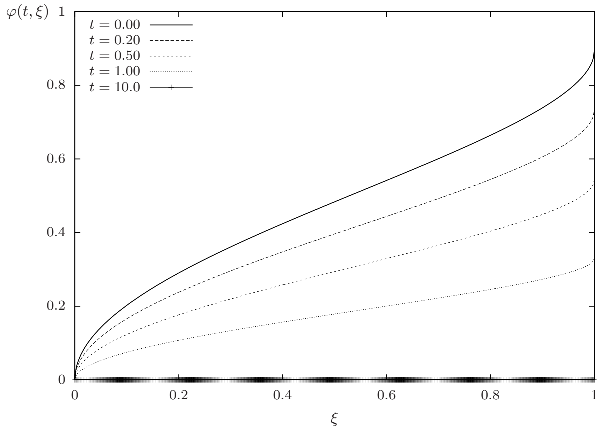


Figure 2.2: Evolution of $\xi \mapsto \varphi(t,\xi)$ for $W(x) = \frac{|x|^2}{2}$ towards total concentration at 0.

$\omega(r, R_{ab})$ satisfies the conditions of Theorems 2.5 and 2.6: $\omega(R_{ab}, R_{ab}) = 0$, $\partial_1\omega(R_{ab}, R_{ab}) \quad 0$, $\mathrm{sign}(\omega(r, R_{ab})) = \mathrm{sign}(R_{ab} - r)$, $\partial_1\omega(0, R_{ab}) \quad 0$.

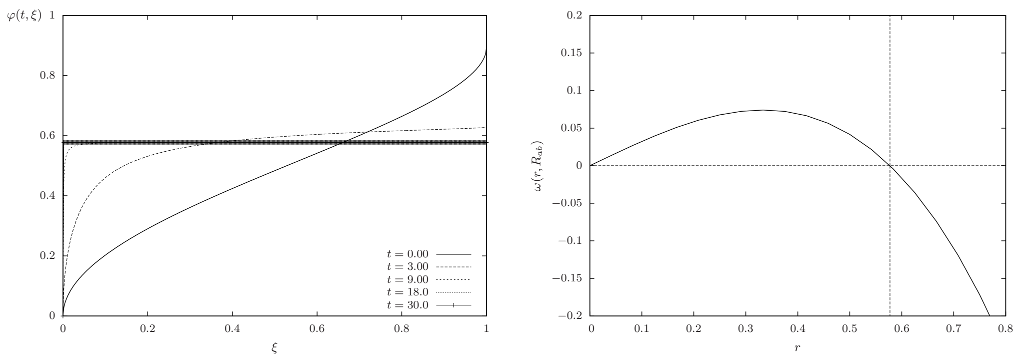


Figure 2.3: Case $W(x) = \frac{|x|^4}{4} - \frac{|x|^2}{2}$. Left: Evolution of $\xi \mapsto \varphi(t,\xi)$ towards the uniform distribution on the sphere of radius $R_{ab} = \frac{\sqrt{3}}{3}$. Right: Velocity eld $r \mapsto \omega(r, \frac{\sqrt{3}}{3})$ with the vertical line pointing out R_{ab}.

Instability Case for the Spherical Shell: $W(x) = \frac{|x|^2}{2} - |x|$.

In this case, the powers are in the instability region of Figure 2.1, below the curve $b = \frac{a}{a-1}$. Then, due to the results in Theorem 2.9, a spherical shell is unstable. One can notice on Figure 2.4 that the function $r \mapsto \omega(r, R_{ab})$ associated to the potential $W(x) = \frac{|x|^2}{2} - |x|$ satisfies $\partial_1\omega(R_{ab}, R_{ab}) \quad 0$, so that the instability condition of Theorem 2.3 is indeed met.

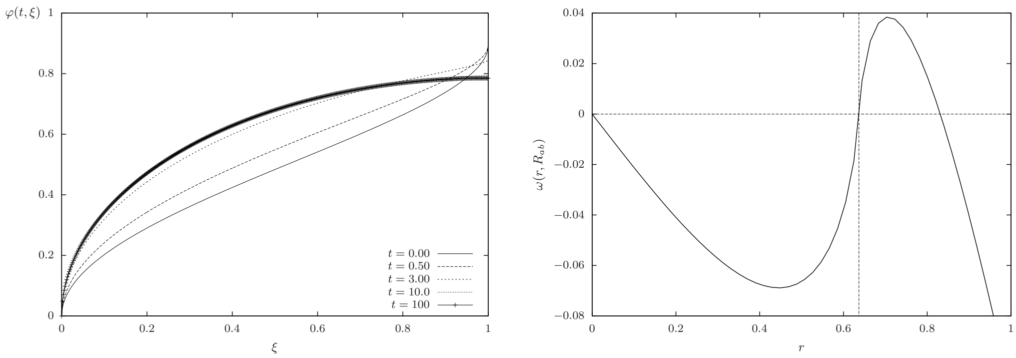


Figure 2.4: Case $W(x) = \frac{|x|^2}{2} - |x|$. Left: Evolution of $\xi \mapsto \varphi(t,\xi)$ towards a stationary pro le, possibly an integrable function. Right: Velocity eld $r \mapsto \omega(r, R_{ab})$ with the vertical line pointing out $R_{ab} \sim 0.6366$.

Figure 2.4 shows that the solution seems to converge to some stationary state which does not have any singular part, i.e., possibly an integrable function. Numerically, this behavior appears for any powers a, b in the instability region of Figure 2.1. We conjecture that in this region there exists integrable radial stationary states which are locally stable under radial perturbations. This has already been proved in the particular case of $b = 2 - N$ and a 2 in [64]. Some numerical simulations using particle systems done in [81] however suggest that these stationary states might be unstable for non radial perturbations.

Energy dissipation

We remind that the energy functional is given by

$$E[\mu(t)] = \int_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)\ (t,x)\ (t,y)\, dy\, dx\,.$$

Using the polar change of coordinates $x = r\sigma$ and $y = s\sigma$ and using the radial symmetry of (t), this energy writes:

$$E[\mu(t)] = \frac{1}{2\sigma_N} \int_{\mathbb{R}^2_+} \int_{\partial B(0,1)} W(r\sigma - se_1) d\sigma\, d\hat{\mu}_t(s)\, d\hat{\mu}_t(r)\,.$$

A formal calculation implies that the derivative w.r.t. time of the energy is negative and given by

$$\frac{d}{dt} E[\mu(t)] = -\int_{\mathbb{R}^+} \hat{v}(t,r)^2\, d\hat{\mu}_t(r)\,,$$

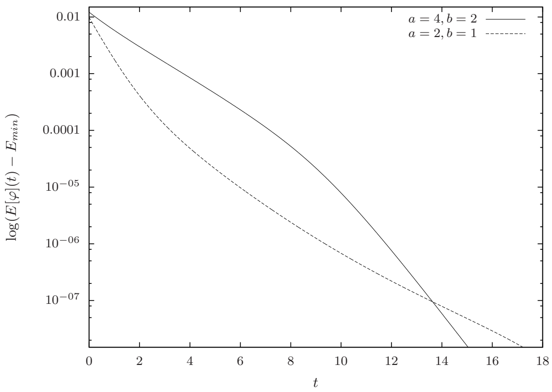


Figure 2.5: Energy decay in logarithmic scale for the regular repulsive-attractive potential, case $a = 4$ and $b = 2$ (solid line) and for the singular repulsive-attractive potential, case $a = 2$ and $b = 1$ (dashed line). Note that E_{min} is the numerical limit of the energy as $t \to \infty$.

the energy should then decrease in time. Using radially symmetric coordinates, the energy functional for the inverse distribution function is given by

$$E[\ ](t) = \frac{1}{2\sigma_N} \int_0^1 \int_0^1 \int_{\partial B(0,1)} W(\ (t,\ )\sigma - \ (t,\ )e_1) d\sigma d\ d\ . \qquad (2.71)$$

We have computed the energy using the formula (2.71) to check numerically, in each case, that the energy decreases. In Figure 2.5 we observe the exponential decay of the energy for the two numerical examples presented above for repulsive-attractive potentials.

2.8 Conclusions

Nonlocal interaction equations with repulsive-attractive potentials arise as minimal models in biological and material sciences where individuals or particles self-organize in order to reach the configuration with the lowest interaction energy. The current work as well as other recent works [81, 127, 126, 64, 63] focus on understanding what type of potentials leads to what type of configurations.

This analysis has implications in the modelling of the potential to make biologically or physically reasonable choices leading to a better approximation of the shape of ock and/or mill profiles in swarming [45, 127] or better approximation of the particle configurations observed in the lab for application to materials science [56, 128, 111, 73, 126].

The simplest type of stationary states arising in nonlocal interaction equations with repulsive-attractive potentials are uniform distributions on a sphere. They appear due to the balance of attractive and repulsive forces from the potential. The linear stability analysis of these spherical shell steady states was performed in [81, 127] for discrete systems of particles in two/three dimensions. In the present chapter we have considered continuum densities of particles in $\mathbb{R}^N$ and we have conducted a fully nonlinear stability analysis of these spherical shell solutions with respect to radial perturbations. We show that under simple conditions on the potential the "fattening instability" is triggered: the spherical shell destabilizes producing a fat shell solution. On the contrary, we prove that the complementary conditions suffice to get asymptotic stability of these spherical shell solutions and we provide exponential convergence rate in term of distances between measures. Additionally, the specific case of power-law potentials was investigated and careful numerical simulations were done to illustrate the main results of this chapter.

The study performed in this chapter about the fattening instability provides insight on the dimensionality of stable steady states: our results indicate that if the repulsive singularity of the potential at the origin is strong enough then the fattening instability kicks in, and instead of a spherical shell of dimension $N-1$, the dynamics leads to a "fat" spherical shell of dimension N. A deeper analysis of how the singularity of the potential dictates the dimensionality of stable steady states is currently under way.

2.9 Appendix

Let us start by some differential geometry facts. For the sake of clarity, we first define the type of hypersurfaces we will work with.

Definition 2.5. *$\mathcal{M} \subset \mathbb{R}^N$ is a C^2 hypersurface (manifold of dimension $N-1$) if for any $x \in \mathcal{M}$ there exists a C^2 chart (U, φ), i.e., a pair of an open connected set and a C^2 diffeomorphism $\varphi : U \longrightarrow \mathbb{R}^N$, with $x \in U \subset \mathbb{R}^N$ such that $\varphi(x) = 0$ and $y \in \mathcal{M} \cap U$ if and only if $\varphi(y) \in \{0\} \times \mathbb{R}^{N-1}$.*

We will need some technical result from differential geometry in order to deal with the regularity of the function ω in (2.3) and its generalizations to any compact hypersurface. Note first that if $\mathcal{M} \subset \mathbb{R}^N$ is a hyperplane then $M \cap \partial B(x,r)$ is a $N-2$ dimensional sphere of radius $(r^2 - \mathrm{dist}(x,\mathcal{M})^2)_+^{1/2}$ and therefore its surface area is

$$|\mathcal{M} \cap \partial B(x,r)|_{\mathcal{H}^{N-2}} = \sigma_{N-1}(r^2 - \mathrm{dist}(x,\mathcal{M})^2)_+^{\frac{N-2}{2}}$$

where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure, and we remind that σ_{N-1} is the surface area of the unit sphere in $\mathbb{R}^{N-1}$.

The following result is a classical consequence of uniform graphs lemmas in differential geometry. They state that a compact regular hypersurface can be covered by graphs with bounds on their derivatives depending only on the uniform bound of the second fundamental form. We refer to [105, Lemma 4.1.1]. This allows to show that the volume elements locally converge to those of a hyperplane in a uniform manner.

Lemma 2.8. *Let $\mathcal{M} \subset \mathbb{R}^N$ be a C^2 compact hypersurface of dimension $N-1$ immersed in $\mathbb{R}^N$. Then there exist small enough $r_0 > 0$ and constants $C, C > 0$ depending on the global bound of the second fundamental form of $\mathcal{M}$ such that for all $0 \leq r \leq r_0$, and all $x \in \mathbb{R}^N$ with dist$(x, \mathcal{M}) \leq r_0$*

$$C\,(r^2 - \mathrm{dist}(x,\mathcal{M})^2)_+^{\frac{N-2}{2}} \leq |\mathcal{M} \cap \partial B(x,r)|_{\mathcal{H}^{N-2}} \leq C\,(r^2 - \mathrm{dist}(x,\mathcal{M})^2)_+^{\frac{N-2}{2}}. \tag{2.72}$$

Remark 2.13. *Let us note that the previous Lemma is trivial in the case of $\mathcal{M} = \partial B(0,\eta)$ for any $\eta > 0$ since the intersection of two $(N-1)$-dimensional spheres of different radius is always a $(N-2)$-dimensional sphere lying on a hyperplane. In fact, we can easily compute that if two spheres $\partial B(0,\eta)$ and $\partial B(x,r)$ intersect, that is $||x| - \eta| \leq r$, then*

$$|\partial B(0,\eta) \cap \partial B(x,r)|_{\mathcal{H}^{N-2}} = \sigma_{N-1} r_1^{N-2}$$

where $r_1 = r_1(\eta, r, \mathrm{dist}(x, \partial B(0,\eta)))$ is the radius of the intersection, which is computable:

$$r_1 = \eta\sqrt{1 - \left(\frac{|x|^2 + \eta^2 - r^2}{2|x|\eta}\right)^2} \sim \sqrt{\frac{\eta}{|x|}}\sqrt{r^2 - \mathrm{dist}(x, \partial B(0,\eta))^2},$$

as $r - \mathrm{dist}(x, \partial B(0,\eta)) \to 0$. The constants r_0, C, and C of Lemma 2.13 can then be taken uniform for variations of the radius in bounded intervals, i.e., for $0 < \eta_1 \leq \eta \leq \eta_2$.

We now can deal with the continuity of the velocity fields generated by probability densities concentrated on manifolds. Recall that $\mathbb{R}_+ = (0, +\infty)$.

Lemma 2.9. *Let $\mathcal{M} \subset \mathbb{R}^N$ be a compact C^2 hypersurface, μ a probability distribution such that $\mu = \delta_{\mathcal{M}}$, where $\in L^\infty(\mathcal{M})$, and $g \in C(\mathbb{R}^N/\{0\})$ a radially symmetric function which is locally integrable on hypersurfaces. Then, the function*

$$(x) = \int_{\mathbb{R}^N} g(x-y)\,d\mu(y)$$

is continuous in $x \in \mathbb{R}^N$. Moreover, the same results hold while replacing $g(x)$ by a non-radially symmetric function $G \in C(\mathbb{R}^N/\{0\})$ such that $|G(x)|$ $|g(x)|$, where g satisfies the properties above.

Proof. It is straightforward to check that (x) is continuous for all $x \notin \mathcal{M}$. Let $x \in \mathcal{M}$ and let r_0 be given by Lemma 2.8. For 0 r_0, let $\in C^\infty(\mathbb{R}_+)$ be a cut-o function, such that $= 1$ on $[0, /2]$, and $= 0$ on $[, \infty)$. The function can then be written as

$$\begin{aligned} (x) = & \int_{\mathcal{M}} g(x-y) \quad (x-y)\, d\mu(y) + \int_{\mathcal{M}} g(x-y)[1 - \quad (x-y)]\, d\mu(y) \\ := & \ _1(x) + \ _2(x). \end{aligned} \tag{2.73}$$

It is clear that $_2$ is continuous on $x \in \mathbb{R}^N$, since g is continuous away from the origin and supp $(\mu) = \mathcal{M}$ is compact. Moreover, given the set $U = \{x \in \mathbb{R}^N : \text{dist}(x, \mathcal{M}) \quad r_0\}$, we can estimate for all $x \in U$

$$\begin{aligned} | \ _1(x)| & \quad \int_{\mathbb{R}^N} |g(x-y)| \quad (x-y)\, d\mu(y) \quad \int_{B(x, \ )} |g(x-y)|\, d\mu(y) \\ & \quad \| \quad \|_{L^\infty(\mathcal{M})} \int_0 |\hat{g}(r)|\, |\{y \in \mathcal{M};\ |y-x| = r\}|_{\mathcal{H}^{N-2}}\, dr \\ & \quad C\| \quad \|_{L^\infty(\mathcal{M})} \int_0 |\hat{g}(r)|\, r^{N-2}\, dr \end{aligned}$$

where (2.72) is used. Moreover, by construction $v_1(x) = 0$ for all $x \notin U$ for r_0. Therefore, due to the integrability over hypersurfaces of g, then

$$\lim_{\to 0} \| \ _1\|_{L^\infty(\mathbb{R}^N)} = 0\,.$$

This is enough to show the continuity of on $\mathcal{M}$: for any δ 0, there exists 0 such that $\| \ _1\|_{L^\infty(\mathbb{R}^N)} \quad \frac{}{2}$. Since $_2$ is continuous, there exists 0 such that $| \ _2(x) - \ _2(x)| \quad \frac{}{2}$ if $|x - x|$. Then, $| \ (x) - \ (x)| \quad \delta$ if $|x - x|$. The last part of the proof is an adaptation of the previous arguments since the integral inside the norm is less or equal than v_1. □

Now, we want to obtain the continuity with respect to the hypersurface for the velocity fields associated to measures concentrated on them. We restrict to the case of spheres since we only need this particular case. The proof uses the transport distance d_∞. We remind the reader that it is introduced in Section 3.

Lemma 2.10. *Let* $\mathcal{M} := \partial B(0,\eta)$ *and* $\mu = \delta_{\mathcal{M}}$ *be probability measures such that* $\in L^{\infty}(\mathcal{M})$ *with* $0 \quad \eta$. *Let* $g \in C^1(\mathbb{R}^N \backslash \{0\})$ *be a radially symmetric function which is locally integrable on hypersurfaces. If the functions are uniformly bounded in* η *and* $d_\infty(\mu \;, \mu \;) \to 0$ *as* $\eta - \eta \to 0$, *then*

$$v(x,\eta) = \int_{\partial B(0,\ )} g(x-y)\, d\mu \ (y)$$

is continuous in $\mathbb{R}^N \times \mathbb{R}_+$. *Moreover, the same result holds while replacing* $g(x)$ *by a non-radially symmetric function* $G \in C^1(\mathbb{R}^N \backslash \{0\})$ *such that* $|G(x)| \quad |g(x)|$ *with the properties above.*

Proof. Lemma 2.9 implies directly the continuity with respect to x for all fixed η. Using the Remark 2.13 and the proof of Lemma 2.9, it can be easily checked that this continuity in x is uniform in η. Indeed $|v_1|$ can be made small uniformly in η and, due to the estimate

$$|\nabla v_2(x)| \quad \sup_{\partial B(x,\ )} |\nabla[g(1 - \quad)]| \,,$$

v_2 is continuous uniformly in η. Therefore, we only need to show the continuity in η of v for a fixed $x \in \mathbb{R}^N$.

As in the proof of Lemma 2.9, let r_0 be as obtained in Remark 2.13 uniform in $0 \quad \eta_1 \quad \eta \quad \eta_2$. We choose again $0 \qquad r_0$ and $\in C^\infty(\mathbb{R}_+)$ a cut-o function, such that $= 1$ on $[0, \ /2]$, and $= 0$ on $[\ , \infty)$. We can write $(x,\eta) = \ _1(x,\eta) + \ _2(x,\eta)$ analogously to (2.73). As in Lemma 2.9 using the properties of g and the uniformity in Remark (2.13), we can easily show that

$$\lim_{\to 0} \| \ _1(\cdot,\eta)\|_{L^\infty(\mathbb{R}^N)} = 0 \,.$$

uniformly in $0 \quad \eta_1 \quad \eta \quad \eta_2$. Therefore, for any $\delta \quad 0$, there exists $r_0 \qquad 0$ such that $\| \ _1(\cdot,\eta)\|_{L^\infty(\mathbb{R}^N)} \quad \overline{4}$ uniformly in $0 \quad \eta_1 \quad \eta \quad \eta_2$. Now, we estimate

$$\begin{aligned} |v(x,\eta) - v(x,\eta)| \quad & \| \ _1(\cdot,\eta)\|_{L^\infty(\mathbb{R}^N)} + \| \ _1(\cdot,\eta)\|_{L^\infty(\mathbb{R}^N)} \\ & + \left| \int_{\mathbb{R}^N} g(x-y)[1 - \quad (|x-y|)]\, d(\mu \ - \mu \ )(y) \right| \\ & \frac{\delta}{2} + \|\nabla[g(1 - \quad)]\|_{L^\infty(\ )}\, d_\infty(\mu \ , \mu \ ), \end{aligned}$$

where is the convex hull of the set $\{x\} - (\text{supp}\ \mu \ ) \cup (\text{supp}\ \mu \ )$. Notice that the set is uniformly bounded in η and η.

This estimate shows the continuity in η since $d_\infty(\mu \ , \mu \ ) \to 0$ as $\eta \to \eta$, and thus, the last term is bounded by $\delta/2$ provided that η is close enough to

η. Again, the final part of this Lemma is a small variation of the previous arguments. □

Finally, we complete the results by showing that if the function is not locally integrable on hypersurfaces then the velocity field is not bounded.

Lemma 2.11. *Let $\mathcal{M} := \partial B(0,\eta)$ and $\mu = \delta_{\mathcal{M}}$ be probability measures such that $(x) \quad _0 \quad 0$ for all $\eta_1 \quad \eta \quad \eta_2$. Let $g \in C(\mathbb{R}^N\backslash\{0\})$ be a nonnegative radially symmetric function which is not locally integrable on hypersurfaces. Then for all $M \quad 0$ there exists $\delta \quad 0$ such that*

$$\mathrm{dist}(x,\mathcal{M}) \quad \delta \Longrightarrow \quad _{\mathbb{R}^N} g(x-y)\, d\mu\, (y) \quad M$$

for all $x \in \mathbb{R}^N$ and for all $\eta_1 \quad \eta \quad \eta_2$.

Proof. Using Lemma 2.8 and Remark 2.13, for $x \in \mathbb{R}^N$ with $\mathrm{dist}(x,\mathcal{M}) \quad r_0$, we get

$$\begin{aligned} _{\mathbb{R}^N} g(x-y)\, d\mu\, (y) &\quad _{|x-y|<r_0} g(x-y)\, d\mu\, (y) \\ &_0 \quad _0^{r_0} \hat{g}(r)\, |\{y \in \mathcal{M}\ ;\ |y-x| = r\}|_{\mathcal{H}^{N-2}}\, dr \\ &_0 C \quad _0^{r_0} \hat{g}(r)\, (r^2 - \mathrm{dist}(x,\mathcal{M})^2)_+^{\frac{N-2}{2}}\, dr\,. \end{aligned}$$

Since $_0^1 \hat{g}(r) r^{N-2}\, dr = +\infty$ and g is continuous and nonnegative on $(0,1]$, we deduce that

$$\lim_{\mathrm{dist}(x,\mathcal{M})\to 0} \quad _0^{r_0} \hat{g}(r)\, (r^2 - \mathrm{dist}(x,\mathcal{M})^2)_+^{\frac{N-2}{2}}\, dr = +\infty\,,$$

by the monotone convergence theorem, which conclude the proof. □

Chapter 3

Confinement for repulsive-attractive kernels

The contents of this chapter appear in:

> *D. Balagué, J. A. Carrillo, Y. Yao, Confinement for Repulsive-Attractive Kernels. To appear in Discrete and Continuous Dynamical Systems - Series A.* [9]

3.1 Introduction

In this chapter, we want to address "confinement" properties of solutions to the nonlocal interaction equation

$$\rho_t = \nabla \cdot (\rho(\nabla W * \rho)), \qquad x \in \mathbb{R}^N, t > 0, \tag{3.1}$$

with compactly supported initial data $\rho(0, \cdot) = \rho_0$ in a functional space to be specified. These nonlocal equations appear in many instances of mathematical biology [97, 119, 120, 66, 78, 109], mathematical physics [15, 41, 121, 86], and materials science [129, 115, 116, 87, 3, 2, 89, 57, 90]. They are minimal models for the interaction of particles/agents through pairwise potentials.

We say that the nonlocal equation (3.1) satisfies a confinement property in certain functional setting if every solution $\rho(t, \cdot)$ to (3.1) in that setting with compactly supported initial data ρ_0 is compactly supported for all times and its support lies in a fixed ball whose radius only depends on ρ_0 and W.

In most of the mentioned applications, particles/agents repel to each other in a short length scale while there is an overall attraction in larger length scales. Therefore, we can typically concentrate on repulsive-attractive potentials W as in [81, 127, 64, 126, 80, 117, 110, 61, 62, 10, 8]. These potentials lead to a rich ensemble of compactly supported steady states whose stability has recently been analyzed [81, 127, 126, 10, 8]. While the existence of these compactly supported stable stationary states is a good indication of confinement properties for these repulsive-attractive potentials, it is not equivalent to confinement. Let us finally mention, that repulsive-attractive potentials have also been used in second order models for swarming [54, 47, 45, 40] where exponential decaying at infinity potentials are more suitable from the modelling viewpoint.

Confinement properties were addressed in [35] taking advantage of the well-posedness theory for weak measure solutions of (3.1) developed in [36]. Using the continuity with respect to initial data in the functional setting of probability measures $\mathcal{P}(\mathbb{R}^N)$, the authors reduced the confinement of the solutions to (3.1) to a similar confinement property for solutions of the associated particle ode system:

$$\dot{x}_i = -\sum_{j\in Z(i)} m_j \nabla W(x_i - x_j), \quad i = 1, \dots, n, \tag{3.2}$$

where $Z(i) = \{j \in \{1, \dots, n\} : j \neq i, x_j(t) \neq x_i(t)\}$, $x_i(0) \in \mathbb{R}^N$ for all $i = 1, \dots, n$, $0 \le m_i \le 1$, and $\sum_i m_i = M > 0$. The authors obtain a confinement property in probability measures assuming that the potential is radial and attractive outside a ball, apart from other technical assumptions related to the well-posedness theory for probability measures. We will improve over the main result in [35] in terms of the assumed attractive strength at infinity. More precisely, we will allow for slower growing at infinity potentials, see Section 2 for the precise hypotheses.

We will later obtain a confinement property for solutions to (3.1) in a smooth functional setting of compactly supported initial data in $\mathcal{W}^{2,\infty}(\mathbb{R}^N)$. The trade-off is to allow more singular repulsive at the origin potentials. In fact, we will concentrate on the particular case of Newtonian repulsion plus smooth attractive potential with certain growth at infinity. In this functional setting, we can deal with smooth solutions obtained by a slight variation of the arguments in [22, 7, 21, 10]. Section 3 shows that the same ideas used for particles and solutions in the functional setting $\mathcal{P}(\mathbb{R}^N)$ apply in the continuum model (3.1) for smooth solutions with this particular potential. The strong Newtonian repulsion at the origin of the potential allows us to derive a priori L^∞ bounds that otherwise are not known. Let us finally mention that confinement for the repulsive Newtonian plus an attractive harmonic potential was obtained in [21].

In the last section of the chapter, Section 4, we perform some numerical computations for (3.2) with different repulsive-attractive potentials and we study the confinement of the stationary states for these cases. The stationary states are found using numerical techniques similar to the ones used in [127, 126]. We do a careful study in each case by computing the radius of the support in order to verify numerically the confinement for the solutions. Our numerical studies indicate that the assumptions we impose on the potentials are not sharp and could possibly be improved.

3.2 Con nement for probability measures

In this section, we will work with the theory developed in [36] in the framework of optimal mass transportation theory applied to (3.1). We remind the reader that the equation (3.1) can be classically understood as the gradient ow of the interaction potential energy [41, 4, 42]. Optimal transport techniques allow to construct a well-posedness theory in the space of probability measures with bounded second moments $\mathcal{P}_2(\mathbb{R}^N)$ at least for smooth potentials [4]. The regularity assumptions on the potential were relaxed in [36] allowing for potentials attractive at the origin with a at most Lipschitz singularity there, i.e., allowing for local behaviors like $W(x) \simeq |x|^a$, with $1 \quad a \quad 2$.

More precisely, we assume that the potential $W(x)$, see [36], satisfies
(NL0) $W \in C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \backslash \{0\})$, $W(x) = W(-x)$, and $W(0) = 0$.
(NL1) W is -convex for a certain $\in \mathbb{R}$, i.e. $W(x) - \frac{}{2}|x|^2$ is convex.
(NL2) There exists a constant $C \quad 0$ such that

$$W(z) \quad C(1 + |z|^2), \quad \text{for all } z \in \mathbb{R}.$$

to derive the well-posedness theory of gradient ow solutions to (3.1) with initial data in $\mathcal{P}_2(\mathbb{R}^N)$.

Under this set of assumptions **(NL0)** **(NL2)**, we can derive from [36, Theorems 2.12 and 2.13] that the mean-field limit associated to the model in (3.1) holds. On one hand, this means that approximating the initial data by atomic measures, we can approximate generic solutions of (3.1) by particular solutions corresponding to initial data composed by finite number of atoms (particle solutions). On the other hand, this also implies that the solution of (3.1) given in [36, Theorems 2.12 and 2.13] coincides with the atomic measure constructed by evolving the locations of the atoms through the ODE system (3.2). In other words, if one is interested in showing a confinement property for (3.1), it su ces to prove the confinement property for the particle system solving (3.2) since the solutions of the particle system (3.2) approximate accurately in finite time intervals the solutions of the partial di erential equation (3.1). All these details are fully explained in [35, Section 3] allowing us to reduce directly to particle solutions.

To show confinement, we need additional assumptions on $W(x)$ as in [35]. Throughout this chapter, we assume that W is radially symmetric, and attractive outside some ball, i.e.,

(NL-RAD) W is radial, i.e. $W(x) = w(|x|)$, and there exists R_a 0 such that $w'(r)$ 0 for r R_a.

It is pointed out in [36, Remark 1.1] that **(NL1)** guarantees that the repulsive force cannot be too strong, more precisely,

$$C_W := \sup_{x \in B(0,R_a)\setminus\{0\}} |\nabla W(x)| \tag{3.3}$$

is finite. Here, we take the convention $C_W = 0$ in case $R_a = 0$.

In order to prove confinement results, we need some other condition to ensure that the attractive strength does not decay too fast at infinity. In addition to **(NL0)-(NL3)** and **(NL-RAD)**, we assume that W satisfies the following confinement condition:

(NL-CONF) $$\lim_{r\to\infty} w'(r)r = +\infty,$$

which is less restrictive than the assumption in [35], namely $\lim_{r\to\infty} w'(r)\sqrt{r} = +\infty$.

Therefore, our goal in this section is to show that if the particles interact under a potential satisfying assumptions **(NL0)-(NL3)**, **(NL-RAD)**, and **(NL-CONF)**, have total mass $M = 1$, center of mass at 0, and are initially confined in $B(0, R_0)$, then they will be confined in some ball $B(0, R)$ for all times, where R is independent of the number of particles n but only depending on the kernel W and the initial support of the cloud of particles R_0. Note that the zero center of mass assumption is possible due to the translational invariance of (3.1) and (3.2). Note also that the solution to the particle system (3.2) in the sense of [35, Remark 2.1] might lead to a finite number of collision times, in which the solution may lose its regularity. Hence when we study the evolution of some quantities in time, we only take the time derivative in the time intervals in which the solution is regular.

The strategy to get confinement for particles is as follows: we need to control quantities that quantifies how much the distribution spreads in time. In [35, Proposition 4.2] the argument was based in following the particle furthest away from the origin and use some energetic arguments to control the mass of the particles nearby pushing the furthest particle. Here we follow a di erent idea. We consider other moments of the particle system to control the spread of the distribution of particles in conjunction with the evolution of the furthest particle from the center of mass. More precisely, we couple the evolution of the third absolute moment of the particle system with the evolution of the furthest particle.

3.2.1 Evolution of the third moment

Let $M_3(t)$ denote the third absolute moment of the particle system (3.2), namely

$$M_3(t) := \sum_{i=1}^{n} m_i |x_i|^3.$$

In this section our main goal is to estimate the time derivative of $M_3(t)$. As we discussed before, there might be a finite number of collision times in which $M_3(t)$ becomes non-differentiable. Nevertheless, since all the particles have finite velocity, $M_3(t)$ is Lipschitz continuous in time even during collision. In all the computation below, the time derivative of M_3 is only taken in the time intervals where $M_3(t)$ is differentiable; and the continuity of $M_3(t)$ ensures that the fundamental theorem of calculus still holds for $M_3(t)$.

Since $|x|^3$ is a convex function on $\mathbb{R}^N$, the following computation shows that every pair of attracting particles would give a negative contribution to dM_3/dt, whereas every repulsing pair gives a positive contribution. We can directly evaluate dM_3/dt as follows:

$$\begin{aligned} \frac{dM_3(t)}{dt} &= 3\sum_i m_i |x_i| \langle \dot{x}_i, x_i \rangle \\ &= 3\sum_i \sum_{j\in \mathcal{Z}(i)} m_i m_j \langle -\nabla W(x_i - x_j), x_i|x_i| \rangle \\ &= \frac{3}{2}\sum_i \sum_{j\in \mathcal{Z}(i)} m_i m_j \langle -\nabla W(x_i - x_j), \big(x_i|x_i| - x_j|x_j|\big) \rangle \\ &= -\frac{3}{2}\sum_i \sum_{j\in \mathcal{Z}(i)} m_i m_j w'(|x_i - x_j|) T_{ij}, \end{aligned} \tag{3.4}$$

with $T_{ij} := \dfrac{x_i - x_j}{|x_i - x_j|} \cdot \big(x_i|x_i| - x_j|x_j|\big)$ and where antisymmetry of $\nabla W(x)$ is used. Elementary manipulations yield that T_{ij} can be rewritten as

$$T_{ij} = (|x_i| + |x_j|) \frac{\frac{1}{2}(|x_i| - |x_j|)^2 + \frac{1}{2}|x_i - x_j|^2}{|x_i - x_j|},$$

which gives the following upper and lower bound for T_{ij}:

$$\frac{1}{2}\big(|x_i| + |x_j|\big)|x_i - x_j| \le T_{ij} \le \big(|x_i| + |x_j|\big)\big|x_i - x_j\big|. \tag{3.5}$$

Next we will find an upper bound for $-w'(|x_i - x_j|)$ in (3.4). Let us define the nearest particles set $N(i)$ as the set of indexes of particles that are possibly repelling the i-th particle, more precisely,

$$N(i) := \Big\{ j \in \{1, \dots, n\} : 0 \le |x_j(t) - x_i(t)| \le R_a \Big\}.$$

Then (3.3) implies that $-w'(|x_i - x_j|) \le C_W$ for all $j \in N(i)$.

For $j \notin N(i)$, **(NL-RAD)** gives that $-w'(|x_i - x_j|) \le 0$. A better bound can be obtained using **(NL-CONF)**: note that for any fixed constant $K_1 > 0$ to be specified momentarily, there exists some $R_{K_1} \ge 2R_a$, such that

$$-w'(r) \le -\frac{K_1}{r} \quad \text{for all } r \ge R_{K_1}.$$

Let us define the set of furthest particles $F(i)$ as the set of indexes of particles whose distance to the i-th particle are larger than R_{K_1}, namely

$$F(i) := \Big\{ j \in \{1, \dots, n\} : |x_j(t) - x_i(t)| \ge R_{K_1} \Big\}.$$

The definitions of $N(i)$ and $F(i)$ are illustrated in Figure 3.1. Then the upper bound for $-w'(|x_i - x_j|)$ can be summarized as following:

$$-w'(|x_i - x_j|) \le \begin{cases} C_W & \text{for } j \in N(i), \\ -\dfrac{K_1}{|x_i - x_j|} & \text{for } j \in F(i), \\ 0 & \text{for } j \notin N(i) \cup F(i). \end{cases} \tag{3.6}$$

By plugging (3.5) and (3.6) into (3.4) and setting $K_1 := 10 C_W R_a$, we obtain

$$\begin{aligned} \frac{dM_3(t)}{dt} &\le \frac{3}{2} \sum_i m_i \Big(\sum_{j \in N(i)} m_j C_W R_a \big(|x_i| + |x_j|\big) - \sum_{j \in F(i)} m_j K_1 \frac{1}{2} (|x_i| + |x_j|) \Big) \\ &\le \frac{3}{2} C_W R_a \sum_i m_i \big(T_r^i - 5 T_a^i \big), \end{aligned} \tag{3.7}$$

with

$$T_r^i := \sum_{j \in N(i)} m_j \big(|x_i| + |x_j|\big) \qquad \text{and} \qquad T_a^i := \sum_{j \in F(i)} m_j (|x_i| + |x_j|)\,.$$

Now we claim that

$$T_r^i \le 4 T_a^i \quad \text{if } |x_i| \ge R_{K_1}. \tag{3.8}$$

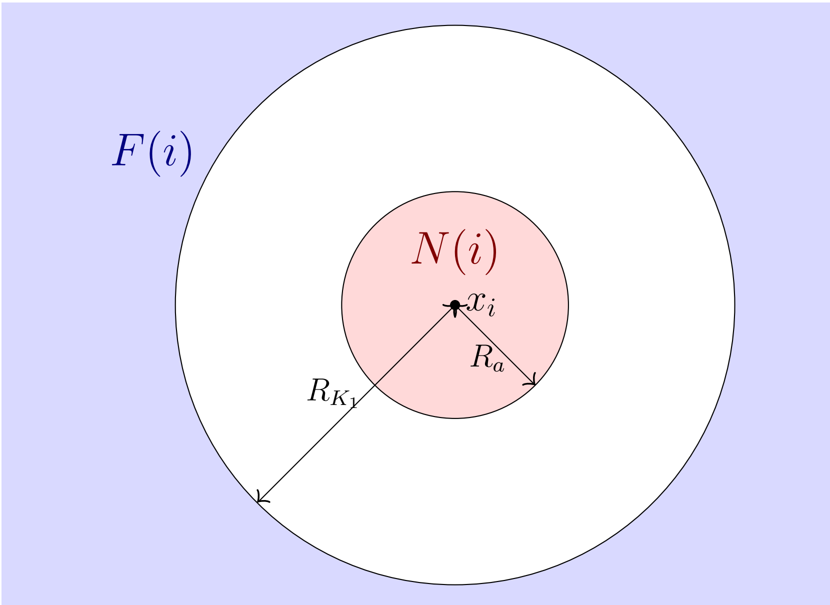


Figure 3.1: Illustration of the sets $N(i)$ and $F(i)$. For the i-th particle, $N(i)$ is defined as the set of indexes of particles in the red region, while $F(i)$ is the set of indexes of particles in the blue region.

Its validity is one of the main reasons for imposing the requirement **(NL-CONF)**. To prove the claim, recall that we assume the center of mass is at 0 at $t = 0$ without loss of generality. Due to the conservation of the center of mass, for any time t, we have $x_j(t)$ satisfies $\sum_j m_j x_j(t) = 0$. Let $e_i \in \mathbb{R}^N$ denote the unit vector pointing in the direction of x_i, then it follows immediately that

$$\sum_{j=1}^{n} m_j x_j \cdot e_i = 0.$$

For $|x_i| \geq R_{K_1}$, we split the above sum into three parts, and get

$$\begin{aligned} \sum_{j\in N(i)} m_j x_j \cdot e_i &= -\sum_{j\in F(i)} m_j x_j \cdot e_i - \sum_{j\notin N(i)\cup F(i)} m_j x_j \cdot e_i \\ &\leq -\sum_{j\in F(i)} m_j x_j \cdot e_i \leq T_a^i, \end{aligned} \tag{3.9}$$

where the first inequality is due to the fact that $x_j \cdot e_i \geq 0$ for all $j \notin F(i)$.

Moreover, recall that in the definition of R_{K_1}, we force it to be bigger than $2R_a$. This is to guarantee that for all $|x_i| \geq R_{K_1} \geq 2R_a$ and $j \in N(i)$, the angle between the vectors x_i and x_j is less than $\arctan 1/2 = \frac{\pi}{6}$. As a result, we have $x_j \cdot e_i \geq \frac{\sqrt{3}}{2}|x_j|$. Notice that for $|x_i| \geq R_{K_1}$ and $j \in N(i)$ we also have $|x_j| \geq |x_i|/2$, which is equivalent with $|x_j| \geq \frac{1}{3}(|x_i| + |x_j|)$. Thus finally we have

$$\sum_{j\in N(i)} m_j x_j \cdot e_i \geq \sum_{j\in N(i)} \frac{m_j}{2\sqrt{3}}(|x_i| + |x_j|) = \frac{T_r^i}{2\sqrt{3}} \quad \text{for } |x_i| \geq R_{K_1},$$

and by combining it with (3.9) we obtain the claim (3.8).

Due to (3.8), we deduce that for any $|x_i| \geq R_{K_1}$, $T_r^i - 5T_a^i \geq -T_a^i$, hence (3.7) becomes

$$\frac{dM_3(t)}{dt} \leq \frac{3}{2} C_W R_a \Big(T_1 - \sum_{|x_i| > R_{K_1}} \sum_{j\in F(i)} m_i m_j (|x_i| + |x_j|) \Big),$$

with

$$T_1 := \sum_{|x_i| \leq R_{K_1}} \sum_{j \in N(i)} m_i m_j (|x_i| + |x_j|).$$

Note that we can easily bound T_1 by a constant only depending on W (since $|x_i| \leq R_{K_1}$ and $|x_j| \leq R_{K_1} + R_a$), thus we can rewrite the above inequality as inequality as

$$\frac{dM_3(t)}{dt} \leq C_1 - C_2 \sum_{|x_i| > R_{K_1}} \Big(m_i |x_i| \sum_{j \in F(i)} m_j \Big), \tag{3.10}$$

where C_1, C_2 only depends on W. At this point we will take a pause on the evolution of M_3; we will revisit the inequality (3.10) soon in Section 3.2.2 to couple it with the evolution of the support.

3.2.2 Coupling with the evolution of the support

For all $t \geq 0$, let $R(t)$ denote the distance of the furthest particle from the center of mass (which we assumed to be 0 without loss of generality), namely

$$R(t) := \max_{i=1,\dots,n} |x_i(t)|. \tag{3.11}$$

It is pointed out in [36, Proposition 4.2] that $R(t)$ is Lipschitz in time. Our goal is to prove that $\limsup_{t\to\infty} R(t) \leq R$, where R only depends on W.

We begin by reminding a claim proved in [35, Proposition 2.2]: Let e be any unit vector. Then

$$\sum_{x_j \cdot e \leq R(t)/2} m_j \geq \frac{1}{3} \sum_j m_j, \tag{3.12}$$

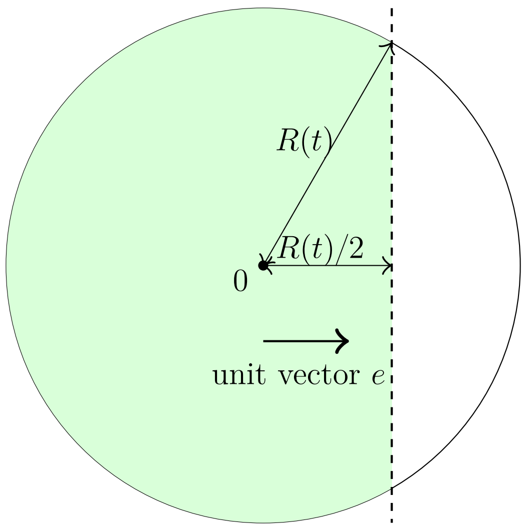


Figure 3.2: For any unit vector e, the green region above contains at least one third of the total mass. Here $R(t)$ is as defined in (3.11).

i.e. the green region in Figure 3.2 contains at least 1/3 of the total mass.

It is argued in the proof of [36, Proposition 4.2] that for all time $t \geq 0$, there is a particle index $i_0(t)$ (here i_0 may depend on t), such that

$$|x_{i_0}(t)| = R(t) \qquad \text{and} \qquad \frac{d^+}{dt}R(t) = \dot{x}_{i_0}(t) \cdot \frac{x_{i_0}(t)}{R(t)}, \tag{3.13}$$

where $\frac{d^+}{dt}$ stands for the right derivative. This technical point is due again to the lack of regularity of $R(t)$ for all t, see [36, Proposition 4.2]. From now on, the index i_0 refers to any index satisfying the previous properties.

To control $\frac{d^+}{dt}R(t)$, it suffices to look at the outward velocity of the i_0-th particle at this time. As argued in [35, Proposition 2.2] and illustrated in Figure 3.3, the red region is possibly pushing it out, but all the green region is pulling it towards the origin. We proceed by estimating the compensation between these two competing effects. Let $G(i_0)$ denote the set of indexes of particles in the green region in Figure 3.3, namely

$$G(i_0) := \left\{ j \in \{1, \ldots, n\} : x_j(t) \cdot \frac{x_{i_0}}{|x_{i_0}|} \leq \frac{R(t)}{2} \right\}.$$

Recall that throughout this section, we assume the total mass is 1 without loss of generality. It then follows from (3.12) that

$$\sum_{j \in G(i_0)} m_j \geq 1/3\,. \tag{3.14}$$

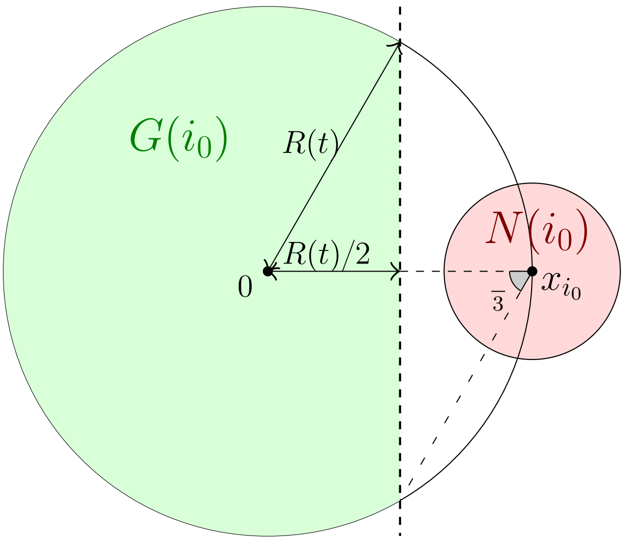


Figure 3.3: When $R(t) > 2R_a$, the particles in the green region are all pulling the particle x_{i_0} towards the origin, while the particles in the red region are possibly pushing it out.

Using (3.13), (3.2), and some simple manipulations, the growth of $R(t)$ is controlled by the following inequality:

$$\begin{aligned}\frac{d^+}{dt}R(t) &\le C_W\Big(\sum_{j\in N(i_0)} m_j - \sum_{j\in G(i_0)} m_j w'(|x_j - x_{i_0}|)\cos(\angle(-x_{i_0}, x_j - x_{i_0}))\Big),\\ &\le C_W\Big(\sum_{j\in N(i_0)} m_j - \frac{1}{2}\sum_{j\in G(i_0)} m_j w'(|x_j - x_{i_0}|)\Big),\end{aligned} \tag{3.15}$$

where $\angle(-x_{i_0}, x_j - x_{i_0})$ denotes the angle between the two vectors $-x_{i_0}$ and $x_j - x_{i_0}$, which is less than $\frac{\pi}{3}$ as shown in Figure 3.3. Due to **(NL-CONF)**, for any large constant K_2, which we will fix later, there exists some radius R_{K_2}, such that $w'(r) \ge K_2/r$ for all $r \ge R_{K_2}$. Hence for all t satisfying $R(t) \ge R_{K_2}$, (3.15) becomes

$$\begin{aligned}\frac{d^+}{dt}R(t) &\le C_W\Big(\sum_{j\in N(i_0)} m_j - \frac{1}{2}\frac{K_2}{\sup_{j\in G(i_0)} |x_j - x_{i_0}|}\sum_{j\in G(i_0)} m_j\Big)\\ &\le C_W\Big(\sum_{j\in N(i_0)} m_j - \frac{K_2}{12R(t)}\Big),\end{aligned} \tag{3.16}$$

where (3.14) was used to obtain the last inequality.

To ensure the coupling between the growth of $M_3(t)$ and the growth of $R(t)$ go smoothly, let us go back to (3.10) and perform some elementary manipulation on it. When $R(t) \ge R_{K_1} + R_a$, for any $j \in N(i_0)$, we have $|x_j| \ge R_{K_1}$, hence

$$\frac{dM_3(t)}{dt} \le C_1 - C_2 \sum_{i\in N(i_0)} \Big(m_i|x_i| \sum_{j\in F(i)} m_j \Big). \tag{3.17}$$

And if in addition we have $R(t) \ge 2(R_{K_1} + R_a)$, then it follows that $G(i_0) \subset F(i)$ for any $i \in N(i_0)$, hence we can replace the $F(i)$ in (3.17) by $G(i_0)$ and obtain

$$\begin{aligned}\frac{dM_3(t)}{dt} &\le C_1 - C_2 \sum_{i\in N(i_0)} \Big(m_i|x_i| \sum_{j\in G(i_0)} m_j \Big) \le C_1 - \frac{1}{3}C_2(R(t) - R_a) \sum_{i\in N(i_0)} m_i \\ &= \Big(C_1 + \frac{1}{3}C_2 R_a \Big) - \frac{1}{3}C_2 R(t) \sum_{i\in N(i_0)} m_i,\end{aligned} \tag{3.18}$$

where we used (3.14) again to obtain the second inequality. Finally, we set $R_1 := \max\{R_{K_2}, 2(R_{K_1} + R_a)\}$, to ensure that both (3.16) and (3.18) hold for $R(t) \ge R_1$.

Finally we are ready to couple $M_3(t)$ with $R(t)$. By putting the estimates on $\frac{d^+}{dt}R(t)$ and $\frac{d}{dt}M(t)$ together, we will show that if $R(t)$ grows from R_1 to some very large number in some time interval $[t_1, t_2]$, then the integral of dM_3/dt is negative over this time interval, i.e., $M_3(t_2) < M_3(t_1)$. On the other hand, we will directly prove that $M_3(t_2)$ must be bigger than $M_3(t_1)$, which causes a contradiction.

Let A_1 be a sufficiently large constant which we will determine later. If the particles start in $B(0, R_0)$ and eventually touch the boundary of $B(0, A_1R_1)$, then there exist $0 < t_1 < t_2$, such that

$$R(t_1) = R_1, \quad R(t_2) = A_1R_1, \quad \frac{d^+}{dt}R(t_2) \ge 0,$$

and

$$R(t) \in [R_1, A_1R_1] \quad \text{for all } t_1 \le t \le t_2.$$

More precisely, by letting $t_2 := \min\{t \ge 0 : R(t) = A_1R_1\} > 0$, and $t_1 := \max\{0 \le t \le t_2 : R(t) = R_1\}$, they would satisfy all the requirements.

Since $R(t_2) > R(t_1)$, we have

$$\int_{t_1}^{t_2} \Big(\frac{d^+}{dt}R(t)\Big) R(t)\, dt = \frac{R^2(t)}{2}\Big|_{t_1}^{t_2} > 0.$$

Using (3.16), the above inequality implies

$$\int_{t_1}^{t_2} C_W \left(\sum_{j \in N(i_0(t))} m_j \right) R(t)\, dt \geq \int_{t_1}^{t_2} \frac{1}{12} K_2\, dt,$$

and by plugging it into the integral version of (3.18) we obtain

$$\int_{t_1}^{t_2} \frac{dM_3}{dt}(t)\, dt \leq \int_{t_1}^{t_2} \left[(C_1 + \frac{1}{3} C_2 R_a) - \frac{C_2}{36 C_W} K_2 \right] dt,$$

hence by choosing $K_2 := 1 + 36 C_W (C_1 + \frac{1}{3} C_2 R_a)/C_2$, which only depends on W, we have $M_3(t_2) \leq M_3(t_1)$.

On the other hand, if $R(t)$ successfully grows from R_1 to $A_1 R_1$, we will show that M_3 indeed has to increase, namely $M_3(t_2) \geq 2M_3(t_1)$ for A_1 sufficiently large. First, we can bound $M_3(t_1)$ above by the very rough bound $R(t_1)^3 = R_1^3$. At time t_2, recall that $\frac{d^+}{dt} R(t_2) \geq 0$, hence (3.16) implies that

$$\sum_{j \in N(i_0(t_2))} m_j \geq \frac{K_2}{12 C_W A_1 R_1}.$$

Finally, by noticing that

$$M_3(t_2) \geq (R(t_2) - R_a)^3 \sum_{j \in N(i_0(t_2))} m_j \geq \frac{K_2}{12 C_W A_1 R_1} (A_1 R_1 - R_a)^3,$$

we obtain $M_3(t_2) \gtrsim (A_1 R_1)^2$. Therefore we can choose A_1 sufficiently large such that $M_3(t_2) \geq 2M_3(t_1)$, which leads to a contradiction with $M_3(t_2) \leq M_3(t_1)$.

Note that the proof above shows that $R(t)$ can never reaches $R := A_1 R_1$, which is a large constant only depend on W, and in particular is independent of the number of particles.

3.3 Confinement for kernels with Newtonian repulsion

In this section, we consider the interaction kernel $W(x)$ given by

$$W(x) = -\mathcal{N}(x) + W_a(x) \tag{3.19}$$

with $N \geq 2$. Here $\mathcal{N}(x)$ is the Newtonian kernel, namely

$$\mathcal{N}(x) = \begin{cases} \frac{1}{2\pi} \ln |x| & N = 2, \\ -\frac{c_N}{|x|^{N-2}} & N \geq 3, \end{cases}$$

where c_N denotes the volume of a unit ball in $\mathbb{R}^N$. Throughout this section we assume that $W_a(x)$ satisfies the following assumptions:

(W1) $W_a \in L^1_{loc}(\mathbb{R}^N)$.
(W2) W_a is bounded in $\mathbb{R}^N \backslash B\ (0)$ for any 0.
(W-RAD) $W_a(x) = w(|x|)$ with $w \in C^1((0,\infty))$ and $w'(r)$ 0 for r 0.
(W-CONF) $\lim\limits_{r\to\infty} w'(r) r^{1/N} = +\infty$.

Our goal in this section is to show that under the above assumptions, if a solution has total mass $M = 1$, center of mass at 0, and are initially confined in $B(0, R_0)$, then it will be confined in some fixed ball centered at 0 for all times, where the radius of the ball only depends on W_a, R_0, the dimension N, and the L^∞ norm of the initial data.

Remark 3.1. *For $N = 1$, the Newtonian kernel becomes $|x|$. Note that in this case the confinement result does not hold under the assumptions above, since the repulsive velocity field between two particles will be a constant regardless of the distance between them, while the attraction may vanish as the distance goes to infinity. We can compensate this difficulty by imposing stronger assumption on the attractiveness of W_a at infinity. More precisely, by replacing* **(W-CONF)** *by $\lim\limits_{r\to\infty}(w'(r) - 1)r = +\infty$, the confinement result will hold with a similar proof as in Section* 2 *carried over at the continuum level.*

Remark 3.2. (W-CONF) *is more restrictive than* **(NL-CONF)**, *especially for large N. In the proof below, one can see that dM_3/dt does not cause a problem at all, indeed it satisfies the same inequality as in the non-singular kernel case in Section* 2. *The problem lies in dR/dt: due to the singular repulsive kernel, we got a worse control of dR/dt than before, see* (3.25).

Remark 3.3. *In this section we focus on the Newtonian repulsive kernel, since the solutions would stay bounded for all times due to the localization arising from Newtonian kernel (see Lemma 3.1). If the repulsive part of W is given by $|x|^{-}$ instead, where 0 $N-2$, i.e., W still has singular repulsion but weaker than Newtonian, we cannot obtain the uniform boundedness of solutions in time as we did in Lemma 3.1. However, if we assume* a priori *that the solution remains bounded for all time, then the rest of the proof will go through. Indeed, in this case* ***(W-CONF)*** *could be replaced by a weaker assumption $\lim_{r\to\infty} w'(r) r^{1-(\ +1)/N} = +\infty$.*

We point out that slight variations of the arguments in [10, Section 5] and [7, 21] give a well-posedness theory for smooth solutions constructed by characteristics. More precisely, for any compactly supported initial data $_0 \in \mathcal{W}^{2,\infty}(\mathbb{R}^N)$, there exists a unique classical solution $\in C^1([0,T] \times \mathbb{R}^N) \cap \mathcal{W}^{1,\infty}_{loc}(\mathbb{R}_+, \mathcal{W}^{1,\infty}(\mathbb{R}^N))$ to (3.1) with W satisfying **(W1)**-**(W2)**. Moreover, the associated velocity field $v(t,x) = -\nabla W *$ is Lipschitz continuous in both space and time, hence the characteristics are well defined:

$$\frac{d}{dt} X_t = -(\nabla W * \ )(t, X_t),$$

and the solution is given by

$$(t,x) = \ _0(X_t^{-1}) \det(D X_t^{-1}).$$

Since the initial data is compactly supported, it remains compactly supported for all time (although the support may grow in time), and its support is obtained through the C^1-characteristic maps X_t.

First we remind a lemma showing L^∞-bounds of the solution. This is a classical argument that can be seen for instance in [62, 21] and [10, Section 5] but we give a short proof for completeness.

Lemma 3.1. *Let W be given by* (3.19)*, with W_a satisfying* **(W1)**-**(W2)**. *Let be a classical solution to* (3.1) *with compactly supported initial data $_0 \in \mathcal{W}^{2,\infty}(\mathbb{R}^N)$. Then $\| \ (t,\cdot)\|_\infty \quad M_0$ for all $t \quad 0$, where M_0 only depends on W_a and $_0$.*

Proof. Due to the assumption **(W1)**, we can find $r_0 \quad 0$ su ciently small, such that

$$\int_{B(0,r_0)} | \ W_a(x)| dx \quad \frac{1}{2}.$$

Then it follows from **(W2)** that $M_W := \sup_{x \in \mathbb{R}^N \setminus B(0,r_0)} | \ W_a(x)|$ is finite. We define M_0 as

$$M_0 := \max\{2 M_W \| \ _0\|_1, \| \ _0\|_\infty\}.$$

Let us denote by $M(t) = \max_{x \in \mathbb{R}^N} \ (t,x)$. If the desired result does not hold, then there exists some $t_1 \quad 0$, such that $M(t_1) \quad M_0$ and $M(t)$ is increasing at $t = t_1$. This enables us to find some $x_1 \in \mathbb{R}^N$, such that $M(t_1) = \ (t_1, x_1)$, and $\frac{\partial}{\partial t} \ (t_1, x_1) \quad 0$. On the other hand, since is a classical solution, we have

$$\begin{aligned} \frac{\partial}{\partial t} \ (t_1, x_1) &= \nabla \ \cdot (\nabla W * \ ) + (\ W * \ ) \\ &= \ (t_1, x_1) \Big((\ * \ W_a)(t_1, x_1) - \ (t_1, x_1) \Big). \end{aligned} \tag{3.20}$$

Now let us split the integral $\ \ast\ W_a$ in $B(0, r_0)$ and outside to get

$$(\ \ast\ W_a)(t_1, x_1) \quad \frac{1}{2}\|\ (t_1, \cdot)\|_\infty + M_W\|\ (\cdot, 0)\|_1 = \frac{1}{2}\ (t_1, x_1) + M_W\|\ {}_0\|_1.$$

Plugging it into (3.20), we have

$$\begin{aligned} \frac{\partial}{\partial t}\ (t_1, x_1) \quad & (t_1, x_1)\Big(M_W\|\ {}_0\|_1 - \frac{1}{2}\ (t_1, x_1)\Big) \\ & (t_1, x_1)\Big(\frac{M_0}{2} - \frac{(t_1, x_1)}{2}\Big) \quad 0, \end{aligned}$$

which contradicts with the assumption that $\frac{\partial}{\partial t}\ (t_1, x_1) \quad 0$. □

Next we present a technical lemma which will be used in the proof of confinement.

Lemma 3.2. *Assume $u \in L^\infty(\mathbb{R}^N)$ with $0 \quad p \quad N$. Then it follows that*

$$\quad_{B(x_0,R)} \frac{u(y)}{|x_0 - y|^p} dy \quad C(N, p)\|u\|_\infty^{p/N} \Big(\quad_{B(x_0,R)} u(y)dy\Big)^{(N-p)/N} \tag{3.21}$$

for all $x_0 \in \mathbb{R}^N$ and all $R \quad 0$.

Proof. First note that it su ces to prove the following inequality holds for all $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$:

$$\quad_{\mathbb{R}^N} \frac{v(y)}{|y|^p} dy \quad C(N, p)\|v\|_\infty^{p/N}\|v\|_1^{(N-p)/N},$$

by letting $v(y) = \ {}_{B(0,R)}(y)u(y + x_0)$, where $\quad$ is the indicator function on a set $\quad \mathbb{R}^N$. We point out that one could use Hölder inequality and interpolation inequality on weak L^p spaces to obtain a slightly weaker inequality than above, but we will use an easier and more elementary approach instead.

Let w be an indicator function taking value $\|v\|_\infty$ on some disk centered at 0 and taking value 0 outside, where the size of the disk is chosen such that w and v have the same L^1 norm. More precisely, w is given by

$$w := \|v\|_\infty\ \ {}_{B(0,r_0)}, \text{ where } r_0 := \Big(\frac{\|v\|_1}{c_N\|v\|_\infty}\Big)^{1/N},$$

here c_N is the volume of the unit ball in $\mathbb{R}^N$. Since $\|v\|_1 = \|w\|_1$, it is straightforward to verify that

$$\quad_{B(0,r)} v(y)dy \quad\quad_{B(0,r)} w(y)dy \text{ for all } r \quad 0. \tag{3.22}$$

Now we start with the left hand side of (3.21), and Fubini's theorem yields that

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{v(y)}{|y|^p}dy &= \int_{\mathbb{R}^N}\int_0^\infty v(y)\,\mathbb{1}_{\{t\leqslant|y|^{-p}\}}dtdy = \int_0^\infty\int_{B(0,t^{-1/p})} v(y)dydt\\ &\leqslant \int_0^\infty\int_{B(0,t^{-1/p})} w(y)dydt = \int_{\mathbb{R}^N}\frac{w(y)}{|y|^p}dy\\ &= \frac{Nc_N\|v\|_\infty}{N-p}|r_0|^{N-p} = \frac{Nc_N^{p/N}\|v\|_\infty^{p/N}}{N-p}\|v\|_1^{(N-p)/N},\end{aligned}$$

where (3.22) was used. □

3.3.1 Evolution of the third absolute moment

Similar to the particle system case, we also start with estimating the time derivative of the third absolute moment M_3. Here the third absolute moment M_3 is given by

$$M_3(t) := \int_{\mathbb{R}^N}\rho(t,x)|x|^3dx,$$

and note that in the continuum setting M_3 is indeed differentiable in time for all $t \geq 0$, since $\rho(t,x)$ is a classical solution. The same computation as (3.4) leads to

$$\frac{dM_3(t)}{dt} = -\frac{3}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho(t,x)\rho(t,y)w'(|x-y|)\frac{x-y}{|x-y|}\cdot(x|x|-y|y|)dydx.$$

Due to the assumptions **(W-RAD)** and **(W-CONF)** on W_a, for any $A > 0$ (which will be fixed at the end of this subsection), there exists some $R_A \geq 1$, such that the following bound for $-w'(r)$ holds, where C_N is some constant only depending on N:

$$-w'(r) \leq \begin{cases}\dfrac{C_N}{r^{N-1}} & \text{for } 0 \leq r \leq R_A,\\[2mm] -\dfrac{A}{r^{1/N}} & \text{for } r > R_A.\end{cases}$$

Using this bound and (3.5), $\frac{dM_3(t)}{dt}$ becomes

$$\begin{aligned}\frac{dM_3(t)}{dt} \leq \frac{3}{2}\int_{\mathbb{R}^N}\rho(t,x)\Big[&\underbrace{\int_{B(x,R_A)}\frac{C_N\,\rho(t,y)}{|x-y|^{N-2}}(|x|+|y|)dy}_{T_r^x} -\\ &\underbrace{\int_{\mathbb{R}^N\setminus B(x,R_A)}\rho(t,y)\frac{A}{2}|x-y|^{(N-1)/N}(|x|+|y|)dy}_{T_a^x}\Big]dx\,.\end{aligned}$$

Similar to Section 2, we again claim that $T_r^x \le T_a^x/2$ for $|x| \ge 2R_A$. We start with controlling T_r^x. It is easy to check that

$$T_r^x \le (2|x| + R_A) \int_{B(x,R_A)} \frac{C_N \rho(t,y)}{|x-y|^{N-2}} dy.$$

Note that the singularity of the Newtonian kernel is more difficult to treat than in Section 2. We compensate this difficulty by using the fact that $\rho(t,x)$ is uniformly bounded by M_0 from Lemma 3.1. Hence for $N \ge 2$, we are able to apply Lemma 3.2 to $\rho(t,\cdot)$, and obtain

$$T_r^x \le C_3(2|x| + R_A) \left(\int_{B(x,R_A)} \rho(t,y) dy \right)^{2/N} \tag{3.23}$$

here C_3 only depends on N and M_0 as obtained in Lemma 3.1.

To simplify notation, from now on, we define by $m(t,x)$ the mass of ρ within radius R_A of x at time t, namely

$$m(t,x) := \int_{B(x,R_A)} \rho(t,y) dy.$$

Recall that in the beginning of this section we assume that ρ_0 integrates to 1, which implies that $\rho(t,\cdot)$ also integrates to 1 for all $t \ge 0$. Then, for any t and x, one of the two following scenarios must be true: either $m(t,x) \ge \frac{1}{2}$, or $1 - m(t,x) \ge \frac{1}{2}$.

If $m(t,x) \ge \frac{1}{2}$ at some $|x| \ge 2R_A$, it follows that $m(t,x)^{2/N}$ is comparable to $m(t,x)$. Hence, using (3.23), we get

$$\begin{aligned} T_r^x &\le C_3\, 2^{1-2/N}\, m(t,x)(2|x| + R_A) \\ &\le 8C_3 \int_{B(x,R_A)} \rho(t,y)(|x| + |y|) dy \quad (\text{since } |x| \ge 2R_A)\,. \end{aligned}$$

Hence by repeating the same argument on the center of mass as in Section 2 (see (3.9) and the paragraph after it), we can choose A to be sufficiently large, then we would obtain that $T_r^x \le T_a^x/2$.

On the other hand, if the opposite scenario is true at some $|x| \ge 2R_A$, i.e.

$$\int_{\mathbb{R}^N \setminus B(x,R_A)} \rho(t,y) dy \ge \frac{1}{2}\,,$$

then one can directly bound T_r^x by $C(W,N)|x|$ by applying Lemma 3.1 and Lemma 3.2. Meanwhile it follows directly from the definition of T_a^x that $T_a^x \ge \frac{A}{4}|x|$, hence by choosing A sufficiently large we obtain that $T_r \le T_a/2$.

Finally, we choose A to be the maximum value needed in the two scenarios. As a result, $T_r \quad T_a/2$ holds for for all $|x| \quad 2R_A$, implying that

$$\frac{dM_3(t)}{dt} \quad C_4 - C_5 \quad {}_{\mathbb{R}^N \setminus B(0,2R_A)} \quad (t,x)|x| \quad {}_{\mathbb{R}^N \setminus B(x,R_A)} \quad (t,y)dydx, \tag{3.24}$$

where C_4, C_5 only depends on W_a, N and $\| \;_0\|_\infty$. Note that this inequality is parallel to the inequality (3.10) for the discrete case.

3.3.2 Coupling with the evolution of the support

Next we will proceed similarly as in Section 3.2.2, where most of the arguments are parallel. We will quickly go through the similar parts in the proof, and emphasize the di erences caused by the Newtonian repulsive kernel.

At time t, we can find $x_0 \in \partial\text{supp}(\;_0)$ depending on t, such that

$$|X_t(x_0)| = R(t) \qquad \text{and} \qquad \frac{d^+R(t)}{dt} \quad -(\nabla W * \quad)(X_t(x_0), t) \cdot \frac{X_t(x_0)}{R(t)},$$

similarly to [18, 7]. Here $\frac{d^+}{dt}$ stands for the right derivative, since $R(t)$ might not be di erentiable in time.

Due to **(W-CONF)**, for any large constant K_3 to be determined later, there exists some radius $R_{K_3} \quad 6R_A$ such that $w'(r) \quad K_3/r^{1/N}$ for all $r \quad R_{K_3}$. Hence whenever $R(t) \quad R_{K_3}$, the growth of $R(t)$ is now controlled by

$$\frac{d^+R(t)}{dt} \quad {}_{B(X_t(x_0),R_A)} \frac{C_N \quad (t,y)}{|y - X_t(x_0)|^{N-1}} dy - \frac{K_3}{12R(t)^{1/N}}, \tag{3.25}$$

where the second term on the right hand side is obtained in the same way as the last term in (3.16), except that the power 1 is replaced by $1/N$ due to **(W-CONF)**.

To deal with the singularity in the first term on the right hand side, recall that $\| \;(t,\cdot)\|_\infty$ is bounded above by M_0 for all time due to Lemma 3.1, which again enables us to apply Lemma 3.2 to obtain

$$\frac{d^+R(t)}{dt} \quad C_6 m(t, X_t(x_0))^{1/N} - \frac{K_3}{12R(t)^{1/N}}, \tag{3.26}$$

where C_6 only depends on N, W_a and $\| \;_0\|_\infty$.

Similar to Section 3.2.2, we can find some time interval $[t_1, t_2]$, such that $R(t)$ increases from R_{K_3} to $A_2 R_{K_3}$ within $[t_1, t_2]$, and $\dot{R}(t_2) \quad 0$. Here A_2 is a sufficiently large number to be determined at the end of this subsection. Then we have

$$\int_{t_1}^{t_2} \frac{d^+R(t)}{dt} R(t)^{1/N} \quad 0,$$

implying that

$$\int_{t_1}^{t_2} \left(C_6 \; m\big(t, X_t(x_0)\big)^{1/N} R(t)^{1/N} - \frac{K_3}{12} \right) dt \quad 0. \tag{3.27}$$

We apply Hölder s inequality on (3.27), and obtain that

$$\int_{t_1}^{t_2} m\big(t, X_t(x_0)\big) R(t) dt \quad \left(\frac{K_3}{12C_6} \right)^N (t_2 - t_1). \tag{3.28}$$

Note that this extra step is needed here but unnecessary in Section 2, due to the different powers in **(NL-CONF)** and **(W-CONF)**.

Now we are ready to couple the growth of M_3 with (3.28). Since $R(t)$ $6R_A$ for all $t \in [t_1, t_2]$ (recall that when defining R_{K_3} we set it to be greater than $6R_A$), we could treat (3.24) in the same way as we did in (3.17) and (3.18), and bound the growth of M_3 as follows:

$$\frac{dM_3(t)}{dt} \quad (C_4 + \frac{1}{3} C_5 R_A) - C_5 m\big(t, X_t(x_0)\big) R(t). \tag{3.29}$$

Then we integrate (3.29) in $[t_1, t_2]$, and it becomes

$$\int_{t_1}^{t_2} \Big(C_5 m\big(t, X_t(x_0)\big) R(t) - (C_4 + \frac{1}{3} C_5 R_A) \Big) dt \quad M_3(t_1) - M_3(t_2).$$

By putting the above inequality together with (3.28), we can fix K_3 to be sufficiently large such that $M_3(t_1) \quad M_3(t_2)$.

Finally, we prove that if A_2 is sufficiently large, we would have $M_3(t_2)$ $M_3(t_1)$, hence causing a contradiction. It follows from (3.26) and $\dot{R}(t_2) \quad 0$ that

$$C_6 m\big(t_2, X_{t_2}(x_0)\big)^{1/N} - \frac{K_3}{12 R(t_2)^{1/N}} \quad 0,$$

implying that

$$M_3(t_2) \quad m\big(t_2, X_{t_2}(x_0)\big) \big(A_2 R_{K_3} - R_{K_3}\big)^3 \gtrsim A_2^3 R_{K_3}^2,$$

which can be made to be greater than $M_3(t_1)$ if A_2 is chosen to be sufficiently large, thus we obtain a contradiction with $M_3(t_1) \quad M_3(t_2)$. This means that $R(t)$ can never reach $A_2 R_{K_3}$, thus implies the confinement of support for all times.

Remark 3.4. *Let us emphasize that for potentials W given by* (3.19)*, we are only able to prove confinement in the continuum setting, not in the particle setting. The reason is that in the coupling method we use, we need to bound the repulsion part of $\frac{d}{dt}R(t)$ using the mass in some neighborhood of the outermost particle. In the continuum setting this is achieved by first obtaining an L^∞ bound on $\rho(t,\cdot)$ in Lemma* 3.1*, then applying Lemma* 3.2 *to arrive to* (3.26)*. However, in the particle setting, we are unable to obtain a bound on the "local density" of the particles that is independent of the particle number, and we are unaware of any such results for repulsive-attractive kernels to the best of our knowledge. Intuitively we do expect the "density" of particles to be bounded, since the singular repulsion would not allow the particles to be densely concentrated. We find it an interesting open problem to prove some non-local version of Lemma* 3.1 *for the particle system* (3.2) *with W given by* (3.19)*.*

3.4 Numerics

In this section we numerically check the confinement properties of several potentials together with the long-time behavior of the corresponding particle systems. Let us remark that in all the cases we have simulated, for which confinement holds, the long time behavior of the system seems to converge toward a compactly supported stationary state. In some of the potentials below, this has not been rigorously proved. This is an interesting theoretical question that will be treated elsewhere. Our objective in this section is to check if the conditions under which confinement has been shown in previous sections are sharp or not. With this aim, we remind, as it was said in Section 2, that equation (3.1) is a gradient flow of the interaction energy

$$E[\rho] = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W(x-y)\,\rho(x)\,\rho(y)\,dx\,dy$$

with respect to the Wasserstein distance. Thus, stable stationary states of (3.1) are local minimizers of the interaction energy.

In Section 2 we have shown that the radius $R(t)$ defined by (3.11) is bounded by a constant $\overline{R}$ that depends only on the potential W and the initial data. Moreover, $\overline{R}$ is independent of the number of particles and under certain additional assumptions, see Section 2, we know that the particle systems are indeed good approximations of the solutions to the continuum model (3.1). For this reason, we have chosen a particle framework to perform our numerical investigation. We also follow the idea of decreasing the energy since stationary states are local minimizers of the energy. Given n particles located at $x_1, \ldots, x_n \in \mathbb{R}^N$ with masses $m_1 = m_2 = \cdots = m_n = 1/n$, their discrete interaction energy is given by

$$E[x_1, \ldots, x_n] = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} W(x_i - x_j).$$

The simulations are done by an explicit Euler scheme leading to a trivial gradient descent method as long as the energy is decreasing at each time step. This method allows to efficiently solve for stationary states of (3.2). In stiffer situations, as for the Morse potentials below, an explicit Runge-Kutta method is used instead. These methods are essentially the same as the ones used in [127, 126] for finding stationary states of different repulsive-attractive potentials. Our stopping criterion is to achieve a numerical steady state. For us, a numerical steady state is a particle distribution for which the discrete l^∞-norm of the velocity field in (3.2) is below some predetermined threshold, which we impose to be $0.001/n$.

The section is divided into three subsections, each one showing the results for a particular chosen potential. The limit growth for the attractiveness of the potential at infinity under condition **(NL-CONF)** is $\log(r)$. For this reason in Section 4.1 we construct a piecewise potential with exact logarithmic attraction at infinity not satisfying **(NL-CONF)**. This selection has been done to check the sharpness of this condition. In Subsection 4.2 we use a repulsive-attractive potential of the type $w(r) = r^\alpha/\alpha - \mathcal{N}(r)$. For this potential we test the sharpness of condition **(W-CONF)** with a particular choice of α. At the end of this section, in Subsection 4.3, we also analyze the case of the Morse potential. This potential is known to be a repulsive-attractive potential under certain choices of the parameters with negligible attractive strength at infinity, i.e., $W(x) \to 0$ as $|x| \to \infty$. These potentials are more interesting in terms of biological relevance as discussed in [54, 40].

As a final remark, we point out that all the used potentials are not singular at the origin and simulations are performed in dimension $N = 2$.

3.4.1 Logarithmic attraction at infinity

We show confinement when the potential has exact logarithmic attraction at infinity. The chosen potential is

$$w(r) = \begin{cases} \frac{95}{2} r^2 - \frac{83}{6} r - 64 r^3 + \frac{239}{6} r^4 - \frac{19}{2} r^5 & 0 \le r \le 1, \\ \log(r) & r \ge 1. \end{cases} \tag{3.30}$$

It can be checked that it is a repulsive-attractive satisfying $w(0) = w(1) = 0$, $w(r) \in C^3(0, +\infty)$, and the repulsion at the origin is $\simeq -r$. The stationary states are shown in Table 3.1. We have chosen initial data in such a way that

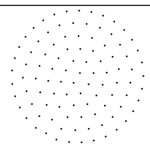
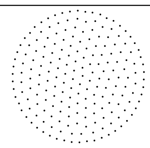
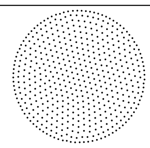
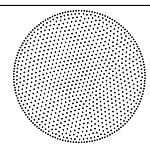
$n = 100$	$n = 200$	$n = 500$	$n = 1000$
			
$r \sim 0.2286$	$r \sim 0.2315$	$r \sim 0.2343$	$r \sim 0.2352$

Table 3.1: Stationary states and radius of their support as a function of the number of particles for the potential $w(r)$ given in equation (3.30).

the particles feel the logarithmic interaction by randomly placing n particles in a centered square in such a way that $|x_i - x_j|$ 1 for some values of $i, j \in \{1, 2, \ldots, n\}$. We have run simulations varying the number of particles n and the initial data. For each simulation the center of mass C_n for the particle system was computed and then

$$r_n(t) = \max_{1 \leqslant j \leqslant n} |x_j(t) - C_n|.$$

It is observed that $r_n(t)$ 1 in all the cases for large times and it converges to some asymptotic value. This fact can be explained because when all the particles are out of the range of the $\log(r)$ part then the radius and the behavior depends only on the polynomial part of the potential. Simulations indicate that there is confinement for this potential and thus, condition **(NL-CONF)** is not sharp. Figure 3.4 shows the evolution of the radius as a function of the particle number n and as a function of time for a particular initial data. In Figure 3.4 we observe the stabilization of the radius around the value 0.2352 for t 350.

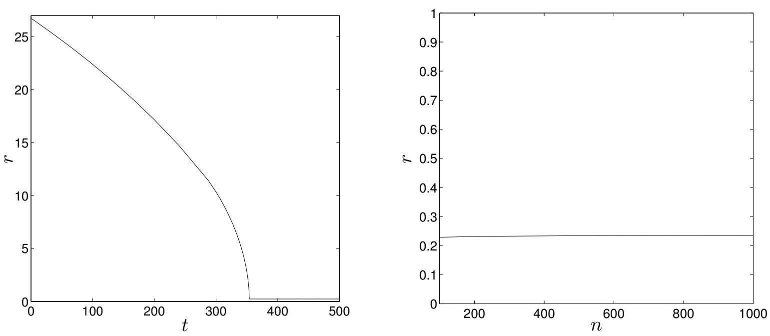


Figure 3.4: Left: Evolution of the radius as a function of t for $n = 1000$. Right: Evolution of the radius as a function of n.

3.4.2 Newtonian repulsion

In this case we consider a potential with Newtonian repulsion at the origin

$$w(r) = \frac{r}{\quad} - \mathcal{N}(r).$$

To test if condition **(W-CONF)** is sharp we choose $\;= 0.5$. We remind that this condition says that

$$\lim_{r\to\infty} w'(r) r^{1/N} = +\infty.$$

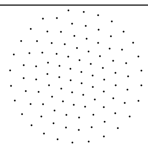
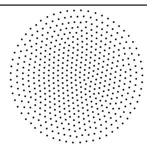
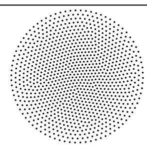
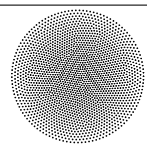
$n = 100$	$n = 500$	$n = 1000$	$n = 2000$
			
$r \sim 0.0289$	$r \sim 0.0307$	$r \sim 0.0312$	$r \sim 0.0314$

Table 3.2: Stationary states and radius for the potential $w(r) = \frac{r^{0\,5}}{0\,5} - \frac{1}{2}\ln(r)$.

When $N = 2$, $\;= 0.5$ is a limit case and does not satisfy the condition. In [64, 63], the authors prove existence and uniqueness of radially symmetric solutions for any power $2 - N \quad q$ with initial data supported on a ball. In our case we are starting with random initial data placed in a square and we observe that the support of these stationary states converges toward a disk. Numerical simulations in Table 3.2 also indicate that **(W-CONF)** is not sharp.

3.4.3 Morse potential

The usual form of this potential is the following

$$U(r) = -C_A e^{-r/l_A} + C_R e^{-r/l_R},$$

where constants C_A and C_R are the attraction and repulsion strength respectively and the constants l_A and l_R are their respective length scales. For our simulations we will take the scaling shown in [54, 40]. That is,

$$U(r) = C_A(V(r) - C\,V(r/l)), \tag{3.31}$$

where $V(r) = -\exp(-r/l_A)$ and $C = C_R/C_A$ and $l = l_R/l_A$. It is known for this potential [54, 40] that for $C \quad 1$ and $l \quad 1$ the potential $U(r)$ is short-range repulsive and long-range attractive with a unique minimum defining a typical distance between particles. Also, in this regime, the condition $Cl^N = 1$ distinguishes between the so-called H-stable and catastrophic regimes. For the definition of H-stable we refer to [114, 54, 47].

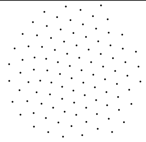
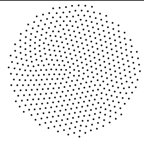
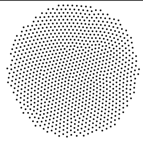
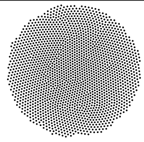
$n = 100$	$n = 500$	$n = 1000$	$n = 2000$
			
$r \sim 15.96$	$r \sim 34.002$	$r \sim 47.35$	$r \sim 64.45$

Table 3.3: Stationary states and radius of their support as a function of the number of particles for the potential $U(r)$ given in equation (3.31).

H-stable case: In our simulations we fix the parameters as $C_A = l_A = 1$, $C_R = 1.9$, and $l_R = 0.8$ leading to $C = 1.9$, $l = 0.8$, and $Cl^2 = 1.216$ 1.

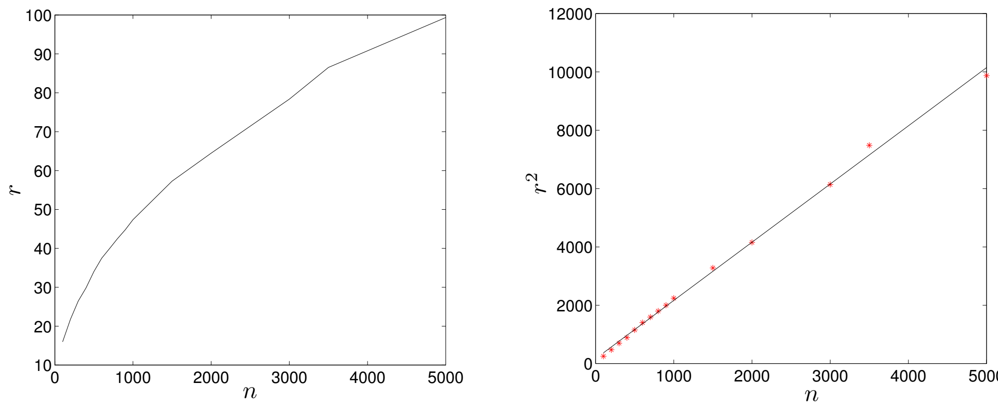


Figure 3.5: H-stable Case. Left: Evolution of the radius as a function of n. Right: Squared radius of the support of the steady state as a function of n and the linear regression curve $y = 170.34 + 1.99n$ computed using the crossed points.

Numerical experiments in Table 3.3 demonstrate that the radius increases by increasing the number of particles, but with a slower rate. As clearly visualized in Figure 3.5(left), the radius appears to grow like a square root function as the number of particles increases. This observation is further supported by numerical evidence in Figure 3.5(right), where we plot the square of the radius versus the number of particles, and the linear regression provides a good fit to the data.

It would be interesting to study the H-stable case in more details, although it is outside of the scope of this work. From these numerical results, we can extract a conjecture that for the continuum system the support of the density would go unbounded over time, i.e., the confinement result should not hold for the H-stable potential.

Catastrophic case: The parameters we choose for the experiments are $C_A = l_A = 1$, $C_R = 1.3$ and $l_R = 0.2$ so that $Cl^2 = 0.052 \quad 1$.

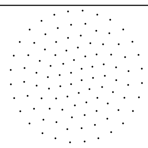
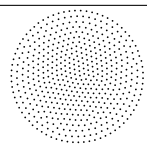
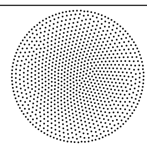
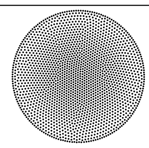
$n = 100$	$n = 500$	$n = 1000$	$n = 2000$
			
$r \sim 0.5269$	$r \sim 0.5480$	$r \sim 0.5518$	$r \sim 0.5540$

Table 3.4: Stationary states and radius of their support as a function of the number of particles for the potential $U(r)$ given in equation (3.31).

The results for this case are shown in Table 3.4. In contrast to the H-stable case, the radius of the support converges to a limiting value. In Figure 3.6(left) we observe how the radius of the support decreases in time to a limiting value with $n = 1000$ particles, and in Figure 3.6(right) we show how the radius of the support of the stationary state increases and converges to a certain value as a function of the number of particles. We conclude that there should be confinement properties for the Morse potential in the catastrophic case.

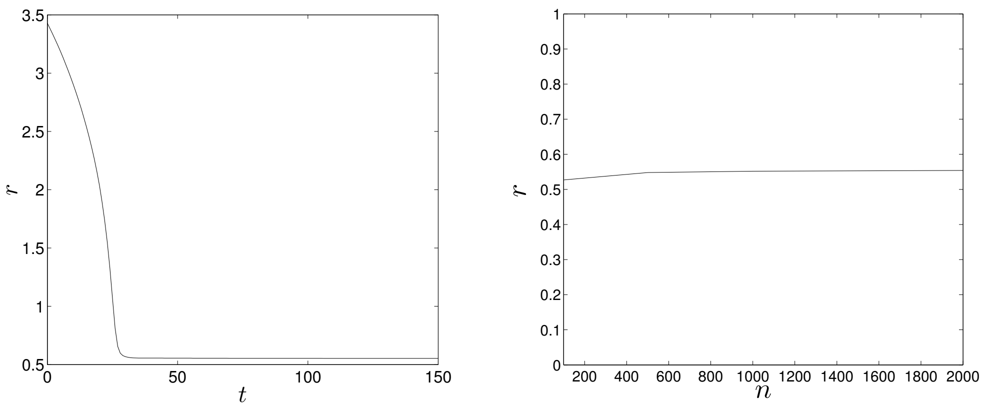


Figure 3.6: Catastrophic Case. Left: Evolution of the radius as a function of t for $n = 1000$. Right: Evolution of the radius as a function of n.

The final goal would be to find replacements for the condition **(NL-CONF)** in order to include the cases where $w(r) \to 0$ as $r \to \infty$. One possibility is to invoke scaling limits for integrable potentials. More precisely, we scale the potential in (3.1) as

$$ _t = \nabla \cdot (\ (\nabla W \ * \ )), \qquad x \in \mathbb{R}^N, t \quad 0, \tag{3.32}$$

in such a way that $W\ (x) = \ ^{-N} W(x/\ )$ approximates a Dirac Delta at 0 with certain weight as $\ \to 0$. Now, if the potential is such that

$$:= \ _{\mathbb{R}^N} W(x)\, dx\,,$$

then equation (3.32) is formally approaching

$$ _t = \ \nabla \cdot (\ \nabla \ ), \qquad x \in \mathbb{R}^N, t \quad 0.$$

In the H-stable case, the Morse potential satisfying $\quad 0$ leads to a limiting nonlinear di usive equation, which is coherent with the no confinement property. In the catastrophic case, the Morse potential satisfying $\quad 0$ leads to a limiting anti-di usive nonlinear equation, which might also be coherent with the confinement property of the potential. We conjecture these integrability conditions might have some implications for confinement properties of potentials.

Chapter 4

Dimensionality of local minimizers of the interaction energy

The contents of this chapter appear in:

> *D. Balagué, J. A. Carrillo, T. Laurent, G. Raoul, Dimensionality of Local Minimizers of the Interaction Energy. To appear in Archive for Rational Mechanics and Analysis.* [8]

4.1 Introduction

In this chapter we consider local minimizers (in the topology of transport distances) of the interaction energy associated to a repulsive-attractive potential. We show how the dimensionality of the support of local minimizers is related to the repulsive strength of the potential at the origin.

We show that if the Laplacian of the potential behaves like $-1/|x|$ around the origin, with 0 N, then the dimension of minimizers is at least and if the Laplacian does not blow up at the origin, then the dimension is zero, see the precise statement in Theorems 4.1 and 4.2. This is illustrated in the case of two dimensions ($N = 2$) in Table 4.1, where we show some local minimizers of E_W with interaction potentials of the form

$$W(x) = -\frac{|x|}{} + \frac{|x|}{} \qquad , \tag{4.1}$$

so that $W(x) \sim -\frac{|x|}{}$ and $W(x) \sim -\frac{1}{|x|}$ with $= 2 -$ as $x \to 0$.

Subfigure (a): = 2.5 and = 15. The support of the minimizer has zero Hausdor dimension in agreement with Theorem 4.2. Actually, in this particular case it is supported just on 3 points.

Subfigure (b) and (c): we consider two examples where the potentials have the same behavior at the origin, = 1.5, but di erent attractive long range behavior (= 7 and 2 respectively). Theorem 4.1 shows that the Hausdor dimension of the support must be greater or equal to = 2 − = 0.5. Indeed, the minimizer for the first example has a one-dimensional support on three curves whereas the minimizer for the second example has two-dimensional support.

Subfigure (d): = 0.5 and = 5. Theorem 4.1 proves that the Hausdor dimension of the support must be greater than = 2 − = 1.5. The numerical simulation demonstrates that it has dimension two.

In our extensive numerical experiments using gradient descent methods we never observed minimizers with a support that might be of non-integer Hausdor dimension.

In most of this chapter, we will consider local minimizers for the topology induced by the transport distance d_∞ (see Chapter 1 for a definition of d_∞). This topology is indeed the natural one to consider. In particular, gradient descent numerical methods based on particles typically lead to local minimizers for the $d_\infty-$topology. Moreover the topology induced by d_∞ is the finest topology among the ones induced by d_p, 1 p ∞ (see Chapter 1 for a definition of d_p). As a consequence local minimizers in the d_p-topology are automatically local minimizers in the d_∞-topology, and thus they are also covered by our study. In Section 4.4 we will discuss in more detail these questions.

Let us finally mention that the gradient ow of the energy E_W in the Wasserstein sense d_2 [41, 4, 42] has been extensively studied in recent years [84, 20, 18, 19, 22, 36, 35, 21, 61, 62, 110, 10, 9]. It leads to the nonlocal interaction equation

$$\frac{\partial \mu}{\partial t} + \operatorname{div}(\mu v) = 0 \quad , \quad v = -\nabla W * \mu \tag{4.2}$$

where $\mu(t, x) = \mu_t(x)$ is the probability or mass density of particles at time t and at location $x \in \mathbb{R}^N$, and $v(t, x)$ is the velocity of the particles. Stability properties of steady states for (4.2) with repulsive-attractive potentials have only been analyzed very recently. In Chapter 2 we gave conditions for radial stability/instability of particular local minimizers. We should also

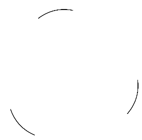
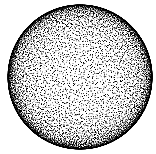
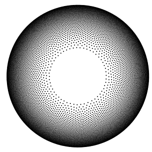
	Dim = 0	Dim = 1	Dim = 2
= 2.5	(a) 		
= 1.5		(b) 	(c) 
= 0.5			(d) 

Table 4.1: Local minimizers of the interaction energy E_W^n for various potentials $W(x)$. In these computations $n = 10,000$. When ΔW does not blow-up at the origin (Case a) the Hausdor dimension of the the support of minimizers is zero. When $\Delta W \sim -1/|x|$ as $x \to 0$, $0 < \beta < N$ (Cases b,c,d) the Hausdor dimension of the the support of minimizers is greater or equal to β.

mention that the one dimensional case was analyzed in detail in [61, 62]. Well-posedness theories for these repulsive-attractive potentials in various functional settings have been provided in [4, 84, 22, 36, 21, 10]. Stable steady states of (4.2) under certain set of perturbations are expected to be local minimizers of the energy functional (11) in a topology to be specified. Actually, this topology should determine the set of admissible perturbations. As already mentioned, the d_∞-stability is the one typically studied by performing equal mass particles simulations.

Finally, we can now interpret our dimensionality result in terms of the nonlocal evolution equation (4.2). The heuristic idea behind the implication: (12) with 0 N implies dimensionality larger than   of the support of local minimizers of E_W; can be understood in terms of the divergence of the velocity field in (4.2). In fact, it is straightforward to check that the divergence of the velocity field generated by a uniform density localized over a smooth manifold of dimension k is $+\infty$ on the manifold if and only if k (this is equivalent to non-integrability of $-\ W$ on manifolds of dimension k). Heuristically, if div $v = -\ W * \mu$ associated to μ diverges on its support the density has a strong tendency to spread, the configuration is not stable and then μ is not a local minimizer. Therefore, we can reinterpret our result in Theorem 1 as follows: local minimizers of (11) have to be supported on manifolds where the divergence of their generated velocity field is not $+\infty$.

The plan of the chapter is as follows. Strongly repulsive potentials are treated in Section 4.2 while mildly repulsive potentials are analyzed in Section 4.3. In Section 4.4, for the smaller subset of local minimizers in the d_2-topology, we show that we can use an Euler-Lagrange approach in the spirit of [16] to derive some properties of these minimizers. Extensive numerical tests as well as details of the algorithm used in order to minimize E_W^n are reported in Section 4.5.

4.2 Lower bound on the Hausdor dimension of the support

In this section we consider potentials which are strongly repulsive at the origin and we prove that if $W \sim -1/|x|$ as $x \to 0$, 0 N, then the Hausdor dimension of the support of local minimizers of the interaction energy is greater than or equal to . Actually our result is slightly stronger: we prove that if μ is a local minimizer then the support of any part of μ has Hausdor dimension greater or equal to . Let us illustrate the importance of controlling not only the dimension of μ, but also the dimension of the parts of μ. Suppose for example that $W \sim -1/|x|$ as $x \to 0$, then our result implies that any part of μ has Hausdor dimension greater or equal to 1. As a consequence μ can not have an atomic part. If $W \sim -1/|x|^{1\ 5}$ as $x \to 0$ then μ can not have a part concentrated on a curve and so on.

4.2.1 Hypotheses and statement of the main result

In this section, we will assume that the potential $W : \mathbb{R}^N \to (-\infty, +\infty]$ satisfies the following hypotheses:

(H1) W is bounded from below.

(H2) W is lower semicontinuous (l.s.c.).

(H3) W is uniformly locally integrable: there exists $M \quad 0$ such that

$$\int_{B(x,1)} W(y)dy \quad M \quad \text{for all} \quad x \in \mathbb{R}^N.$$

In order to state the main results of this section we will also need the following two definitions:

De nition 4.1 (Generalized Laplacian)**.** *Suppose $W : \mathbb{R}^N \to (-\infty, +\infty]$ is locally integrable. The approximate Laplacian of W is defined by*

$$- \quad W(x) := \frac{2(N+2)}{2} \left(W(x) - - \quad W(y)dy \right),$$
$$B(x, \ )$$

where $-_{B(x_0,r)} f(x)dx$ stands for the average of f over the ball of radius r centered at x_0, and the generalized Laplacian of W is defined by

$$- \quad {}^0W(x) := \liminf_{n\to\infty} \quad - \quad {}^{(1/n)}W(x)\} .$$

Let us emphasize the fact that the function W in the above definition is defined pointwise in $\mathbb{R}^N$ (it possibly takes the value $+\infty$), and is Borel measurable. As a consequence the functions $- \quad W$ and $- \quad {}^0W$ are also Borel measurable, defined pointwise in $\mathbb{R}^N$, and take values in $(-\infty, +\infty]$ and $[-\infty, +\infty]$ respectively.

De nition 4.2 (-repulsive potential)**.** *Suppose $W : \mathbb{R}^N \to (-\infty, +\infty]$ is locally integrable. W is said to be -repulsive at the origin if there exists $\quad 0$ and $C \quad 0$ such that*

$$- \quad {}^0W(x) \quad \frac{C}{|x|} \qquad \textit{for all } 0 \quad |x| \tag{4.3}$$

$$- \quad {}^0W(0) = +\infty. \tag{4.4}$$

By doing a Taylor expansion one can easily check that ${}^0W(x) = \quad W(x)$ wherever W is twice di erentiable. In particular if W is twice di erentiable away from the origin as it is often the case for potentials of interest, then (4.3) simply means that $- \quad W(x) \quad C/|x|$ for all $0 \quad |x| \quad$. The terminology -repulsive is justified by the fact that the rate at which ${}^0W(x)$ goes to $-\infty$ as x approaches the origin quantifies the repulsive strength of the potential at the origin, therefore the greater is the more repulsive the potential is around the origin. Additionally to hypotheses (H1) (H3), we will need the following technical assumption on the potential W:

(H4) There exists C^* 0 such that

$$W(x) \quad C^* \quad \forall x \in \mathbb{R}^N \text{ and } \forall \;\; \in (0,1).$$

We are now ready to state the main theorems of this section:

Theorem 4.1. *Suppose W satisfies* (H1) (H4) *and let μ be a compactly supported local minimizer of the interaction energy with respect to the topology induced by d_∞. If W is -repulsive at the origin,* 0 N*, then the Hausdorff dimension of the support of any part of μ is greater or equal to .*

Remark 4.1. *Observe that* (H3) *and* (H4) *are conditions which restrict the growth of the potential and its derivatives at ∞. For instance, a potential growing algebraically at ∞ does not satisfy those assumptions. However, if we are only interested in the dimensionality of the support for compactly supported local minimizers, Theorem* 4.1 *holds under weaker assumptions not restricting the growth of the potential at ∞. Namely,* (H3) *and* (H4) *can be substituted by* (H3-loc) *and* (H4-loc)*:*

(H3-loc) *W is locally integrable.*

(H4-loc) *For every compact subset K of $\mathbb{R}^N$ there exists C_K^** 0 *such that*

$$W(x) \quad C_K^* \quad \forall x \in K \text{ and } \forall \;\; \in (0,1),$$

with obvious changes in the proof.

Remark 4.2. *In Theorem* 4.1 *(resp. Remark* 4.1*) potential W is assumed to be -repulsive at the origin and to satisfy hypotheses* (H1) (H4) *(resp.* (H1)-(H2)-(H3-loc)-(H4-loc)*). Whereas hypotheses* (H1) (H3) *(resp.* (H1)-(H2)-(H3-loc)*) are easily verified for a given potential, hypotheses* (H4) *or* (H4-loc) *and the -repulsivity are not as transparent. To clarify the meaning of these more technical assumptions let us consider the case where W is smooth away from the origin and satisfies*

$$- \;\; W(x) \quad \frac{C}{|x|} \qquad \text{for all } 0 \quad |x| \tag{4.5}$$

for some 0 N*. Such a potential satisfies* (4.3) *as pointed out in the comment after Definition* 4.2*. Moreover, most potentials of interest satisfying* (4.5) *will also satisfy* (4.4) *and either* (H4) *or* (H4-loc)*, but of course this need to be checked case by case. In subsection* 3.3 *we consider some typical repulsive-attractive potentials satisfying* (4.5) *and we show that they satisfy* (4.4) *and either* (H4) *or* (H4-loc) *depending on their behavior at infinity.*

4.2.2 Proof of Theorem 4.1

First note that without loss of generality we can replace hypothesis $(H1)$ by

(H1) W is nonnegative

since adding a constant to the potential W does not a ect the local minimizers of E_W. The following lemma is classical:

Lemma 4.1. *Suppose W satisfies* (H1) *and* (H2) *and let $\mu \in \mathcal{P}(\mathbb{R}^N)$. Then the function $V : \mathbb{R}^N \to [0, +\infty]$ defined by*

$$V(x) = (W * \mu)(x) = \int_{\mathbb{R}^N} W(x-y) d\mu(y)$$

is lower semicontinuous.

Proof. Suppose $x_n \to x$, then by Fatou s lemma we have

$$V(x) = \int_{\mathbb{R}^N} W(x-y)\, d\mu(y) \qquad \int_{\mathbb{R}^N} \liminf_n W(x_n - y)\, d\mu(y)$$
$$\liminf_n \int_{\mathbb{R}^N} W(x_n - y)\, d\mu(y) = \liminf_n V(x_n)$$

as desired. □

Suppose now that W satisfies (H1) (H4). Note that hypothesis (H4) implies that $- \;{}^0W \quad -C^*$ and recall that by definition $- \;{}^0W$ is a Borel measurable function which is defined pointwise in $\mathbb{R}^N$. As a consequence, for any $\mu \in \mathcal{P}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, the integral

$$\begin{aligned}(- \;{}^0W * \mu)(x) &:= \int_{\mathbb{R}^N} (- \;{}^0W)(x-y) d\mu(y) \\ &= \int_{\mathbb{R}^N} \left[(- \;{}^0W)(x-y) + C^*\right] d\mu(y) - C^*\end{aligned}$$

is well defined and belongs to $[-C^*, +\infty]$. The function $- \;{}^0W * \mu$ is therefore defined pointwise in $\mathbb{R}^N$ and takes values in $[-C^*, +\infty]$. Note that the integral against $\mu \in \mathcal{P}(\mathbb{R}^N)$ makes sense since $- \;{}^0W$ is Borel measurable. Indeed the measurability of W and $- \;{}^0W$ also follows from (H2), since any l.s.c. function can be seen as the decreasing limit of continuous functions.

Lemma 4.2. *Suppose that W satisfies* (H1) (H4) *and let $\mu \in \mathcal{P}(\mathbb{R}^N)$. If x_0 is a local min of $V = W * \mu$, in the sense that there exists ${}_0 \quad 0$ such that*

$$V(x_0) \quad V(x) \text{ for almost every } x \in B(x_0, {}_0), \tag{4.6}$$

then $(\;{}^0W * \mu)(x_0) \quad 0$.

Proof. Assume that x_0 satisfies (4.6). Note first that $V\ (x_0)\quad +\infty$, since

$$
\begin{aligned}
V\ (x_0) \quad & \text{ess inf}_{B(x_0,\ _0)} V \quad \frac{1}{|B(x_0,\ _0)|} \ _{B(x_0,\ _0)} V\ (x)\, dx \\
& \frac{1}{|B(x_0,\ _0)|} \ _{\mathbb{R}^N} \ _{B(x_0-y,1)} W(z)\, dz\, d\mu(y) \\
& \frac{M}{|B(x_0,\ _0)|} \quad +\infty,
\end{aligned}
$$

where we have used Fubini s Theorem and Hypothesis (H3). Now, for $\quad _0$ we have

$$
\begin{aligned}
0 \quad & \frac{2(N+2)}{^2} \left(-_{B(0,\ )} V\ (x_0+x)dx - V\ (x_0) \right) \\
= & \frac{2(N+2)}{^2} \left(\ _{\mathbb{R}^N} -_{B(0,\ )} W(x_0+x-y)dx d\mu(y) - \ _{\mathbb{R}^N} W(x_0-y)d\mu(y) \right) .
\end{aligned}
\tag{4.7}
$$

Note that hypothesis (H4) is equivalent to

$$
-_{B(0,\ )} W(x_0+x-y)dx \quad W(x_0-y) + \frac{C^{*\ 2}}{2(N+2)}.
$$

Since $V\ (x_0)\quad +\infty$, the functions $y \mapsto W(x_0-y)$ and $y \mapsto -_{B(0,\ )} W(x_0+x-y)dx$ are μ-integrable and the di erence of the integrals in (4.7) is equal to the integral of the di erence. Therefore we have:

$$
\begin{aligned}
0 \quad & \ _{\mathbb{R}^N} \frac{2(N+2)}{^2} \left(-_{B(0,\ )} W(x_0-y+x)dx - W(x_0-y) \right) d\mu(y) \\
= & \ _{\mathbb{R}^N} \quad W(x_0-y)d\mu(y) .
\end{aligned}
\tag{4.8}
$$

Because of hypothesis (H4), we have that $- \quad W + C^* \quad 0$ for all $\quad \in (0,1)$. Therefore using Fatou s Lemma and (4.8):

$$
\begin{aligned}
& \ _{\mathbb{R}^N} \liminf_{n\to\infty} \ - \ ^{(1/n)} W(x_0-y) + C^* \} \, d\mu(y) \\
& \qquad \liminf_{n\to\infty} \ _{\mathbb{R}^N} \left[- \ ^{(1/n)} W(x_0-y) + C^* \right] d\mu(y) \quad C^* ,
\end{aligned}
$$

that is, $(\ ^0 W * \mu)(x_0) \quad 0$. □

Proposition 4.1. *Suppose that W satisfies* (H1)-(H2)-(H3)*. Let μ be a local minimizer of the interaction energy with respect to the d_∞ and assume that $E[\mu] \quad +\infty$. Then any point $x_0 \in$ supp (μ) is a local minimizer of V , in the sense that there exists $\ _0 \quad 0$ such that*

$$V(x_0) \quad V(x) \text{ for almost every } x \in B(x_0, \ _0).$$

Proof. We argue by contradiction. Assume that there exists $x_0 \in \mathrm{supp}(\mu)$ which is not a local minimum of V . Fix $\quad 0$. Then there exists a set A $B(x_0, \ )$ of positive Lebesgue measure, such that for $x \in A$, $V(x) \quad V(x_0)$. The set A can be written as follows:

$$A = \cup_{n=1}^{\infty}\{x \in A;\, V(x) \quad V(x_0) - 1/n\},$$

that is A is an increasing union of measurable sets. Thanks to the continuity from below of the Lebesgue measure, it implies that

$$0 \quad |A| = \lim_{n\to\infty} |\{x \in A;\, V(x) \quad V(x_0) - 1/n\}|,$$

and there exists n_0 such that $A := \{x \in A;\, V(x) \quad V(x_0) - 1/n_0\}$ is of positive Lebesgue measure. Thanks to the lower semicontinuity of V , there exists $\eta \in (0, \ )$ such that

$$\inf_{B(x_0, \ )} V \quad V(x_0) - \frac{1}{2n_0} \quad \sup_A V \ + \frac{1}{2n_0}. \tag{4.9}$$

Notice that $x_0 \in \mathrm{supp}(\mu)$ implies $\mu(B(x_0, \eta)) \quad 0$. We can therefore define the probability measures μ_0, μ_A by

$$\mu_0(B) = \frac{1}{m_0}\mu(B \cap B(x_0, \eta)), \quad \mu_A(B) = \frac{1}{|A|}|B \cap A|$$

for any Borel set $B \in \mathcal{B}(\mathbb{R}^N)$, where $m_0 := \mu(B(x_0, \eta))$. Let us now write μ as a convex combination $\mu = m_0\mu_0 + m_1\mu_1$, and define the curve of measures

$$\begin{aligned} \mu_t &= (m_0 - t)\mu_0 + t\mu_A + m_1\mu_1 \\ &= m_0\left[\left(1 - \frac{t}{m_0}\right)\mu_0 + \frac{t}{m_0}\mu_A\right] + m_1\mu_1. \end{aligned}$$

It is clear by construction that $\mu_t \in \mathcal{P}(\mathbb{R}^N)$ for $t \in [0, m_0]$, and since $\mu_0 := \left(1 - \frac{t}{m_0}\right)\mu_0 + \frac{t}{m_0}\mu_A$ is supported in $B(x_0, \eta)$, Lemma 1.1 implies that $d_\infty(\mu, \mu_t) \quad 2\ $. Inequality (4.9) shows that the function V is greater on the region $B(x_0, \eta)$ than on the region A, therefore one would expect that transporting mass from one region to the other will decrease the interaction energy. Indeed we will show that $E[\mu_t] \quad E[\mu]$ for t small enough. Since was arbitrary, this will imply that we can always find a probability measure arbitrarily close to μ (in the sense of the d_∞) with strictly smaller energy. This is a contradiction concluding the proof.

We are left to show that $E[\mu_t] \quad E[\mu]$ for t small enough. Since 0 $E[\mu] \quad \infty$ and it is given by

$$E[\mu] = m_0^2 E[\mu_0] + 2m_0 m_1 B[\mu_0, \mu_1] + m_1^2 E[\mu_1] \,,$$

the three terms $E[\mu_0], B[\mu_0, \mu_1]$ and $E[\mu_1]$ are all positive and finite. Note that $E[\mu_A]$ is also finite: indeed, since W is locally integrable by (H3), we have

$$\begin{aligned} E[\mu_A] = & \quad \quad W(x-y)\, d\mu_A(x)\, d\mu_A(y) \\ & {}_{\mathbb{R}^N \times \mathbb{R}^N} \\ & \frac{1}{|A|^2} {}_{B(x_0,\ ) \times B(x_0,\ )} \quad W(x-y)\, dx\, dy \quad +\infty. \end{aligned}$$

From (4.9) and the fact that $B[\mu, \mu_0] \quad \frac{1}{m_0} E[\mu] \quad +\infty$, we also have that

$$B[\mu, \mu_A] + \frac{1}{2n_0} \quad B[\mu, \mu_0] \quad +\infty. \tag{4.10}$$

Using all these, we can show that all combinations of the bilinear form $B[\cdot, \cdot]$ for the measures μ_0, μ_1, and μ_A are finite:

$$E[\mu_0] \quad +\infty \quad , \quad E[\mu_1] \quad +\infty \quad , \quad E[\mu_A] \quad +\infty \quad , \quad B[\mu_1, \mu_0] \quad +\infty \,, \tag{4.11}$$

$$B[\mu_A, \mu_0] \quad \frac{1}{m_0} B[\mu_A, \mu] \quad +\infty \quad , \quad B[\mu_A, \mu_1] \quad \frac{1}{m_1} B[\mu_A, \mu] \quad +\infty \,, \tag{4.12}$$

where we have used (4.10) in order to obtain (4.12). Note that in (4.12) we have assumed $m_1 \neq 0$. If $m_1 = 0$ then μ_1 can be chosen to be zero and therefore $B[\mu_A, \mu_1] \quad +\infty$ trivially holds. Using (4.11)-(4.12), we are allowed to expand $E[\mu_t]$ as:

$$
\begin{aligned}
E[\mu_t] &= E[(m_0 - t)\mu_0 + m_1\mu_1 + t\mu_A] \\
&= (m_0 - t)^2 E[\mu_0] + m_1^2 E[\mu_1] + t^2 E[\mu_A] \\
&\quad + 2(m_0 - t)m_1 B[\mu_0, \mu_1] + 2(m_0 - t)tB[\mu_0, \mu_A] + 2m_1 tB[\mu_1, \mu_A] \\
&= m_0^2 E[\mu_0] + 2m_0 m_1 B[\mu_0, \mu_1] + m_1^2 E[\mu_1] \\
&\quad + 2t\Big(m_0 B[m_0, \mu_A] + m_1 B[\mu_1, \mu_A]\Big) \qquad (4.13) \\
&\quad - 2t\Big(m_0 B[\mu_0, \mu_0] + m_1 B[\mu_0, \mu_1]\Big) \\
&\quad + t^2 E[\mu_0] + t^2 E[\mu_A] - 2t^2 B[\mu_0, \mu_A] \\
&= E[\mu] + 2t\Big(B[\mu_A, \mu] - B[\mu_0, \mu]\Big) + t^2 T[\mu_0, \mu_A]\,. \qquad (4.14)
\end{aligned}
$$

Note that in the above computations we have only used the bilinearity of $B[\cdot,\cdot]$ on the space of positive measures. However, a formal computation using the bilinearity of $B[\cdot,\cdot]$ on the space of signed measures leads to the same result in a much simpler way:

$$
E[\mu_t] = E[\mu - t\mu_0 + t\mu_A] = E[\mu] + 2t\Big(B[\mu_A, \mu] - B[\mu_0, \mu]\Big) + t^2 T[\mu_0, \mu_A].
$$

To conclude the proof note that because of (4.10) the term $B[\mu_A, \mu] - B[\mu_0, \mu]$ appearing in (4.14) is strictly negative and since the term $T[\mu_0, \mu_A] = E[\mu_0] - 2B[\mu_0, \mu_A] + E[\mu_A]$ is finite we can choose t small enough so that $E[\mu_t]$ $E[\mu]$. This concludes the proof. □

Under the additional assumption that W is not singular at the origin, we can obtain a slightly stronger version of Proposition 4.1 which will be needed in section 4.

Proposition 4.2. *Assume that W and μ satisfy the same hypotheses than in Proposition* 4.1*. Assume moreover that $W(0)$ $+\infty$. Then any point $x_0 \in \mathrm{supp}(\mu)$ is a local minimizer of V in the classical sense and V is constant on any connected compact set K* $\mathrm{supp}(\mu)$.

Proof. The proof of the first statement is similar to the proof of Proposition 4.1. We argue by contradiction: assume that $\mu \in \mathcal{P}(\mathbb{R}^N)$ is a local minimizer of $E[\cdot]$ and that there exists $x_0 \in \mathrm{supp}(\mu)$ which is not a (classical) local minimum of V. Fix 0, then there exists $x_a \in B(x_0, \ )$ such that $V(x_a)$ $V(x_0)$. But since V is l.s.c. there exists 0 η such that

$$
V(x_a) \quad V(x_0) \quad V(x) + \frac{V(x_0) - V(x_a)}{2} \quad \text{for all } x \in B(x_0, \eta). \qquad (4.15)
$$

We then define μ_0 and μ_1 as in the proof of Proposition 4.1. The di erent idea now is to send mass from μ_0 to a Dirac Delta at the location x_a instead of distributing it evenly over a set A of nonzero Lebesgue measure: instead of letting $\mu_t = (m_0 - t)\mu_0 + t\mu_A + m_1\mu_1$ as before, we now define $\mu_t = (m_0 - t)\mu_0 + t\delta_{x_a} + m_1\mu_1$. The same expansion leads to

$$E[\mu_t] = E[\mu - t\mu_0 + t\delta_{x_a}] = E[\mu] + 2t\Big(B[\delta_{x_a}, \mu] - B[\mu_0, \mu]\Big) + t^2 T[\mu_0, \delta_{x_a}]\,.$$

From (4.15) we obtain that the term $B[\delta_{x_a}, \mu] - B[\mu_0, \mu]$ is strictly negative. In order to conclude the argument we need the term $T[\mu_0, \delta_{x_a}] = E[\mu_0] - 2B[\mu_0, \delta_{x_a}] + E[\delta_{x_a}]$ to be finite. Note that $E[\delta_{x_a}] = W(0)/2$ therefore it is necessary for $W(0)$ to be finite in order to conclude the proof.

We now prove the second statement. We follow classical arguments from potential theory, see [103, Proposition 0.4] for instance. Let K be a connected compact set contained in $\mathrm{supp}(\mu)$ and consider the sets $A = \{x \in K : V(x)$ $\inf_K V\}$ and $B = \{x \in K : V(x) = \inf_K V\}$. Since V is l.s.c. the set A is open relative to K. Let us show that B is also open relative to K. We argue by contradiction. Suppose there exists $x_a \in B$ such that for every $\quad 0$ there exists $x_{0,} \in K$ with $|x_a - x_{0,}|$ and $V(x_a)$ $V(x_{0,})$. Then following the exact same steps as in the proof of the first statement we can construct a probability measure with lower energy than μ and whose d_∞ distance to μ is smaller than , therefore leading to a contradiction and proving that B is open relative to K. Since K is connected then either A or B must be empty. But since V is l.s.c it has to reach its minimum on compact sets and therefore $A = \varnothing$ and $B = K$. □

Remark 4.3. *Since* $\mathrm{supp}(\mu)$ *is closed, the connected components of* $\mathrm{supp}(\mu)$ *are also closed. So the second statement of Proposition* 4.2 *implies that* V *is constant on any bounded connected component of* $\mathrm{supp}(\mu)$*. In particular if* μ *is compactly supported then* V *is constant on any connected component of* $\mathrm{supp}(\mu)$.

Combining Lemma 4.2 and Proposition 4.1 we obtain:

Corrolary 4.1. *Suppose that* W *satisfies* (H1) (H4). *If* μ *is local minimizer of the interaction energy with respect to* d_∞ *and* $E[\mu]$ $+\infty$, *then* $(\ {}^0W * \mu)(x)$ $\ 0$ *for all* $x \in \mathrm{supp}(\mu)$.

We recall the following result from [60, Theorem 4.13], see also [91, Chapter 8].

Proposition 4.3. *Let A be a Borel subset of $\mathbb{R}^N$, and $s \quad 0$. If there exists a probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ supported on A such that*

$$\underset{\mathbb{R}^N\times\mathbb{R}^N}{} \frac{d\mu(x)\, d\mu(y)}{|x-y|^s} \quad \infty,$$

then $dim_H A \quad s$, with dim_H being the Hausdorff dimension of A.

We are now ready to prove the main theorem.

Proof of Theorem 4.1. Let be a nonzero part of μ, that is $\mu = \quad + \quad$ for some nonnegative measure . Let $A = \mathrm{supp}(\ )$ and let us show that $\dim_H A \quad$. Choose small enough so that (4.3) holds, choose $x_0 \in A$ and define the measure

$$\mu_0(B) = \quad (B \cap B(x_0,\ /2)).$$

Clearly μ can be written $\mu = \mu_0 + \mu_1$, where μ_0 and μ_1 are two (nonnegative) measures of mass $m_0 \quad 0$ and $m_1 \quad 0$ and where μ_0 is supported in $A \cap B(x_0,\ /2)$. Then from (4.3) we get:

$$C \underset{\mathbb{R}^N\times\mathbb{R}^N}{} \frac{d\mu_0(x)\, d\mu_0(y)}{|x-y|} \qquad \underset{\mathbb{R}^N\times\mathbb{R}^N}{} - \quad {}^0W(x-y)d\mu_0(x)d\mu_0(y)$$

$$= \underset{\mathbb{R}^N\times\mathbb{R}^N}{} - \quad {}^0W(x-y)d\mu(x)d\mu_0(y)$$

$$- \underset{\mathbb{R}^N\times\mathbb{R}^N}{} - \quad {}^0W(x-y)d\mu_1(x)d\mu_0(y)$$

$$= - \underset{\mathbb{R}^N\times\mathbb{R}^N}{} (\quad {}^0W * \mu)(y)d\mu_0(y)$$

$$+ \underset{\mathbb{R}^N\times\mathbb{R}^N}{} \quad {}^0W(x-y)d\mu_1(x)d\mu_0(y)$$

$$\underset{\mathbb{R}^N\times\mathbb{R}^N}{} \quad {}^0W(x-y)d\mu_1(x)d\mu_0(y) \quad C^* m_1 m_0 \quad +\infty.$$

We have used the fact that $\quad {}^0W * \mu$ is nonnegative on the support of μ from Corollary 4.1 and that $\quad {}^0W(x) \quad C^*$ by hypothesis (H4). We then apply Proposition 4.3 to the probability measure μ_0/m_0, which is supported on A, to obtain $\dim_H A \qquad$. □

4.2.3 Example of potentials satisfying the hypotheses of Theorem 4.1

In this subsection we consider the class of potentials

$$W\ (x) = c\,h\ (x) + \ (x) \tag{4.16}$$

where $\in C^3(\mathbb{R}^N)$ is bounded from below, c 0 and $h\ : \mathbb{R}^N \to (-\infty, \infty]$ is the power-law function:

$$h\ (x) = -|x|\ /$$

for $x \neq 0$ and $\in \mathbb{R}$ with the convention $h_0(x) = -\log|x|$. We define $h\ (0) = 0$ if 0 or $h\ (0) = +\infty$ if 0. The potentials W are typical examples of repulsive-attractive potentials behaving like $-|x|\ /$ around the origin. It is trivial to check that W satisfies (H1)-(H2)-(H3-loc) for any $-N$ (in the case 0 the function need to grow fast enough at infinity for hypothesis (H1) to hold). Note also that for $x \neq 0$ we have

$$-\ \ W\ (x) = c\frac{(\ + N - 2)}{|x|^{2-}} - \ \ (x) \tag{4.17}$$

and therefore if $+ N - 2$ 0 then W satisfies (4.3) from the definition of -repulsivity with $= 2 -$. The goal of this subsection is to show that W also satisfies (4.4) and (H4-loc).

We start by checking (4.4). An explicit computation gives

$$\begin{aligned} -\ \ h\ (0) &= \frac{2(N+2)}{2}\left(h\ (0) - -_{B(0,\ )}\ h\ (y)dy\right) \\ &= \begin{cases} 2(N+2)\frac{N}{N+}\ \frac{-2}{} & \text{if } \quad 0 \\ +\infty & \text{if } 2 - N \qquad 0 \end{cases} \end{aligned}$$

where we have used the fact that $h\ (0) = 0$ for 0 and $h\ (0) = +\infty$ for 0. Letting $\to 0$ and using the fact that (0) is finite we obtain

$$-\ \ ^0W(0) = +\infty \quad \text{for all} \qquad 2. \tag{4.18}$$

Combining (4.17) and (4.18) we see that for $2 - N$ 2 the potential W is -repulsive with $= 2 - \ \in (0, N)$.

We now show that for $2 - N$ the potentials W satisfies hypothesis (H4-loc). The key point is that the functions h are superharmonic for $2 - N$. Let us recall the definition of superharmonicity:

De nition 4.3. *A lower semicontinuous function* $h : \mathbb{R}^N \to (-\infty, \infty]$ *is said to be superharmonic on the connected open set* *if it is not identically equal to* $+\infty$ *on* *and if*

$$h(x) \quad -_{B(x,r)} \quad h(y)dy$$

for all $x \in$ *and* $r \quad 0$ *such that* $B(x,r) \quad .$

We also recall that if $h \in C^2(\;\;)$, then h is superharmonic on if and only if $h(x) \quad 0$ for all $x \in$. To see that the functions h are superharmonic for $\quad 2-N$, first note that for $x \neq 0$

$$h \;\; (x) = -\frac{(\;\; + N - 2)}{|x|^{2-}} \quad 0.$$

Therefore h is superharmonic on $\mathbb{R}^N \backslash \{0\}$ and it can be easily checked that it satisfies the super-mean value property at the origin [103, Definition 2.1]. Both together imply that it is actually superharmonic on the full space $\mathbb{R}^N$, [103, Corollary 2.1]. As a consequence we directly obtain from the definition of the approximate Laplacian that $h \;\; (x) \quad 0$ and therefore

$$W \;\; = c \quad h \;\; + \qquad\qquad .$$

To conclude we note that since $\in C^3(\mathbb{R}^N)$ a simple Taylor expansion shows that converges uniformly to on compact sets. Indeed, the expansion gives

$$\begin{aligned}
(x) &= \frac{2(N+2)}{2} -_{B(0,\;)} \quad (x+y) - \quad (x)\; dy \\
&= \frac{2(N+2)}{2} -_{B(0,\;)} \quad y^T \nabla \quad (x) + y^T H \quad (x) y + O(\;^3)\; dy && (4.19) \\
&= \frac{2(N+2)}{2} \left(\quad (x) -_{B(0,\;)} \quad y_1^2 dy + O(\;^3) \right) && (4.20) \\
&= \quad (x) + O(\;)
\end{aligned}$$

To go from (4.19) to (4.20) we have used the fact that most of the terms in the Taylor expansion are equal to zero after integrating on spheres due to symmetry. The only remaining terms are the ones involved in the Laplacian. Note that since the partial derivative of order 3 of are bounded on compact subsets of $\mathbb{R}^N$, then the error term is uniform for x in compact sets. Since is bounded on compact sets, we conclude that for $\quad 2-N$ the potential W satisfies (H4-loc).

We summarize this discussion in the following proposition:

Proposition 4.4. *If* $2 - N \quad 2$ *and if* $\in C^3(\mathbb{R}^N)$ *then* W *is* $(2 - \ )$-*repulsive around the origin and satisfies* (H4-loc).

Finally let us give examples of repulsive-attractive potentials $W(x) = w(|x|)$ that satisfy (H1) (H4) rather than (H1)-(H2)-(H3-loc)-(H4-loc). In order to provide these examples we will use the already constructed potential W and ensure that they behave well as $|x| \to \infty$.

Since $W(x) = w(|x|)$ is assumed to be repulsive-attractive, the function w is increasing for r greater than some $r_0 \quad 0$. Condition (H3) therefore implies that w is bounded on $(r_0, +\infty)$. On the other hand condition (H4) implies some bound on the derivatives of W. If $2 - N \quad 0$ then $h \ (x) \to 0$ as $|x| \to \infty$. It can then be easily checked that if the function in (4.16) and its partial derivatives up to order three are bounded in $\mathbb{R}^N$, then W satisfies (H1) (H4). If $0 \quad 2$ one can construct a family of potentials W that satisfies (H1) (H4) by letting $W \ = W$ in some ball $B(0, R)$, and extending W outside of this ball in such a way that:

1) W is three time continuously di erentiable away from the origin,

2) W and its derivative up to order three are bounded in $\{x \in \mathbb{R}^N : |x| \quad R/2\}$.

4.3 Mild Repulsion implies 0-dimensionality

In this section, we will show that if the potential is mildly repulsive, meaning that it behaves locally near zero like $-|x|$ with $\quad 2$, then the support of the measure cannot contain measures concentrated on smooth manifolds of any dimension except 0-dimensional sets.

De nition 4.4. *Let* $1 \quad k \quad N$. *A probability measure* $\mu \in \mathcal{P}(\mathbb{R}^N)$ *is said to have a regular* k*-dimensional part if it can be written*

$$\mu = \mu_1 + \mu_2$$

where μ_1 *is a nonnegative measure on* $\mathbb{R}^N$ *and* μ_2 *is defined by*

$$\int_{\mathbb{R}^N} (x) d\mu_2(x) = \int_{\mathcal{M}} (x) f(x) d\sigma(x) \quad \forall \quad \in C_0(\mathbb{R}^N)$$

for some C^2 *manifold* $\mathcal{M}$ *of dimension* k *embedded in* $\mathbb{R}^N$ *and a non identically zero nonnegative function* $f : \mathcal{M} \to (0, +\infty]$ *integrable with respect to the surface measure* $d\sigma(x)$ *on* $\mathcal{M}$. *Moreover to avoid pathological situations, we assume that there exists* $x_0 \in \mathcal{M}$, $c, \quad 0$ *such that*

$$f(x) \quad c \qquad \forall x \in \mathcal{M} \cap B(x_0, \ ). \tag{4.21}$$

We now state the main result of this section:

Theorem 4.2. *Let $W \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which is equal to $-|x|$ / in a neighborhood of the origin. If 2 then a local minimizer of the interaction energy with respect to d_∞ cannot have a k-dimensional component for any 1 k N.*

For the above theorem to be true it is not necessary for the potential to be exactly equal to a power-law $-|x|$, 2, around the origin. It is enough for the potential to behave like $-|x|$, 2, at the origin in a precise convexity sense, see Theorem 4.3.

4.3.1 Preliminaries on convexity

To prove Theorem 4.2, we need some convex analysis concepts, see [42, 4] and the references therein. The term *modulus of convexity* refers to any convex function on the positive reals satisfying

($_0$) : $[0, \infty) \longrightarrow \mathbb{R}$ is continuous, $(0) = 0$, and $(x) \neq 0$ for x 0.

($_1$) (x) $-kx$ for some k ∞.

Now, we can quantify the convexity of certain functions in terms of a modulus of convexity.

De nition 4.5. *A function $h : [0, +\infty) \to \mathbb{R}$ is $-$uniformly convex on (a, b) if there exists a modulus of convexity such that*

$$h\left(\frac{r_1 + r_2}{2}\right) \quad \frac{1}{2}\left(h(r_1) + h(r_2)\right) - \frac{1}{4}\int_0^{|r_1 - r_2|} (t)\, dt, \tag{4.22}$$

for all $r_1, r_2 \in (a, b)$.

A function $h : [0, +\infty) \to \mathbb{R}$ is $-$convex on (a, b) if it is -uniformly convex with $(s) =$ s and $\in \mathbb{R}$.

Note that if h is $-$convex, then (4.22) reads

$$h\left(\frac{r_1 + r_2}{2}\right) \quad \frac{1}{2}h(r_1) + \frac{1}{2}h(r_2) - \frac{}{8}(r_1 - r_2)^2, \tag{4.23}$$

for all $r_1, r_2 \in (a, b)$. It is equivalent to assume that the function $h(r) - \frac{}{2}r^2$ is convex on (a, b). The following proposition can be easily proven:

Proposition 4.5 (Convexity properties of power-laws)**.**

(i) *If* $q \in (1, 2]$*, then* $h(r) = r^q$ *is* $\;$ $-$*convex on* $[0, R]$ *for* $\quad = \inf_{(0,R)} h'' = q(q-1)R^{q-2} \quad 0$*, and thus, uniformly convex on* $[0, R]$*.*

(ii) *If* $q \quad 2$*, then* $h(r) = r^q$ *is* $\;$ $-$*uniformly convex on* $\mathbb{R}_+$*, with* $\;(t) = 2^{2-q}t^{q-1}/q$*. That is*

$$h\left(\frac{r_1 + r_2}{2}\right) \quad \frac{1}{2}\left(h(r_1) + h(r_2)\right) - 2^{-q}|r_1 - r_2|^q, \tag{4.24}$$

for $r_1, r_2 \quad 0$*.*

4.3.2 Proof of Theorem 4.2

Theorem 4.3. *Let* $W(x) = w(|x|)$ *be continuously differentiable, bounded from below, and decreasing as a function of* $|x|$ *in a neighborhood of the origin. Assume moreover that* W *behaves like the power-law* $-|x|\;$ *,* $\quad 2$*, near the origin, in the sense that for some* $C^* \quad 0$ *and* $R \quad 0$ *small enough,* $w(r) = -h(r^2)$ *satisfies:*

if $\;\in (2, 4]$*,* h *is* $\;-$*convex on* $[0, R]$ *with* $\; = C^* R^{\;/2-2}$*.*

if $\;\in (4, \infty)$*,* h *is* $\;-$*uniformly convex on* $[0, R]$*, with* $\;(t) = C^* t^{\;/2-1}$*,*

and $C^*|w'(r)| \quad r^{\;-1}$ *on* $[0, R]$*. Then a local minimizer of the interaction energy with respect to* d_∞ *cannot have a* k*-dimensional part for any* $1 \quad k \quad N$*.*

Theorem 4.2 is a direct consequence of Theorem 4.3, thanks to Proposition 4.5.

We first provide an explicit formula for how the energy changes when perturbing a local minimizer:

Lemma 4.3. *Suppose that* $W : \mathbb{R}^N \to (-\infty, +\infty]$ *is symmetric, l.s.c. and bounded from below with* $W(0) \quad +\infty$*. Let* $\mu \in \mathcal{P}(\mathbb{R}^N)$ *be a local minimizer of the interaction energy with respect to* d_∞ *and* $E[\mu] \quad +\infty$*. Given a connected domain* $\quad \sim \operatorname{supp}(\mu)$*, a Borel map* $\;:\; \to$ $\;$ *and a convex decomposition* $\mu = m_1\mu_1 + m_2\mu_2$ *with* $\operatorname{supp}(\mu_1)$ $\;$*, we have that*

$$E[m_1(\;\#\mu_1) + m_2\mu_2] - E[m_1\mu_1 + m_2\mu_2] = m_1^2 T[\;\#\mu_1, \mu_1]\,. \tag{4.25}$$

Proof. Since B in (1.3) is a bilinear form,

$$\begin{aligned} E[m_1(\;\#\mu_1) + m_2\mu_2] &- E[m_1\mu_1 + m_2\mu_2] \\ &= m_1^2 B[\;\#\mu_1, \;\#\mu_1] + 2m_1m_2B[\;\#\mu_1, \mu_2] \\ &\quad - m_1^2 B[\mu_1, \mu_1] - 2m_1m_2B[\mu_1, \mu_2]. \end{aligned} \tag{4.26}$$

We now use the fact that μ is a local minimizer to express the terms involving μ_2 as terms involving only μ_1. Proposition 4.2 implies that the function $V\ (x)$ is constant on the connected domain . Since ( ) , we have:

$$\int_{\mathbb{R}^N} W(\ (x) - y)\, d\mu(y) = \int_{\mathbb{R}^N} W(x - y)\, d\mu(y) \quad \forall x \in \ .$$

and therefore, since $\mu = m_1\mu_1 + m_2\mu_2$,

$$m_1 \int_{\mathbb{R}^N} W(\ (x) - y)\, d\mu_1(y) + m_2 \int_{\mathbb{R}^N} W(\ (x) - y)\, d\mu_2(y)$$
$$= m_1 \int_{\mathbb{R}^N} W(x - y)\, d\mu_1(y) + m_2 \int_{\mathbb{R}^N} W(x - y)\, d\mu_2(y)$$

for all $x \in$. Since $\text{supp}(\mu_1)$ we can integrate both sides with respect to $d\mu_1(x)$ and obtain, after multiplication by m_1:

$$m_1^2 \int_{\mathbb{R}^N\times\mathbb{R}^N} W(\ (x) - y)\, d\mu_1(y) d\mu_1(x) + m_1 m_2 \int_{\mathbb{R}^N\times\mathbb{R}^N} W(\ (x) - y)\, d\mu_2(y)\, d\mu_1(x)$$
$$= m_1^2 \int_{\mathbb{R}^N\times\mathbb{R}^N} W(x - y)\, d\mu_1(y) d\mu_1(x) + m_1 m_2 \int_{\mathbb{R}^N\times\mathbb{R}^N} W(x - y)\, d\mu_2(y)\, d\mu_1(x)$$

or equivalently, using the B-notation,

$$2m_1^2 B[\mu_1,\ \#\mu_1] + 2m_1 m_2 B[\mu_2,\ \#\mu_1] = 2m_1^2 B[\mu_1, \mu_1] + 2m_1 m_2 B[\mu_2, \mu_1]$$

and therefore rearranging the terms, we can express the terms involving μ_2 in terms of the ones involving only μ_1:

$$2m_1 m_2 \Big[B[\ \#\mu_1, \mu_2] - B[\mu_1, \mu_2]\Big] = 2m_1^2 \Big[B[\mu_1, \mu_1] - B[\mu_1,\ \#\mu_1]\Big].$$

The desired identity (4.25) is readily obtained by plugging the last equality into (4.26) recalling the definition (1.4) of $T[\mu,\ ]$ in Chapter 1. □

De nition 4.6. *Let* $1 \quad k \quad N$. *We denote by* D^k *the the* k*-dimensional disk of radius* :

$$D^k = \{(x_1, \ldots, x_N) :\ x_1^2 + \ldots + x_k^2 \quad {}^2 \text{ and } x_{k+1} = \cdots = x_N = 0\},$$

and by $_{,k} \in \mathcal{P}(\mathbb{R}^N)$ *the uniform probability distribution on* D^k, *i.e., the probability measure defined as*

$$\int_{\mathbb{R}^N} (x) d \ _{,k}(x) = \frac{1}{|D^k|} \int_{x_1^2 + \ \ + x_k^2 \leqslant \ ^2} (x_1, \ldots, x_k, 0, \ldots, 0)\, dx_1 \ldots dx_k$$

for all $\in C^0(\mathbb{R}^N)$, *where* $|D^k|$ *is the Lebesgue measure of dimension* k *of* D^k, *that is* $|D^k| = \sigma_k \ ^k$ *with* σ_k *being the area of the unit* k*-dimensional ball.*

The following Lemma combined with Lemma 4.3 shows that if a flat k-dimensional disk is contained in the support of a local minimizer, then the energy can be reduced by concentrating all the mass contained in the disk into a single point. As a consequence the support of a local minimizer cannot contain a flat k-dimensional disk.

Lemma 4.4. *Suppose that $W(x) = -h(|x|^2)$ satisfies the assumptions of Theorem 4.3. Then, there exists $c_{k,\epsilon} \geq 0$ such that for ϵ small enough,*

$$T[\delta_0, \mu_{\epsilon,k}] = B[\delta_0, \delta_0] - 2B[\delta_0, \mu_{\epsilon,k}] + B[\mu_{\epsilon,k}, \mu_{\epsilon,k}] \leq -c_{k,\epsilon} \quad .$$

Proof. Since W is bounded from below and $W(0) \leq +\infty$, we can assume without loss of generality that $W(0) = 0$ by adding to W a suitable constant. Then, the first term $B[\delta_0, \delta_0]$ is equal to zero. Symmetrizing the integral involved in second term we obtain:

$$\begin{aligned} B[\delta_0, \mu_{\epsilon,k}] &= -\frac{1}{2}\int_{\mathbb{R}^N} h(|y|^2) d\mu_{\epsilon,k}(y) = -\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} h(|y|^2)\, d\mu_{\epsilon,k}(x) d\mu_{\epsilon,k}(y) \\ &= -\frac{1}{4}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \big[h(|x|^2) + h(|y|^2)\big]\, d\mu_{\epsilon,k}(x) d\mu_{\epsilon,k}(y). \end{aligned}$$

Since the density of the measure $\mu_{\epsilon,k}$ is symmetric by definition, we can also symmetrize the third term and obtain:

$$\begin{aligned} B[\mu_{\epsilon,k}, \mu_{\epsilon,k}] &= -\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} h(|x-y|^2)\, d\mu_{\epsilon,k}(x) d\mu_{\epsilon,k}(y) \\ &= -\frac{1}{4}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \big[h(|x-y|^2) + h(|x+y|^2)\big]\, d\mu_{\epsilon,k}(x) d\mu_{\epsilon,k}(y). \end{aligned}$$

Combining the three terms we find

$$T[\delta_0, \mu_{\epsilon,k}] = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} A(x,y)\, d\mu_{\epsilon,k}(x) d\mu_{\epsilon,k}(y) \tag{4.27}$$

with

$$A(x,y) := h(|x|^2) + h(|y|^2) - \frac{h(|x-y|^2) + h(|x+y|^2)}{2}.$$

Under the assumptions of Theorem 4.2, h is convex on $(0, 2\epsilon^2)$ and since $h(0) = 0$, we deduce

$$h(r_i^2) \leq \frac{r_i^2}{r_1^2 + r_2^2} h(r_1^2 + r_2^2),$$

for $r_i \geq 0$, $i = 1, 2$. Using the above inequalities for $i = 1, 2$ we get

$$h(|x|^2) + h(|y|^2) \leq h(|x|^2 + |y|^2) = h\left(\frac{1}{2}|x+y|^2 + \frac{1}{2}|x-y|^2\right).$$

In the rest of this Lemma, C will denote some generic constant that will change from step to step. For $\in (2,4]$, h is $-$convex on $(0, 2\ ^2)$ with $= C\ ^{-4}$, so that plugging into (4.23), we obtain

$$h(|x|^2) + h(|y|^2) \quad \frac{1}{2} h\left(|x+y|^2\right) + \frac{1}{2} h\left(|x-y|^2\right) - C\ ^{-4}(x \cdot y)^2. \quad (4.28)$$

Combining (4.27) and (4.28) together with the change of variables $(x, y) =$ (x, y), we get dropping the tildes:

$$T[\delta_0, \ _{,k}] \quad -C\ ^{-4} \ _{\mathbb{R}^N \times \mathbb{R}^N} (x \cdot y)^2 \ d \ _{,k}(x) d \ _{,k}(y)$$
$$= -C \ _{\mathbb{R}^N \times \mathbb{R}^N} (x \cdot y)^2 \ d \ _{1,k}(x) d \ _{1,k}(y) \, .$$

For $\ 4$, h is $-$convex with $(t) = Ct^{\ /2-1}$, so that plugging into (4.24), we obtain

$$h(|x|^2) + h(|y|^2) \quad \frac{1}{2} h\left(|x+y|^2\right) + \frac{1}{2} h\left(|x-y|^2\right) - C|x \cdot y|^{\ /2}. \quad (4.29)$$

Combining (4.27) and (4.29) with a -dilation change of variables, we similarly get:

$$T[\delta_0, \ _{,k}] \quad -C \ _{\mathbb{R}^N \times \mathbb{R}^N} |x \cdot y|^{\ /2} \ d \ _{,k}(x) d \ _{,k}(y)$$
$$= -C \ _{\mathbb{R}^N \times \mathbb{R}^N} |x \cdot y|^{\ /2} \ d \ _{1,k}(x) d \ _{1,k}(y) \, .$$

□

The last Lemma combined to Lemma 4.3 shows that the support of a local minimizer cannot contain a at k-dimensional disk of radius . To conclude the proof of Theorem 4.3, we need to introduce some di erential geometry tools. Let $R \quad 0$, and $g : D_R^k \to \mathbb{R}^{N-k}$ a C^2-function such that $g(0) = 0$, $\nabla g(0) = 0$. We define the parameterisation P_g of the graph of g as follows:

$$\begin{array}{rcl} P_g : D_R^k & \longrightarrow & \mathbb{R}^N, \\ (x', 0) & \mapsto & (x', g(x')) \end{array} \quad (4.30)$$

where $x' = (x_1, \ldots, x_k)$ stands for the k first coordinates. Let us remark that classical di erential geometry implies that any C^2-manifold can be locally parameterized by such graphs by choosing conveniently the axis and reordering of variables. Moreover, this can be done in such a way that the volume element of the graph J_g is as close as desired to the unit volume element of the at tangent space by taking R small enough, see [105]. More precisely, there exists a constant C_g depending only on the second derivatives of g on D^k_R such that

$$\|J_g - 1\|_{L^\infty(D^k)} \quad C_g \ , \tag{4.31}$$

for 0 $\quad R$ small enough.

Lemma 4.5. *If W satisfies the assumptions of Theorem* 4.3*, and if $g \in C^2(D^k_R, \mathbb{R}^{N-k})$ satisfies $g(0) = 0$, $\nabla g(0) = 0$, then for* $\quad 0$ *small enough,*

$$T[\delta_0, P_g\# \quad _{,k}] - T[\delta_0, \quad _{,k}] \quad \frac{2 \ ^{-1}}{C^*} \|\nabla g\|_{L^\infty(D^k)}.$$

Proof. Note that by continuity for $\quad$ 0 small enough, we have $\|\nabla g\|_{L^\infty(D^k)}$ 1. We first point out that

$$T[\delta_0, P_g\# \quad _{,k}] - T[\delta_0, \quad _{,k}] = \quad _{\mathbb{R}^N \times \mathbb{R}^N} A(x,y) d \quad _{,k}(x) d \quad _{,k}(y)$$

with

$$A(x,y) = \frac{w(|P_g(x) - P_g(y)|) - w(|x-y|)}{2} - \Big[w(|P_g(x)|) - w(|x|) \Big].$$

Thanks to the definition of the parameterisation P_g, $|P_g(x) - P_g(y)| \quad |x-y|$. Moreover since w is decreasing in a neighborhood of the 0, the first term in $A(x,y)$ is negative for $\max(|x|,|y|)$ $\quad$ small enough. To estimate the second term, we use the mean value theorem for g around $x' = 0$, remembering that $g(0) = 0$:

$$|P_g(x) - (x', 0)| = |g(x') - g(0)| \quad \|\nabla g\|_{L^\infty(D^k)},$$

since $C^*|w'(r)| \quad r \ ^{-1}$, we conclude

$$w(|P_g(x)|) - w(|x|) \quad \|w'\|_{L^\infty([0,2 \ ])} \ \|\nabla g\|_{L^\infty(D^k)} \quad \frac{2 \ ^{-1}}{C^*} \|\nabla g\|_{L^\infty(D^k)} \ .$$

□

Proof of the Theorem 4.2. Assume that μ is a local minimizer of E in d_∞ and that it has a regular k-dimensional part in the sense of Definition 4.4. Let $\mathcal{M}$ be the C^1-submanifold on which this component is supported, and f be the density on $\mathcal{M}$ of this component. Let $x_0 \in \mathcal{M}$, and c, $\quad$ 0 satisfying (4.21). As discussed above and without loss of generality, we can assume that x_0=0 and that $\mathcal{M}$ is locally the graph of a C^2-function $g : D^k_R \to \mathbb{R}^{N-k}$, for some $\quad R \quad$ 0, such that $g(0) = 0$, $\nabla g(0) = 0$.

Let P_g be the parameterisation defined in (4.30). Note that for $\quad R$, $\mu_1 := P_g\# \quad_{,k} \in \mathcal{P}(\mathbb{R}^N)$ is absolutely continuous with respect to the volume element on $\mathcal{M}$ with a density denoted still by μ_1, and satisfying (by (4.31))

$$\|\mu_1\|_{L^\infty(\mathcal{M},d\quad)} \quad \frac{1}{|D^k|\, I} \quad \frac{1}{|D^k|\,(1 - C_g \quad)}, \quad \text{with } I \;= \inf_{x\in D^k} J_g(x).$$

where we used (4.31). Therefore, choosing $m_1 = \frac{c}{2}|D^k|\,(1-C_g \quad)$, then $f(x)$ $m_1\mu_1$ on $x \in D^k$, and we can decompose μ as a convex combination

$$\mu = m_1\mu_1 + m_2\mu_2,$$

where $\mu_2 \in \mathcal{P}(\mathbb{R}^N)$.

We are going to send now all mass from μ_1 to a Dirac Delta at $x_0 = 0$. Let us define $\quad : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ by $\quad \equiv 0$ and $\mu \; := m_1 \; \#\mu_1 + m_2\mu_2$ (note that $\#\mu_1 = \delta_0$). μ is then a small perturbation of μ in d_∞:

$$d_\infty(\mu, \mu \;) \qquad (1 + \|\nabla g\|_{L^\infty(D^k)}). \tag{4.32}$$

To check this just take a map $\mathcal{T}$ in Definition 1.2 such that $\mathcal{T}(x) = x$ for all $x \in \mathcal{M}/P_g(D^k)$ and such that $\mathcal{T}(x) = 0$ for $x \in P_g(D^k)$. Thus, the maximum displacement produced by the transport map $\mathcal{T}$ is bounded by the maximum of $|P_g(x)|$ for $x \in P_g(D^k)$ leading to (4.32) using that $g(0) = 0$ and the mean value theorem.

Since μ_1 has a connected support that contains $\quad(\text{supp}(\mu_1)) = \{0\}$, we can apply Lemma 4.3 to get:

$$\begin{aligned} E[\mu \;] - E[\mu] &= m_1^2 T[\; \#\mu_1, \mu_1] \\ &= m_1^2 T[\; \#\mu_1, \quad_{,k}] + m_1^2 \left(T[\; \#\mu_1, \mu_1] - T[\; \#\mu_1, \quad_{,k}]\right) \end{aligned}$$

Since $\#\mu_1 = \delta_0$, we can use Lemma 4.4 to estimate the first term. Moreover, since $\mu_1 = P_g\# \quad_{,k}$, we can use Lemma 4.5 to estimate the last two terms, so that we finally conclude

$$E[\mu \;] - E[\mu] \quad m_1^2 \left[-c_{k,} \qquad + C \quad \|\nabla g\|_{L^\infty(D^k)}\right].$$

Note that $g \in C^1(D^k_R)$ and $\nabla g(0) = 0$ imply that $\|\nabla g\|_{L^\infty(D^k)} \to 0$ as $\quad \to 0$; thus, if $\quad 0$ is small enough, we get that $E[\mu \;] - E[\mu] \quad 0$.

Thus, μ is a better competitor in the minimization of E for arbitrary small. This leads to a contradiction with the fact that μ is a local minimizer of E showing Theorem 4.2. □

4.4 Euler-Lagrange approach to study local minimizers in the d_2-topology

So far we have used transport plans to build perturbed measures. This enabled us to study local minimizers of the interaction energy with respect to the d_∞-topology. To study local minimizers with respect to the d_2-topology it is actually possible to use a more classical Euler-Lagrange approach as we will present in this section. The Euler-Lagrange conditions that we will derive were formally obtained in [16] by perturbing densities inside and outside their support. Here, we provide a rigorous proof in the case of probability measures endowed with the distance d_2.

Theorem 4.4. *Consider an interaction potential W satisfying* (H1) (H2)*. Let us consider $\mu \in \mathcal{P}_2(\mathbb{R}^N)$ a local minimizer of E with respect to d_2 such that $E[\mu] \quad \infty$. Then,*

(i) $(W * \mu)(x) = 2E[\mu]$ *μ-a.e.*

(ii) $(W * \mu)(x) \quad 2E[\mu]$ *for all* $x \in \operatorname{supp}(\mu)$.

(iii) $(W * \mu)(x) \quad 2E[\mu]$ *for a.e.* $x \in \mathbb{R}^N$.

Proof. As usual, we assume that $W \quad 0$ without loss of generality. Lemma 4.1 implies that $W * \mu$ is well defined, lower semicontinuous, and non-negative.

In order to prove the first two items, let us choose $\in C_0^\infty(\mathbb{R}^N)$ to define

$$= \left(\quad - \int_{\mathbb{R}^N} \quad d\mu \right) \mu := a(x)\mu$$

and $\mu \quad = \mu + \quad = (1 + \quad a(x))\mu$ with $\quad 0$ to be specified. It is clear that $\mu \ (\mathbb{R}^N) = 1$ since $a(x)$ has zero integral with respect to μ. Moreover, since $a(x) \quad -2\| \quad \|_{L^\infty}$ then $\mu \quad 0$ for $\quad \frac{1}{2\| \quad \|_{L^\infty}} = \quad$. Therefore, $\mu \ \in \mathcal{P}(\mathbb{R}^N)$ for all $\quad$. It is easy to check that $\mu \ \in \mathcal{P}_2(\mathbb{R}^N)$, that $\mu \ \rightharpoonup \mu$ weakly-$*$ as measures, and

$$\int_{\mathbb{R}^N} |x|^2 d\mu \ \to \int_{\mathbb{R}^N} |x|^2 d\mu \,.$$

Therefore, we have that

$$d_2(\mu \ , \mu) \to 0 \quad \text{as} \quad \to 0.$$

Note that it is not true that $d_\infty(\mu \ , \mu) \to 0$ as $\ \to 0$. Consider for instance $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ and $\ |_{[0,1]} = x$, then, $\mu \ = (\frac{1}{2} - \ )\delta_0 + (\frac{1}{2} + \ )\delta_1$, and $d_\infty(\mu \ , \mu) = 1$ for any $\quad 0$.

Now, since μ is a local minimizer in d_2 then $E[\mu\ ] \quad E[\mu]$ for small enough. Moreover, since μ has finite energy, then $E[\mu\ ] \quad \infty$ and we can expand it as

$$\frac{E[\mu\ ] - E[\mu]}{} = \quad \underset{\mathbb{R}^N\times\mathbb{R}^N}{} W(x-y)d\ \ (x)d\mu(y)$$

$$+ \frac{}{2} \underset{\mathbb{R}^N\times\mathbb{R}^N}{} W(x-y)d\ \ (x)d\ \ (y) \quad 0\,,$$

with both integral terms well-defined. As $\ \to 0$, we easily get

$$\underset{\mathbb{R}^N\times\mathbb{R}^N}{} W(x-y)d\ \ (x)d\mu(y) \quad 0$$

or equivalently,

$$[(W*\mu)(x) - 2E[\mu]]\, d\mu(x) \quad 0$$

for all $\ \in C_0^\infty(\mathbb{R}^N)$. Since one can take either $\ $ or $-\ $ as test functions, we deduce

$$[(W*\mu)(x) - 2E[\mu]]\, d\mu(x) = 0$$

for all $\ \in C_0^\infty(\mathbb{R}^N)$, and thus (i) is satisfied a.e. μ.

Let us now prove (ii). Take $x \in \operatorname{supp}(\mu)$ then there exists $\{x_n\}_{n\in\mathbb{N}} \to x$ with $x_n \in \operatorname{supp}(\mu)$, such that $(W*\mu)(x_n) = 2E[\mu]$. The existence of such a sequence is ensured since $\mu(B(x,\ ))\quad 0$ for all $\quad 0$ by definition of the support of μ. Then, by lower semicontinuity of $W*\mu$ we get

$$(W*\mu)(x) \quad \liminf_{n\to\infty}(W*\mu)(x_n) = 2E[\mu]\,.$$

and then (ii) is satisfied.

In order to show (iii), we consider di erent variations to the ones constructed above. Take $\ \in \mathcal{C}_0^\infty(\mathbb{R}^N),\quad 0$ and then take

$$= \quad - \left(\underset{\mathbb{R}^N}{} \quad dx \right) \mu.$$

Again, defining $\mu\ = \mu + \ $, then it verifies $\mu\ (\mathbb{R}^N) = 1$ and if $\quad 1/\quad dx$ then $\mu\quad 0$. As previously, it is easy to check that

$$d_2(\mu\ , \mu) \to 0 \quad \text{as} \quad \to 0.$$

Note again that it is not true in general that $d_\infty(\mu\ ,\mu) \to 0$ as $\ \to 0$.

Proceeding similarly as in point (i), we get

$$\ _{\mathbb{R}^N\times\mathbb{R}^N} W(x-y)d\ \ (y)d\mu(x) \quad 0$$

taking $\ \to 0$ in $E[\mu\ ] \quad E[\mu]$. Therefore, we conclude that

$$\ _{\mathbb{R}^N} ((W*\mu)(x) - 2E[\mu]) \quad dx \quad 0\,,$$

for all $\ \in C_0^\infty(\mathbb{R}^N)$, $\quad 0$. This readily implies (iii). □

Remark 4.4. *Note that putting together (i), (ii), and (iii) in previous theorem, we conclude that*

$$\begin{cases} (W*\mu)(x) & = 2E[\mu] \quad \textit{for a.e.} \quad x \in \operatorname{supp}(\mu) \\ (W*\mu)(x) & \quad 2E[\mu] \quad \textit{for a.e.} \quad x \in \mathbb{R}^N \backslash \operatorname{supp}(\mu). \end{cases}$$

if μ is absolutely continuous with respect to the Lebesgue measure. These two properties are the Euler-Lagrange conditions that were found for densities in [16].

Remark 4.5. *Let us now clarify the differences between local minimizers in the d_2-topology and local minimizers in the $d_\infty-$topology. Following* [61]*, let us consider as an example the interaction potential $W(x) := -x^2 + \frac{x^4}{2}$ in one dimension. Then,*

$$\ _m = m\delta_0 + (1-m)\delta_1$$

is a critical point of the interaction energy for any $m \in [0,1]$. Theorem 3.1 *in* [61] *shows that the measure $\ _m$ is a local minimizer in the $d_\infty-$topology as soon as $m \in (1/3, 2/3)$. Indeed, what is proven is the stronger statement that $\ _m$ is locally asymptotically stable for the aggregation equation* (4.2) *with respect to any perturbation in the $d_\infty-$topology. However, $E(\ _m) = \frac{1}{2}\left(m - \frac{1}{2}\right)^2 - \frac{1}{8}$, so that only one of them, namely $\ _{1/2}$, can be a local minimizer of the energy in the d_2-topology (and one can prove that it actually is).*

This shows that the set of local minimizers with respect to the d_2-topology is strictly contained in the set of local minimizers with respect to the $d_\infty-$topology. Moreover, numerical simulations suggest that, for $m \in (1/3, 2/3)$, $\ _m$ is actually stable (although not asymptotically stable) with respect to small d_2-perturbations. As a consequence, when using a gradient flow approach to compute numerically minimizers of the interaction energy via particles, one obtains $d_\infty-$local minimizers which typically are not d_2-local minimizers (see e.g. Figure 2 *of* [61]*).*

4.5 Numerical experiments

In this section we conduct a numerical investigation of the local minimizers of the discrete interaction energy

$$E_W^n[X_1, \ldots, X_n] = \frac{1}{2n^2} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} W(X_i - X_j). \tag{4.33}$$

with high number of particles. The gradient ow of (4.33) is given by the system of ODEs:

$$\dot{X}_i = - \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \nabla W(X_i - X_j). \tag{4.34}$$

In order to e ciently find local minimizers of (4.33), we solve (4.34) by an explicit Euler scheme with an adaptive time step chosen as the largest possible such that the discrete energy (4.33) decreases. This scheme is nothing else than a gradient descent method for the discrete energy (4.33). Although this method might not be accurate enough for the dynamics, it is e cient to find local minimizers of the discrete energy. In sti er situations an explicit Runge-Kutta method is used instead. These methods are essentially the ones proposed in [127, 126]. The results of these simulations in two dimensions with power- law potentials were presented in the introduction, see Table 4.1. In Subsection 4.5.1 we discuss similar computations in three dimensions. We also provide numerical experiments suggesting that for some potentials, there are local minimizers of the interaction energy with mixed dimensionality, that is, local minimizers that are the sum of measures whose support have di erent Hausdor dimension.

In Subsection 4.5.2 we show how our numerical results can be further understood by using the results from [81, 127, 10], where a careful stability analysis of ring solution (in 2D) and spherical shell solution (in 3D) was conducted. We also show how this stability analysis connects to the analytical results presented in this chapter.

4.5.1 Numerical experiments in 3D

We first compute numerically local minimizers of E_W^n where W is the power-law potential defined by (4.1). Recall that $W(x) \sim -1/|x|$ with $= 2 -$ as $x \to 0$. The computations are performed with $n = 2,500$ particles. The results are shown in Table 4.2 and are discussed below:

- Subfigure (a): = 2.5 and = 5. The support of the minimizer has zero Hausdor dimension in agreement with Theorem 4.2. Actually, in this particular case it is supported on 4 points forming a tetrahedron.

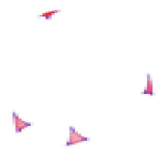
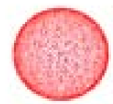
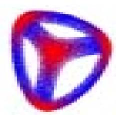
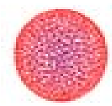
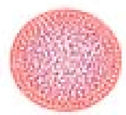
	Dim = 0	Dim = 1	Dim = 2	Dim = 3
= 2.5	(a) 			
= 1.25		?	(b) 	(c) 
= 0.5			(d) 	(e) 
= −0.5				(f) 

Table 4.2: Minimizers of E_W^n in $\mathbb{R}^3$ for various power-law potentials with $n = 2,500$.

- Subfigure (b) and (c): the two potentials have the same behavior at the origin, = 1.25, but di erent attractive long range behavior (= 15 and = 1.4 respectively). Theorem 4.1 shows that the Hausdor dimension of the support must be greater or equal to = 2 − = 0.75. Numerically, we observe that the local minimizer for the first example has a two-dimensional support and the minimizer for the second example has a three-dimensional support. We did not choose the value = 1.5 because we were not able to obtain a change of dimensionality of the stable steady states varying . Note that = 1.5 is always above the instability curve for radial perturbations which meets line = at the point $(\sqrt{2}, \sqrt{2})$. See Figure 4.4 and Subsection 6.2 for more details.

- Subfigure (d) and (e): the two potentials have the same behavior at the origin, = 0.5, but di erent attractive long range behavior (= 23 and = 1.4 respectively). Theorem 4.1 shows that the Hausdor dimension of the support must be greater or equal to = 2 − = 1.5. Numerically, we observe that the local minimizer for the first example has a two-dimensional support and the local minimizer for the second example has a three-dimensional support.

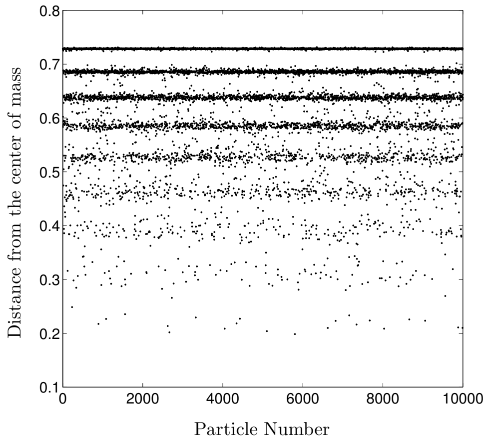


Figure 4.1: Distances of the particles from the center of mass for the power-law potential with $\alpha = -0.5$, $\gamma = 5$ in 3D. Case (f) in Subsection 6.1 in Table 4.2 with $n = 10,000$.

- Subfigure (f): = −0.5 and = 5. Theorem 4.1 proves that the Hausdor dimension of the support must be greater than = 2 − = 2.5, which can also be observed numerically. In Figure 4.1, we have represented the radius of particles to the center of mass. The particles seem to organize into successive two dimensional layers. Such lattices were also observed in [80], and it is related to the finite number of particles used in the simulations.

Notice that we were not able to find examples of interaction potentials leading numerically to a local minimizer with one dimensional support. We could however observe such situations with an additional asymmetric confining potentials, we thus believe it should be possible to produce such cases.

A natural question following Tables 4.1 and 4.2 is whether it is possible to produce local minimizers that are a sum of two measures whose support have di erent Hausdor dimensions. A possible candidate was already observed in [127]. Here, we analyze it more carefully with much larger number of particles. From our simulations, it seems that the interaction potential $W(x) = w(|x|)$ with w defined by

$$-w'(r) = \tanh((1-r)a) + b, \quad a = 5, \quad b = 0.5,$$

leads numerically to a local minimizer consisting in a ball (Hausdor dimension three) inside a spherical shell (Hausdor dimension two), see Figure 4.2.

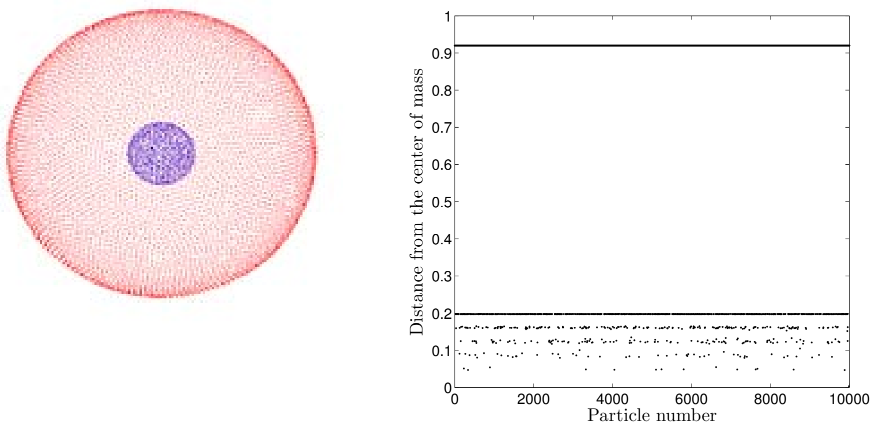


Figure 4.2: Left: Local minimizer in 3D with $n = 10,000$. Right: Distance of the particles from the center of mass.

The distance of each particle to the center of mass is displayed on the right part of Figure 4.2. The inner ball appears to be composed of five equally spaced layers of particles. This is most likely due to the fact that particles are organized into a lattice configuration, and therefore the distances between the particles and the origin do not form a continuum. It is instructive to compare the distribution of the radius of the particles in the right subplot of Figure 4.2 with the one in Figure 4.1 for the case of an approximated local minimizer with three dimensional support, i.e., Case (f) of Table 4.2. Although Theorem 4.1 guarantees that the support of the local minimizer corresponding to Figure 4.1 has Hausdor dimension greater or equal to 2.5, we can also observe that particles arrange themselves in layers. Notice that in dimension N=2, such artifacts also appear in simulations using a finite number of particles, see Figure 4 in [80].

4.5.2 Relationship with previous works on ring and shell solutions

An important characteristic of the analysis performed in the main theorems of this work is that we do not assume a specific shape on the local minimizers. If on the contrary, one is interested by the special case of delta ring minimizers (in 2D), or spherical shell minimizers (in 3D), perturbative methods provide more detailed results.

In [81] the local stability of discrete ring solutions, made of N-particle equally distributed in a circle, was studied for the N-particle system (4.34). The authors considered the power-law interaction potentials (4.1), and led a formal linear stability analysis for the continuum ring solution as steady state of the aggregation equation (4.2) by taking $N \to \infty$. Those predictions were then confirmed numerically. They could not obtain nonlinear stability of the ring solution particularly because there is no spectral gap as $N \to \infty$, i.e, the largest negative eigenvalue tends to 0 when $N \to \infty$. In [10], the nonlinear stability of the ring solutions was proved for radial perturbations, corroborating some of the formal results of [81], together with the instability due to fattening in the complementary set of parameters.

We have represented this set of parameters in Figure 4.3, as well as all the parameters used in the two dimensional numerical simulations of this article (Tables 4.1 and 4.3). As the caricature presented in Table 4.3 shows, crossing the lower border of this set, curve $= /(- 1)$, leads to a fattening of the delta ring, that is to minimizers with dimensionality 2, see [81, 10]. On the other hand, crossing its upper border, given by the curve marked with 3, does not modify the dimensionality of the stable steady states as long as 2 (they remain one dimensional), but leads to a shape instability towards a triangular configuration that breaks the ring into 3 connected one

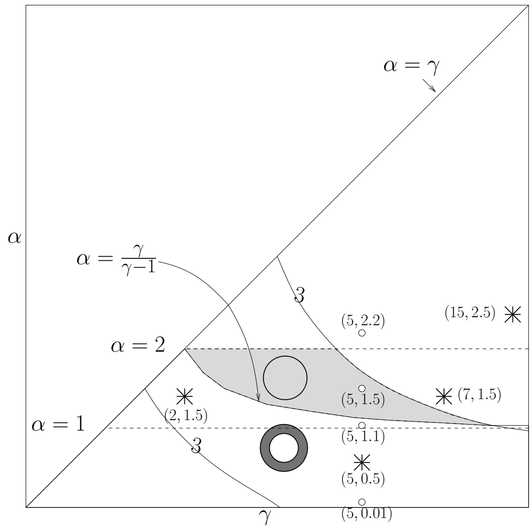


Figure 4.3: Sketch of all the computed cases in dimension $N = 2$. The parameters (γ, α) used in Table 4.1 are marked with $*$, while those used in Table 4.3 are marked with $\circ$. Notice that $\alpha < \gamma$ is necessary for the interaction potential to be con ning. In dark gray is represented the set of parameters such that a delta ring could be a local minimizer.

dimensional components as in case (b) of Table 4.1.

Finally, if   2, local minimizers become of dimensionality 0, as predicted by Theorem 4.2, whereas if   1, all the minimizers are of dimensionality 2, as shown by Theorem 4.1. In three dimensions, a linear stability

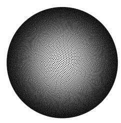
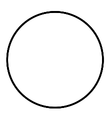
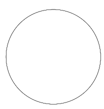
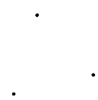
			
= 0.01	= 1.1	= 1.5	= 2.2

Table 4.3: Evolution of local minimizers when $\alpha > 0$ increases, while $\gamma = 5$ remains constant. The computations were done with $n = 10,000$ particles.

analysis of discrete spherical shell solutions is also possible but it leads to more cumbersome instability curves, see [127, 126]. Again, the results in [10] give the fattening instability curve dividing instability from stability under radial perturbations. In Figure 4.4, we have only represented the set of parameters such that the spherical shells are not local minimizers for spherically symmetric perturbations, as well as all the parameters (,) used for 3D numerical simulations in this article in Table 4.2. Just as we have observed in the 2D case, crossing the lower border of this set leads to a fattening" instability of the spherical shell.

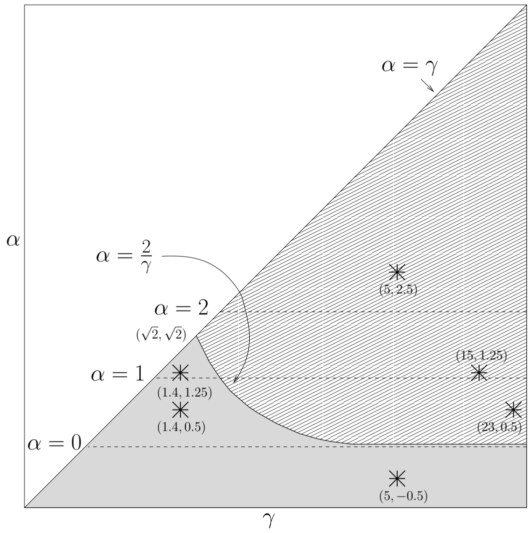


Figure 4.4: Sketch of all the computed cases in dimension $N = 3$. The parameters (γ, α) used in Table 4.2 are marked with $*$. Notice that $\alpha < \gamma$ is necessary for the interaction potential to be con ning. The curve is the limit between parameters leading to spherical shell solutions for radial perturbations (above the curve) and to minimizers of dimensionality 3 (below the curve).

Notice finally that it is also possible to modify the dimensionality of the local minimizers with other perturbations of power-law potentials. As an example, in Table 4.4, we consider the following perturbations of the power-law potential (4.1):

$$W(x) = -\frac{|x|}{} + \frac{|x|}{} + \frac{3}{2p}\cos(px), \qquad , \quad p = 3, 5. \tag{4.35}$$

In Table 4.4 we have represented the power-law case in the first column, and the perturbations in the next two. For (,) = (2, 1.5), the unperturbed power-law potential leads to a local minimizer with Hausdor dimension two. When we add the perturbation $p = 3$, the dimension of the minimizer changes to one. Notice that the perturbation does not alter the local behavior of the potential at the origin or at infinity, suggesting that Theorem 4.1 is probably sharp at least in terms of natural dimensions.

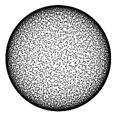
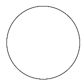
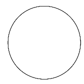
	Powers	$p = 3$	$p = 5$
$(\ , \ ) =$ $(2, 1.5)$			

Table 4.4: Local minimizers with the power-law potential (4.1) and the perturbed potential (4.35), $n = 10,000$.

Chapter 5

Fine asymptotics of pro les and relaxation to equilibrium for growth-fragmentation equations with variable drift rates

The contents of this chapter appear in:

> *D. Balagué, J. A. Cañizo, P. Gabriel, Fine asymptotics of profiles and relaxation to equilibrium for growth-fragmentation equations with variable drift rates. Kinetic and Related Models 6(2):219-243, June 2013* [6]

5.1 Introduction

In this chapter we are interested in the long-time behavior of the growth-fragmentation equation, commonly used as a model for cell growth and division and other phenomena involving fragmentation [106, 93, 65]. There are a number of works which study the existence and other properties of the first eigenfunctions (also called *profiles*) of the growth-fragmentation operator and its dual [94, 55, 59] and the convergence of solutions to equilibrium [83, 108, 96, 95, 30, 107]. These eigenfunctions are fundamental since they give the asymptotic shape of solutions (i.e., the stationary solution of the rescaled equation) and a conserved quantity of the time evolution. However, precise estimates on their behavior close to 0 and $+\infty$ are usually not given, are very rough, or are restricted to a particular kind of growth or fragmentation coe cients. Our first objective is to give accurate estimates on the first eigenfunctions, valid for a wide range of growth and fragmentation coe cients which include most cases in which they behave like power laws. We give, in most cases, the first-order behavior of both first eigenfunctions (of the growth-fragmentation operator and its dual); detailed results are given later in this introduction.

Our second objective is to use these estimates to show that the growth-fragmentation operator has a spectral gap (in a certain natural Hilbert space) for a wide choice of the coefficients, which is interesting because it readily implies exponential convergence to equilibrium of solutions. For this we follow the techniques in [30], which require careful estimates on the profiles which were previously available only for particular growth rates (constant and linear). Our results on exponential convergence to equilibrium are valid for general coefficients behaving like power laws, improving or complementing known results applicable to constant or linear total fragmentation rates [83, 108, 30]. However, our results still impose some restrictions on the fragment distribution (which must be bounded below) and the decay of the total fragmentation rate for small sizes.

Let us introduce the equation under study more precisely and state our main results. The *growth-fragmentation equation* is given by

$$\partial_t g_t(x) + \partial_x(\tau(x) g_t(x)) + \lambda g_t(x) = \mathcal{L}[g_t](x), \tag{5.1a}$$

$$(\tau g_t)(0) = 0 \qquad (t \geq 0), \tag{5.1b}$$

$$g_0(x) = g_{\mathrm{in}}(x) \qquad (x \geq 0). \tag{5.1c}$$

The unknown is a function $g_t(x)$ which depends on the time $t \geq 0$ and on $x \geq 0$, and for which an initial condition g_{in} is given at time $t = 0$. The positive function τ represents the *growth rate*. The symbol $\mathcal{L}$ stands for the fragmentation operator (see below), and λ is the largest eigenvalue of the operator $g \mapsto -\partial_x(\tau g) + \mathcal{L}g$, acting on a function $g = g(x)$ depending only on x. The main motivation for the study of equation (5.1) is the closely related

$$\partial_t n_t(x) + \partial_x(\tau(x) n_t(x)) = \mathcal{L}[n_t](x), \tag{5.2}$$

with the same initial and boundary conditions. Solutions of the two are related by $n_t(x) = e^{\lambda t} g_t(x)$, and n_t represents the size distribution at a given time t of a population of cells (or other objects) undergoing growth and division phenomena. The population grows at an exponential rate determined by $\lambda > 0$, called the *Malthus parameter*, and approaches an asymptotic shape for large times. Equation (5.1) has a stationary solution and is more convenient for studying its asymptotic behavior, which is why it is commonly considered. Of course, results about (5.1) are easily translated to results about (5.2) through the simple change $n_t(x) = e^{\lambda t} g_t(x)$.

The *fragmentation operator* $\mathcal{L}$ acts on a function $g = g(x)$ as

$$\mathcal{L}g(x) := \mathcal{L}_+ g(x) - B(x)g(x),$$

where the positive part $\mathcal{L}_+$ is given by

$$\mathcal{L}_+ g(x) := \int_x^\infty b(y,x)g(y)\,dy.$$

The coefficient $b(y,x)$, defined for $y \geq x \geq 0$, is the *fragmentation coefficient*, and $B(x)$ is the *total fragmentation rate* of cells of size $x \geq 0$. It is obtained from b through

$$B(x) := \int_0^x \frac{y}{x}\, b(x,y)\,dy \qquad (x \geq 0).$$

The *eigenproblem* associated to (5.1) is the problem of finding both a stationary solution and a stationary solution of the dual equation, this is, the first eigenfunction of the growth-fragmentation operator $g \mapsto -(\tau g)' + \mathcal{L}(g)$ and of its dual $\phi \mapsto \tau\phi' + \mathcal{L}^*(\phi)$. If λ is the largest eigenvalue of the operator $g \mapsto -(\tau g)' + \mathcal{L}g$, the associated eigenvector G satisfies

$$(\tau(x)\,G(x))' + \lambda G(x) = \mathcal{L}(G)(x), \tag{5.3a}$$

$$\tau(x)G(x)\big|_{x=0} = 0, \tag{5.3b}$$

$$G \geq 0, \quad \int_0^\infty G(x)\,dx = 1. \tag{5.3c}$$

Of course, the eigenvector G is an equilibrium (i.e., a stationary solution) of equation (5.1). The associated dual eigenproblem reads

$$-\tau(x)\,\phi'(x) + (B(x) + \lambda)\,\phi(x) = \mathcal{L}_+^*\phi(x), \tag{5.4a}$$

$$\phi \geq 0, \quad \int_0^\infty G(x)\,\phi(x)\,dx = 1, \tag{5.4b}$$

where

$$\mathcal{L}_+^*\phi(x) := \int_0^x b(x,y)\,\phi(y)\,dy.$$

This dual eigenproblem is interesting because ϕ gives a conservation law for (5.1):

$$\int_0^\infty \phi(x)\,g_t(x)\,dx = \int_0^\infty \phi(x)\,g_{\mathrm{in}}(x)\,dx = \mathrm{Cst} \qquad (t \geq 0).$$

In this chapter we always denote by G, ϕ and λ the solution to (5.3) and (5.4).

In the rest of this introduction we describe the assumptions used throughout the chapter and state our main results. In Section 5.2 we give the proof of our estimates on the stationary solution G, and Section 5.3 is devoted to estimates of the dual eigenfunction . Our results on the spectral gap of the growth-fragmentation operator are proved in Section 5.4, and we also include two appendices: one, Appendix 5.5.1, on different approximation procedures that may be used for G and , and which are more convenient in some of our arguments; and Appendix 5.5.2, which gives asymptotic estimates of some of the expressions involving the positive part $\mathcal{L}^+$ of the fragmentation operator, and are used for our large-x estimates of G.

5.1.1 Assumptions on the coefficients

For proving our results we need some or all of the following assumptions. First of all, we assume that the fragmentation coefficient b is of self-similar form, which is general enough to encompass most interesting examples and still allows us to obtain accurate results on the asymptotics of G and :

Hypothesis 1 (Self-similar fragmentation rate)**.** *The coefficient $b(x, y)$ is of the form*

$$b(x,y) = B(x)\frac{1}{x}\,p\Big(\frac{y}{x}\Big) \tag{5.5}$$

for some locally integrable $B : (0, +\infty) \to (0, +\infty)$, and some nonnegative finite measure p on $[0, 1]$ satisfying the mass preserving condition

$$\int_0^1 z\,p(z)\,dz = 1,$$

and also the condition

$$\int_0^1 p(z)\,dz \quad 1.$$

(When writing the integral of a measure it is always understood that the integration limits are included in the integral.)

The measure p gives the distribution of fragments obtained when a particle of a certain size breaks.

Remark 5.1. *Define, for $k \quad 0$, the moment*

$$_k := \int_0^1 z^k p(z)\,dz.$$

We have from Hypothesis 1 that $_1 = 1$ *and* $_0 \quad 1$. *Physically,* $_0$ *represents the mean quantity of fragments produced by the fragmentation of one particle. Because of the strict inequality* $_0 \quad _1$, *one can deduce that* p *is not concentrated at* $z = 1$ *(i.e.* $p \neq {}_0\delta_1$*). As a consequence we have that* $_k \quad 1$ *if* $k \quad 1$ *and* $_k \quad 1$ *if* $k \quad 1$.

Hypothesis 2. *The growth rate is a continuous and strictly positive function* $: (0, +\infty) \to (0, +\infty)$.

Our next assumption says that the growth rate and total fragmentation rate have a power-law behavior for large and small sizes:

Hypothesis 3 (Asymptotics of fragmentation and drift rates)**.** *Assume that for some constants* $B_0, B_\infty, {}_0, {}_\infty \quad 0$ *and* $_0, \ , \ _0, \quad \in \mathbb{R}$

$$B(x) \sim B_0 \, x^{\ 0} \quad \text{as } x \to 0, \tag{5.6}$$

$$B(x) \sim B_\infty \, x \quad \text{as } x \to +\infty, \tag{5.7}$$

$$(x) \sim {}_0 \, x^{\ 0} \quad \text{as } x \to 0, \tag{5.8}$$

$$(x) \sim {}_\infty \, x \quad \text{as } x \to +\infty. \tag{5.9}$$

We also impose the conditions

$$_0 - {}_0 + 1 \quad 0, \tag{5.10}$$

$$- \quad + 1 \quad 0, \tag{5.11}$$

to ensure the existence of a solution to the eigenproblem (see [55, 94]).

Likewise, we impose that the distribution of small fragments behave like a power law:

Hypothesis 4 (Behavior of p close to 0)**.** *There exist* $p_0 \quad 0$ *and* $\mu \quad 0$ *such that*

$$p(z) = p_0 \, z^{\ -1} + o(z^{\ -1}) \quad \text{as } z \to 0, \tag{5.12}$$

with the condition

$$\mu - {}_0 + 1 \quad 0. \tag{5.13}$$

Remark 5.2. *When* $p_0 \quad 0$ *condition (5.12) is the same as*

$$p(z) \sim p_0 \, z^{\ -1} \tag{5.14}$$

as $z \to 0$. *We prefer to write it as given in order to allow for* $p_0 = 0$, *which is usually not allowed in the notation (5.14). For instance, if* $p(z)$ *is equal to* 0 *in a neighborhood of* 0 *(such as for the mitosis case, see below), then (5.12) holds with* $p_0 = 0$, *but (5.14) does not make sense.*

To find the asymptotic behavior of the function G when $x \to \infty$ the following hypothesis will also be needed.

Hypothesis 5 (Asymptotics to second order)**.** *Assume that is a $\mathcal{C}^1$ function and that, for some δ 0 and −1,*

$$B(x) = B_\infty x \ + O(x^{\ -\ }) \quad \text{as } x \to +\infty, \tag{5.15}$$

$$(x) = \ _\infty x \ + O(x^{\ -\ }) \quad \text{as } x \to +\infty, \tag{5.16}$$

$$p(z) = p_1(1 - z) \ + O((z - 1)^{\ +\ }) \quad \text{as } z \to 1. \tag{5.17}$$

Finally, to prove the entropy-entropy dissipation inequality, we will need an additional restriction on the fragmentation coe cient. It essentially says that p is uniformly bounded below by some constant $\underline{p}$ 0, and that it behaves like a constant at the endpoints 0 and 1:

Hypothesis 6. *There exist positive constants $\underline{p}$, p_0, p_1 0 such that*

$$\forall\, z \in (0, 1), \quad p(z) \quad \underline{p} \qquad \textit{(in the sense of measures)},$$

$$p(z) \xrightarrow[z \to 0]{} p_0, \qquad p(z) \xrightarrow[z \to 1]{} p_1,$$

and $_0$ 2 *(which is nothing but condition* (5.13) *in the case* $\mu = 1$*).*

The reader may check that [55, Theorem 1], which gives existence and uniqueness of G, , satisfying (5.3) and (5.4) is applicable under Hypotheses 1 4. We assume at least these hypotheses throughout the chapter in order to ensure the existence of profiles.

Let us give some common examples of coe cients satisfying the above assumptions:

Power coe cients If we set

$$b(x, y) = 2x^{\ -1} \text{ for } x \quad y \quad 0, \qquad (x) = x \quad \text{for } x \quad 0,$$

then all our hypotheses are satisfied when − + 1 0 and 2. Observe that in this case $B(x) = x$ and $p(z) \equiv 1$, which satisfies Hypotheses 1, 4 with $\mu = 1$ and = 0, and also 6. Since (x) is a power, it satisfies Hypothesis 2. Hypotheses 3 and 5 are also satisfied.

Self-similar fragmentation The previous case with $(x) = x$ is referred to as the *self-similar fragmentation equation.* It is closely related to the fragmentation equation $\partial_t g_t = \mathcal{L}(g_t)$ (see [59, 30]).

Mitosis Cellular division by equal mitosis is modeled by a distribution of fragments p concentrated at a size equal to one half:

$$p(z) = 2\,\delta_{z=\frac{1}{2}}.$$

This measure p satisfies Hypothesis 4 with $p_0 = p_1 = 0$ (the value of μ, being irrelevant). In order to make the theory work, one has to choose B and such that the rest of Hypotheses are satisfied. For instance, $B(x) = x$ and $(x) = x$ with $- \quad + 1 \quad 0$ (and then defining $b(x, y)$ through (5.5)) are valid choices for the same reasons as before.

5.1.2 Summary of main results

Estimates on the pro les. We describe the asymptotics of the profile G and give accurate bounds on the eigenvector . Define

$$(x) := \int_1^x \frac{+ B(y)}{(y)} dy$$

and

$$:= \begin{cases} p_1 & \text{if} \quad 0 \text{ and} \quad = 0, \\ p_1 \frac{B_\infty}{+ B_\infty} & \text{if} \quad = \quad = 0, \\ 0 & \text{if} \quad 0 \text{ and} \quad 0, \text{ or} \quad 0 \text{ and} \quad -1 + \frac{+1-}{1-}, \end{cases}$$

where the parameters are the ones appearing in the previous hypotheses. In Section 5.2.2 we prove the following result, which improves previous estimates of the profile G given in [59, 30, 55]

Theorem 5.1. *Assume Hypotheses 1–5. There exists $C \quad 0$ such that*

$$G(x) \underset{x \to +\infty}{\sim} C e^{-\ (x)} x^{\ -\ } . \tag{5.18}$$

This result works for all the examples given before. For all of them, it shows that the profile G decays exponentially for large sizes, with a precise exponential rate given by (x). We observe that (x) behaves like $x^{\ +-\ +1}$ (with $_+ := \max\{\ , 0\}$), which is always a positive power of x. There are some observations about this that match intuition: the equilibrium profile decays faster when the total fragmentation rate is stronger for large sizes, and it decays slower when the growth rate is larger for large sizes. Also, it is interesting to notice that does not depend on the fragment distribution (this is, p), but only on the total fragmentation rate B.

The additional power x^{-} which gives a correction to the exponential behavior, in turn, depends only on the behavior of the distribution of fragments $p(z)$ close to $z = 1$, this is, on fragments of size close to the size of the particle that breaks. In the mitosis case, for example, $= 0$ since we obtain no fragments of similar size when a particle breaks.

The behavior of $G(x)$ for x close to 0 depends on the power $_0$ from Hypothesis 3 and the distribution of small fragments that result when a particle breaks. The following result is proven in Section 5.2.3:

Theorem 5.2. *Assume Hypotheses 1–4 with* p_0 0*. If* $_0$ 1*, there exists* C 0 *such that*

$$G(x) \underset{x \to 0}{\sim} C\, x^{\ -\ _0}.$$

If $_0$ 1*, there exists* C 0 *such that*

$$G(x) \underset{x \to 0}{\sim} C\, x^{\ -1}.$$

This shows that G is (roughly) more concentrated close to 0 the weaker the growth is for smaller sizes; and is less concentrated when there are fewer smaller fragments resulting from breakage. This result includes cases in which $G(x)$ blows up as $x \to 0$, cases in which it behaves like a constant, and cases in which it tends to 0 like a power. We recall that the boundary condition is $(x)G(x) \to 0$ as $x \to 0$, which is always ensured by μ 0 from Hypothesis 4.

For the profile we derive the following estimates, proved in section 5.3, by the use of a maximum principle (Lemma 5.2):

Theorem 5.3. *Assume Hypotheses 1–4. If* 0*, there are two positive constants* C_1 *and* C_2 *such that*

$$C_1\, x \quad (x) \quad C_2\, x, \qquad \forall x \quad 1. \tag{5.19}$$

If 0 *and under the additional assumption that* $\mu = 1$ *and* p_0 0 *in Hypothesis 4, there exist two positive constants* C_1 *and* C_2 *such that*

$$C_1\, x^{\ -1} \quad (x) \quad C_2\, x^{\ -1}, \qquad \forall x \quad 1. \tag{5.20}$$

Estimates of are significantly harder than those of G, and they have to be obtained through comparison arguments. To our knowledge, this is the first result in which can be bounded above and below by the same power (except for the cases in which can be found explicitly). This improves the results in [30] also in that it is valid for a general power-law behavior of .

We do not include the case $= 0$ in the above theorem (this is, $B(x)$ asymptotic to a constant as $x \to +\infty$), but we remark that in the case of $B(x)$ *equal* to a constant (and with the very mild condition that $b(x,y)\,dy$ is equal to a constant independent of x), then $\equiv 1$. The case $(x) = {}_0 x$ is also explicit: in that case, $= {}_0$ and $(x) = Cx$ for some number $C \quad 0$.

As for the behavior at zero, we prove the following result:

Theorem 5.4. *Assume Hypotheses 1–4. Then there exists a constant $C \quad 0$ such that*

$$(x) \underset{x\to 0}{\sim} Ce^{\ (x)}.$$

We remark that the behavior of (x) for small x is determined by whether $(B(x) + \ )/\ (x)$ is integrable close to $x = 0$. Since $B(x)/\ (x)$ is always integrable close to $x = 0$ by hypothesis (as ${}_0 - {}_0 \quad -1$), we deduce that:

1. If ${}_0 \quad 0$, then (x) tends to a positive constant as $x \to 0$.
2. If ${}_0 \quad 0$, then there are three possible cases:
   (a) If ${}_0 \quad 1$, then again (x) tends to a positive constant as $x \to 0$.
   (b) If ${}_0 = 1$, then (x) behaves like a positive power of x as $x \to 0$.
   (c) If ${}_0 \quad 1$, then (x) decays exponentially fast as $x \to 0$.

Spectral gap. The estimates of the previous theorems allow us to prove a spectral gap inequality. The *general relative entropy principle* [95, 96] applies here and we have

$$\frac{d}{dt}\int_0^\infty (x)G(x)H(u(x))\,dx = \int_0^\infty\int_y^\infty (y)b(x,y)G(x) \times (H(u(x)) - H(u(y)) + H'(u(x))(u(y) - u(x)))\,dx\,dy,$$

where H is any function and we denote

$$u(x) := \frac{g(x)}{G(x)} \quad (x \quad 0).$$

In the particular case of $H(x) := (x-1)^2$ we define

$$H[g|G] := \int_0^\infty G(u-1)^2\,dx \qquad (5.21)$$

$$D[g|G] := \int_0^\infty\int_x^\infty (x)G(y)b(y,x)(u(x) - u(y))^2\,dy\,dx, \qquad (5.22)$$

and obtain that

$$\frac{d}{dt}H[g|G] = -D[g|G] \quad 0.$$

The next result shows that H is in fact bounded by a constant times D:

Theorem 5.5. *Assume that the coefficients satisfy Hypotheses 1–6 with one of the following additional conditions on the exponents* $_0$ *and* $_0$*:*

either $_0 \quad 1$*,*

or $_0 = 1$ *and* $_0 \quad 1 + \ / \ _0$*,*

or $_0 \quad 1$ *and* $_0 \quad 2 - \ _0$*.*

Consider also that we are in the case $\neq 0$*. Then the following inequality holds*

$$H[g|G] \quad CD[g|G], \tag{5.23}$$

for some constant $C \quad 0$ *and for any nonnegative measurable function* $g : (0, \infty) \to \mathbb{R}$ *such that* $g = 1$*. Consequently, if* g_t *is a solution of problem* (5.1) *the speed of convergence to equilibrium is exponential in the* L^2*-weighted norm*
$\| \cdot \| = \| \cdot \|_{L^2(G^{-1} \ dx)}$*, i.e.,*

$$H[g_t|G] \quad H[g_0|G]\, e^{-Ct} \quad \textit{for } t \quad 0.$$

Remark that in general we do not know the value of the eigenvalue which appears in the assumption on $_0$ for the case $_0 = 1$. Nevertheless in the case of the self-similar fragmentation equation (*i.e.* $(x) \equiv \ _0 x$) we know by integration of equation (5.3a) multiplied by x that $= \ _0$ and the condition on $_0$ becomes $_0 \quad 2$. Thus Theorem 5.5 includes the result of the first part of [30, Theorem 1.9].

The main restrictions on the coe cients needed for Theorem 5.5 to hold are the following. First, we require Hypothesis 6, which says that the fragment distribution p should be bounded below. Consequently, this does not include the mitosis case and other cases in which the fragment distribution has gaps ; we refer to [83] for a proof that exponential decay does hold in that case, at least for a constant total fragmentation rate. Second, the exponent $_0$ cannot be too large in order to ensure that the term $b(x, y)$ which appears in the entropy dissipation is not too small and can be bounded below by our methods. (An exception to this is the case $_0 \quad 1$: in this case (x) decays exponentially fast as $x \to 0$, and this allows us to remove the upper bound on $_0$.) This restriction on $_0$ might be a shortcoming of the arguments we are using; we do not know if there is a spectral gap when it is removed.

On the other hand, it is remarkable that Theorem 5.5 does not place any restrictions on the behavior of the fragmentation or growth coe cients for large sizes. This is a significant improvement over [30], where the behavior at 0 and $+\infty$ of the coe cients was taken to be the same power of x, and results were restricted to the cases in which is constant or linear.

5.2 Estimates of the pro le G

5.2.1 Estimates of the moments of G

When Hypothesis 3 is satisfied, we define

$$:= \lim_{x\to+\infty} x^{\quad - \quad +} \frac{B(x)+}{(x)} = \frac{\mathbb{1}_{\leqslant 0} + B_\infty \mathbb{1}_{\geqslant 0}}{\infty} = \begin{cases} \frac{B_\infty}{\infty} & \text{if} \quad 0, \\ \frac{+B_\infty}{\infty} & \text{if} \quad = 0, \\ \frac{}{\infty} & \text{if} \quad 0, \end{cases} \tag{5.24}$$

where $_+ = \max\{0, \}$. Remark that, for 0, we have the relation

$$= p_1 \frac{B_\infty}{\infty} \,^{-1}. \tag{5.25}$$

Lemma 5.1. *Assume Hypotheses 1–4. For any $m \quad 1+$ it holds that*

$$\int_1^\infty G(x)\, e^{\ (x)}\, x^{\ -m}\, dx \quad +\infty.$$

Proof. As usual, we carry out a priori estimates which can be rigorously justified by an approximation procedure (such as the truncated equation (5.50)). As G is integrable, it is enough to prove the convergence of the above integral on $(x_0, +\infty)$ for a su ciently large $x_0 \quad 0$. Hence, take any $x_0 \quad 0$, multiply (5.3a) by $x^{1-m} e^{\ (x)}$ with $m \quad 1+$ and integrate on $(x_0, +\infty)$ to obtain

$$- G(x_0) e^{\ (x_0)}\ (x_0) x_0^{1-m} + (m-1) \int_{x_0}^\infty G(x) e^{\ (x)}\ (x) x^{-m}\, dx \\ = \int_{x_0}^\infty G(y) \int_{x_0}^y e^{\ (x)} x^{1-m} b(y,x)\, dx\, dy \tag{5.26}$$

where we have done an integration by parts on the last term.

We first consider the case 0 (this is, 0 and = 0). From Equation (5.9) we have that for any 0 there exists x_0 0 such that

$$(m-1) \quad \int_{x_0}^{\infty} G(x)e^{\ (x)} \ (x)x^{-m}\, dx$$

$$(m-1)(1- \ ) \ _\infty \int_{x_0}^{\infty} G(x)e^{\ (x)}x^{\ -m}\, dx, \quad (5.27)$$

and, applying Lemma 5.8, also such that

$$\int_{x_0}^{\infty} G(y) \int_{x_0}^{y} e^{\ (x)}x^{1-m}b(y,x)\, dx\, dy$$

$$(1+ \ )B_\infty p_1 \ ^{-1} \int_{x_0}^{\infty} G(y)e^{\ (y)}y^{\ -m}\, dy \quad (5.28)$$

(observe that we have used 0 and = 0 here). Using (5.27) and (5.28) we obtain from (5.26) that

$$\Big((m-1)(1- \ ) \ _\infty - (1+ \ )B_\infty p_1 \ ^{-1}\Big) \int_{x_0}^{\infty} G(x)e^{\ (x)}x^{\ -m}\, dx$$

$$G(x_0)e^{\ (x_0)} \ (x_0)x_0^{1-m}.$$

When $(m-1)(1- \ ) \ _\infty - (1+ \ )B_\infty p_1 \ ^{-1}$ 0 this gives a bound for the integral on the left hand side. If m $1+$ we can always choose small enough for this to be true, because of relation (5.25), and it proves the result.

The remaining case is = 0, this is, $-1 + \frac{+1-}{+ +1-}$. In this case we have to substitute (5.28) by the following, according to Lemma 5.8:

$$\int_{x_0}^{\infty} G(y) \int_{x_0}^{y} e^{\ (x)}x^{1-m}b(y,x)\, dx\, dy$$

$$(1+ \ )B_\infty p_1 \ ^{-1-} \ (1+ \ ) \int_{x_0}^{\infty} G(y)e^{\ (y)}y^{1-m+ \ -(\ + - \ +1)(1+ \ )}\, dy. \quad (5.29)$$

Since $-1 + \frac{+1-}{+ +1-}$, we have

$$1-m+ \ -(\ _+ - \ +1)(1+ \ ) \quad -m+ \ .$$

Thus the exponent of y on the right hand side of (5.29) is strictly smaller than $-m$, so we can always find x_0 large enough so that

$$\int_{x_0}^{\infty} G(y) \int_{x_0}^{y} e^{\ (x)}x^{1-m}b(y,x)\, dx\, dy \qquad \int_{x_0}^{\infty} G(y)e^{\ (y)}y^{\ -m}\, dy.$$

Using this and (5.27) in (5.26) we may follow a similar reasoning as before to obtain the result. □

5.2.2 Asymptotic estimates of G as $x \to +\infty$

In this section we prove Theorem 5.1 by using the moment estimates in Section 5.2.1.

Proof of Theorem 5.1. We divide the proof in two steps:

Step 1: proof that the limit is nite. Again, we carry out a priori estimates on the solution which can be fully justified by using the approximation (5.50). Let us first prove that $x^{\ -}\ G(x)e^{\ (x)}$ has a finite limit $C \quad 0$ as $x \to +\infty$, and later we will show that $C \quad 0$. We use equation (5.3a) to obtain

$$(x^{-}\ \ (x)G(x)e^{\ (x)})' = -\ \ x^{-\ -1}\ \ (x)G(x)e^{\ (x)} \\ + x^{-}\ e^{\ (x)} \int_x^\infty b(y,x)G(y)\,dy.$$

Let us show that the right hand side of this last expression is integrable on $(x_0, +\infty)$ for some $x_0 \quad 0$. Once we have this the result is proved, since then $x^{\ -}\ G(x)e^{\ (x)}$ must have a limit as $x \to +\infty$. Integrating the right hand side we obtain:

$$-\int_{x_0}^\infty x^{-\ -1}\ \ (x)G(x)e^{\ (x)}\,dx + \int_{x_0}^\infty x^{-}\ e^{\ (x)} \int_x^\infty b(y,x)G(y)\,dy\,dx \\ = \int_{x_0}^\infty G(x)\left(\int_{x_0}^x y^{-}\ b(x,y)e^{\ (y)}\,dy - \ \ x^{-\ -1}\ \ (x)e^{\ (x)}\right)dx. \quad (5.30)$$

We just need to show that the parenthesis is of the order of $e^{\ (x)}x^{\ -\ -1-}$ for some $\quad 0$, and then Lemma 5.1 shows that the above integral is finite.

The case $\quad 0$. Let us start considering the case $\quad 0$ (this is, $\quad 0$ and $= 0$). Using Lemma 5.8

$$\int_{x_0}^x y^{-}\ b(x,y)e^{\ (y)}\,dy = p_1 B_\infty\ ^{-1}\,x^{\ -\ -1}e^{\ (x)} + O(x^{-\ +\ -1-}\ )e^{\ (x)}, \quad (5.31)$$

for some $\quad 0$. From (5.16) we also have

$$x^{-\ -1}\ \ (x)e^{\ (x)} = \ _\infty\ \ x^{\ -\ -1}e^{\ (x)} + O(x^{\ -\ -1-}\ )e^{\ (x)}. \quad (5.32)$$

Using (5.31)-(5.32) and the relation (5.25), the parenthesis in (5.30) is, in absolute value, less than $Cx^{\ -\ -1-}\ e^{\ (x)}$ for some constant $C \quad 0$. Hence by Lemma 5.1 the integral in (5.30) is finite, and we conclude that $x^{\ -}\ G(x)e^{\ (x)}$ has a finite limit as $x \to +\infty$ when $\quad 0$.

The case $= 0$. In this case we have from Lemma 5.8

$$\int_{x_0}^{x} b(x,y)e^{\ (y)}\,dy \sim p_1 B_\infty^{\ -1-} \ (1+\ )x^{\ -(\ +-\ +1)(1+\ )}e^{\ (x)}.$$

Using the same reasoning as at the end of the proof of Lemma 5.1 we have that, when $= 0$,

$$-(\ _+ -\ +1)(1+\ ) \qquad -1,$$

which then shows that the right hand side of (5.30) is finite due to Lemma 5.1.

Step 2: proof that C 0. In order to show that C 0 in (5.18) set $F(x) := \ (x)G(x)e^{\ (x)}$ and obtain the following from (5.3a):

$$F'(x) = e^{\ (x)} \int_x^\infty b(y,x)G(y)\,dy. \tag{5.33}$$

In particular, F is nondecreasing, and this is enough to conclude in the case $= 0$ (since then $(x)G(x)e^{\ (x)}$ must converge to a positive quantity, so the same must be true of $x\ \ G(x)e^{\ (x)}$). In the case $\ \ 0$ we may bound

$$F'(x) = e^{\ (x)} \int_x^\infty b(y,x)\frac{1}{\ (y)}e^{-\ (y)}F(y)\,dy$$
$$F(x)e^{\ (x)} \int_x^\infty b(y,x)\frac{1}{\ (y)}e^{-\ (y)}\,dy,$$

which implies that

$$F(x) \qquad F(x_0)\exp\left(\int_{x_0}^{x} S(w)\mathrm{w},\right)$$

with

$$S(w) := e^{\ (w)} \int_w^\infty b(y,w)\frac{1}{\ (y)}e^{-\ (y)}\,dy.$$

Due to equation (5.16) we have

$$\frac{1}{\ (x)} = \frac{1}{\ _\infty x} + R_1(x),$$

with $R_1(x) = O(x^{- \; -})$. Using this, and due to Lemma 5.8,

$$\int_{x_0}^{x} S(w)\,\mathrm{w} \quad \int_{x_0}^{x} e^{\;(w)} \int_{w}^{x} b(y,w)\frac{1}{(y)}e^{-\;(y)}\,dy\,\mathrm{w}$$
$$= \int_{x_0}^{x} \frac{1}{(y)}e^{-\;(y)} \int_{x_0}^{y} e^{\;(w)} b(y,w)\,\mathrm{w}\,\mathrm{y}$$
$$= p_1 B_\infty^{\;\;-1} \int_{x_0}^{x} \frac{1}{(y)}(y^{\;-1} + R_2(y))\,\mathrm{y}$$
$$= \int_{x_0}^{x} \frac{1}{y}\mathrm{y} + \int_{x_0}^{x} R_3(y)\,\mathrm{y} \qquad \log(y) + C_1,$$

with $R_2(y) = O(y^{\;-1-})$, $R_3(y) = O(y^{-1-})$, and $C_1 \in \mathbb{R}$ some real number. As a consequence,

$$F(x) \quad F(x_0)x \; e^{C_1},$$

which shows that $\lim_{x\to+\infty} F(x)x^{-}$ (which we know exists) must be strictly positive. This finishes the proof. □

5.2.3 Asymptotic estimates of G as $x \to 0$

Proof of Theorem 5.2. Define

$$F(x) := \;(x)G(x)e^{\;(x)}.$$

We know from [55] that $F(x) \to 0$ when $x \to 0$ and more precisely that $F(x) \quad C\,x$. The derivative of F, as noted in (5.33), is

$$F'(x) = e^{\;(x)} \int_{x}^{\infty} b(y,x)G(y)\,dy \quad 0$$

so F is increasing.

Case $_0$ 1 In this case, $(x) \to \;(0) \quad 0$. Choose 0 such that p is a function on $[0, \;)$ (the fact that this can be done for small enough is implicit in Hypothesis 4), and call $p_* = p\,\mathbb{1}_{[0,\;]}$. Then, from Hypothesis 4,

$$x^{1-}\; p_*\left(\frac{x}{y}\right) \to p_0\, y^{1-} \qquad \text{as } x \to 0, \tag{5.34}$$

with the above convergence being pointwise in y. We may additionally choose $\in (0,1)$ and $C \quad 0$ such that

$$p(z) \quad Cz^{\;-1} \qquad \text{for all } z \in (0, \;). \tag{5.35}$$

Now we write

$$
\begin{aligned}
x^{1-} & {}_{x}^{\infty} b(y,x)G(y)\,dy \\
&= x^{1-} {}_{x}^{\frac{x}{}} \frac{B(y)}{y}G(y)p\Big(\frac{x}{y}\Big)\,dy + x^{1-} {}_{\frac{x}{}}^{\infty} \frac{B(y)}{y}G(y)p\Big(\frac{x}{y}\Big)\,dy \\
&= x^{1-} {}^{1} B\Big(\frac{x}{z}\Big)G\Big(\frac{x}{z}\Big)p(z)\frac{dz}{z} + x^{1-} {}_{\frac{x}{}}^{\infty} \frac{B(y)}{y}G(y)p_*\Big(\frac{x}{y}\Big)\,dy.
\end{aligned}
$$

For the first term in the r.h.s., we use that $B(y) \underset{y\to 0}{\sim} B_0 y^{\ 0}$ and $G(y) \quad Cy^{\ -\ 0}$ (see [55]) to write

$$
x^{1-} {}^{1} B\Big(\frac{x}{z}\Big)G\Big(\frac{x}{z}\Big)p(z)\frac{dz}{z} \quad Cx^{\ 0+1-\ 0} {}^{1} z^{\ 0-\ -\ 0-1}p(z)\,dz
$$

and conclude that it tends to zero when $x \to 0$ since $_0 + 1 - \ _0 \quad 0$. For the second term, we use (5.34) and (5.35) to obtain by dominated convergence

$$
x^{1-} {}_{\frac{x}{}}^{\infty} \frac{B(y)}{y}G(y)p_*\Big(\frac{x}{y}\Big)\,dy \xrightarrow[x\to 0]{} p_0 {}_{0}^{\infty} B(y)y^{-}\ G(y)\,dy.
$$

This limit is strictly positive and finite, since $G(y) \quad Cy^{\ -\ 0}$ and $_0 - \ _0$ -1. Finally, we have deduced that there is a positive constant $C \quad 0$ such that

$$
F'(x) \underset{x\to 0}{\sim} C\,x^{\ -1},
$$

which by integration gives

$$
(x)G(x) \sim C\,x
$$

and so

$$
G(x) \underset{x\to 0}{\sim} C\,\frac{x}{(x)} \underset{x\to 0}{\sim} C\,x^{\ -\ 0}.
$$

Case $_0 \quad 1$ In this case we necessarily have $\quad 0$ and

$$
(x) \sim -C\,x^{1-\ 0}.
$$

As a consequence, following a similar reasoning as for the previous case, we have

$$
F'(x) \sim C_1\,x^{\ -1}e^{-C_2\,x^{1-\ 0}}
$$

and consequently

$$
F(x) \sim C_1 {}_{0}^{x} y^{\ -1}e^{-C_2\,y^{1-\ 0}}dy \sim C_3\,x^{\ 0+\ -1}e^{-C_2\,x^{1-\ 0}}
$$

due to l Hôpital s rule. This finally gives $G(x) \underset{x\to 0}{\sim} C\,x^{\ -1}$. □

5.3 Estimates of the dual eigenfunction

5.3.1 Asymptotic estimates of as $x \to 0$

We first give the proof of Theorem 5.4, which is rather direct:

Proof of Theorem 5.4. Define

$$(x) := \quad (x)e^{- \quad (x)}.$$

This function is decreasing since it satisfies

$$'(x) = -\frac{1}{(x)} \int_0^x b(x,y) \quad (y)\, dy\, e^{- \quad (x)} \qquad 0.$$

Moreover it is a positive function, so to prove Theorem 5.4 we only have to prove that is bounded at $x = 0$. Consider, for $\eta \quad 0$, as defined in the approximation procedure (see (5.49) in Appendix 5.5.1). Then denote by , and the corresponding functions. First we know from [55] that converges locally uniformly to when $\eta \to 0$. We have, for $\eta \quad 0$, that $- \quad (x) = \int_x^1 \frac{+B(y)}{(y)} dy$ is bounded at $x = 0$ and this is why it is useful to consider this regularization. We have for any $x_0 \quad 0$,

$$\begin{aligned}
\sup_{\mathbb{R}^+} \quad = \quad (0) = \quad & (x_0) + \int_0^{x_0} \frac{1}{(y)} \int_0^y b(y,z) \quad (z)\, dz\, e^{- \quad (y)}\, dy \\
& (x_0) + \int_0^{x_0} \frac{1}{(y)} \int_0^y b(y,z) \quad (z)\, e^{- \quad (z)}\, dz\, dy \\
& (x_0) + \sup \quad \int_0^{x_0} \frac{1}{(y)} \int_0^y b(y,z)\, dz\, dy \\
= \quad & (x_0) + \sup \quad \int_0^{x_0} \frac{B(y)}{(y)} \int_0^y p\left(\frac{z}{y}\right) \frac{dz}{y}\, dy \\
= \quad & (x_0) + \ _0 \sup \quad \int_0^{x_0} \frac{B(y)}{(y)}\, dy.
\end{aligned}$$

Now, because $\frac{B}{}$ is integrable at $x = 0$, we can choose $x_0 \quad 0$ such that $_0 \int_0^{x_0} \frac{B(y)}{(y)} dy$ $= \quad 1$ and we obtain

$$(1 - \quad) \sup \quad (x) \qquad (x_0) \xrightarrow[\to 0]{} \quad (x_0).$$

So is uniformly bounded when $\eta \to 0$ and thus the limit (x) is bounded.

□

5.3.2 A maximum principle

For finding the bounds on the dual eigenfunction at $x \to +\infty$ we use comparison arguments, valid for each truncated problem on $[0, L]$ (see Appendix 5.5.1 for details on the truncation). Then we pass to the limit, as the bounds we obtain are independent of L. The function $\phi_L(x)$ satisfies the equation

$$\mathcal{S}\phi_L(x) = 0 \quad (x \in (0, L)),$$

where $\mathcal{S}$ is the operator given by

$$\mathcal{S}w(x) := -\tau(x)w'(x) + (B(x) + \lambda_L)w(x) - \int_0^x b(x, y)w(y)dy,$$

defined for all functions $w \in W^{1,\infty}(0, L)$ and for $x \in (0, L)$. This operator satisfies

$$\forall w \in W^{1,\infty}(0, L) \ s.t. \ w(L) = 0, \qquad \int_0^L \mathcal{S}w(x)\, G_L(x)\, dx = 0 \tag{5.36}$$

where G_L is the eigenfunction of the truncated growth-fragmentation operator. We recall the concept of *supersolution*:

Definition 5.1. *We say that $w \in W^{1,\infty}(0, L)$ is a* supersolution *of $\mathcal{S}$ on the interval $I \sim (0, L)$ when*

$$\mathcal{S}w(x) \geq 0 \ (x \in I).$$

Maximum principles were a powerful tool for proving the existence of sub and supersolutions for the growth-fragmentation models as in [94, 55]. For our case, we recall the maximum principle given in [55].

Lemma 5.2 (Maximum principle for $\mathcal{S}$)**.** *Assume Hypotheses 1-3. There exists $A \geq 0$, independent of L, such that if w is a supersolution of $\mathcal{S}$ on (A, L), $w \geq 0$ on $[0, A]$ and $w(L) \geq 0$ then $w \geq 0$ on $[A, L]$.*

Proof. We start from the fact w is a supersolution on (A, L)

$$-\tau(x)w'(x) + (B(x) + \lambda_L)w(x) - \int_0^x b(x, y)w(y)\, dy =: f(x) \geq 0.$$

Testing this equation against $\mathbb{1}_{w \leqslant 0}$ we obtain on (A, L)

$$-\tau(x)w_-'(x) + (B(x) + \lambda_L)w_-(x) = \mathbb{1}_{w(x)<0} \int_0^x b(x, y)w(y)\, dy + f(x)\mathbb{1}_{w(x)\leqslant 0}$$
$$\leq \int_0^x b(x, y)w_-(y)\, dy + f(x)\mathbb{1}_{w(x)\leqslant 0}.$$

Extend f by zero on $[0, A]$. Since $w_-(x) = 0$ on $[0, A]$ by assumption, the latter inequality holds true on $(0, L)$ and it writes

$$\forall x \in (0, L), \quad \mathcal{S}w_-(x) \quad f(x)\mathbb{1}_{w(x)\leqslant 0}.$$

Testing this last inequality against G_L, we obtain using (5.36)

$$0 \quad \int_0^L f(x)\mathbb{1}_{w(x)\leqslant 0} G_L(x)\, dx = \int_A^L f(x)\mathbb{1}_{w(x)\leqslant 0} G_L(x)\, dx.$$

Because f and G_L are positive on (A, L), this is possible only if $\mathbb{1}_{w\leqslant 0} \equiv 0$ on (A, L) and it ends the proof. □

5.3.3 Asymptotic estimates of as $x \to +\infty$

Now we prove the results concerning the asymptotic behavior of (x) when $x \to +\infty$, Theorem 5.3. For these results, we still assume that Hypotheses 1-4 are satisfied and, in the case 0, we additionally assume that $\mu = 1$ and p_0 0 (so that $p(z) \xrightarrow[z\to 0]{} p_0$ 0). We recall that Hypothesis 3 says that $B(x)$ behaves like a -power of x and (x) like an -power of x, with $+ 1 -$ 0.

Proof of Theorem 5.3. The proof is done in two cases, and each case is proved in two steps. In the first step we give particular supersolutions and prove the upper bound, and in the second one we do the corresponding for lower bounds.

Case 1: 0.

Step 1: Upper bounds. We claim that for any C 0, there exists A 0 and L_* 0 such that

$$v(x) := Cx + 1 - x^k$$

is a supersolution on $[A, L]$ for any L L_*, provided that max(0, $-$) k 1. First we recall that $+ 1 -$ 0 by assumption, so $-$ 1 and we can find $k \in (- , 1)$. Then

$$\mathcal{S}v(x) = - (x)(C - kx^{k-1}) + (Cx - x^k + 1) + (_k - 1)B(x)x^k - (_0 - 1)B(x)$$

and the right hand side is positive for x large enough because the dominant term is $C x + (_k - 1)B(x)x^k \sim C x + (_k - 1)B_\infty x^{ +k}$. Indeed $_k$ 1 because k 1 (see Remark 5.1) and the dominant power is $+ k$ because k 0 and $+ k$.

Now we prove that there exists $C \quad 0$ such that

$$\forall x \quad 0, \qquad (x) \quad C(1+x).$$

First we can choose C such that $v(x) = Cx + 1 - x^k$ is bounded below by a positive constant. Moreover we take an approximation $_L$ of such that $_L(L) = 0$. Then, choosing $K \quad 0$ large enough, we have that $Kv(x) \qquad (x)$ on $[0, A]$ because is bounded uniformly in L on $[0, A]$, and $Kv(L) = KCL + K - KL^k \quad 0$ for L large enough. So, using the maximum principle and the previous lemma, we obtain that

$$\forall x \quad 0, \qquad (x) \quad Kv(x) \quad C(1+x).$$

Step 2: Lower bounds. For the lower bounds we first prove that $v(x) := x + x^k - 1$ is a subsolution for $\max(0, 1- \ ) \quad k \quad 1$.
The idea is to use x^k to transform x which is a supersolution into a subsolution.

$$\mathcal{S}v(x) = - \ (x)(1 + kx^{k-1}) + \ (x + x^k - 1) - (\ _k - 1)B(x)x^k + (\ _0 - 1)B(x)$$

where $_k \quad 1$ since $k \quad 1$. Due to Assumption (5.7), $B(x)x^k \sim B_\infty x^{\ +k}$ and $v(x)$ is a subsolution for x large because $k \quad 0$ and $\ + k \quad 1$.

For $\quad 0$, there exists $C \quad 0$ such that

$$\forall x \quad 0, \qquad (x) \quad C(x-1)_+.$$

We know that is positive, so for C small enough, $C(x + x^k - 1) - \ (x) \quad 0$ on $[0, A]$. Moreover, taking an approximation $_L$ of such that $_L(L) = L$, we have $Cv(L) - \ (L) \quad 0$ for $C \quad 1$ and L large enough. Finally we use the lemmas on the maximum principle and the subsolution to conclude that there exists $C \quad 0$ such that

$$\forall x \quad 0, \qquad (x) \quad C(x + x^k - 1)$$

and the result follows.

Case 2: $\quad 0$.

Step 1: Upper bounds. We start by proving that for any $\eta \quad \left(\frac{-}{B_\infty p_0}\right)^{\frac{1}{}}$, $v(x) = (\eta + x)^{\ -1}$ is a supersolution. We compute

$$\mathcal{S}v(x) = (1 - \ ) \ (x)(\eta + x)^{\ -2} + (\ + B(x))(\eta + x)^{\ -1} - \int_0^x b(x,y)(\eta + y)^{\ -1}\,dy$$

and to estimate the last term in the r.h.s. we proceed similarly as in the proof of Theorem 5.2. We write, for $\in (0,1)$,

$$\int_0^x b(x,y)(\eta+y)^{\;-1}\,dy = \frac{B(x)}{x}\int_0^x (\eta+y)^{\;-1}p\Big(\frac{y}{x}\Big)\,dy$$
$$+\,B(x)\int^1 (\eta+zx)^{\;-1}p(z)\,dz.$$

Then, choosing such that (5.35) is satisfied (for this we use Hypothesis 4), we obtain by dominated convergence from (5.34) that

$$\frac{B(x)}{x}\int_0^x (\eta+y)^{\;-1}p\Big(\frac{y}{x}\Big)\,dy \underset{x\to+\infty}{\sim} \frac{B(x)}{x}\frac{p_0\eta}{-}.$$

On the other hand we have

$$B(x)\int^1 (\eta+zx)^{\;-1}p(z)\,dz \underset{x\to+\infty}{\sim} x^{\;-1}B(x)\int^1 z^{\;-1}p(z)\,dz.$$

Since 0, we obtain

$$\int_0^x b(x,y)(\eta+y)^{\;-1}\,dy \underset{x\to+\infty}{\sim} \frac{B_\infty p_0\eta}{-}x^{\;-1}$$

and finally

$$\mathcal{S}v(x) \underset{x\to+\infty}{\sim} \Big(\;-\frac{B_\infty p_0\eta}{-}\Big)x^{\;-1}$$

because $(x) \sim {}_\infty x$ and -1 0. So $v(x)$ is a supersolution for x large when η $\big(\frac{-}{B_\infty p_0}\big)^{\frac{1}{}}$.

Now, we claim that there exist C 0 and 0 such that

$$\forall\, x \quad 0, \qquad (x) \quad C(\eta+x)^{\;-1}.$$

The proof of this fact follows from the maximum principle and taking the an approximation $_L$ of such that $_L(L)=0$ and that $v(x)$ is a supersolution.

Step 2: Lower bounds. For the lower bounds we define

$$v(x) := \begin{cases} 0 & \text{for } 0 \quad x \quad , \\ (x-\;)x^{\;-2} & \text{for } x \quad . \end{cases}$$

Then for $\quad \left(\frac{(\ -1)}{B_\infty p_0}\right)^{\frac{1}{\ }}$, v is a subsolution. Indeed we have for x

$$\mathcal{S}v(x) = \ (x)(x^{\ -2} + (\ -2)(x- \ )x^{\ -3}) + (\ + B(x))(x- \ )x^{\ -2}$$
$$- \int^{x} b(x,y)(y- \ )y^{\ -2}\,dy$$

and, reasoning as in Step 1, we obtain that

$$\mathcal{S}v(x) \underset{x\to+\infty}{\sim} (\ - B_\infty p_0 C\ )x^{\ -1}.$$

Finally, there exist $C \quad 0$ and $\quad 0$ such that

$$\forall\, x \quad 0, \qquad (x) \quad Cx^{\ -2}(x- \ )_+.$$

Again, choosing an approximation $_L$ of such that $_L(L) = L$, the proof uses the maximum principle and the fact that $v(x)$ is a subsolution. □

5.4 Entropy dissipation inequality

As it was seen in [95, 96, 83, 30] the *general relative entropy principle* applies to solutions of (5.1). We remind that we use the entropy $H[g|G]$ defined in (5.21), with dissipation $D[g|G]$ given by (5.22). We recall that

$$\frac{d}{dt}H[g|G] = -D[g|G] \quad 0.$$

For the proof of the entropy inequality we will use [30, Lemma 2.2] with $(x) \equiv 1$. We need to check its hypotheses.

Lemma 5.3. *Assume that Hypotheses 1-3 and 6 are satisfied with* $\neq 0$. *Given* $M \quad 1$ *there exists* $K \quad 0$ *and* $R \quad 1$ *such that the profiles and* G *satisfy the relations*

$$0 \quad G(x) \quad K \quad (x \quad 0), \tag{5.37}$$

$$\int_{Rx}^{\infty} G(y)\ (y)\,dy \quad KG(x) \quad (x \quad M), \tag{5.38}$$

$$(y) \quad K\ (z) \quad (\max\{2RM, Rz\} \quad y \quad 2Rz). \tag{5.39}$$

Proof. The bound (5.37) on G is true because of Theorem 5.1 and Theorem 5.2 with $\mu = 1$. For the second bound, we have due to l Hôpital s rule and using Theorem 5.1

$$\int_{Rx}^{\infty} G(y)\ (y)\,dy \quad K \int_{Rx}^{\infty} y^{1+\ -\ } e^{-\ (y)}\,dy \sim Kx^{1+\ -\ } e^{-\ (Rx)}$$
$$Kx^{\ -\ } e^{-\ (x)} \sim KG(x).$$

Finally, (5.39) is a consequence of Theorem 5.3. □

Moreover, for proving the entropy-entropy dissipation inequality, we will need the following bounds, similar to those required in [30, Theorem 2.4].

Lemma 5.4. *Suppose that the coefficients satisfy Hypotheses* 1-3 *and 6 with one of the following additional conditions on the exponents* $_0$ *and* $_0$*:*

either $_0 \quad 1$,

or $_0 = 1$ *and* $_0 \quad 1 + \; / \; _0$,

or $_0 \quad 1$ *and* $_0 \quad 2 - \; _0$.

Let G *and* *be the stationary profiles for the problems* (5.3) *and* (5.4)*. Then we can choose constants* $K, M \quad 0$ *and* $R \quad 1$ *such that the profiles* *and* G *satisfy*

If $\quad 0$ *then*

$$G(x) \; (y) \quad Kb(y,x) \qquad (0 \quad x \quad y \quad \max\{2Rx, 2RM\}), \tag{5.40}$$

$$y^{-1} \quad Kb(y,x) \qquad (y \quad M, \; y \quad x \quad 0). \tag{5.41}$$

If $\quad 0$ *then*

$$G(x) \; (y) \quad Kb(y,x) \qquad (0 \quad x \quad y). \tag{5.42}$$

Proof. **Step 1:** $\quad 0$**,** $y \quad 2RM$ **and** $x \quad y$. We need to estimate $G(x) \; (y)$ at the limit $x \quad y \to 0$. Assume for the moment that $_0 \quad 1$. Using Theorem 5.2 ($G(x) \sim Cx^{1- \; _0}$, notice that due to Hypothesis 6 one has $\mu = 1$) and Theorem 5.4 to bound $G(x)$ and (y) respectively, we have

$$G(x) \; (y) \quad Cx^{1- \; _0} e^{\; (y)} \quad C' y^{1- \; _0}$$

since $_0 \quad 1$. Then under the condition $_0 \quad 2 - \; _0$ and from Hypothesis 6, we get

$$G(x) \; (y) \quad Cy^{\; _0 - 1} \quad Kb(y,x). \tag{5.43}$$

When $_0 = 1$, we can do better since in this case we have necessarily $_0 \quad 0$ and

$$(y) \underset{y \to 0}{\sim} \frac{}{_0} \ln(y).$$

Thus we can write

$$G(x) \; (y) \quad Cy^{\; / \; _0}$$

and we obtain the bound $G(x) \; (y) \quad Kb(y,x)$ from Hypothesis 6 as soon as $_0 \quad 1 - \frac{}{_0}$.

When $_0$ 1 we know that (y) decays exponentially as $y \to 0$, and one easily sees that the bound (5.43) holds without any restriction on $_0$.

Step 2: 0 and $2RM$ y $2Rx$. We need to estimate $G(x)$ (y) at the limit $2Rx$ $y \to +\infty$. Using (5.19) and (5.18) we have

$$\begin{aligned} G(x) \; (y) \quad & C(1+y)x^{\;-} \; e^{-\;(x)} \\ & C'(1+y)y^{\;-} \; e^{-\;(y/2R)} \\ & C''y^{\;-1} \end{aligned}$$

where C'' depends on , and R. We conclude by using Hypothesis 6.

Step 3: 0, y M **and** y x 0. Since $\mu = 1$ by assumption, we know from Theorems 5.2 and 5.1 that $G(x)$ is bounded. When 0, we observe first that y^{-1} $Cy^{\;-1}$ and we conclude that (5.41) holds true by using Hypothesis 6.

Step 4: 0. When 0 we have from Theorems 5.3, 5.4 and Hypothesis 6 that (y) $Cy^{\;-1}$ $Kb(y,x)$ for all 0 x y. □

At this point, we have all the tools to prove the entropy - entropy dissipation inequality.

Proof of Theorem 5.5. From [30, Lemma 2.1] one can rewrite the entropy as follows

$$D_2[g|G] := \int_0^\infty \int_x^\infty (x)G(x) \; (y)G(y)(u(x)-u(y))^2 \, dy \, dx = H[g|G]. \quad (5.44)$$

If one looks at the integrand, one realizes that D and D_2 have both (x) and $G(y)$ as a common terms. So we would like to compare and check that

$$G(x) \; (y) \quad Kb(y,x). \quad (5.45)$$

We will denote by C any constant depending on G, , K, M, or R, but not on g. We now distinguish two cases.

Case 0. The relation (5.45) is satisfied due to (5.42). So we can compare pointwise the integrands of $D_2[g|G]$ with $D[g|G]$ and the inequality (5.23) holds.

Case 0. For proving the case 0 we follow the same argument as in [30, Theorem 2.4]. We start by rewriting $D_2[g|G]$ as follows:

$$D_2[g|G] = D_{2,1}[g|G] + D_{2,2}[g|G],$$

where

$$D_{2,i} := \int\int_{A_i} (x)G(x) \; (y)G(y)(u(x) - u(y))^2 \, dy \, dx$$

with $A_1 := \{(x, y) \in \mathbb{R}^2_+ \, : \, y \quad x \, , \, y \quad RM$ or $y \quad Rx\}$ and $A_2 = A_1^c$. For the first term and thanks to (5.40) one has

$$\begin{aligned} D_{2,1}[g|G] & \quad \int_0^\infty \int_x^{\max\{2Rx, 2RM\}} Kb(y, x) \; (x)G(y)(u(x) - u(y))^2 \, dy \, dx \\ & \quad KD[g|G]. \end{aligned} \tag{5.46}$$

For the other term, what we have is

$$\begin{aligned} D_{2,2}[g|G] & \quad C \int_0^\infty \int_{\max\{x,M\}}^\infty y^{-1} \; (x)G(y)(u(x) - u(y))^2 \, dy \, dx \\ & \quad C\, K \int_0^\infty \int_{\max\{x,M\}}^\infty b(y, x) \; (x)G(y)(u(x) - u(y))^2 \, dy \, dx \\ & \quad C\, K\, D[g|G], \end{aligned} \tag{5.47}$$

where in the first inequality we applied [30, Lemma 2.2] with the bounds given in Lemma 5.4 and for the second one we used (5.41). The proof concludes by gathering (5.46) and (5.47). □

5.5 Appendix

5.5.1 Approximation procedures

To prove the estimates on the dual eigenfunction , we use a truncated problem. More precisely, we use alternatively one of the following ones, which di er only in their boundary condition

$$\begin{cases} - \; (x)\partial_x \; _L(x) + (B(x) + \; _L) \; _L(x) = \mathcal{L}^*_+(\; _L)(x) \quad \text{for } x \in (0, L), \\ _L(L) = 0 \quad \text{or} \quad _L(L) = \delta \quad 0 \quad \text{or} \quad _L(L) = \delta L, \\ _L \quad 0, \qquad \int_0^L G(x) \; _L(x) \, dx = 1. \end{cases} \tag{5.48}$$

The following lemma ensures that these truncations converge to the accurate limit when $L \to +\infty$.

Lemma 5.5. *There exists L_0 0 such that for each L L_0 the problem (5.48) has a unique solution ($_L$, $_L$) with $_L$ 0 and $_L \in W^{1,\infty}_{loc}(\mathbb{R}_+)$. Moreover we have*

$$_L \xrightarrow[L\to+\infty]{} \quad ,$$

$$\forall A \quad 0, \qquad _L \xrightarrow[L\to+\infty]{} \qquad \textit{uniformly on } [0, A).$$

Proof. We start with the case $_L(L) = 0$ by following the method in [55]. Define for η 0

$$(x) := \begin{cases} \eta & \text{for } 0 \quad x \quad \eta, \\ \quad (x) & \text{for } x \quad \eta. \end{cases} \tag{5.49}$$

Then consider for 0 and L 0 the truncated (and regularized) eigenvalue problem on $[0, L]$

$$\begin{cases} \dfrac{\partial}{\partial x}(\quad (x)G_L(x)) + (B(x) + \quad _L)G_L(x) = \displaystyle\int_0^L b(y,x)G_L(y)\,dy, \\ G_L(0) = \int_0^L G_L(y)\,dy, \qquad G_L(x) \quad 0, \qquad \int_0^L G_L(x)dx = 1, \\ - \quad (x)\dfrac{\partial}{\partial x} \quad _L(x) + (B(x) + \quad _L) \quad _L(x) - \displaystyle\int_0^L b(x,y) \quad _L(y)\,dy = \quad (0) \quad _L(0), \\ \quad _L(L) = 0, \qquad _L(x) \quad 0, \qquad \int_0^L \quad _L(x)G_L(x)dx = 1. \end{cases} \tag{5.50}$$

Notice that in this problem, the eigenelements ($_L, G_L,$ $_L$) depend on η, and and should be denoted ($_L^{,}$, $G_L^{,}$, $_L^{,}$). We forget here the superscripts for the sake of clarity.

The existence of a solution to Problem (5.50) is proved in the Appendix of [55] by using the Krein-Rutman theorem. Then we need to pass to the limit η, $\to 0$. The uniform estimates in [55] allow to do so, provided that $_L^{,}$ is positive for all η, . In [55] this condition is ensured for L large enough under the constraint that L is a fixed constant, which means that $L = L(\ )$ tends to $+\infty$ as $\to 0$. Here we want to pass to the limit $\to 0$ for a fixed positive value of L. For this we prove the existence of a constant L_0 0 such that $_L^{,}$ 0 for all η, 0 and all L L_0.

Assume by contradiction that $_L$ 0. Then we have by integration of the direct eigenequation between 0 and $x \quad L$

$$
\begin{aligned}
0 \quad & {}_0^x G(y)\,dy \\
&= - \quad (x)G(x) - {}_0^x B(y)G(y)\,dy + {}_0^x {}_z^L b(y,z)G(y)\,dy\,dz \\
&= - \quad (x)G(x) + (\ _0 - 1) {}_0^x B(y)G(y)\,dy + {}_x^L \left({}_0^x b(y,z)\,dz \right) G(y)\,dz.
\end{aligned}
$$

We assume that $b(y,x) = \frac{B(y)}{y} p\left(\frac{x}{y}\right)$ with ${}_0^1 p(h)\,dh = \ _0 \quad 1$. Thus, for p bounded, there exists $s \in (0,1)$ such that ${}_0^s p(h)\,dh \quad _0 - 1$. For $L \quad y \quad x \quad sL$, we have

$$
\begin{aligned}
{}_0^x b(y,z)\,dz = B(y) & {}_0^{\frac{x}{y}} p(h)\,dh \\
& B(y) {}_0^{\frac{sL}{y}} p(h)\,dh \quad B(y) {}_0^s p(h)\,dh \quad (\ _0 - 1)B(y),
\end{aligned}
$$

so for all $x \quad sL$

$$
0 \quad - \quad (x)G(x) + (\ _0 - 1) {}_0^L B(y)G(y)\,dy
$$

which leads to

$$
B(x)G(x) \quad (\ _0 - 1)\frac{B(x)}{(x)} {}_0^L B(y)G(y)\,dy \quad (\ _0 - 1)\frac{B(x)}{(x)} {}_{sL}^L B(y)G(y)\,dy
$$

and finally, by integration on $[sL, L]$,

$$
(\ _0 - 1) {}_{sL}^L \frac{B(y)}{(y)}\,dy \quad 1. \tag{5.51}
$$

We have from Hypothesis 3 that

$$
\exists A \quad 0, \qquad \forall x \quad A, \quad \frac{xB(x)}{(x)} \quad \frac{1}{(\ _0 - 1)|\ln(s)|}
$$

so, for $L \quad \frac{A}{s}$, we obtain

$$
(\ _0 - 1) {}_{sL}^L \frac{B(y)}{(y)}\,dy \quad \frac{1}{|\ln(s)|} {}_{sL}^L \frac{1}{y}\,dy = 1
$$

which contradicts (5.51). Finally, $_L$ 0 for all L $L_0 := \frac{A}{s}$.

We have proved the existence of solution for Problem (5.48) in the case $_L(L) = 0$ and we know that

$$_L \xrightarrow[L \to +\infty]{} \quad ,$$

$$G_L \xrightarrow[L \to +\infty]{} G \qquad \text{in } L^1(\mathbb{R}_+),$$

$$\forall A \quad 0, \qquad _L \xrightarrow[L \to +\infty]{} \qquad \text{uniformly on } [0, A).$$

Now we use this result to treat the cases $_L(L) = \delta$ 0 and $_L(L) = \delta L$. Since δ 0, we can prove by using the Krein-Rutman theorem the existence of a solution to Problem (5.48). To prove the convergence of $_L$ to , we integrate the equation on $_L$ multiplied by G and we obtain

$$- \quad _L = \quad (L)G(L) \quad _L(L).$$

We know from estimates on G that $(L)G(L)L \to 0$ when $L \to +\infty$, which ensures the convergence of . Because 0, it also ensures the existence of L_0 such that $_L$ 0 for L L_0, which allows to prove the convergence of $_L$ to locally uniformly (see [55] for details). □

5.5.2 Laplace s method

Laplace s method (see [130, II.1, Theorem 1] for example) gives a way to calculate the asymptotic behavior of integrals which contain an exponential term with a large factor in the exponent. We give here a result of this kind, with conditions which are adapted to the situation encountered in Section 5.2.

Lemma 5.6. *Take $x_0, D_0 \in \mathbb{R}$. Assume that $g : [x_0, +\infty) \to \mathbb{R}$ is a measure and $h : [x_0, +\infty) \times [D_0, +\infty) \to \mathbb{R}$ a measurable function satisfying*

$$g(x) \sim g_0(x - x_0) \qquad \text{as } x \to x_0, \text{ for some } g_0 \neq 0 \text{ and } \sigma \quad -1, \tag{5.52}$$

$$h(x, D) - h(x_0, D) \sim h_0(x - x_0) \qquad \text{as } x \to x_0 \text{ and } D \to +\infty, \text{ for some } h_0, \omega \quad 0, \tag{5.53}$$

$$\int_{x_0}^{\infty} |g(x)| e^{-D_0 h(x,D)}\, dx \quad C_0 \qquad \text{for some } C_0 \quad 0 \text{ and all } D \quad D_0. \tag{5.54}$$

Assume also that for all D D_0, the function $x \mapsto h(x, D)$ (with D fixed) attains its unique global minimum at $x = x_0$, in the following strong sense:

there exists a nondecreasing strictly positive function $\ : (0,+\infty) \to (0,+\infty)$ *such that*

$$h(x,D) - h(x_0,D) \qquad (x - x_0) \quad \textit{for all } x \quad x_0 \textit{ and all } D \quad D_0. \tag{5.55}$$

Then, as $D \to +\infty$,

$$\int_{x_0}^{\infty} e^{-Dh(x,D)} g(x)\,dx \sim g_0 D^{\frac{-1-}{}} e^{-Dh(x_0,D)} \int_0^{\infty} x \ e^{-h_0 x} \ dx. \tag{5.56}$$

The constants implicit in (5.56) *depend only on the constants implicit or explicit in* (5.52)*–*(5.55)*.*

Some remarks on the conventions used above are in order. Although g is a measure we denote it as a function in the expressions in which it appears. For example, integrals in which g appears should be considered as integrals with respect to the measure g. Also, in equation (5.52), it is understood that close to x_0 the measure g is equal to a function, and the asymptotic approximation (5.52) holds.

Proof. First of all, by translating g and h we may consider always that $x_0 = 0$. We may also assume that $h(x_0, D) = 0$ for all $D \quad D_0$, as otherwise one obviously obtains the additional factor $e^{-Dh(x_0,D)}$.

An important part of the argument is based on the observation that if one excludes a small region close to 0, then the rest of the integral decreases fast as $D \to +\infty$: from (5.53) and (5.55) we deduce that for some $\quad 0$

$$h(x,D) \qquad \min\{1, x \ \} \quad \text{for all } x \quad 0,\ D \quad D_0. \tag{5.57}$$

Then, for $D \quad D_0$ and $0 \qquad 1$ we have from (5.57):

$$\left| \int^{\infty} g(x) e^{-Dh(x,D)}\,dx \right| \qquad e^{-(D-D_0)} \qquad \int^{\infty} |g(x)| e^{-D_0 h(x,D)}\,dx \quad C_0 e^{-(D-D_0)} \quad ,$$

due to (5.54). If we take $\ := D^{-\frac{1}{2}}$ then for all $D \quad D_0$ we have

$$\left| \int_{D^{-\frac{1}{2}}}^{\infty} g(x) e^{-Dh(x,D)}\,dx \right| \quad C_0 e^{-\left(\sqrt{D} - \frac{D_0}{\sqrt{D}}\right)} \ .$$

This quantity decreases faster than any power of D as $D \to +\infty$.

For the remaining part of the integral, since we are integrating in a region which is closer and closer to 0 it is easy to see due to (5.52) and (5.53) that for all 0 there exists D 0 such that

$$\int_0^{D^{-\frac{1}{2}}} (1-)g_0 x \ e^{-D(1+)h_0 x} \ dx \quad \int_0^{D^{-\frac{1}{2}}} g(x)e^{-Dh(x,D)}\,dx$$
$$\int_0^{D^{-\frac{1}{2}}} (1+)g_0 x \ e^{-D(1-)h_0 x} \ dx \quad (5.58)$$

for all D D . Through the change of variables $z = xD^{1/}$ we see that

$$\int_0^{D^{-\frac{1}{2}}} (1-)g_0 x \ e^{-D(1+)h_0 x} \ dx$$
$$= (1-)g_0 D^{\frac{-1-}{}} \int_0^{D^{\frac{1}{2}}} z \ e^{-(1+)h_0 z} \ dz$$
$$\sim (1-)g_0 D^{\frac{-1-}{}} \int_0^{\infty} z \ e^{-(1+)h_0 z} \ dz,$$

where the $\sim$ sign denotes asymptotics as $D \to +\infty$. Carrying out a similar calculation for the last integral in (5.58) and letting $\to 0$ we deduce our result. □

For the next result we recall that ${}_+ = \max\{0, \}$ and is defined by (5.24).

Lemma 5.7. *Assume Hypotheses 1–5. There is* 0 *such that*

$$\int_{x_0}^{x} e^{\ (y)} y^k \, dy = {}^{-1} x^{k- \ _+ +} \ e^{\ (x)} + \underset{x\to+\infty}{O}(x^{k- \ _+ + \ -})e^{\ (x)}.$$

Proof. We use l Hôpital s rule to calculate the limit as $x \to +\infty$ of

$$\frac{\int_{x_0}^{x} e^{\ (y)} y^k \, dy - {}^{-1} x^{k- \ _+ +} \ e^{\ (x)}}{x^m e^{\ (x)}}.$$

Di erentiating both the numerator and denominator we find that this limit is the same as the limit as $x \to +\infty$ of

$$\frac{x^k - {}^{-1}(k- {}_+ +)x^{k- \ _+ + \ -1} - {}^{-1} x^{k- \ _+ +} \frac{B(x)+}{(x)}}{mx^{m-1} + x^m \frac{B(x)+}{(x)}} =: \frac{T_N(x)}{T_D(x)}. \quad (5.59)$$

Using (5.15) and (5.16), we obtain that

$$x^k - \;{}^{-1}x^{k- \;+ +}\;\frac{B(x)+}{(x)} = O(x^{k-}\;)$$

for some δ 0. Observing that $_+ -\; + 1$ 0 and calling := $\min\{\delta, \;_+ - \;+ 1\}$ 0 we have

$$T_N(x) = O(x^{k-}\;).$$

In a similar way,

$$T_D(x) = x^{m+\;+-} \;+ O(x^{m+\;+-\;-}\;),$$

so from (5.59) we obtain that the limit is 0 whenever

$$m + \;_+ - \quad k - \;, \qquad \text{i.e., } m \quad k - \;_+ + \;- \;.$$

This shows the result. □

We now use this to prove an estimate which is needed in Section 5.2:

Lemma 5.8. *Assume Hypotheses 1–4 with* p_1 0*, and take* $k \in \mathbb{R}$*. Then,*

$$\int_{x_0}^{x} e^{\;(y)} y^k b(x,y)\,dy \underset{x\to+\infty}{\sim} p_1 B_\infty \;{}^{-1-}\;(\;+1)x^{k+\;-(\;+-\;+1)(1+\;)}e^{\;(x)}. \quad (5.60)$$

If we also assume Hypothesis 5 and = 0 *(and now we allow any* p_1 0*) then there is* 0 *such that*

$$\int_{x_0}^{x} e^{\;(y)} y^k b(x,y)\,dy$$
$$= p_1 B_\infty \;{}^{-1}\, x^{k+\;-\;+ +\;-1} e^{\;(x)} + \underset{x\to+\infty}{O}(x^{k+\;-\;+ +\;-1-}\;)e^{\;(x)}. \quad (5.61)$$

Proof. We call p_* and p^*, respectively, the parts of the measure p on the intervals $[0,1/2)$ and $[1/2,1]$, i.e., $p_* := p\,\mathbb{1}_{[0,1/2)}$ and $p^* := p\,\mathbb{1}_{[1/2,1]}$. With this we break the integral we want to estimate in two parts:

$$I(x) := \int_{x_0}^{x} e^{\;(y)} y^k p\left(\frac{y}{x}\right)dy$$
$$= \int_{x_0}^{x} e^{\;(y)} y^k p_*\left(\frac{y}{x}\right)dy + \int_{x_0}^{x} e^{\;(y)} y^k p^*\left(\frac{y}{x}\right)dy =: I_*(x) + I^*(x).$$

The first part, I_*, can be estimated by

$$|I_*(x)| = \int_{x_0}^{x} e^{\ (y)} y^k p_* \left(\frac{y}{x}\right) dy \quad e^{\ (x/2)} \int_{x_0}^{x} y^k p_* \left(\frac{y}{x}\right) dy$$
$$e^{\ (x/2)} \max\{x^k, x_0^k\} \int_{x_0}^{x} p_* \left(\frac{y}{x}\right) dy \qquad {}_0\, x\, e^{\ (x/2)} \max\{x^k, x_0^k\}.$$

Since we will show that $I^*(x)$ behaves as given in the statement, this term is of lower order (since (x) is asymptotic to a positive power of x as $x \to +\infty$) and can be disregarded.

For I^* we make the change of variables $z = y/x$ and denote $D := x^{\ +-\ +1}$ to obtain

$$I^*(x) := \int_{x_0}^{x} e^{\ (y)} y^k p^* \left(\frac{y}{x}\right) dy = x^{k+1} \int_{\max\left\{\frac{x_0}{x}, \frac{1}{2}\right\}}^{1} e^{\ (xz)} z^k p\,(z)\ dz$$
$$= x^{k+1} \int_{\max\left\{\frac{x_0}{x}, \frac{1}{2}\right\}}^{1} \exp\left(-Dh(z,D)\right) g(z)\, dz \qquad (5.62)$$

with

$$h(z,D) := -\frac{1}{D}\ (xz) = -\frac{1}{D}\ \left(D^{\frac{1}{+-\ +1}} z\right), \qquad g(z) := z^k p\,(z)\,.$$

Now, the asymptotics in D of the integral in (5.62) can be obtained from Lemma 5.6 with $x_0 = 1$. Let us see that h and g indeed satisfy the needed hypotheses. The property (5.52) is satisfied with $g_0 = p_1$ and $\sigma =$ due to Hypothesis 4, and to show (5.53) we write (with asymptotics notation understood to be for $z \to 1$ and $D \to +\infty$)

$$h(z,D) - h(1,D) = \frac{1}{D}\left(\ (x) - \ (xz)\right) = \frac{1}{D} \int_{xz}^{x} \frac{+ B(u)}{(u)}\, du$$
$$\sim \frac{}{+-\ +1} \frac{1}{D} x^{\ +-\ +1}(1 - z^{\ +-\ +1}) = \frac{}{+-\ +1}(1 - z^{\ +-\ +1}) \sim\ (1-z), \qquad (5.63)$$

which corresponds to $h_0 =\ $, $\omega = 1$ in Lemma 5.6. For (5.54) we write

$$\int_{\max\left\{\frac{x_0}{x}, \frac{1}{2}\right\}}^{1} \exp\left(-D_0 h(z,D)\right) g(z)\, dz$$
$$\int_{\frac{1}{2}}^{1} \exp\left(\frac{D_0}{D}\ (xz)\right) z^k p(z)\, dz$$
$$\exp\left(\frac{D_0}{D}\ (x)\right) \int_{\frac{1}{2}}^{1} z^k p(z)\, dz \qquad C_0$$

for some C_0 0 (which in particular depends on k), since $x \mapsto$ $(x)/D =$ $(x)/x^{\ +- \ +1}$ is bounded for x 1. This gives (5.54). Obviously $z \mapsto h(z, D)$ attains its minimum at $z = 1$, and (5.55) is a consequence of (5.63) and the fact that $h(z, D) - h(1, D)$ is decreasing in z for all D.

We may then apply Lemma 5.6 to obtain

$$I^*(x) \sim p_1 x^{k+1} D^{-1-} \ e^{\ (x)} \int_0^\infty z \ e^{- \ z} \, dz$$
$$= p_1 \ ^{-1-} \ (1+ \ ) x^{k+1-(\ +- \ +1)(1+ \ )} e^{\ (x)}.$$

Since $I_*(x)$ was shown to be of lower order, this is enough to show (5.60).

Finally, in order to show (5.61), we have

$$\int_{x_0}^x e^{\ (y)} y^k p\left(\frac{y}{x}\right) dy = \int_{x_0}^x e^{\ (y)} y^k \left(p\left(\frac{y}{x}\right) - p_1\right) dy + p_1 \int_{x_0}^x e^{\ (y)} y^k \, dy.$$

For the first term, using (5.17) and (5.60) we have

$$\int_{x_0}^x e^{\ (y)} y^k \left(p\left(\frac{y}{x}\right) - p_1\right) dy = O(x^{k+1-(\ +- \ +1)(1+ \ )}) e^{\ (x)},$$

and for the second term, Lemma 5.7 gives

$$\int_{x_0}^x e^{\ (y)} y^k \, dy = \ ^{-1} x^{k- \ ++} \ e^{\ (x)} + O(x^{k- \ ++ \ -}) e^{\ (x)}.$$

Since $_+ -$ $+1$ 0, this shows the result. □

Chapter 6

Stability analysis of ock and mill rings for 2nd order models in swarming

The contents of this chapter appear in:

> *G. Albi, D. Balagué, J. A. Carrillo, J. von Brecht. Stability Analysis of Flock and Mill rings for 2nd Order Models in Swarming. Preprint.* [1]

6.1 Introduction

Individual-based models (IBMs) appear in biology, mathematics, physics, and engineering. They describe the motion of a collection of N individual entities, so the system is defined on a microscopic scale. IBMs are good models for some applications when the number of particles is reasonable. Nonetheless, when the number is large, it is more reasonable to use a continuum model. Some continuum models, like the one in [47, 45], are derived as a mean-field particle limit leading to a mesoscopic kinetic description of the problem. At this level, one looks at the probability density of finding particles at a certain position and velocity at a given time. Several models have been proposed to describe the ocking of birds [31, 104, 11, 88], the formation of ant trails [58], the schooling of fish [75, 24, 12], swarms of bacteria [79], etc.

These models can include some rules that describe the behavior of each individual of the system. Such mechanisms can help to describe the in uence of each individual on the others, depending on their relative position and velocity. For instance, one example is the classical three zone model [5, 76].

A three zone model describes how social the individual is in the following sense. If two individuals are too close, they want to have their own space (repulsion). When one individual is far from the group, it wants to go back and socialize (attraction). And finally, in the group, each individual tries to mimic the behavior of the others (orientation). Other models just consider rules for orientation, like the Vicsek model [122, 50]. In this case, there is a mechanism of self-propulsion in which each individual moves with constant speed and adopts the average direction among their local neighbors.

We focus our study in the analysis of two particular examples of IBMs. The first one is a self-propelled interacting particle model that was introduced in [85] and extensively studied in [54, 47]:

$$\begin{cases} \dot{x}_j = v_j \\ \dot{v}_j = S(|v_j|)v_j + \frac{1}{N}\sum_{\substack{l=1 \\ l\neq j}}^{N} \nabla W(x_l - x_j) \end{cases} , \quad j = 1, \ldots, N \tag{6.1}$$

We are going to consider the same *self-propulsion/ friction* term used in [54, 47],

$$S(|v_j|) = \quad - \quad |v_j|^2, \qquad , \quad 0.$$

Note that such a term gives us an asymptotic speed for the particles, equal to $\sqrt{\ /\ }$. In these references, the authors study (6.1) with pairwise interaction given by the so-called Morse potential

$$U(r) = C_A e^{-r/l_A} - C_R e^{-r/l_R},$$

with C_A, C_R denoting the attractive and repulsive strengths and l_A, l_R their respective length scales. They find and describe several patterns for the asymptotic behavior in 2D. They observed ocking behavior, mill on a ring, and clustering when particles are milling. In [32], a well-posedness theory is developed for (6.1) proving the mean-field limit under smoothness assumptions on the potential. The authors show convergence of the particle model toward a measure solution of the kinetic equation.

We perform an analysis on the stability of ock rings and the mill rings as asymptotic solutions for (6.1). The ring solution was recently studied in [81, 23] where the authors in this work do a careful general linear analysis of the rings for the first order model

$$\dot{X}_j = \sum_{\substack{l=1 \\ l\neq j}}^{N} \nabla W(X_j - X_l), \quad j = 1, \ldots, N. \tag{6.2}$$

Part of the analysis of (6.2) is used to study the the stability of mill rings in (6.1), described in [54]. Related pattern formation in the associated first order model has been studied in [127, 80].

Another second order model that we are going to study is

$$\begin{cases} \dot{x}_j = v_j \\ \dot{v}_j = \dfrac{1}{N}\displaystyle\sum_{l=1}^{N} H(x_j - x_l)(v_l - v_j) + \dfrac{1}{N}\displaystyle\sum_{\substack{l=1 \\ l\neq j}}^{N} \nabla W(x_l - x_j) \end{cases}, \quad j = 1, \ldots, N \tag{6.3}$$

with $x_j, v_j \in \mathbb{R}^2$ where the velocity v_j is described by the Cucker-Smale alignment term H and the pairwise interaction by a repulsive-attractive radial potential $W(x) = k(|x|)$.

Even if the analysis has been done in full generality for the parameter functions of the model H and W, we will emphasize the results in some relevant cases. For instance, we consider the case of power-law repulsive-attractive potentials [81, 10]

$$k(r) = \frac{r^a}{a} - \frac{r^b}{b}, \qquad a \quad b \quad 0. \tag{6.4}$$

For the Cucker-Smale alignment [48, 49, 72, 71, 37], a relevant case is $H(x) = g(|x|)$ with

$$g(r) = \frac{1}{(1 + r^2)}, \qquad\qquad 0.$$

The main results of this work show that the flock ring is unstable in the second order models (6.1) and (6.3) if and only if its spatial shape is unstable in the first order model (6.2). We use the same kind of strategy as in [23].

The main idea is to study the stability of the system of ODEs (6.1) by analyzing the eigenvalues of a suitable linearization with restricted perturbations. We consider particular perturbations of the flock rings in such a way that translational invariance is avoided while preserving the mean velocity. This is really needed since the linearized system associated to (6.1) is always linearly unstable due to translations. Translational invariance implies the existence of a generalized eigenvector associated to the zero eigenvalue of the matrix defining the linearized system. We characterize all cases in which the linearized system has eigenvalues with zero real part and their consequences in the instability condition. An analysis of the stability of the family of flock solutions is under way in [39].

In addition to flock rings, other spatial shapes are possible as asymptotic solutions. One can also observe flocks on annuli (fattening), lines or points (clustering). These patterns can be explained due to the results in [8].

For the mill ring analysis, we start from the results of [23] to explore other mill configurations that appear with repulsive-attractive potentials. We numerically investigate the formation of fat mills due to the repulsive force and the formation of clusters when varying the asymptotic speed. In addition, we show some switching behaviors between ock and mill rings.

The structure of the paper is as follows. In Section 6.2 we study the microscopic and the mesoscopic models and we give the definitions of the main objects in our studies, the ock and mill rings. In Section 6.3 we do a linear stability analysis on the ock rings for models (6.1) and (6.3). We also explore the fattening and cluster formation. Finally, in Section 6.4 we do a similar study on the stability for mill rings. In all sections we have performed several numerical tests supporting our theoretical results.

6.2 Microscopic & Mesoscopic models

Let us introduce some particular solutions of the particle model (6.1) and its continuum counterpart.

6.2.1 Flock and mill solutions: microscopic model

De nition 6.1. *We call a* ock ring, *the solution of* (6.1) *such that* $\{x_j\}_{j=1}^N$ *are equally distributed on a circle with a certain radius,* R *and* $\{v_j\}_{j=1}^N = u_0$, *with* $|u_0| = \sqrt{\ /\ }$.

De nition 6.2. *We call a* mill ring, *the solution of* (6.1) *such that* $\{x_j\}_{j=1}^N$ *are equally distributed on a circle with a certain radius,* R *and* $\{v_j\}_{j=1}^N = u_j^0 = \sqrt{\ /\ }\ x_j^\perp/|x_j|$ *with* $x_j^\perp$ *the orthogonal vector.*

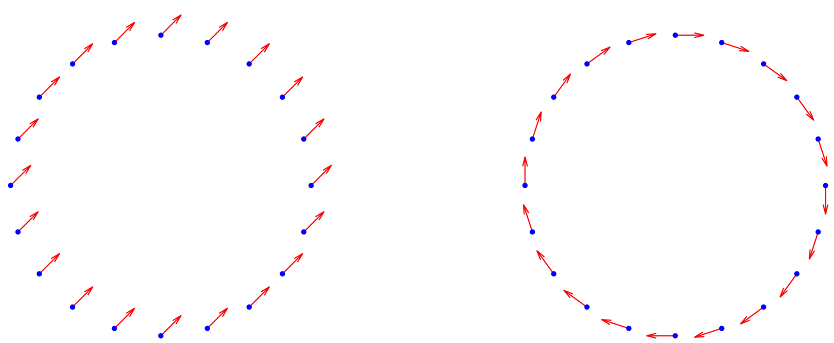


Figure 6.1: Flock and mill ring solutions.

By abuse of notation, we will write $|u_0|$ for $\left|u_j^0\right|$ since $\left|u_j^0\right| = \sqrt{\ \ /\ \ }$ for all $j = 1, \ldots, N$. Moreover, we will make use of notation $|u_0|$ for both ock and mill rings indistinctly.

All over the paper, we will identify $e^{i} \equiv (\cos\ , \sin\ )$ and use x to identify the vector and the complex numbers indistinctly

$$x_j(t) = R\left(\cos\left(\frac{2}{N}j + \omega t\right), \sin\left(\frac{2}{N}j + \omega t\right)\right) = Re^{i\frac{2\ j}{N}}e^{i\ \ t}. \tag{6.5}$$

In the case of mill rings, we are looking for a solution of the form (6.5). The case of a ock ring in the comoving frame is equivalent to looking for a solution of the form (6.5) with $\omega = 0$. Plugging (6.5) into (6.1), we obtain

$$\sum_{\substack{l=1\\ l\neq j}}^{N} \nabla W(x_j - x_l) = 0;$$

therefore, R is determined only by the repulsive-attractive potential.

In the case of mill rings, we have

$$v_j(t) = \dot{x}_j(t) = R\omega i e^{i\ \ j} e^{i\ \ t}, \qquad {}_j = \frac{2\ \ j}{N},$$

and thus, $R^2\omega^2 = -$. Moreover, by taking the derivative

$$\dot{v}_j = -\omega^2 x_j = -\omega^2 \frac{1}{N} \sum_{\substack{l=1\\ l\neq j}}^{N} (x_l - x_j) = 0.$$

Plugging this into (6.1), we get

$$\sum_{\substack{l=1\\ l\neq j}}^{N} \left[\nabla W(x_l - x_j) - \omega^2 (x_l - x_j)\right] = \sum_{\substack{l=1\\ l\neq j}}^{N} \nabla W(x_l - x_j) = 0,$$

with $W(x) = W(x) - \omega^2 \frac{|x|^2}{2}$. Thus in order to find the radius for ock and mill rings, we need to solve the same equation. This expression implies that the spatial shape has to balance attraction versus repulsion and centrifugal forces. Now, a direct computation yields

$$|x_j - x_l| = 2R \sin\left(\frac{(l-j)}{N}\right)$$

for all times. One can easily compute that

$$x_l - x_j = 2R\sin\left(\frac{p}{N}\right)\begin{pmatrix} -\sin\left(\frac{p}{N}\right) & \cos\left(\frac{p}{N}\right) \\ \cos\left(\frac{p}{N}\right) & \sin\left(\frac{p}{N}\right)\end{pmatrix}\begin{pmatrix}\cos(\ _j) \\ \sin(\ _j)\end{pmatrix}, \quad p = l - j,$$

and

$$\begin{aligned}\sum_{\substack{l=1\\ l\neq j}}^{N} \nabla W(|x_j - x_l|) &= \sum_{\substack{l=1\\ l\neq j}}^{N} (x_j - x_l)\frac{k'(|x_j - x_l|)}{|x_j - x_l|} \\ &= \sum_{\substack{l=1\\ l\neq j}}^{N} \begin{pmatrix} -\sin\left(\frac{p}{N}\right)\cos(\ _j) + \cos\left(\frac{p}{N}\right)\sin(\ _j) \\ \cos\left(\frac{p}{N}\right)\cos(\ _j) + \sin\left(\frac{p}{N}\right)\sin(\ _j)\end{pmatrix} k'(|x_j - x_l|).\end{aligned}$$

By symmetry we can assume $j = N$, and thus

$$\sum_{\substack{l=1\\ l\neq j}}^{N} \nabla W(|x_j - x_l|) = \sum_{p=1}^{N-1} \begin{pmatrix} -\sin\left(\frac{p}{N}\right) \\ \cos\left(\frac{p}{N}\right)\end{pmatrix} k'(|x_j - x_l|),$$

changing j to $N - j$, we finally obtain

$$\sum_{\substack{l=1\\ l\neq j}}^{N} \nabla W(|x_j - x_l|) = \begin{pmatrix} -\sum_{p=1}^{N-1} \sin\left(\frac{p}{N}\right) k'\left(2R\sin\left(\frac{p}{N}\right)\right) \\ 0 \end{pmatrix}. \tag{6.6}$$

As a conclusion, the radius of a ock or mill ring solution is characterized by

$$\sum_{p=1}^{N-1} \sin\left(\frac{p}{N}\right) k'\left(2R\sin\left(\frac{p}{N}\right)\right) = 0.$$

For general potentials there can be more than one ock or mill solution. In the case of the power-law potentials (6.4), there is only one solution. Condition (6.6) reads

$$(2R)^{a-1}\frac{1}{N}\sum_{p=0}^{N-1}\sin^a\left(\frac{p}{N}\right) - (2R)^{b-1}\frac{1}{N}\sum_{p=0}^{N-1}\sin^b\left(\frac{p}{N}\right) - 2R\omega^2\frac{1}{N}\sum_{p=0}^{N-1}\sin^2\left(\frac{p}{N}\right) = 0. \tag{6.7}$$

To prove uniqueness, we notice that the function $f(r) = C_1 r^a - C_2 r^b - C_3$ with $a \quad b \quad 0$ and $C_1 \quad C_2 \quad 0$, $C_3 \quad 0$, has only one zero. Computing the first derivative and looking for critical points, we obtain $r_1 = 0$ and $r_2^{a-b} = \frac{C_2 b}{C_1 a}$. Taking the second derivative and evaluating at r_2, one obtains $f''(r_2) = r_2^{b-2} C_2 b(a-b) \quad 0$, so r_2 is a local minimum. For $0 \quad r \quad r_2$, one has $f'(r) \quad 0$ whereas for all $r \in (r_2, +\infty)$, one has $f'(r) \quad 0$. Then we conclude that $f(r)$ has a unique zero. Notice that the solution to (6.7) depends on the number of particles and we will use the notation $R = R(N)$.

6.2.2 Flock and mill solutions: mesoscopic model

In this subsection we characterize the radius of ock and mill rings. We introduce the function

$$(s) = \frac{1}{\quad} \int_0 (1 - s\cos \quad)(1 + s^2 - 2s\cos \quad)^{\frac{-2}{2}} \, d \quad ,$$

already analyzed in [10]. A change of variables in the previous function shows that

$$(1) = \frac{2^{\quad -1}}{\quad} B\left(\frac{\quad + 1}{2}, \frac{1}{2}\right). \tag{6.8}$$

Lemma 6.1. *If $N \to \infty$ then $R(N) \to R_{ab}(|u_0|)$ where $R_{ab}(|u_0|)$ is the solution of the following equation:*

$$_a(1)R^{a-1} - \quad _b(1)R^{b-1} - \omega^2 R = 0.$$

Proof. We first take $2^{a-1}\frac{1}{N} \sum_{p=0}^{N-1} \sin^a(\frac{p}{N})$. Multiplying and dividing by we obtain the following equality

$$\lim_{N\to\infty} 2^{a-1}\frac{1}{\quad}\left(\frac{\quad}{N}\sum_{p=0}^{N-1}\sin^a\left(\frac{p}{N}\right)\right) = 2^{a-1}\frac{1}{\quad}\int_0 \sin^a(x)\,dx$$
$$= 2^{a-1}\frac{1}{\quad}\left(2\int_0^{/2}\sin^a(x)\,dx\right). \tag{6.9}$$

Now, we use again the expression for the Beta function given in (2.69), with $x = \frac{1}{2}$, $y = \frac{a+1}{2}$, and using that $B(x,y) = B(y,x)$ in (6.9) together with (6.8) to obtain

$$\lim_{N\to\infty} 2^{a-1}\frac{1}{\quad}\left(\frac{\quad}{N}\sum_{p=0}^{N-1}\sin^a\left(\frac{p}{N}\right)\right) = 2^{a-1}\frac{1}{\quad}B\left(\frac{a+1}{2},\frac{1}{2}\right) = \quad _a(1).$$

The same reasoning works by changing a for b in the second term in (6.7). For the third term we use the fact that we can compute the exact sum

$$\frac{2}{N}\sum_{p=0}^{N-1}\sin^2\left(\frac{p}{N}\right) = 1.$$

□

Remark 6.1. *In the case of flock rings $\omega = 0$, their radius is determined by the radius of the aggregation ring found in* [10]

$$R(N) \to R_{ab} = \frac{1}{2}\left(\frac{B(\frac{b+1}{2},\frac{1}{2})}{B(\frac{a+1}{2},\frac{1}{2})}\right)^{\frac{1}{a-b}} \quad as \quad N \to \infty.$$

Remark 6.2. *Let $W(x) = k(|x|)$ be a general interaction potential. Call $f(r) = -k'(r)/r$. Then the radius of the ring is determined by*

$$\int_0^{\overline{2}} f(2R\sin(\ ))\sin^2(\ )\, d\ \ = 0,$$

as shown in [23].

Remark 6.3. *The corresponding mesoscopic model to the particle system* (6.1)*, as proven in* [32]*, is given by the kinetic equation*

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + \mathrm{div}_v[(\ \ -\ \ |v|^2)vf)] - \mathrm{div}_v[(\nabla_x W *\ \ )f] = 0, \qquad (6.10)$$

where

$$(t,x) = \int_{\mathbb{R}^2} f(t,x,v)\, dv.$$

It was shown in [45] *that singular solutions of the type*

$$f(t,x,v) = \ (t,x)\delta\left(v - u_0\right), \qquad f(t,x,v) = \ (t,x)\delta\left(v - \sqrt{-\frac{x^\perp}{x}}\right),$$

with (t,x) the uniform distribution on a ring, are weak solutions of the kinetic model (6.10)*, called the flock and mill ring continuous solutions respectively.*

6.3 Linear stability analysis for ock rings

We will now focus on the stability analysis of ock rings for some particular perturbations in terms of the parameters of the model (a, b, u_0). We take advantage of the careful stability analysis of the ring solutions of the aggregation equation performed in [23].

6.3.1 Stability of ock solutions without the Cucker-Smale term

We consider the model (6.1) and we perform the change of variables to the comoving frame

$$\begin{cases} y_j(t) = x_j(t) - u_0 t \\ z_j(t) = v_j(t) - u_0 \end{cases} \qquad j = 1, \dots, N, \tag{6.11}$$

where u_0 is the asymptotic velocity of a fixed ock ring. Therefore the system (6.1) reads

$$\begin{cases} \dfrac{d}{dt} y_j = v_j - u_0 = z_j \\ \dfrac{d}{dt} z_j = \underbrace{(\ - \ |z_j + u_0|^2)}_{S_0(|z_j|)}(z_j + u_0) - \dfrac{1}{N} \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{N} \nabla W(y_j - y_k) \end{cases}, \quad j = 1, \dots, N.$$

A ock ring can then be characterized as a stationary solution of the form $\left(y_j^0, z_j^0\right) = \left(Re^{i \ j}, 0\right)$, where $_j = \frac{2 \ j}{N}$ for $j = 1, \dots, N$. This stationary solution satisfies

$$S_0(|z_j^0|) = 0, \quad \nabla S_0(|z_j^0|) = -2 \ u_0 u_0^*, \quad 0 = \sum_{k \neq j} k'(|y_j^0 - y_k^0|) \frac{(y_k^0 - y_j^0)}{|y_k^0 - y_j^0|}.$$

As in [23], we restrict the set of possible perturbations of the ock solution to those of the form

$$y_j(t) = Re^{i \ j}(1 + h_j(t)),$$

where $h_j \in \mathbb{C}$, such that $|h_j| \ll 1$ and

$$\sum_{j=1}^{N} h_j(t) = \sum_{j=1}^{N} h_j'(t) = 0. \tag{6.12}$$

The first restriction is to avoid the zero eigenvalue due to translations. The second one comes from the fact that the mean velocity of the perturbed system should be u_0. More general perturbations will generically lead to other ock solutions with di erent asymptotic velocity u_0. Their orbital stability will be analyzed elsewhere [39]. Therefore, the perturbed system reads

$$\begin{cases} \dfrac{d}{dt} y_j(t) = Re^{i \ j} h_j' = z_j \\ \dfrac{d}{dt} z_j(t) = Re^{i \ j} h_j'' = S_0(|z_j|) - \dfrac{1}{N} \displaystyle\sum_{\substack{l=1 \\ l \neq j}}^{N} \nabla W(y_j - y_l) \end{cases}, \quad j = 1, \dots, N.$$

The linearization of the system around the ock solution (y_j^0, z_j^0) reads as

$$\begin{cases} \dfrac{d}{dt} y_j(t) = Re^{i \ j} h_j' = z_j \\ Re^{i \ j} h_j'' = -2 \ u_0 u_0^* - \dfrac{1}{N} \displaystyle\sum_{\substack{l=1 \\ l \neq j}}^{N} \nabla W(y_j - y_l) \end{cases}, \quad j = 1, \ldots, N.$$

From the previous equation, we can characterize h_j'' as

$$h_j'' = \sum_{l \neq j} \left[G_1(\ /2)(h_j - e^{i \ } h_l) + G_2(\ /2)(h_l - e^{i \ } h_j) \right] - 2 \ u_0 u_0^T h_j',$$

where $\ = \frac{2 \ (l-j)}{N}$ and

$$G_1(\ ) = \frac{1}{2N} \left[-a(2R|\sin \ |)^{a-2} + b(2R|\sin \ |)^{b-2} \right],$$
$$G_2(\ ) = \frac{1}{2N} \left[-(a-2)(2R|\sin \ |)^{a-2} + (b-2)(2R|\sin \ |)^{b-2} \right],$$

for the power-law potentials. The details of these previous computations for general potentials can be found in [23].

Let us consider the following ansatz for perturbations h_j

$$h_j = \ _+(t) e^{im \ j} + \ _-(t) e^{-im \ j}, \qquad m = 2, 3, \ldots, \tag{6.13}$$

which satisfies conditions (6.12). We need to exclude the case $m = 1$ since it leads to a zero eigenvalue due to the rotational invariance of the system. Following the same strategy as in [23] and some computations, we finally deduce

$$\begin{pmatrix} \ _+'' \\ \ _-'' \end{pmatrix} = \underbrace{\begin{pmatrix} I_1(m) & I_2(m) \\ I_2(m) & I_1(-m) \end{pmatrix}}_{M} \begin{pmatrix} \ _+ \\ \ _- \end{pmatrix} - 2 \ u_0 u_0^T \begin{pmatrix} \ _+' \\ \ _-' \end{pmatrix},$$

with I_1 and I_2 real functions given by

$$I_1(m) = \sum_{l \neq j} G_1(\ /2)(1 - e^{i(m+1) \ }) = 4 \sum_{p=1}^{N/2} G_1 \left(\frac{\ p}{N} \right) \sin^2 \left(\frac{(m+1) \ p}{N} \right), \tag{6.14}$$

$$\begin{aligned} I_2(m) &= \sum_{l \neq j} G_2 (\ /2) \left(e^{im \ } - e^{i \ } \right) \\ &= 4 \sum_{p=1}^{N/2} G_2 \left(\frac{\ p}{N} \right) \left[\sin^2 \left(\frac{\ p}{N} \right) - \sin^2 \left(\frac{m \ p}{N} \right) \right]. \end{aligned} \tag{6.15}$$

The previous system can be written also in the following form

$$\frac{d}{dt}\begin{pmatrix} _{+} \\ _{-} \\ \eta_+ \\ \eta_- \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id} \\ M & -2 \quad u_0 u_0^T \end{pmatrix} \begin{pmatrix} _{+} \\ _{-} \\ \eta_+ \\ \eta_- \end{pmatrix} = L \begin{pmatrix} _{+} \\ _{-} \\ \eta_+ \\ \eta_- \end{pmatrix}, \tag{6.16}$$

where $(\eta_+, \eta_-) = (\ '_+, \ '_-)$.

If we do not assume (6.12) and (6.13), then we cannot reduce the analysis to a 4×4 system. An arbitrary perturbation for general ocks leads instead to a matrix of the form

$$L = \begin{pmatrix} 0 & \mathrm{Id} \\ & \\ \mathbf{M} & -2 \quad U \end{pmatrix},$$

where the partition into $2N \times 2N$ sub-blocks re ects the distinction between position and velocity contributions to the Jacobian: The symmetric matrix $\mathbf{M}$ is the $2N \times 2N$ Hessian that results from linearizing the first order system (6.2) about a given ocking configuration, whereas U denotes a block-diagonal matrix with N blocks of the 2×2 matrix $u_0 u_0^T$ along the diagonal. By rotational invariance we can reduce to the case $u_0 = e_1 = (1, 0)$, so that the block matrix U acts on $\mathbf{x} = (x_1, \ldots, x_N)^T \in \mathbb{R}^{2N}$, $x_i \in \mathbb{R}^2$, according to the relation

$$(U\mathbf{x})_i = \begin{pmatrix} \langle x_i, e_1 \rangle \\ 0 \end{pmatrix}.$$

We now turn to the task of characterizing the eigenvalues of L in terms of the eigenvalues of $\mathbf{M}$. In other words, we aim to characterize the stability of a ock in terms of the stability of its spatial shape as a solution to the first order model. To fix the notation, we write the eigenvalue problem for the ock as

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathbf{M} & -2 \quad U \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} = L \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix}, \tag{6.17}$$

where the matrix $\mathbf{M}$ determines the stability of the ocking configuration as a solution of the first order model. For any given eigenvector $(\mathbf{x}, \mathbf{v}) \in \mathbb{C}^{2N} \times \mathbb{C}^{2N}$ of the full system (6.17), we always assume the normalization $\mathbf{x}^* \mathbf{x} = 1$. Substituting the first equation $\ \mathbf{x} = \mathbf{v}$ into the second equation yields the equivalent statement

$$^2\mathbf{x} + 2 \quad U\mathbf{x} - \mathbf{M}\mathbf{x} = 0. \tag{6.18}$$

Let $|\mathbf{x}|_2$ denote the semi-norm on $\mathbb{C}^{2N}$ defined according to

$$|\mathbf{x}|_2^2 := \sum_{i=1}^{N} |\langle x_i, e_1 \rangle|^2,$$

and let $E^N \cong \mathbb{C}^N$ denote the subspace

$$E^N := \quad \mathbf{x} \in \mathbb{C}^{2N} : |\mathbf{x}|_2 = 0\big\} = \ker(U).$$

Premultiplying by $\mathbf{x}^*$, the fact that $\mathbf{x}^* U \mathbf{x} = |\mathbf{x}|_2^2$, the normalization on $\mathbf{x}$ and the quadratic formula combine to imply the key identity

$$= - \quad |\mathbf{x}|_2^2 \pm \sqrt{\quad {}^2|\mathbf{x}|_2^4 + \mathbf{x}^*\mathbf{M}\mathbf{x}}. \tag{6.19}$$

As $\mathbf{M}$ is symmetric, we may write its $2N$ real eigenvalues and corresponding normalized ($\mathbf{x}^*\mathbf{x} = 1$) eigenvectors as

$$\mu_{2N} \quad \mu_{2N-1} \quad \cdots \quad \mu_2 \quad \mu_1 \qquad \mathbf{M}\mathbf{x}_i = \mu_i \mathbf{x}_i.$$

The notation $a_L(\ )$, $a_{\mathbf{M}}(\mu)$ will denote the algebraic multiplicities of $\ ,\mu$ as eigenvalues of their respective matrices. The bulk of the analysis lies in characterizing the eigenvalues of the full system (6.17) that have $(\ ) = 0$.

Lemma 6.2. *Let denote an eigenvalue of* (6.17)*. Then* $(\ ) = 0$ *and* $(\ ) \neq 0$ *if and only if* $= \pm i\sqrt{-\mu_k}$ *for some k with* $\mu_k \quad 0$ *and* $\mathbf{x}_k \in E^N$*. The eigenspace consists only of eigenvectors.*

Proof. If $\mathbf{x}_k \in E^N$ then (6.18) reads ${}^2\mathbf{x}_k = \mathbf{M}\mathbf{x}_k$, or equivalently ${}^2 = \mu_k$. To have $(\ ) \neq 0$ then requires $\mu_k \quad 0$. Conversely, if $(\mathbf{x},\ \mathbf{x})$ denotes an eigenvector with $(\ ) = 0$ and $(\ ) \neq 0$, the formula (6.19) implies that necessarily $\mathbf{x} \in E^N$, and therefore $\mathbf{M}\mathbf{x} = \ {}^2\mathbf{x}$. Thus ${}^2 = \mu_k$ for some $\mu_k \quad 0$.

To show the last statement, suppose a generalized eigenvector existed that is not an eigenvector. Then there exists an eigenvector $(\mathbf{x},\ \mathbf{x})$ with $\mathbf{x} \in E^N$ so that the system of equations

$$\begin{pmatrix} -\ \mathrm{Id} & \mathrm{Id} \\ \mathbf{M} & -2\ U - \ \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \ \mathbf{x} \end{pmatrix} \tag{6.20}$$

has a non-trivial solution. Substituting the first equation $\mathbf{w} = \ \mathbf{u} + \mathbf{x}$ into the second equation, then pre-multiplying by $\mathbf{x}^*$ demonstrates

$$\mathbf{M}\mathbf{u} - 2\ U\mathbf{w} = 2\ \mathbf{x} + \ {}^2\mathbf{u}$$
$$\mathbf{x}^*\mathbf{M}\mathbf{u} = 2\ + \ {}^2\mathbf{x}^*\mathbf{u}.$$

The last line follows as $\mathbf{x}^*\mathbf{x} = 1$ and $\mathbf{x} \in E^N = \ker(U)$. The symmetry of $\mathbf{M}$ and the fact that $\mathbf{M}\mathbf{x} = \ {}^2\mathbf{x}$ combine to show $\mathbf{x}^*\mathbf{M}\mathbf{u} = \ {}^2\mathbf{x}^*\mathbf{u}$. Thus $= 0$, leading to a contradiction. □

Lemma 6.3. *Let* $\;\; 0$. *Then* $\; = 0$ *is an eigenvalue of* (6.17) *and* $(\mathbf{x}, \mathbf{0})$ *is a corresponding eigenvector if and only if* $\mathbf{M}\mathbf{x} = \mathbf{0}$. *If* $\mathbf{x} \in E^N$ *then* $(\mathbf{x}, \mathbf{0})$ *generates a single generalized eigenvector, whereas if* $\mathbf{x} \notin E^N$ *then* $(\mathbf{x}, \mathbf{0})$ *generates no generalized eigenvectors.*

Proof. The first statement follows trivially from (6.18). To see the second statement, consider the system of equations (6.20) with $\; = 0$. This reduces to the equations $\mathbf{w} = \mathbf{x}$ and

$$\mathbf{M}\mathbf{u} = 2 \; U\mathbf{x},$$

which by premultiplying by $\mathbf{x}^*$ as before and using the fact that $\mathbf{M}\mathbf{x} = \mathbf{0}$ necessitates $\mathbf{x} \in E^N$ as $\;\; 0$. If indeed $\mathbf{x} \in E^N$ then any $\mathbf{u} \in \ker(\mathbf{M})$ su ces. Without loss of generality, take $\mathbf{u} = \mathbf{x}$ itself. If $(\mathbf{x}, \mathbf{0})$ generates a second generalized eigenvector then the system of equations

$$\begin{pmatrix} 0 & \mathrm{Id} \\ \mathbf{M} & -2 \; U \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{x} \end{pmatrix}$$

has a non-trivial solution. As then $\mathbf{w} = \mathbf{x}$ and $\mathbf{x} \in E^N$ this reads $\mathbf{M}\mathbf{u} = \mathbf{x}$. Premultiplying one last time by $\mathbf{x}$, the facts that $\mathbf{M}\mathbf{x} = \mathbf{0}$ and $\mathbf{x}^*\mathbf{x} = 1$ combine to produce the contradiction $0 = 1$. □

This lemma yields, as a corollary, the algebraic multiplicity $a_L(0)$ of zero as an eigenvalue of the second order system.

Corrolary 6.1. *Let* $\;\; 0$. *Then*

$$a_L(0) = \dim(\ker(\mathbf{M}) \cap E^N) + \dim(\ker(\mathbf{M})).$$

Let $a_{\mathbf{M},\perp}(0) := \dim(\ker(\mathbf{M}) \cap E^N)$, so that $a_L(0) = a_{\mathbf{M},\perp}(0) + a_{\mathbf{M}}(0)$. Note that neither quantity depends on $\;$, and the conclusion holds whenever $\;$ is positive. Thus, if $\; \in (0, \infty)$ it follows that $a_L(0)$ is constant. Moreover, Lemma 6.2 holds uniformly in $\;$ as well. Let $i_1 \;\; i_2 \;\; \cdots \;\; i_k \;\; 2N$ denote those (possibly non-existent) indices where $\mu_{i_j} \;\; 0$ has an eigenvector $\mathbf{x}_{i_j} \in E^N$. The two lemmas then combine to show:

Corrolary 6.2. *Let* $\;\; 0$. *Then*

$$\det(L - \;\mathrm{Id}) = \; {}^{a_{\mathbf{M}\;\perp}(0)+a_{\mathbf{M}}(0)} \; {}_{j=1}^{k}(\; {}^{2} - \mu_{i_j})p \; (\;).$$

The roots of the polynomial $p \; (\;)$ *all have non-zero real part.*

This corollary, along with the formula (6.19), su ce to establish the desired result:

Theorem 6.1. *The linearized second order system around the flock ring solution* (6.1) *has an eigenvalue with positive real part if and only if the linearized first order system around the ring solution has a positive eigenvalue. As a consequence, the flock ring solution is unstable for m-mode perturbations for the second order model* (6.1) *if and only if the ring solution is unstable for m-mode perturbations for the first order model* (6.2).

Proof. Suppose first that μ_1 0. Then $\mathbf{x}^*\mathbf{M}\mathbf{x}$ 0 for any $\mathbf{x}$, whence all eigenvalues of L have non-positive real part due to (6.19). Conversely, suppose μ_1 0 and let $\mathcal{A}$ denote the set

$$\mathcal{A} := \left\{ \quad \in [0,\infty) : \max_{\in \ (L)} \ (\ ) \quad 0 \right\}.$$

Note that $0 \in \mathcal{A}$ due to (6.19). Indeed, then $(\mathbf{x}_1, \sqrt{\mu_1}\mathbf{x}_1)$ defines an eigenvector with eigenvalue $= \sqrt{\mu_1}$ 0. By continuous dependence of the eigenvalues of L on , it follows that $\mathcal{A}$ is relatively open. To show that it is also relatively closed, let $_l \in \mathcal{A}$ and $_l \to \ _0 \in (0,\infty)$. Up to extraction of subsequences, it follows that there exists a corresponding sequence $_l$ of eigenvalues with $(\ _l)$ 0 converging to some $_0$ with $(\ _0)$ 0. Moreover, by continuous dependence of the coe cients of p ( ) on , the roots of $p \ _l(\ )$ converge to roots of $p \ _0(\ )$. Thus $p \ _0(\ _0) = 0$. As no such root can have zero real part by corollary 6.2, $(\ _0)$ 0 and $_0 \in \mathcal{A}$. As $\mathcal{A} \neq \emptyset$ it follows that $\mathcal{A} = [0,\infty)$ as desired. The last part of the theorem is a direct application of the first part to the $4x4$ m-mode perturbation matrix in (6.16). □

Remark 6.4. *As an artifact of translation invariance in the first order model, the vector defined by* $\mathbf{e}_2 := (0,1,\ldots,0,1)^T \in \mathbb{R}^{2N}$ *always defines an eigenvector of* $\mathbf{M}$ *with eigenvalue zero. Due to the fact that* $\mathbf{e}_2 \in E^N$, *Lemma* 6.3 *implies that* $(\mathbf{e}_2, \mathbf{e}_2)$ *furnishes a generalized eigenvector with eigenvalue zero, so that the flock is always linearly unstable for the model* (6.1).

6.3.2 Numerical validations

In this section, we perform some numerical computations to show stability regions for the ock ring. Moreover, we will show the formation of clusters and the fattening instability. Due to Theorem 6.1 and Corollary 6.4, applied to the 4×4 matrix in (6.16), we are reduced to study the determinant of the matrix M for the ock stability. Note that for fixed values of N and m the determinant of M is a function of the parameters a and b and such that

$$D(a,b) := \det(M) = I_1(m)I_1(-m) - (I_2(m))^2,$$
$$T(a,b) := \mathrm{trace}(M) = I_1(m) + I_1(-m).$$

Remark 6.5. *Using the results of* [23, Theorem 3.1] *one is able to estimate the asymptotic value of the determinant of M. In our case, using* $W(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}$ *one obtains that*

$$\det(M) \sim Cm^{-b+1} \quad as \quad m \to \infty,$$

where $C \quad 0$ *and* $b \in (1,2) \cup (4,6) \cup (8,10) \cup \cdots$. *In these cases* $\det(M) \quad 0$ *and* $\mathrm{trace}(M) \quad 0$. *Moreover, this result shows that there is no spectral gap for large modes m since* $\det(M) \to 0$ *as* $m \to \infty$.

In Figure 6.2 we recover some results on the stability already shown in [81] . Since $I_1(m)$ and $I_2(m)$ depend on the powers a and b of the power-law potential, we plot in the parameter region $\{(a,b) : a \quad b \quad 0\}$ the stability and instability regions depending on the determinant of M and its trace. We show the cases $m = 3, 4, 5$ for a fixed $N = 100000$.

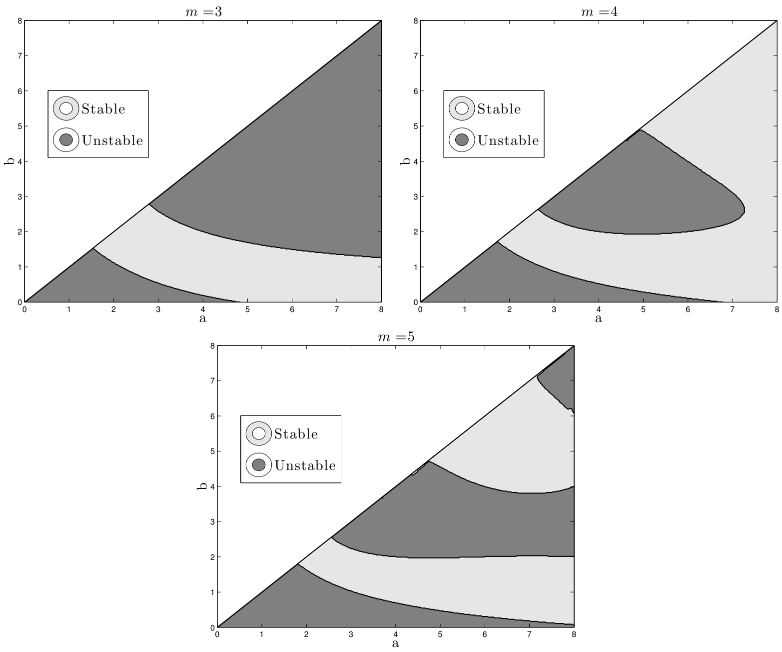


Figure 6.2: Stability regions for di erent perturbations modes $m = 3, 4, 5$. Stability regions correspond to a couple (a, b) of parameters where $D(a, b) > 0, \quad T(a, b) < 0$.

In Figure 6.3 we compute the stability area as a function of a and b. To do so, we compute the intersection of all stability areas for $m \quad 2$. It can be observed from our tests that the stability area shrinks when the number of particles increases. Moreover, it is observed that in the limit when $N \to \infty$, the lower boundary of the stability region converges to the dashed line. The red dashed line is the curve $b = \frac{a}{a-1}$ that corresponds to the $m = +\infty$ mode. This curve is the separatrix of the ins/stability regions for the continuous delta ring of the first order continuum model, studied in [81, 10].

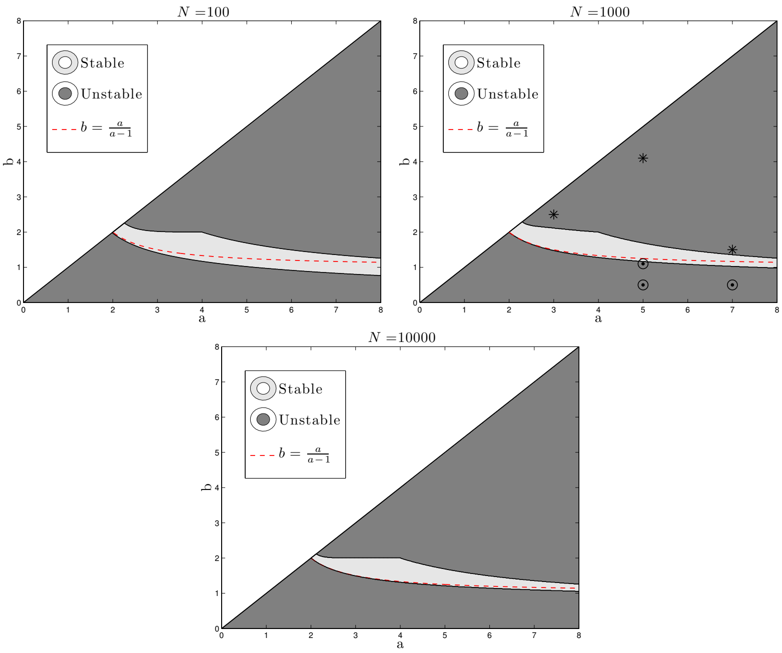


Figure 6.3: Stability areas for ock ring solutions for di erent values of N. From left to right: $N = 100, 1000, 10000$. Markers ($\ast$) and ($\odot$) indicate the explored parameters respectively in Table 6.1 and Table 6.2.

Cluster formation

The formation of clusters occurs when the repulsion strength is small. In other words, this phenomenon depends on how singular the potential is at the origin. We show the bifurcation diagram for the phase transition between equally distributed ock and ock with cluster formation. Figure 6.4 is performed using $N = 1000$ particles equally distributed on the stable circle with all the velocities aligned. We let them evolve until $T_f = 500$. We fix the parameters $|u_0| = 2.5$, $a = 5$ and vary b along the axis. The vertical axis represents the increment of the relative errors

$$\|\mu^N\|_{rel} = \frac{\|\mu^N - \mu_0^N\|_2}{\|\mu_0^N\|_2}$$

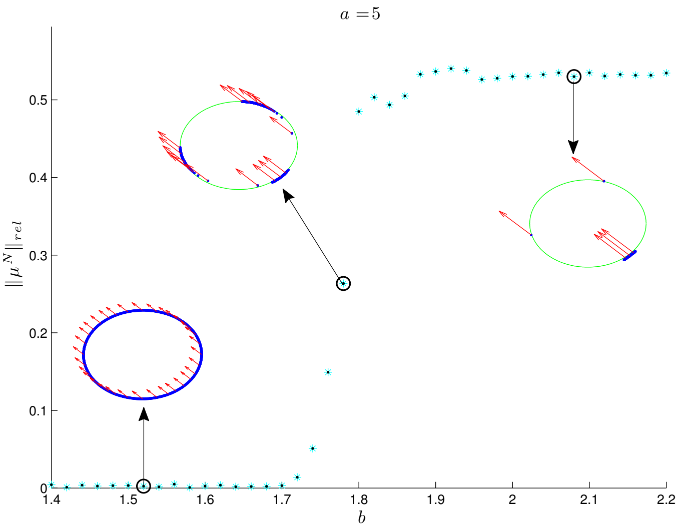


Figure 6.4: Bifurcation diagram for cluster formation at $T_f = 500$, with N=1000 particles, $a = 5$, $|u_0| = 2.5$.

with increasing b, where μ_0^N is the uniform distribution along the stable ring of N particles and μ^N the distribution at time T_f. Simulations are performed with *MATLAB* and the evolution of the system of odes is solved with the *ode45* routine with adaptive time step. Table 6.1 illustrates di erent possible final states for di erent choices of the parameters a, b.

$a = 3, b = 2.5$	$a = 5, b = 4.1$	$a = 7, b = 1.5$

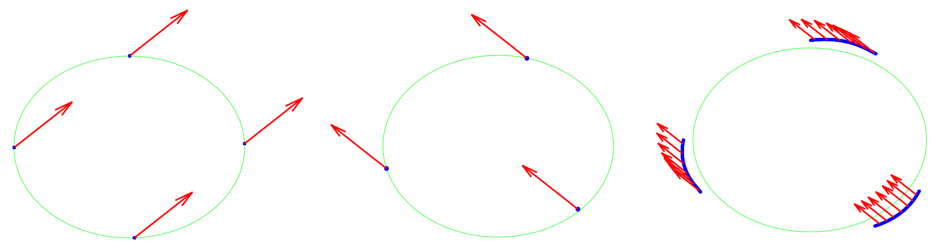


Table 6.1: Long time simulations with $N = 1000$ particles. The location of parameter values are marked as $(\ast)$ points in the central plot of Figure 6.3.

Fattening formation

We show the transition diagram between a ock on a ring and a ock on an annulus. In this case, the fattening phenomenon occurs when the parameters of the potential cross the lower boundary of the stability region. We numerically characterize this behavior in a similar way as in the previous subsection.

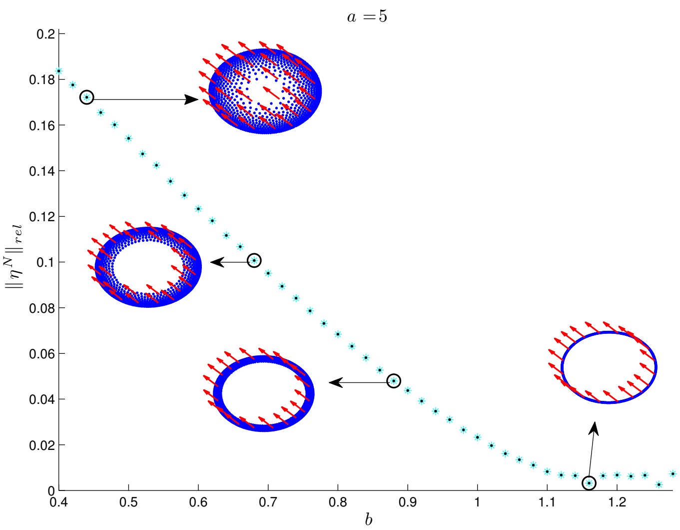


Figure 6.5: Bifurcation diagram for fattening instability at $T_f = 500$ with N=1000 particles, $a = 5$, $|u_0| = 2.5$.

Figure 6.5 is performed using $N = 1000$ particles equally distributed on the stable ring already in the steady state with all the velocities aligned. We then let them evolve until $T_f = 500$. We fix the parameters $|u_0| = 2.5$, $a = 5$ and vary b along the axis. The vertical axis represents the increment of the relative errors

$$\|\eta^N\|_{rel} = \frac{\|\eta^N - \eta_0^N\|_2}{\|\eta_0^N\|_2}$$

for increasing b, where η_0^N represents the average distance from the center of mass for N particles in a ock ring formation, i.e., $\eta_0^N = R$ and η^N is the average distance from the center of mass at time T_f. Table 6.2 shows the final states for some particular choices of parameters after stabilization.

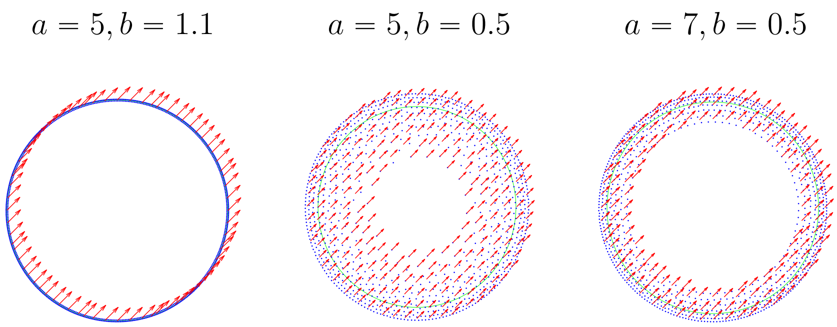


Table 6.2: Long time simulations with $N = 1000$ particles. The location of parameter values are marked as $(\odot)$ points in the central plot of Figure 6.3.

6.3.3 Stability of ock solutions with the Cucker-Smale alignment term

As in Section 6.3.1 we perform a study on the ock stability for model (6.3). If we use the same change of variables as in (6.11), then system (6.3) reads

$$\begin{cases} \dot{y}_j = v_j - u_0 = z_j \\ \dot{z}_j = \dfrac{1}{N}\displaystyle\sum_{l=1}^{N} H(|y_l - y_j|)(z_l - z_j) + \dfrac{1}{N}\displaystyle\sum_{\substack{l=1 \\ l\neq j}}^{N} \nabla W(y_l - y_j) \end{cases}, \quad j = 1, \ldots, N \tag{6.21}$$

We give a characterization of the ock solution in the complex plane for (6.21) with $(y_j^0, z_j^0) = (Re^{i\ _j}, 0)$, where $_j = \frac{2\ j}{N}$. We consider then the perturbed solution

$$y_j(t) = Re^{i\ _j}(1 + h_j(t)),$$

with h_j such that $|h_j| \ll 1$ and satisfying (6.12). Considering the following relations

$$\begin{aligned} y_l - y_j =& Re^{i\ _j}\left(e^{\ _p} h_l - h_j\right), \\ |y_l - y_j| \simeq& 2R\left|\sin\left(\frac{\ _p}{2}\right)\right| \\ &+ \frac{R}{4\left|\sin\left(\frac{\ _p}{2}\right)\right|}\left[\left(1 - e^{i\ _p}\right)\left(h_l - \overline{h_j}\right) + \left(1 - e^{-i\ _p}\right)\left(\overline{h_l} - h_j\right)\right], \\ z_l - z_j =& Re^{i\ _j}\left(e^{\ _p} h_l' - h_j'\right), \end{aligned}$$

where ${}_p = 2\ (l-j)/N = 2\ p/N$. We linearize the Cucker-Smale alignment term around the solution up to first order, leading to

$$
\begin{aligned}
H(|y_l - y_j|) \simeq & H(2R\,|\sin(\ {}_p/2)|) \\
& + H'(2R\,|\sin(\ {}_p/2)|)\frac{R}{4\left|\sin\left(\frac{\ {}_p}{2}\right)\right|} \\
& \times \left[\left(1-e^{i\ {}_p}\right)\left(h_l-\overline{h_j}\right)+\left(1-e^{-i\ {}_p}\right)\left(\overline{h_l}-h_j\right)\right].
\end{aligned}
$$

Substituting the linearization in (6.21) and neglecting the second order terms, we obtain the following characterization of h_j''

$$
\begin{aligned}
h_j'' = & \frac{1}{N}\sum_{l=1}^{N} H(2R|\sin\ {}_p|)\left[e^{i\ {}_p}h_l' - h_j'\right] \\
& + \frac{1}{N}\sum_{\substack{l=1\\ l\neq j}}^{N}\left[G_1(\ {}_p/2)(h_j - e^{i\ {}_p}h_l) + G_2(\ {}_p/2)(\overline{h_l} - e^{i\ {}_p}\overline{h_j})\right].
\end{aligned}
\tag{6.22}
$$

In order to study the behavior of the perturbations h_j, we reduce the complexity of the problem, assuming that h_j satisfies the following relation

$$
h_j = \ {}_+(t)e^{im\ {}_j} + \ {}_-(t)e^{-im\ {}_j}, \quad h_j' = \ {}'_+(t)e^{im\ {}_j} + \ {}'_-(t)e^{-im\ {}_j}, \qquad m \in \mathbb{N}.
$$

Therefore, we can express h_l in terms of h_j as

$$
h_l = \ {}_+(t)e^{im\ {}_j}e^{im\ {}_p} + \ {}_-(t)e^{-im\ {}_j}e^{-im\ {}_p}, \qquad m \in \mathbb{N}.
$$

Inserting the previous expressions in (6.22) and gathering terms in $e^{i\ {}_j m}$ and $e^{-i\ {}_j m}$, we can characterize ${}_+$ and ${}_-$ as

$$
\begin{aligned}
{}''_+ = & \frac{1}{N}\sum_{l=1}^{N} H(2R|\sin\ {}_p|)\left[e^{i\ {}_p(m+1)}-1\right]\ {}'_+ + I_1(m)\ {}_+ + I_2(m)\overline{\ }_-, \\
\overline{\ }''_- = & \frac{1}{N}\sum_{l=1}^{N} H(2R|\sin\ {}_p|)\left[e^{i\ {}_p(m-1)}-1\right]\overline{\ }_- + I_2(m)\ {}_+ + I_1(-m)\overline{\ }_-,
\end{aligned}
$$

where I_1 and I_2 are defined in (6.14) and (6.15). Through a simple manipulation of the sum for the linearized Cucker-Smale term, we obtain that the expression

$$
\begin{aligned}
& \frac{1}{N}\sum_{l\neq j} H(2R|\sin\ {}_p|)\left[e^{i\ {}_p(m\pm1)}-1\right] = \\
& \frac{1}{N}\sum_{l\neq j} H(2R|\sin\ {}_p|)\left[\cos(\ {}_p(m\pm1))-1\right] + \frac{i}{N}\sum_{l\neq j} H(2R|\sin\ {}_p|)\sin(\ {}_p(m\pm1)),
\end{aligned}
$$

is real. Actually, $H(2R|\sin\theta_p|)$ and $\sin(\theta_p(m\pm1))$ are respectively symmetric and antisymmetric with respect to the values of θ_p, so the imaginary part vanishes. Recalling the definition of θ_p, we conclude

$$\begin{aligned} J_\pm(m) =& \frac{1}{N}\sum_{k=1}^{N} H\left(2R\left|\sin\left(\frac{2\pi p}{N}\right)\right|\right)\left[\cos\left(\frac{2\pi p}{N}(m\pm1)\right)-1\right] \\ =& -\frac{4}{N}\sum_{k=1}^{N/2} H\left(2R\left|\sin\left(\frac{2\pi p}{N}\right)\right|\right)\left[\sin^2\left(\frac{\pi p}{N}(m\pm1)\right)\right]. \end{aligned}$$

Therefore, we reduce the stability analysis to the following system

$$\begin{pmatrix} \xi_+'' \\ \bar{\xi}_-'' \end{pmatrix} = \underbrace{\begin{pmatrix} I_1(m) & I_2(m) \\ I_2(m) & I_1(-m) \end{pmatrix}}_{M} \begin{pmatrix} \xi_+ \\ \bar{\xi}_- \end{pmatrix} + \underbrace{\begin{pmatrix} J_+(m) & 0 \\ 0 & J_-(m) \end{pmatrix}}_{J} \begin{pmatrix} \xi_+' \\ \xi_-' \end{pmatrix}.$$

Taking the conjugate in the second equation and relabeling $\bar{\xi}_-$ with ξ_- as in [23], the previous system is equivalent to

$$\frac{d}{dt}\begin{pmatrix} \xi_+ \\ \xi_- \\ \eta_+ \\ \eta_- \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ I_1(m) & I_2(m) & J_+(m) & 0 \\ I_2(m) & I_1(-m) & 0 & J_-(m) \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \\ \eta_+ \\ \eta_- \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id} \\ M & J \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \\ \eta_+ \\ \eta_- \end{pmatrix}, \tag{6.23}$$

where $\eta_\pm = \xi_\pm'$.

At this point, we will do a stability analysis based on the eigenvalues of the matrix of the previous system in a similar way as in Section 6.3.1. If instead of the self-propelled/ friction term we use a Cucker-Smale type alignment term

$$-\sum_{l=1}^{N} g\left(|x_j - x_l|\right)(v_j - v_l),$$

where $g(r)$ denotes any strictly positive function, the corresponding stability matrix L_{CS} for the flock reads

$$L_{\mathrm{CS}} = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathbf{M} & -G \end{pmatrix}.$$

As before, $\mathbf{M}$ denotes stability matrix of the first order model. As the alignment term is linear in the velocity, the matrix G acts on $\mathbf{v} = (v_1, \ldots, v_N)^T$, $v_j \in \mathbb{R}^2$, according to the relation

$$(G\mathbf{v})_j = \sum_{l=1}^{N} g\left(|x_j - x_l|\right)(v_j - v_l).$$

In particular, if we denote $||v_j - v_l||_2^2 := (v_j - v_l)^*(v_j - v_l)$ then this relation implies that

$$\mathbf{v}^* G \mathbf{v} = \frac{1}{2} \sum_{j,l=1}^{N} g\left(|x_j - x_l|\right) ||v_j - v_l||_2^2.$$

Consequently, G is positive semi-definite and $G\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{v}$ is “constant” in the sense that $v_j \equiv w$ for some fixed $w \in \mathbb{R}^2$. In other words, $\ker(G) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$. By translation invariance of the first order model, both $\mathbf{e}_1 \in \ker(\mathbf{M})$ and $\mathbf{e}_2 \in \ker(\mathbf{M})$ as well.

Note that the eigenvalue problem for L_{CS} is again equivalent to the following quadratic eigenvalue problem for $\mathbf{x} \in \mathbb{C}^{2N}$: $\lambda^2\mathbf{x} + \lambda G\mathbf{x} - \mathbf{M}\mathbf{x} = \mathbf{0}$. Assuming the normalization $\mathbf{x}^*\mathbf{x} = 1$, the quadratic formula then implies that

$$\lambda = \frac{-\mathbf{x}^* G\mathbf{x} \pm \sqrt{(\mathbf{x}^* G\mathbf{x})^2 + 4\mathbf{x}^*\mathbf{M}\mathbf{x}}}{2}.$$

From this relation and the fact that $\ker(G) \subset \ker(\mathbf{M})$ we conclude that $\ker(L_{\mathrm{CS}}) = \ker(\mathbf{M})$, and moreover that $\lambda(\mathbf{x}) = 0 \Leftrightarrow \mathbf{x}^*\mathbf{M}\mathbf{x} = 0$. Furthermore, $\mathbf{e}_1$ and $\mathbf{e}_2$ generate a single generalized eigenvector whereas each remaining $\mathbf{x} \in \ker(\mathbf{M})$ generates no generalized eigenvectors. Indeed, corresponding to each $\mathbf{x} \in \ker(\mathbf{M})$ the system of equations

$$\begin{pmatrix} 0 & \mathrm{Id} \\ \mathbf{M} & -G \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}.$$

has a solution if and only if $G\mathbf{x} = \mathbf{0}$ and $\mathbf{u} \in \ker(\mathbf{M})$. Additionally, if $\mathbf{x} = \mathbf{e}_i$ then for any $\mathbf{u} \in \ker(\mathbf{M})$ the system of equations

$$\begin{pmatrix} 0 & \mathrm{Id} \\ \mathbf{M} & -G \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{u} \\ \mathbf{x} \end{pmatrix}$$

has no solutions. This follows by multiplying the second equation by $\mathbf{x}^*$, then using the facts that $\mathbf{e}_i \in \ker(G) \subset \ker(\mathbf{M})$ and the facts that G and $\mathbf{M}$ are symmetric. In other words, if $g(r)$ is any strictly positive function then

$$\det(L_{\mathrm{CS}} - \lambda\,\mathrm{Id}) = \lambda^{2+\dim(\ker(\mathbf{M}))} p_g(\lambda),$$

for some polynomial $p_g(\lambda)$ that has non-zero roots. Since this equation holds for any strictly positive function $g(r)$, we may follow the proof of Theorem 6.1 to conclude that the second order model has an eigenvalue with positive real part if and only if the first order system has a positive eigenvalue. Moreover, the vectors $(\mathbf{e}_i, \mathbf{e}_i)$ for each $i = 1, 2$ furnish generalized eigenvectors with eigenvalue zero, so the ring flock is always linearly unstable. As a summary, we have shown:

Theorem 6.2. *The linearized second order system* (6.3) *around the flock ring solution has an eigenvalue with positive real part if and only if the linearized first order system around the ring solution has a positive eigenvalue. As a consequence, the flock ring solution is unstable for m-mode perturbations for the second order model* (6.3) *if and only if the ring solution is unstable for m-mode perturbations for the first order model* (6.2).

The linear stability analysis of the previous system leads to the characterization of the stability areas in Figure 6.6. We show that the stability parameter regions for di erent values of N, and $\ \ = 1$ coincide with the ones in Figure 6.3.

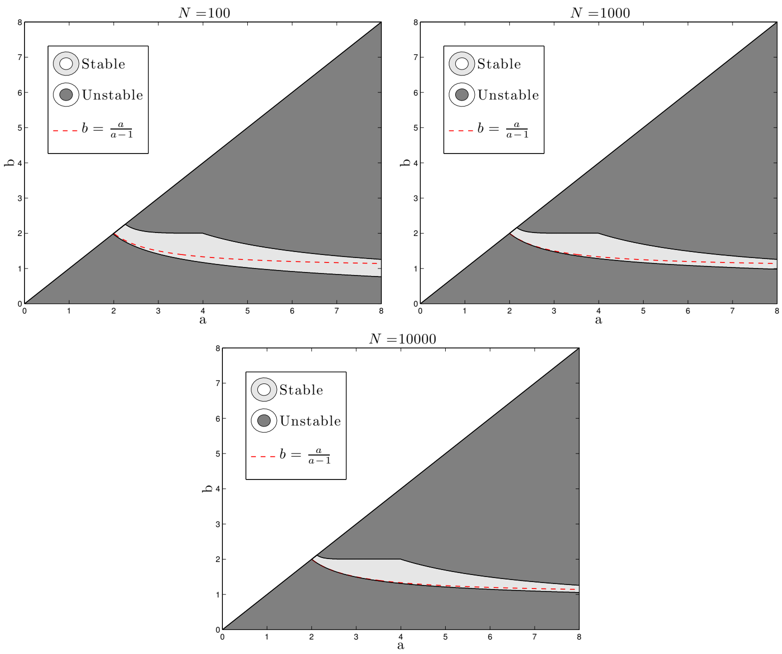


Figure 6.6: Stability regions for ock ring solutions with the Cucker-Smale term for di erent values of N.

We also investigate the behavior of the eigenvalue with the largest real part, $(\ _1)$, of the linearized system (6.23) against the increasing value of communication strength . In Figure 6.7, as the potential gets more repulsive at the origin, we see the change from stability to instability, and the rate of convergence to equilibrium depending on .

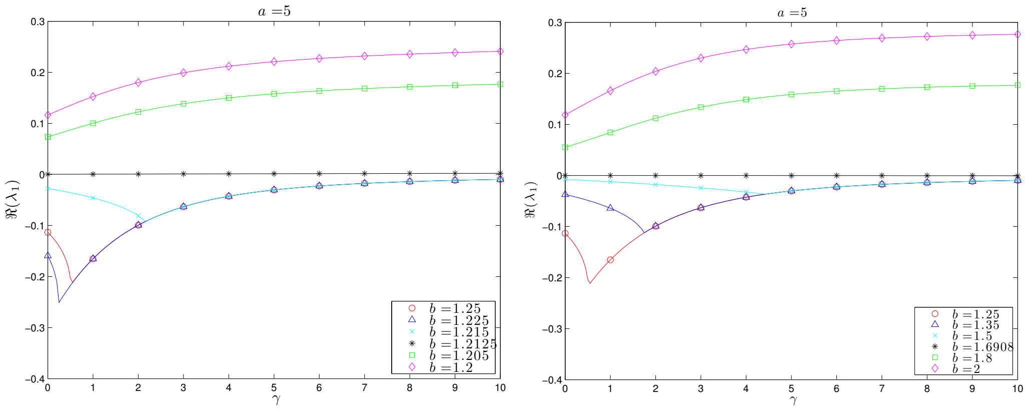


Figure 6.7: The magnitude of (λ_1) is in uenced by γ, for di erent values of b and xed $a = 5$, $N = 10000$.

6.4 Stability for mill solutions

This section is devoted to complement the results in [23] by analyzing the stability of mill ring solutions with repulsion.

6.4.1 Linear stability analysis

Let us consider the transformation

$$\begin{cases} y_j(t) = O(t)x_j(t) \\ z_j(t) = O(t)v_j(t) \end{cases}, \quad j = 1, \dots, N$$

where $O(t)$ is the rotation matrix defined as

$$O(t) = e^{St}, \quad S = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \quad \text{and} \quad \dot{O}(t) = Se^{St}.$$

Evaluating $\dot{y}_j(t)$ and $\dot{z}_j(t)$ and after some straightforward computations, we get

$$\begin{cases} \dot{y}_j(t) = Sy_j(t) + z_j(t) \\ \dot{z}_j(t) = Sz_j(t) + (\quad - \quad |z_j|^2)z_j(t) - \dfrac{1}{N} \displaystyle\sum_{\substack{l=1 \\ l \neq j}}^{N} \nabla W(y_l - y_j) \end{cases}, \quad j = 1, \dots, N.$$

A linear stability analysis for mill rings was performed in [23]. Actually, for a fixed number of particles, we have a mill ring solution given by $(y_j^0, z_j^0) =$

$(Re^{i\ j}, 0)$, where $_j = \frac{2\ j}{N}$, and R determined by equation (6.7). With the same notation as in Section 6.3, the analysis in [23] leads to the linear system

$$\begin{pmatrix} \xi'_+ \\ \xi'_- \\ \eta'_+ \\ \eta'_- \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega i\alpha + \omega^2 + I_1(m) & -\omega i\alpha + I_2(m) & -\alpha - 2\omega i & \alpha \\ \omega i\alpha + I_2(m) & \omega i\alpha + \omega^2 + I_1(-m) & \alpha & -\alpha + 2\omega i \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \\ \eta_+ \\ \eta_- \end{pmatrix}. \tag{6.24}$$

Let us remind that the perturbations are of the form $y_j(t) = Re^{i\ j}(1 + h_j(t))$, with $h_j = \ _+(t)e^{im\ j} + \ _-(t)e^{-im\ j}$, $m = 2, 3, \ldots$, such that $|h_j| \ll 1$ and satisfying (6.12), with $(\eta_+, \eta_-) = ('_+, '_-)$. We will make use of (6.24) to study the stability of mill rings with repulsion.

6.4.2 Numerical tests

Unlike the case of ock solutions where the asymptotic speed does not play role in the linear stability, we will show that the asymptotic speed $|u_0|$ can be used as a bifurcation parameter for mills.

In Table 6.5 we numerically investigate the behavior of the stability region for a fixed number of particles, $N = 1000$, and for increasing values of the asymptotic speed $|u_0|$. We observe that the stability region shrinks with respect to a and gets larger with respect to b. Each stability region in Table 6.5 is computed out of the intersection of the stable areas for the system (6.24) for each perturbation mode $m\ \ 2$. Note that for $|u_0| = 0$ the stability region coincides with the one for the the first order model (6.2) and for the ock ring solution.

A similar analysis, as done in Subsection 6.3.2, can be performed to study the formation of fat mills and clustered mill solutions. We show how both the fattening and the clustering instability are triggered by tunning the asymptotic speed for a choice of the interaction potential (a and b).

In the case of ock ring solutions we observe cluster solutions or annulus solutions when parameters a and b are chosen respectively below" or above" the stability region. In the case of mill solutions, a similar behavior is observed, but this will depend also on the chosen value of $|u_0|$. As an example, we fix $(a, b) = (5, 0.5)$, marked as ($\odot$) in Table 6.5, and we observe the behavioral change of the system for increasing values of the asymptotic speed.

Table 6.3 exhibits this switching behavior from a fat mill to a cluster pattern along with the increment of the asymptotic speed. We observe that for small values of the asymptotic speed fat mill solutions are stable patterns, but when increasing the value of $|u_0|$ the stable solutions form a clustered mill. The first row shows the evolution of the system with asymptotic speed $|u_0| = 0.25$ towards an annulus mill. In the second row we take $|u_0| = 0.5$ the previous stable solution is reshaped to a fat clusters pattern. The speed in the third row is switched to $|u_0| = 5$ and clusters on lines emerge as a stable configuration. Increasing the speed to $|u_0| = 50$ in the fourth row we can observe that clusters on points are stable solutions.

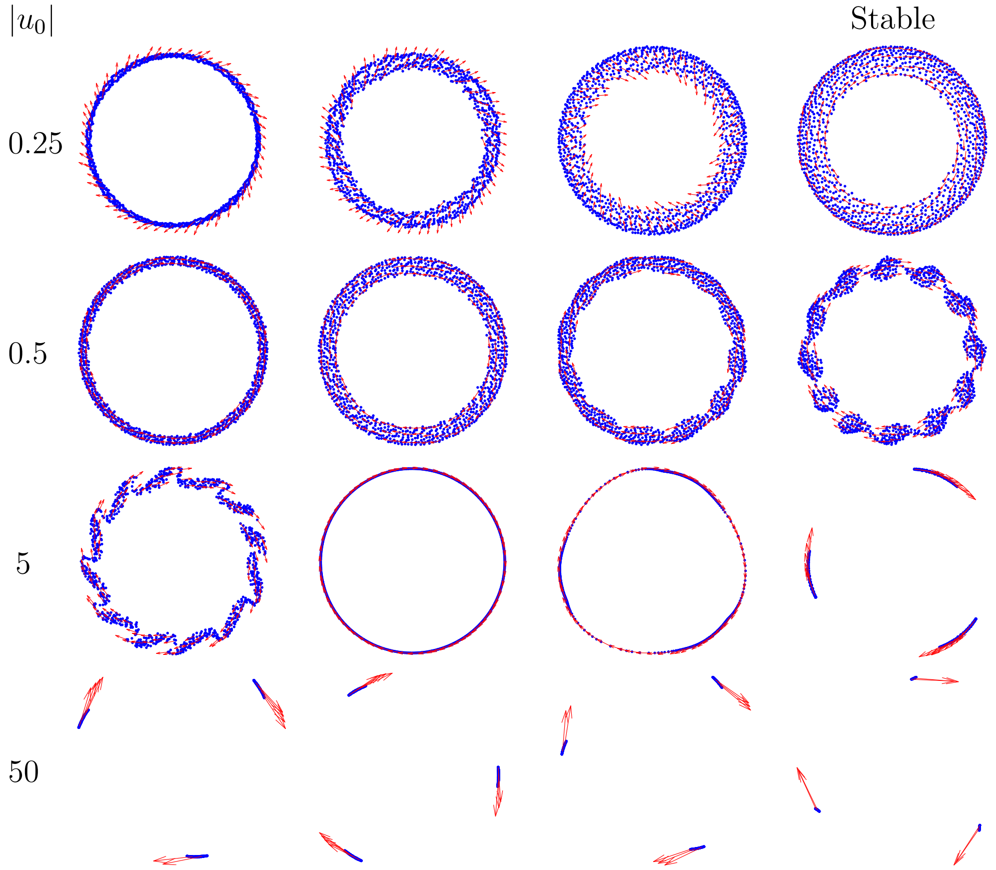


Table 6.3: $N = 1000$ particles, $a = 5, b = 0.5$. The table shows the evolution of a mill ring for increasing values of the speed $|u_0|$. Each row depicts the behavior of the system for a xed speed, until a stable state is reached. The evolution of the second, third and fourth row is computed starting from the stable pattern of the previous line.

In Figure 6.8 we numerically show how the stability region looks like in terms of $(|u_0|, b)$, with a fixed at 0.5, and we enlighten with marker points ($\odot$) the parameter choices of Table 6.3, first and second lines.

For the sake of completeness, we enrich the analysis fixing $|u_0| = 0.5$ and considering di erent values of b, in order to cross the stability region.

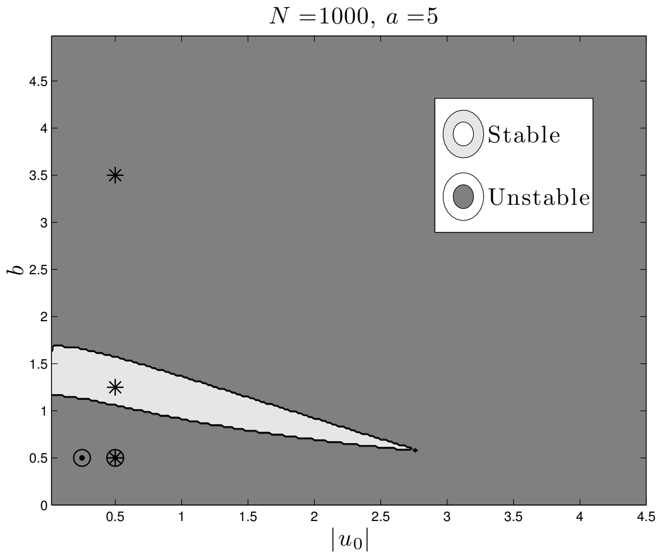


Figure 6.8: Stability region for mill ring solution with parameter $a = 5$, $N = 1000$. Markers ($\odot$) and ($\circledast$) depict the parameter choices respectively for Table 6.3 and Table 6.4.

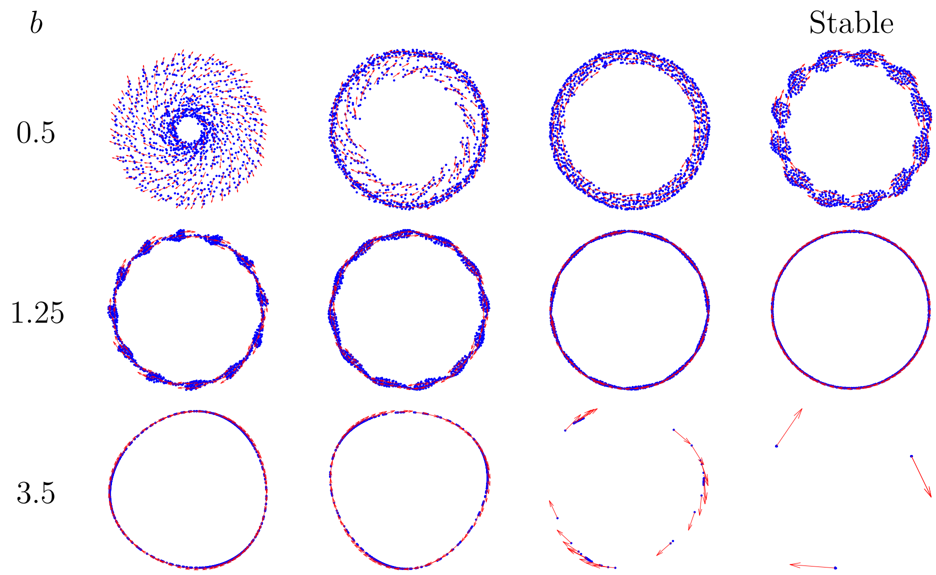


Table 6.4: $N = 1000$ particles, $a = 5, |u_0| = 0.5$. The table shows the evolution of a mill ring for increasing values of b, i.e. decreasing repulsion. The evolution of the second and the third row is computed starting from the stable pattern of the previous line.

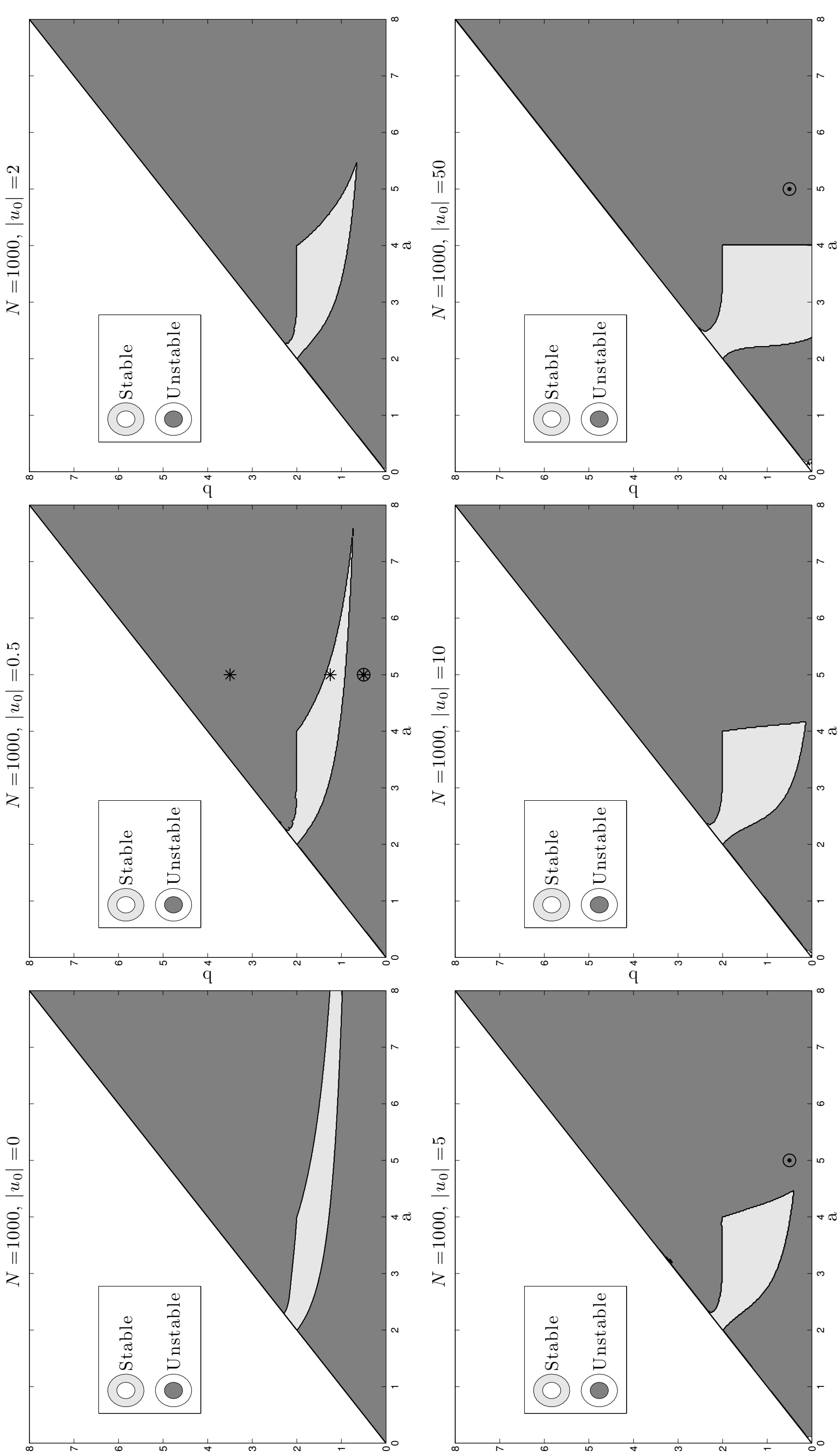


Table 6.5: Stability region for $N = 1000$ and di erent values of the asymptotic speed $|u_0|$. Markers ($\odot$) and ($\ast$) correspond to the explored parameters in Table 6.3 and Table 6.4.

Therefore in Table 6.4 we show the evolution of a mill ring solution with b taken subsequently equal to $0.5, 1.25, 3.5$, parameter choices are marked as $(\circledast)$ in Figure 6.8. The first line of Table 6.4 shows the convergence to the same stable state as the one in second line of Table 6.3, but, since the system starts to evolve directly from a ring mill solution, the transient behavior is di erent. Parameters in second line belong to the stability region, see Figure 6.8. Therefore, the stable state becomes a mill ring solution. Finally, in the third line we increase b and a three point cluster solution is observed as stable pattern.

Mill to Flock and Flock to Mill behavior

We numerically investigate the stability of mill and ock ring solutions for small values of the asymptotic speed, $|u_0|$, and the parameter b, which corresponds to a strong repulsion condition.

We perform two representative simulations showing that for a particular choice of the parameters, mill ring solutions can switch to fat ock solutions and conversely ock mill solutions switch to fat mill patterns.

In Table 6.6 we take $N = 100$ particles and we fix $a = 4, b = 0.0005$ and $|u_0| = 0.01$. The frames in the first row show the instability of mill ring solutions for this choice of parameters. The system initially evolves to an almost chaotic state, then particles start to organize rotating around the center of mass. This rotation actually causes the alignment of the agents and the final fat ock configuration described in the second row.

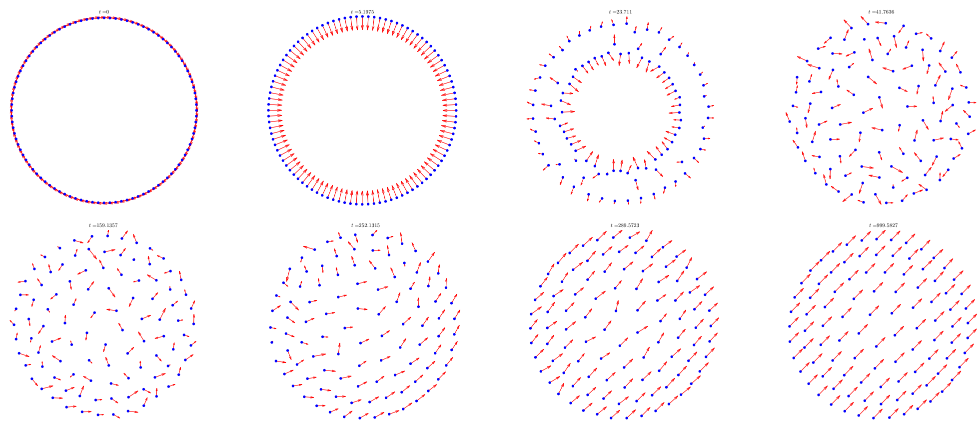


Table 6.6: System with $N = 100$ agents, parameters are xed $a = 4$ and $b = 0.0005$ and $|u_0| = 0.01$. The rst row shows that the initial mill ring con guration is unstable. The second row outlines the self organization of the system in a fat ock con guration.

In Table 6.7 we consider as initial state a ock ring solution. The parameters of the model are $N = 100$, $a = 4$, $b = 0.001$ and $|u_0| = 0.1$. The first row of the table illustrates that the initial configuration is not a stable solution. Therefore, the symmetry of the ock ring is broken and the system exhibits a chaotic behavior. In the second row a rotating dynamic emerges out of the disordered state and finally the system stabilizes to a fat mill solution. These

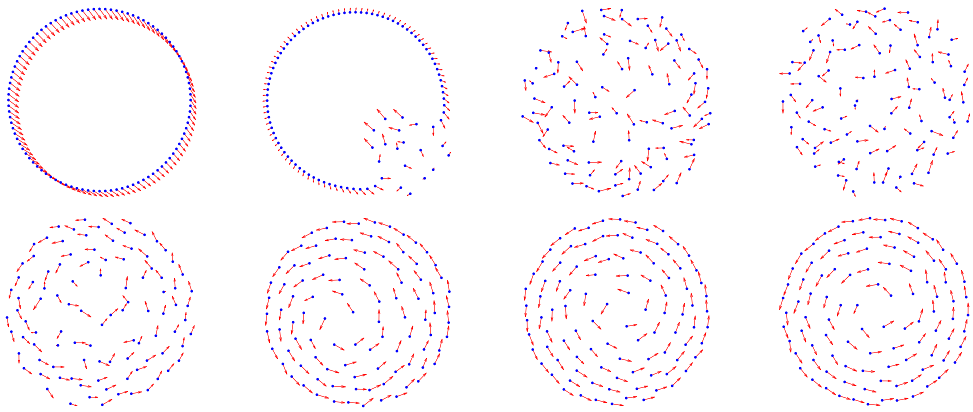


Table 6.7: System with $N = 100$ agents, parameters $a = 4$ and $b = 0.001$ and $|u_0| = 0.1$. The rst row shows the instability of the ock ring solution while the second exhibits the convergence to a fat mill type solution.

numerical tests show surprisingly that it is possible, with a particular choice of the parameters, to obtain mill configurations out of perturbations of initial ock solutions and ock solutions out of perturbations of mill ring solutions. These heteroclinic-kind solutions have not been previously reported. We also remark that the parameter choice is connected to the number of agents we are considering; changing N means finding another set of parameters for which the same switching behavior occurs.

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